Problem Set 13

Note: It is very important that you solve the problems first and check the solutions afterwards.

Problem 1

All text examples in section 4.3.

Problem 2

Consider the following function

\[ F(u, v) = \begin{cases} 
0, & u + v \leq 1 \\
1, & u + v > 1.
\end{cases} \]

Is this a valid joint CDF. Why or why not? Prove your answer and show your work.

Solution

Suppose that \( F \) is the CDF of \((U, V)\). Consider the rectangle \( R \) with vertices \(\{(0,0), (0,2), (2,0), (2,2)\}\).

Then,

\[ P\{(U, V) \in R\} = F(2, 2) - F(2, 0) - F(0, 2) + F(0, 0) = -1. \]

Negative probability implies that our assumption that \( F \) is a CDF is wrong.

Furthermore, \( F \) is not right continuous for any point \((u, v)\) such that \(u + v = 1\). Note that even if we make \( F \) right-continuous by letting

\[ F(u, v) = \begin{cases} 
0, & u + v < 1 \\
1, & u + v \geq 1,
\end{cases} \]

it is still not valid because of the first reason.

Problem 3

Suppose that two cards are drawn at random from a deck of 52 cards. Let \( X \) be the number of queens obtained and let \( Y \) be the number of spades obtained.

a) Find the joint probability mass function of \( X \) and \( Y \), the marginal probability mass function of \( X \), and the marginal probability mass function of \( Y \).

b) Find \( P(X = Y) \).

c) Find \( P(X \leq Y) \).

d) Find \( P(X = 2 | Y = 2) \).
Solution

a) Some of probabilities are easy to find directly. Others may be more easily obtained by conditioning on $X$.

$$P(Y = i, X = 0) = \begin{cases} \frac{36 \cdot 35}{52 \cdot 51}, & i = 0, \\ \frac{2(12 \cdot 36)}{52 \cdot 51}, & i = 1, \\ \frac{12 \cdot 11}{52 \cdot 51}, & i = 2. \end{cases}$$

For $X = 1$, we consider two cases: the queen is the queen of spades, or it is not.

$$P(Y = i, X = 1, Q\spadesuit) = \begin{cases} \frac{2(1 \cdot 36)}{52 \cdot 51}, & i = 1, \\ \frac{2(1 \cdot 12)}{52 \cdot 51}, & i = 2, \end{cases}$$

$$P(Y = i, X = 1, Q\blackspadesuit) = \begin{cases} \frac{2(3 \cdot 36)}{52 \cdot 51}, & i = 0, \\ \frac{2(3 \cdot 12)}{52 \cdot 51}, & i = 1, \end{cases}$$

where $Q\spadesuit$ is the event that the queen of spades is chosen. Hence,

$$P(Y = i, X = 1) = \begin{cases} \frac{2(3 \cdot 36)}{52 \cdot 51}, & i = 0, \\ \frac{2(3 \cdot 12)}{52 \cdot 51}, & i = 1, \\ \frac{2(1 \cdot 12)}{52 \cdot 51}, & i = 2. \end{cases}$$

Finally,

$$P(Y = i, X = 2) = \begin{cases} \frac{3 \cdot 2}{52 \cdot 51}, & i = 0, \\ \frac{3 \cdot 1}{52 \cdot 51}, & i = 1. \end{cases}$$

So we get

<table>
<thead>
<tr>
<th></th>
<th>$X = 0$</th>
<th>$X = 1$</th>
<th>$X = 2$</th>
<th>Marginal of $Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y = 0$</td>
<td>$\frac{11}{442}$</td>
<td>$\frac{43}{442}$</td>
<td>$\frac{169}{442}$</td>
<td>$\frac{210}{442}$</td>
</tr>
<tr>
<td>$Y = 1$</td>
<td>$\frac{114}{442}$</td>
<td>$\frac{24}{442}$</td>
<td>$\frac{21}{442}$</td>
<td>$\frac{144}{442}$</td>
</tr>
<tr>
<td>$Y = 2$</td>
<td>$\frac{22}{442}$</td>
<td>$\frac{4}{442}$</td>
<td>$\frac{2}{442}$</td>
<td>$\frac{26}{442}$</td>
</tr>
</tbody>
</table>

Note that the marginals can be either found directly or by summing up rows and columns.

b) $P(X = Y) = \frac{210 + 24 + 0}{442} = \frac{234}{442}$

c) $P(X \leq Y) = \frac{210 + 144 + 22 + 24 + 4 + 0}{442} = \frac{404}{442}$

d) $P(X = 2|Y = 2) = \frac{P(X = Y = 2)}{P(Y = 2)} = \frac{\frac{22}{442}}{\frac{26}{442}} = 0$. 
Problem 4

The jointly continuous random variables $X$ and $Y$ have joint pdf:

\[
f_{X,Y}(u,v) = \begin{cases} 
1.5, & 0 \leq u < 1, \ 0 \leq v < 1, \ 0 \leq u + v < 1, \\
0.5, & 0 \leq u < 1, \ 0 \leq v < 1, \ 1 \leq u + v < 2, 
\end{cases}
\]

and zero elsewhere.

a) Find the marginal pdf of $Y$.

b) Find $P(X + Y \geq 3/2)$.

c) Find $P(X^2 + Y^2 \leq 1)$.

Solution

The support is the square with vertices $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. On the triangle with vertices $\{(0, 0), (0, 1), (1, 0)\}$, the pdf is 1.5 and on the triangle with vertices $\{(0, 1), (1, 0), (1, 1)\}$, it is 0.5. Sketching the pdf and marking the triangles is helpful for understanding the solution.

a) For $v \in [0, 1]$, we have $f_Y(v) = \int_0^1 f_{X,Y}(u,v) \, du = \int_0^{1-v} \frac{3}{2} \, du + \int_{1-v}^1 \frac{1}{2} \, du = \frac{3(1-v) + 1 - (1-v)}{2} = \frac{3-2v}{2}$. For $v \notin [0, 1]$, we have $f_Y(v) = 0$.

b) $P(X + Y \geq \frac{3}{2}) = \frac{1}{2} \times \text{area of the triangle with vertices } \{\left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, 1\right), (1, 1)\} = \frac{1}{16}$.

c) Let $A$ be the area of the triangle with vertices $\{(0, 0), (0, 1), (1, 0)\}$ and let $O$ be the area of the unit circle. Then,

\[
P(X^2 + Y^2 \leq 1) = \frac{3}{2} A + \frac{1}{2} \left(\frac{\pi}{4} - A\right) = \frac{3}{4} + \frac{1}{2} \left(\frac{\pi}{4} - \frac{1}{2}\right) = 0.89.
\]