Problem 1

(3 × 8 pts)
Consider random variables $X$ and $Y$ with joint pdf

$$f_{X,Y}(x, y) = \begin{cases} 
\frac{1}{3}, & 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \\
\frac{2}{3}, & 0 \leq x \leq 1, \quad -1 \leq y < 0, \\
0, & \text{else}
\end{cases}$$

Determine the following.

a) The marginal pdf of $Y$.
b) The conditional pdf $f_{X|Y}(x|y)$.
c) $P(X^2 + Y^2 \leq 1)$.

Solution

a)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx = \begin{cases} 
\int_0^1 \frac{1}{3} \, dx = \frac{1}{3}, & 0 \leq y \leq 1, \\
\int_0^{2/3} \frac{2}{3} \, dx = \frac{2}{3}, & -1 \leq y < 0, \\
0, & \text{else}
\end{cases}$$

b) The conditional distribution is defined only for $-1 \leq y \leq 1$.

For $-1 \leq y < 0$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \begin{cases} 
\frac{2/3}{2/3} = 1, & 0 \leq x \leq 1, \\
\frac{0}{2/3} = 0, & \text{else}
\end{cases}$$

For $0 \leq y \leq 1$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \begin{cases} 
\frac{1/3}{1/3} = 1, & 0 \leq x \leq 1, \\
\frac{0}{1/3} = 0, & \text{else}
\end{cases}$$
So, for \(-1 \leq y \leq 1,
\)
\[ f_{X|Y}(x|y) = \begin{cases} 
1, & 0 \leq x \leq 1, \\
0, & \text{else}. 
\end{cases} \]

\( P(X^2 + Y^2 \leq 1) = \frac{\pi}{4} \left( \frac{1}{3} \right) + \frac{\pi}{4} \left( \frac{2}{3} \right) = \frac{\pi}{4}. \)

**Problem 2**

(3 \times 8 pts)
Suppose arrival of calls to a call center can be modeled with a Poisson process with arrival rate \( \lambda \) calls per minute.

a) Find the ML estimate of \( \lambda \) if it is observed that the second call arrives at time \( t = 4 \) min.

b) For this part and the following part of the problem assume \( \lambda = 2 \). What is the probability that in the first 2 minutes at least 3 calls arrive.

c) What is the conditional probability that two calls arrive in the first two minutes given that two calls arrive in the first minute?

**Solution**

a) The arrival time of the second call, \( T_2 \), has the following Erlang distribution
\[ f_{T_2}(t) = \begin{cases} 
\lambda^2 t e^{-\lambda t}, & t \geq 0 \\
0, & \text{else}. 
\end{cases} \]

The the likelihood is
\[ f_{T_2}(4) = \lambda^2 4e^{-4\lambda}. \]

To maximize, we take log, differentiate, and set equal to zero:
\[ f_{T_2}(4) \ln 2 \ln \lambda + \ln 4 - 4 \lambda - \frac{4}{\lambda} - 4 = 0 \Rightarrow \hat{\lambda}_{ML} = \frac{1}{2}. \]

b) The desired probability is \( P(N_2 \geq 3). \)
\[ P(N_2 \geq 3) = 1 - P(N_2 = 0) - P(N_2 = 1) - P(N_2 = 1) \\
= 1 - e^{-2\lambda} - e^{-2\lambda} (2\lambda) - e^{-2\lambda} (2\lambda)^2/2 \\
= 1 - e^{-4} (1 + 4 + 8) = 1 - 13e^{-4}. \]

c) Let \( X (= N_1) \) denote the number of calls arriving in the first minute and \( Y = (N_2 - N_1) \) denote the number of calls arriving in the second minute.
\[ P(X + Y = 2|X = 2) = \frac{P(X + Y = 2, X = 2)}{P(X = 2)} = \frac{P(X = 2, Y = 0)}{P(X = 2)} \\
= \frac{P(X = 2) P(Y = 0)}{P(X = 2)} = P(Y = 0) = e^{-2}, \]
where the third equality follows from the independence of the number of arrivals in non-overlapping intervals.

**Problem 3**

\((2 \times 8\) pts\)

Let \(X\) be a uniform random variable over the interval \([-1, 1]\) and let \(Y = e^{X^2}\).

a) Determine the support of \(Y\).

b) Determine the pdf of \(Y\) over its support.

