On the evolution of continuous types under replicator and gradient dynamics: two examples

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June 5, 2015

Abstract

This paper illustrates techniques for assessing the dynamic stability of games where a continuum of types might be present by re-analyzing two models under incomplete information, the Lohman, Oechssler, and Warneryd (2001) public goods game and the Kopel, Lamantia and Szidarovszky (2014) Cournot duopoly game. The evolution of continuous types follows either replicator dynamics (Oechssler and Riedel, 2001, 2002) or gradient dynamics (Friedman and Ostrov, 2010, 2013). The techniques rely on a system of partial differential equations. Numerical solutions obtained through replicator and gradient dynamics highlight the differences and the similarities that arise under both approaches. In the public goods game, the dynamic system affects the stationary distribution of types while in the Cournot duopoly model, the types evolve to a single mass point regardless of the dynamics used. Lastly, these techniques allow us to endogenize the distribution of player types.

Keywords: Population games, Evolutionary dynamics, Corporate social responsability, Mixed oligopoly, Public goods
JEL Codes: C72, C73, L13, L21, M14, H41

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1 Introduction

What types of players survive in the long-run? How does their behavior affect the evolution of the stationary distribution? In this paper, I illustrate plausible dynamic systems that answer both questions when the set of types is drawn from a continuous space.

Previous literature focused either on the evolution of (generally two) discrete types and their respective actions, or on the dynamics of actions in a continuous space. However, there are situations in which a continuum of types is more appropriate. I develop a system of partial differential equations that seeks to model both components (actions and types) in games of incomplete information.

The evolution of player strategies follows the principle of sign-preserving dynamics. This fundamental evolutionary principle implies that strategies with higher payoffs become more prevalent over time. In a continuous action space, Friedman and Ostrov (2010, 2013) assume that actions adjust according to the fitness gradient. They argue that gradient dynamics are natural when the topology of the continuous action space matters —e.g., it is easier to evolve (or cheaper to adjust) to a nearby action than to a distant one— and in particular when the payoff is given by the expectation of the current distribution of actions encountered during interactions.

Following standard arguments in literature on fluid dynamics, Friedman and Ostrov write down a partial differential equation governing the distribution of actions and note under which conditions the distribution converges asymptotically to a mass point or to a stationary density. In this paper, I also use gradient dynamics to study the evolution of player types in a continuous space.

It is reasonable to assume that it is more costly to adjust to distant traits compared to those in the neighborhood.[1] Hence gradient dynamics is suitable for modeling the evolution of player types. The dynamic system consists of partial differential equations in which the evolution of actions usually adjusts faster relative to the evolution of player types (e.g. see Güth and Yaari, 1992 and Ok and Vega-Redondo, 2001).

I take no strong position as to which underlying process governs the evolution of types and actions. It is intuitive for economists to consider entry and exit as governing the evolution of types, but other social scientists may regard individual types subject to gradual change due to the influence of role models and peers. For a given type of individual, the distribution of actions can adapt via personal experience of success and failure, via social learning, or even via entry and exit. The underlying process determines the precise dynamic specification (e.g. see Rabanal and Friedman, 2014).

The dynamic system proposed in this paper can accommodate different families of

[1] Similar ideas have been applied to the evolution of preferences in a discrete case in Rabanal and Friedman (2014).

dynamics. Using two examples, I compare the resulting stationary distributions of types based on gradient and replicator dynamics, the most common dynamics used in the literature (Oechssler and Riedel, 2001, 2002). The main idea behind replicator dynamics is that the proportion of more successful types in a population increases over time at the expense of unsuccessful types. This process is generally explained by entry and exit or the influence of role models and peers.[2]

The first example is a public goods game proposed by Lohman, Oechssler, and Warneryd (2001, henceforth LOW01). LOW01 study a simple game in which players decide whether to contribute to the public good. All players have a subjective intrinsic cost of contributing, which is different than a fixed material cost. Applying replicator dynamics to the evolution of the distribution of types, they find that the marginal contributor approaches the value of the fixed material cost for most initial distributions. The stationary equilibrium is such that players with lower subjective costs compared to the marginal type contribute fully to the public account, while types with higher subjective costs contribute nothing. Using gradient dynamics, I similarly find that the marginal contributor approaches the value of the fixed material cost. However, the stationary distribution and the dynamics are quite different under the two approaches.

