Problem Set 15

Note: It is very important that you solve the problems first and check the solutions afterwards.

Problem 1

Example 4.6.4 and Example 4.7.6.

Problem 2

Assume that \(X\) and \(Y\) are independent exponential random variables, both with parameter \(\lambda\) and support \((0, \infty)\). Find the joint pdf of the random variables \(W = (X - Y)\) and \(Z = \ln (X + Y)\).

Solution

The support for \((X, Y)\) is \((0, \infty)^2\) and the support for \((W, Z)\) is \((0, \infty)^2\). Over the support of \((X, Y)\), the mapping is one-to-one, so

\[
f_{W,Z}(w, z) = \frac{f_{X,Y}(x, y)}{|\det J|}
\]

where \((w, z) = g(x, y) = (g_1(x, y), g_2(x, y)) = (x - y, \ln (x + y))\) and the \(J\) is the Jacobian of \(g\).

\[
J = \begin{bmatrix}
\frac{1}{x+y} & -1 \\
\frac{1}{x+y} & 1 \\
\end{bmatrix} \Rightarrow |\det J| = \frac{2}{x+y}.
\]

We also need to find \(x, y\) in terms of \(w, z\).

\[
\begin{cases}
w = x - y \\
z = \ln (x + y)
\end{cases} \Rightarrow \begin{cases}
w = x - y \\
e^z = x + y \\
y = (e^z - w) / 2
\end{cases}
\]

So for \(w, z > 0\),

\[
f_{W,Z}(w, z) = \frac{f_{X,Y}(x, y)}{|\det J|} = \frac{\lambda^2 e^{-\lambda(x+y)}}{2} = \frac{1}{2} \lambda^2 e^{\lambda} e^{-\lambda e^z}
\]

and \(f_{W,Z}(w, z)\) equals 0 elsewhere.

Problem 3

Suppose \(X\) and \(Y\) are jointly continuous with joint pdf

\[
f_{X,Y}(u, v) = \begin{cases}
u e^{-(1+u)v}, & u, v \geq 0 \\
0, & \text{else.}
\end{cases}
\]

a) Find the marginal pdfs, \(f_X\) and \(f_Y\).

b) Find the conditional pdfs, \(f_{Y|X}\) and \(f_{X|Y}\): Be sure to indicate where these functions are well defined, and where they are zero, as well as giving the nonzero values.
c) $E[X|Y]$ is defined as follows. First define the function $g(v) = E[X|Y = v]$. Then define $E[X|Y]$ as $g(Y)$. Note that $E[X|Y]$ is a random variable.
Find $E[X|Y]$ and $E[Y|X]$.

d) Find the joint CDF, $F_{X,Y}(u,v)$

e) Are $X$ and $Y$ independent? Justify your answer.

Solution

a) For $u \geq 0$,

$$f_X(u) = \int_0^\infty v \exp\left(-(1+u)v\right) dv$$

$$= \frac{1}{(1+u)^2} \int_0^\infty t e^{-t} dt \quad \text{(we have let } t = (u+1)v \text{)}$$

$$= \frac{1}{(1+u)^2} \left[ t (t+1) \right]_0^\infty = \frac{1}{(1+u)^2}$$

and $f_X(u) = 0$ otherwise.

For $v \geq 0$,

$$f_Y(v) = \int_{u=0}^\infty v \exp\left(-(1+u)v\right) du$$

$$= \left[ - \exp\left(-(1+u)v\right) \right]_{u=0}^\infty$$

$$= \exp(-v)$$

$f_Y(v) = 0$ otherwise.

b) If $u, v \geq 0$,

$$f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)}$$

$$= (1+u)^2 v \exp\left(-(1+u)v\right)$$

If $u \geq 0, v < 0$, then $f_{Y|X}(v|u) = 0$. Otherwise $f_{Y|X}(v|u)$ is undefined.

If $u \geq 0, v \geq 0$

$$f_{X|Y}(u|v) = \frac{f_{X,Y}(u,v)}{f_Y(v)}$$

$$= v \exp(v) \exp\left(-(1+u)v\right)$$

$$= v \exp(-uv)$$

If $u < 0, v \geq 0$, then $f_{X|Y}(u|v) = 0$. Otherwise $f_{X|Y}(u|v)$ is undefined.

c) We find $E[X|Y = v]$. For $v > 0$,

$$E[X|Y = v] = \int_0^\infty u f_{X|Y}(u|v) du$$

$$= \int_{u=0}^\infty uv \exp(-uv) du$$

$$= \frac{1}{v}$$

For $u \geq 0$,

$$E [Y|X = u] = \int_0^\infty v f_{Y|X}(v|u) dv$$

$$= \int_0^\infty v^2(1 + u)^2 \exp(-(1 + u)v) dv$$

$$= \frac{2}{1 + u} \quad u \geq 0$$

Otherwise, $E [Y|X = u]$ is undefined. So $E [Y|X] = 2/(1 + X)$.

d) For $u, v \geq 0$,

$$F_{X,Y}(x, y) = \int_{v=0}^y \int_{u=0}^x v \exp(-(1 + u)v) dudv$$

$$= \int_{v=0}^y [-\exp(-(1 + u)v)]_{u=0}^x dv$$

$$= \int_{v=0}^y 1 - \exp(-(1 + x)v) dv$$

$$= \left[v + \frac{1}{1 + x} \exp(-(1 + x)v)\right]_{v=0}^y$$

$$= y + \frac{1}{1 + x} (\exp(-(1 + x)y) - 1)$$

Otherwise, $F_{X,Y}(x, y) = 0$.

e) Since $f_{X,Y} \neq f_X f_Y$, $X$ and $Y$ cannot be independent. The joint distribution cannot be factored implies the same result.