**Solution**

a) The support of \(Y\) is \([e^0, e^1] = [1, e]\).

b) Consider the equation \(y_0 = g(x) = e^{x^2}\). The solutions are

\[
\begin{align*}
  x_1 &= \sqrt{\ln y_0} \\
  x_2 &= -\sqrt{\ln y_0}
\end{align*}
\]

So,

\[
f_Y (y_0) = \frac{f_X (x_1)}{|g' (x_1)|} + \frac{f_X (x_2)}{|g' (x_2)|} = \frac{1}{2x_1 e^{x_1^2}} + \frac{1}{2x_2 e^{x_2^2}} = \frac{1}{2y_0 \sqrt{\ln y_0}}.
\]

**Problem 4**

\((3 \times 8\) pts\)

Consider the following binary hypothesis testing problem.

Under hypothesis \(H_1\), \(X\) is a Normal random variable with mean 0 and variance 1, that is,

\[
f_1 (x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\]

Under hypothesis \(H_0\), \(X\) is uniform over the interval \([-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}]\), that is,

\[
f_0 (x) = \begin{cases} 
\frac{1}{\sqrt{2\pi}} & -\sqrt{\frac{\pi}{2}} \leq x \leq \sqrt{\frac{\pi}{2}} \\
0 & \text{else}
\end{cases}
\]

Determine the following. (Hint: sketch both pdfs. For sketching purpose only, you may use \(\sqrt{\frac{\pi e}{2}} \approx 2, \sqrt{2\pi e} \approx 4, 1/\sqrt{2\pi} \approx 0.40, 1/\sqrt{2\pi e} \approx 0.25\); your solutions must be in terms of \(\pi\) and \(e\).)

a) Find the maximum likelihood (ML) decision rule.

b) Find \(p_m\) for the ML decision rule in terms of the \(Q\) function with positive arguments.

c) Define \(\pi_1 = P (H_1)\) and \(\pi_0 = P (H_0)\). Find the minimum value of \(\frac{\pi_1}{\pi_0}\) such that MAP always declares \(H_1\) to be true.
Solution

a) The following figure is helpful for understanding the solution.

![Figure 1: $f_1$ and $f_0$](image)

Clearly, for $|X| > \sqrt{\frac{\pi e}{2}}$, ML declare $H_1$ to be true. Let $\gamma$ be a non-negative value such that

$$f_1(\gamma) = f_0(\gamma).$$

Hence,

$$\frac{1}{\sqrt{2\pi}} e^{-\gamma^2/2} = \frac{1}{\sqrt{2\pi e}} \iff \gamma = 1.$$  

So, the ML rule becomes

$$\begin{aligned}
  &\begin{cases}
    X < -\sqrt{\frac{\pi e}{2}}, & \text{dec. } H_1 \\
    -\sqrt{\frac{\pi e}{2}} < X < -1, & \text{dec. } H_0 \\
    -1 < X < 1, & \text{dec. } H_0 \\
    1 < X < \sqrt{\frac{\pi e}{2}}, & \text{dec. } H_1 \\
    \sqrt{\frac{\pi e}{2}} < X, & \text{dec. } H_1
  \end{cases}
\end{aligned}$$

b) We have

$$p_m = P\left(1 < |X| < \sqrt{\frac{\pi e}{2}} \big| H_1\right) = 2Q\left(1 - 2Q\left(\sqrt{\frac{\pi e}{2}}\right)\right).$$

c) $\pi_1$ and $\pi_0$ should be such that $\pi_1 f_1$ is always at least as large as $\pi_0 f_0$ as seen in the figure below. The minimum value of $\frac{\pi_1}{\pi_0}$ for which this condition is satisfied can be obtained as

$$\frac{\pi_0}{\sqrt{2\pi e}} = \pi_1 \frac{1}{\sqrt{2\pi}} e^{-\pi e/4} \iff \frac{\pi_1}{\pi_0} = e^{\frac{\pi e}{4} - 1/2}.$$
Problem 5

(12 pts)
An airline sells 162 tickets for a plane with 120 seats. Each passenger actually shows up at the airport with probability $\frac{2}{3}$. Using Gaussian approximation with continuity correction, what is the probability that there are not enough seats on the plane for all passengers who show up?

Solution

Let $X$ denote the number of passengers who actually show up. We have

$$EX = 162 \left(\frac{2}{3}\right) = 108,$$

$$\text{STD}(X) = \sqrt{162 \left(\frac{2}{3}\right) \left(\frac{1}{3}\right)} = 6$$

Then, the desired probability is

$$P(X \geq 121) = P(X \geq 120.5) = P \left( \frac{X - 108}{6} \geq 2.08 \right) = Q(2.08) = 0.0188.$$