Note: It is very important that you solve the problems first and check the solutions afterwards.

**Problem 1**

Examples 4.8.2, 4.8.5, 4.9.1, 4.9.2, 4.11.4, 4.11.5.

**Problem 2**

Random variables $X$ and $Y$ have a uniform joint density on the square bounded by the following four corners: $(1, 0), (0, 1), (-1, 0), \text{ and } (0, -1)$.

a) Calculate the marginal pdfs of $X$ and $Y$. Are $X$ and $Y$ independent? Are they uncorrelated?

b) Let $Z = X + Y$ and $S = X - Y$. Are $Z$ and $S$ uncorrelated or independent or neither of the two?

c) Compute $E[X]$ and $\text{Var}[X]$.

**Solution**

a) Let $R$ be the square with corners \{(1, 0), (0, 1), (1, 0), (0, 1)\}. The area $|R|$ of $R$ is $\sqrt{2}^2 = 2$. So

$$f_{X,Y} (x, y) = \begin{cases} 
\frac{1}{2}, & (x, y) \in R \\
0, & \text{else.}
\end{cases}$$

The marginal density of $X$ is given by

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x,y) dy = \begin{cases} 
\int_{y=x-1}^{1-x} \frac{1}{2} dy, & 0 \leq x \leq 1; \\
\int_{y=x+1}^{1+x} \frac{1}{2} dy, & -1 \leq x \leq 0; \\
0, & \text{else.}
\end{cases}$$

$$= \begin{cases} 
(1-x), & 0 \leq x \leq 1; \\
(1+x), & -1 \leq x \leq 0 \\
0, & \text{else.}
\end{cases}$$

$$= \begin{cases} 
1 - |x|, & -1 \leq x \leq 1 \\
0, & \text{else.}
\end{cases}$$

By symmetry, the marginal density of $Y$ is given by

$$f_Y(y) = \begin{cases} 
1 - |y|, & -1 \leq y \leq 1 \\
0, & \text{else.}
\end{cases}$$

Independence: The support is not a product set, thus $X$ and $Y$ are not independent. Alternatively

$$f_X(x)f_Y(y) = (1 - |x|)(1 - |y|) \neq \frac{1}{2} = f_{X,Y}(x,y)$$
Uncorrelated: Since \( f_X \) (and \( f_Y \)) is symmetric about \( x = 0 \) (and \( y = 0 \)), \( E[X] = E[Y] = 0 \). The covariance of \( X \) and \( Y \) is:

\[
\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 0 - 0
\]

Thus, \( X \) and \( Y \) are uncorrelated. Alternatively, we can compute

\[
E[XY] = \int_{-1}^{1} \int_{-1}^{1} xy f_{X,Y}(x, y) \, dy \, dx
\]

\[
= \int_{-1}^{1} \left( \int_{y=|x|-1}^{1-|x|} \frac{xy}{2} \, dy \right) \, dx
\]

\[
= \int_{-1}^{1} \frac{xy^2}{4} \left( \frac{y=1-|x|}{y=|x|-1} \right) \, dx
\]

\[
= \int_{-1}^{1} \frac{x}{4} \left( (|x| - 1)^2 - (1 - |x|)^2 \right) \, dx = 0
\]

And,

\[
E[X] = \int_{-1}^{1} xf_X(x) \, dx
\]

\[
= \int_{x=1}^{0} x(1 + x) \, dx + \int_{x=0}^{1} x(1 - x) \, dx
\]

\[
= -\left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) = 0
\]

Also, by symmetry \( E[Y] = 0 \). Hence, \( E[XY] = E[X]E[Y] = 0 \). So, \( X \) and \( Y \) are uncorrelated.

b) The transformation between \((Z, S)\) and \((X, Y)\) is given by

\[
\begin{pmatrix}
Z = X + Y \\
S = X - Y
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
X = \frac{Z + S}{2} \\
Y = \frac{Z - S}{2}
\end{pmatrix}
\]

The transformed region \( T \) in the space of \( Z \) and \( S \) corresponding to region \( R \) in the space of \( X \) and \( Y \) can be represented as

\[
T = \{ (z, s) : -1 \leq z \leq 1 \text{ and } -1 \leq s \leq 1 \}
\]

The joint density of \( Z \) and \( S \) is given by

\[
f_{Z,S}(z, s) = f_{X,Y} \left( \frac{z + s}{2}, \frac{z - s}{2} \right) \left| \begin{array}{cc}
\frac{dz}{dx} & \frac{dz}{ds} \\
\frac{dx}{dz} & \frac{dx}{ds}
\end{array} \right|
\]

\[
= \frac{1}{2} \begin{vmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{vmatrix}, \quad (z, s) \in T
\]

\[
= \frac{1}{4}, \quad (z, s) \in T
\]