The second example is based on the recent work of Kopel, Lamantia and Szidarovszky (2014, henceforth KLS14). KLS14 analyze the evolutionary stability of two types of firms in a Cournot duopoly model. The firms are either standard profit maximizers or socially responsible, meaning that they consider consumer welfare in their decisions.[3] KLS14 develop a dynamic system to explain the evolution of production and types. Firm production is driven by best-response dynamics, while the share of types follows the asynchronous updating mechanism (Hofbauer and Weibull, 1996). KLS14 show that socially responsible firms are evolutionary stable only when consumers are willing to pay more for these products.

I extend the Cournot duopoly model of KLS14 to include a continuum of firm types and allow the model to endogenously determine the importance of the consumer surplus on the firm's (subjective) profits.[4] Assuming that the willingness to pay for socially responsible firm products is directly related to the level of awareness of consumer welfare, I find that

[2]Related to my work, Heifetz, Shannon and Spiegel (2007) apply replicator dynamics to the evolution of continuous types. Unfortunately, I cannot apply gradient dynamics to Heifetz et al. (2007) since the gradient term is invariant to the types.

[3]Königstein and Müller (2001) also study a variant of the standard Cournot duopoly model. In their model, there is a continuum of types whose subjective profits are dependent on both the monetary profits and consumer welfare. The firms have perfect information about the counterparty type. Using static analysis, they find that firms that consider consumer welfare are evolutionary stable.

[4]In a complete information environment, Kopel and Brand (2012) assume that a manager can decide whether the firm should be profit maximizer, or should consider consumer welfare. Their conclusions are similar to Königstein and Müller (2001), showing that socially responsible firms can do better than purely profit maximizing firms.

the stationary distribution is a single mass point, in which only the socially responsible firms survive.

I build on the previous work of Rabanal and Friedman (2014) to study the evolution of a continuum of types in a game of incomplete information. The authors develop a dynamic system to study the evolution of actions and types in a discrete space. They show how standard specifications of replicator and gradient dynamics can be combined to computationally assess the stability of games of incomplete information in the context of preference evolution. In the discrete case, the system consists of ordinal differential equations, while in the continuous case, the system uses partial differential equations.

This paper is also related to literature that studies other evolutionary dynamics in continuous action space. Hofbauer, Oechssler and Riedel (2009) study inertial best response dynamics (also called BNN dynamics), Lahkar and Riedel (2015) analyze logit dynamics, Perkins and Leslie (2014) study stochastic fictitious play, and Cheung (2014) focuses on pairwise comparison dynamics.

The exposition of this paper begins in Section 2 with an introduction to the basic game based on LOW01. Section 2.1 sketches the main assumptions of the dynamic setting and the expected payoff calculations, while Section 2.2 describes the adjustment dynamics. Section 2.3 demonstrates that for reasonable parameter values, the model achieves a stationary equilibrium under both replicator and gradient dynamics.

Although the numerical results achieve the predictions of LOW01 using both models, the dynamics process to the stationary equilibrium is different. Under replicator dynamics, the density of types with subjective cost smaller than the marginal type increases over time, while the density of types with higher subjective costs decreases. Under gradient dynamics, the adjustment is smooth and local. The stationary distribution exhibits higher density around types whose subjective cost is equal to the material cost. This gain occurs at the expense of local types with higher costs.

Section 3 presents the Cournot duopoly example based on KLS14 and section 3.1 describes the system of partial differential equations used to endogenize player types and the predictions under replicator and gradient dynamics. The numerical results in section 3.2 confirm the theoretical prediction that the stationary distribution is a single mass point, with only socially responsible firms surviving. The last section summarizes the findings and discusses their implications, arguing why gradient dynamics might be more relevant in the Cournot duopoly model.

