Problem 1

(9+8 pts)
Consider the following pdf for a random variable \( X \).

\[
    f_X(u) = \begin{cases} 
        \frac{24}{u^4}, & u \geq a \\
        0, & u < a
    \end{cases}
\]

a) Find the value of \( a \) such that \( f_X \) is a valid pdf.
b) Find \( E[X] \).

Solution

a) The value of \( a \) should be chosen such that \( \int_{-\infty}^{\infty} f_X(u) \, du = 1 \). We have

\[
    \int_{-\infty}^{\infty} f_X(u) \, du = \int_{a}^{\infty} \frac{24}{u^4} \, du = 8 \left( -\frac{1}{u^3} \right)_{a}^{\infty} = \frac{8}{a^3},
\]

so we let \( a = 2 \).

b) \[ E[X] = \int_{2}^{\infty} u \frac{24}{u^4} \, du = \int_{2}^{\infty} \frac{24}{u^3} \, du = 12 \left( -\frac{1}{u^2} \right)_{2}^{\infty} = \frac{12}{4} = 3. \]

Problem 2

(8+2+8 pts)

A sports competition is held among 16 European national teams, among them are Spain, Germany, Portugal, and Italy. The teams are to be divided into four groups, each with four teams. Suppose teams are grouped randomly, with all possibilities being equally likely.

a) What is the probability that Spain and Italy are in the same group?
b) Which is larger? Indicate your answer by circling the appropriate item. (You can either guess or use the result of part c)

i) the probability that Spain and Italy are in the same group, or

ii) the conditional probability that Spain and Italy are in the same group given that Portugal and Germany are in the same group.

c) What is the conditional probability that Spain and Italy are in the same group given that Portugal and Germany are in the same group?

Solution

a) Spain is in some group. That group has three other members. So the desired probability is $\frac{3}{15}$ where 15 is the number of remaining positions. Hence, the final answer is $\frac{3}{15} = \frac{1}{5}$.

b) ii is larger. See part c.

c) There are two disjoint cases: Either

- all four teams are in the same group: The probability corresponding to this case is
  $$\frac{2}{14} \cdot \frac{1}{13}$$

- Spain and Italy in one group and Portugal and Germany in another group. The probability corresponding to this case is
  $$\frac{12}{14} \cdot \frac{3}{13}$$

The total probability is thus $\frac{2+12\cdot 3}{14\cdot 13} = \frac{38}{182} = \frac{19}{91}$. Note that $\frac{19}{91} > \frac{1}{5}$.

Problem 3

(9 pts)

We are told of the following experiment. First a fair coin is flipped $n$ times and it is noted that Heads is observed 3 times. Next, the same coin is flipped $n$ times once again and this time Heads is observed 6 times. The value of $n$ is unknown to us. With the information given, find the Maximum Likelihood estimate of $n$. (You may find this useful: $\sqrt{12} \simeq 3.46$).

Solution

The number of Heads in the first round, $X_1$, is a Binomial with parameters $(n, 1/2)$. For the second round, the number of Heads, $X_2$, is again a Binomial with parameters $(n, 1/2)$. These two random variables are independent. So the likelihood function is

$$l(n) = P(X_1 = 3, X_2 = 6) = P(X_1 = 3) P(X_2 = 6) = \binom{n}{3} \left(\frac{1}{2}\right)^n \binom{n}{6} \left(\frac{1}{2}\right)^n.$$ 

By definition of ML estimate, $\hat{n}_{ML}$ is the value of $n$ that maximizes $l(n)$. To find $\hat{n}_{ML}$, we find

$$\frac{l(n)}{l(n-1)} = \frac{\binom{n}{3} \left(\frac{1}{2}\right)^n \binom{n}{6} \left(\frac{1}{2}\right)^n}{\binom{n-1}{3} \left(\frac{1}{2}\right)^{n-1} \binom{n-1}{6} \left(\frac{1}{2}\right)^{n-1}} = \frac{n^2}{4(n^2 - 9n + 18)}.$$
Comparing the ratio with 1 yields
\[
\frac{n^2}{4(n^2 - 9n + 18)} < 1 \iff n^2 < 4n^2 - 36n + 72 \iff 3(n^2 - 12n + 24) > 0
\iff n^2 - 12n + 24 > 0 \iff (n - 6)^2 > 12 \iff 6 - \sqrt{12} < n < 6 + \sqrt{12}.
\]
We know that \( n \geq 6 \). In the region of \( n \geq 6 \), \( l(n) \) is increasing for \( n < 6 + \sqrt{12} \). Thus \( \hat{n}_{ML} = \lfloor 6 + \sqrt{12} \rfloor = 9 \).