Hence, \( Z \) and \( S \) are uniformly jointly distributed over the square given by \( \{ (z, s) : -1 \leq z \leq 1 \text{ and } -1 \leq s \leq 1 \} \) of area 4.
Marginal pdf of $Z$ and $S$:

$$f_Z(z) = \int_{s=-1}^{1} f_{Z,S}(z,s) ds$$
$$= \int_{s=-1}^{1} \frac{1}{4} ds \quad -1 \leq z \leq 1$$
$$= \frac{1}{2} \quad -1 \leq z \leq 1$$

By symmetry, $f_S(s) = \frac{1}{2}, -1 \leq s \leq 1$.

Independence: $Z$ and $S$ are independent because,

$$f_Z(z)f_S(s) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = f_{Z,S}(z,s)$$

Uncorrelated: Independence implies uncorrelated. Hence, $Z$ and $S$ are uncorrelated.

c) From part (a),

$$E[X] = 0$$

Variance of $X$ can be computed as

$$Var(X) = E[X^2] - (E[X])^2$$
$$= \int_{x=-1}^{1} x^2(1-|x|)dx - 0$$
$$= 2 \int_{x=0}^{1} x^2(1-x)dx$$
$$= 2 \left( \frac{1}{3} - \frac{1}{4} \right) = \frac{1}{6}$$

Problem 3

Suppose $X$ and $Y$ are jointly Gaussian random variables with $E[X] = 2$, $E[Y] = 4$, $\text{Var}(X) = 9$, $\text{Var}(Y) = 25$, and $\rho = 0.2$. Let $W = X + 2Y + 3$.

a) Find $E[W]$ and $\text{Var}(W)$.

b) Find the correlation and covariance of $X$ and $W$.

Solution

a) 

$$= 13$$
From the problem description, we know the following:

\[
egin{align*}
E [X^2] &= \text{Var}(X) + E[X]^2 = 13 \\
E [Y^2] &= \text{Var}(Y) + E[Y]^2 = 41 \\
E[XY] &= 0.2 \sqrt{\text{Var}(X) \text{Var}(Y)} + E[X]E[Y] = 11 \\
\text{Var}(W) &= E[W^2] - E[W]^2 \\
&= 121
\end{align*}
\]

b)

\[
\begin{align*}
corr(X,W) &= E[XW] \\
&= E[X^2 + 2XY + 3X] \\
&= 41 \\
Cov(X,W) &= corr(X,W) - E[X]E[W] \\
&= corr(X,W) - 26 \\
&= 15
\end{align*}
\]

**Problem 4**

If you drop a raw egg onto a concrete floor, what is the probability that you crack it?

**Solution**

Virtually zero; a concrete floor is very hard to crack.

**Problem 5**

This problem is concerned with minimum mean square error estimators.

a) Find the constant minimum mean square error estimator of the random variable \(3X\), where \(X\) has mean \(E[X] = 3\) and \(\text{Var}(X) = 4\).

b) Find the linear minimum mean square error estimator of the random variable \(2X\), where \(X\) has mean \(E[X] = 3\) and \(\text{Var}(X) = 4\), given an independent random variable \(Y\) with mean 2.

**Solution**

a) 
\[
\delta = E[3X] = 3E[X] = 9.
\]

b) 
\[
L^*(Y) = \mu_{2X} + \sigma_{2X} \rho_{Y,2X} \frac{Y - \mu_Y}{\sigma_Y} \\
= \mu_{2X} \\
= 6.
\]
Problem 6

Let $X \sim N(0,a^2)$ and $Y \sim N(0,b^2)$ and suppose $X,Y$ are jointly Gaussian with correlation coefficient $\rho$. Define $Z = X + Y$. Is the pair $(Z,X)$ jointly Gaussian? Find $\text{Var}(Z)$ and $\text{Cov}(Z,X)$.

Solution

Since

$$
\begin{pmatrix} X \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},
$$

$(X, Z)$ are jointly Gaussian.

$$
\text{Cov}(X,Z) = \text{Cov}(X,X+Y) = \text{Cov}(X,X) + \text{Cov}(X,Y) = a^2 + \rho ab,
$$

$$
\text{Var}(Z) = \text{Cov}(X+Y,X+Y) = \text{Var}(X) + 2\text{Cov}(X,Y) + \text{Var}(Y) = a^2 + b^2 + 2\rho ab.
$$