2 Public goods game

The public goods game is based on LOW01. There are $n \geq 2$ players who have the option to contribute to a public good. Players receive a unit of utility when at least one player contributes and zero otherwise. Each player has a subjective cost $c \in [0,1]$, which may be different from the material (fixed) cost $k \in (0,1)$, a variable that is identical for all players. Suppose that the players do not know the subjective costs of their neighbors, but do know that they are independently drawn from a continuous cumulative distribution $F(c)$.

The decision to contribute to the public good is driven by the subjective payoffs (i.e subjective costs) while the survival of types depends on material payoffs (e.g. see the literature on the indirect evolution approach, Güth and Yaari, 1992). Let us denote x as the probability of an individual contributing to the public good. A player with subjective cost c contributes if the payoff of contributing is greater than or equal to the payoff of not contributing

$$\begin{aligned} 1-c &\geq (1-x)^{n-1} \cdot 0 + [1-(1-x)^{n-1}] \cdot 1 \\ c &\leq (1-x)^{n-1} \end{aligned} \tag{1}$$

where $(1-x)^{n-1}$ is the probability of no one contributing and therefore $[1-(1-x)^{n-1}]$ is the probability that at least one player contributes (of the remaining $n-1$ players).

In equilibrium, all players act rationally. Each player's belief about the probability of the other contributing must be consistent with the actual probabilities of contribution. The equilibrium probability x^* of an individual contributing to the public good is therefore given by the fixed-point equation

$$\begin{aligned} x^* &= Prob(c \leq (1-x^*)^{n-1}) \\ x^* &= F\left((1-x^*)^{n-1}\right) \end{aligned}$$

The symmetric equilibrium threshold cost is defined as $c^* \equiv (1-x^*)^{n-1}$, using equation (1), such that all players with cost lower than c^* contribute, which is given by $x^* = Prob(c < c^*) = F(c^*)$. Using this last condition in from equation (1), c^* satisfies

$$c^* = (1-F(c^*))^{n-1} \tag{2}$$

Given that $(1-F(c))^{n-1}$ is a continuous and decreasing function, by the intermediate value theorem it has a unique interior fixed point c^*. Next, I look at the fitness (material payoff) of a contributor, which is defined as $1-k$, while the fitness of a noncontributor is one if the public good is provided and zero otherwise. Altruists are defined as players with $c < k$ since they have a subjective bias to contribute even when it hurts their material payoff.

LOW01 extend this simple game by introducing plausible dynamics that explain the evolution of types. In the next section, I complement their work by adding adjustment dynamics to the evolution of player actions. Prior to discussing the dynamics, I describe the expected (subjective and material) payoffs.

2.1 Expected Payoffs

Suppose a large (i.e. infinite) population is randomly matched into subgroups of size n and then asked to play the public goods game. Each player knows the actual population distribution $F(c,t)$. I denote the probability of each player contributing as $P(c,t) \in (0,1)$. Therefore, the probability that at least one player contributes is $\mu = 1-(1-\rho)^{n-1}$, where $\rho = \int_0^1 P(c,t)f(c,t)dc$. The individual average fitness ($\phi(c,t)$) can then be written as

$$\begin{aligned}
\phi(c,t) &= E\left(c \mid F(c,t), P(c,t)\right) \\
&= E \begin{cases} 1-k & \text{with probability } P(c,t) \\ 1 & \text{with probability } [1-P(c,t)]\cdot\mu \\ 0 & \text{with probability } [1-P(c,t)]\cdot(1-\mu) \end{cases} \\
\phi(c,t) &= (1-k)P(c,t) + [1-P(c,t)]\cdot\mu \qquad (3)
\end{aligned}$$

Notice that if players with $c \leq c^*$ contribute with probability one, then $\rho = F(c^*)$ and the probability of no one contributing is

$$1-\mu = (1-\rho)^{n-1} \equiv (1-F(c^*))^{n-1} \qquad (4)$$

Average fitness of the population (denoted as $\Phi(F,t)$) is computed as the integral of the individual average fitness over the whole population with density $f(c,t)$. That is

$$\begin{aligned}
\Phi(F,t) &= \int_0^1 \phi(c,F)f(c,t)dc \\
&= (1-k-\mu)\rho + \mu \qquad (5)
\end{aligned}$$