**Problem 4**

(7+6+6 pts)

The random variable \( X \) is defined as follows:

A biased coin, with \( P(\text{Heads}) = \frac{1}{3} \), is tossed. If Heads shows, we let \( X = 0 \). If Tails shows, we randomly and uniformly choose the value of \( X \) from the interval \([0,1]\).

a) Sketch the CDF of \( X \). Clearly mark important points and values. Use full and empty circles to indicate the value of the CDF at discontinuities.

b) Find \( E[X] \). Hint: You can use the law of total probability for expectation.

c) Find \( \text{Var}[X] \).

**Solution**

a) \[
F_X(c) = \begin{cases} 
0, & c < 0 \\
\frac{2}{3}c + \frac{1}{3}, & 0 \leq c \leq 1 \\
1, & c > 1 
\end{cases}
\]
Discontinuity at \( c = 0 \).

b) Let \( H \) denote the event that the coin shows Heads and \( T \) denote the event that the coin shows Tails.

\[
E[X] = E[X|T]P(T) + E[X|H]P(H) = \frac{1}{2} \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{1}{3}
\]

c) To find the variance, we find \( E[X^2] \).

\[
E[X^2] = E[X^2|T]P(T) + E[X^2|H]P(H) = \frac{2}{3} \int_0^1 u^2 du + 0 \cdot \frac{1}{3} = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}
\]

So \( \text{Var}[X] = \frac{2}{9} - \left( \frac{1}{3} \right)^2 = \frac{1}{9} \).

**Problem 5**

(10+9 pts)

We are given two boxes. In each box there are 5 black marbles and 4 white marbles. We pick a marble at random from the first box and put it in the second box. Then, we pick a marble randomly from the second box.

Let \( B_i \), for \( i = 1, 2 \), be the event that the marble picked from the \( i \)th box is black. For example \( B_2 \) is the event that the marble picked from the second box is black. Similarly, define \( W_i \) as the event that the marble picked from the \( i \)th box is white. Find
a) \( P(B_2) \).
b) \( P(W_1|B_2) \).

**Solution**

a)
\[
P(B_2) = P(B_2|W_1) P(W_1) + P(B_2|B_1) P(B_1) = \frac{5}{10} \cdot \frac{4}{9} + \frac{6}{10} \cdot \frac{5}{9} = \frac{2}{9} + \frac{3}{9} = \frac{5}{9}.
\]
b)
\[
P(W_1|B_2) = \frac{P(B_2,W_1)}{P(B_2)} = \frac{P(B_2|W_1) P(W_1)}{P(B_2)} = \frac{2/9}{5/9} = \frac{2}{5}.
\]

**Problem 6**

(9+9 pts)

In the following network, all links \( L_i \) have capacity \( C_i = 1 \). Each link \( L_i \) may fail independently of others with probability \( p_i \). The source is node \( s \) and the destination is node \( t \). Let the capacity of the network be denoted by \( X \).

\[s \xrightarrow{L_1} L_2 \xrightarrow{L_3} L_6 \xrightarrow{L_4} L_5 \xrightarrow{L_5} t\]

a) Find \( P(X = 3) \).
b) Assume that \( p_4 = 1 \). This is equivalent to assuming that \( L_4 \) is not in the network. Use union bound to obtain an upper-bound on the probability of network outage (network failure).

**Solution**

a) Let \( F_i \) be the event that link \( L_i \) fails. For \( X = 3 \), all links must work except \( L_4 \). So
\[
P(X = 3) = P(F_1^c F_2^c F_3^c F_5^c F_6^c) = (1 - p_1) (1 - p_2) (1 - p_3) (1 - p_5) (1 - p_6).
\]
b) Let \( F \) be the event that the network fails.
\[
P(F) = P(F_1 (F_2 \cup F_5) (F_3 \cup F_6)) = P(F_1) P(F_2 \cup F_5) P(F_3 \cup F_6) \leq P(F_1) (P(F_2) + P(F_5)) (P(F_3) + P(F_6)) = p_1 (p_2 + p_5) (p_3 + p_6).
\]