Recall that the contribution decision is driven by subjective payoffs. A player of type c receives a payoff of $1-c$ if she contributes (with probability $P(c,t)$) and of one if at least one other member of the population contributes (with probability μ) when she does not contribute (with probability $1-P(c,t)$). The subjective payoff $\phi^s(c,t)$ can then be written as

$$\begin{aligned}\phi^s(c,t) &= (1-c)P(c,t)+[1-P(c,t)]\cdot\mu \\ &= [1-c-\mu]\cdot P(c,t)+\mu \\ &= [(1-\rho)^{n-1}-c]\cdot P(c,t)+\mu \end{aligned} \tag{6}$$

In expression (6), I use equation (4) to replace $1-\mu$. The main difference between equations (3) and (6) is that the former depends on the material cost k, while the latter depends on the subjective cost c. In the following section, I introduce plausible dynamics that drive the evolution of player actions (P) as well as the distribution of types (F).

2.2 Adjustment dynamics

The evolution of actions $P(c,t)$ follows sign-preserving dynamics. These dynamics express the fundamental evolutionary principle that strategies with higher payoffs become more prevalent over time. Specifically, I adopt the approach used in Rabanal and Friedman (2014) which relies on gradient dynamics.[5] When modeling the adjustment of shares or mixture probabilities in $[0,1]$ space for two (pure) alternatives, there are many smooth monotone (or sign-preserving) dynamic specifications to choose among. As cataloged in Weibull (1997) and Sandholm (2010), these include BNN, perturbed best response and various sorts of learning dynamics. I adopt gradient dynamics as a familiar specification that is a reasonable approximation for dynamics arising from a wide variety of evolutionary processes. The techniques illustrated below can be easily tailored to other specifications.

Under gradient dynamics, $P_t \lesseqgtr 0$ (the time derivative of P) as long as $\frac{\partial\phi^s(c,t)}{\partial P} \lesseqgtr 0$. Recall that contribution choices depend on the subjective payoffs (ϕ^s). The gradient term can be computed by taking a derivative of equation (6) with respect to P, which results in $(1-\rho)^{n-1}-c$. Because the probability of contribution is bounded by zero and one, it is appropriate to include binomial variance terms $(1-P(c,t))\cdot P(c,t)$ in the sign-preserving dynamics expressed in equation (8) such that $P(c,t)\in[0,1]$.

Following the work of LOW01 and Oechssler and Riedel (2001, 2002), I assume that the distribution of types (c) adjusts according to continuous replicator dynamics. In this case, the difference between individual and population payoffs, defined as $\sigma(c,t)$, plays a

[5] Ritzberger and Weibull (1995) use the term sign-preserving dynamics when the growth rate of strategy i is positive (negative) and the payoff of strategy i is greater than (less than) the average payoff in the population. Rabanal and Friedman (2014) point out that gradient dynamics follow the aforementioned sign-preserving dynamics when there are only two possible choices. In this case, given a particular type c, the dynamics become a replicator.

key role

$$\begin{aligned}\sigma(c,t) &= \phi(c,t)-\Phi(c,t)\\ &= (1-k-\mu)P(c,t)-(1-k-\mu)\rho\\ &= (1-k-\mu)(P(c,t)-\rho)\\ &= ((1-\rho)^{n-1}-k)(P(c,t)-\rho) \end{aligned}\tag{7}$$

Combining the equations that explain the evolution of actions and types, under replicator dynamics, the system of partial differential equations with state variables $P(c,t)$ and $F(c,t)$ is

$$\begin{aligned}P_t(c,t) &= \beta(1-P(c,t))\cdot P(c,t)\cdot\frac{\partial\phi^s(c,t)}{\partial P}\\ &= \beta(1-P(c,t))\cdot P(c,t)[(1-\rho)^{n-1}-c] \end{aligned}\tag{8}$$

$$\begin{aligned}F_t(c,t) &= \beta_F\int_o^c \sigma(c,t)f(c)dc\\ &= \beta_F\int_o^c ((1-\rho)^{n-1}-k)(P(c,t)-\rho)f(c,t)dc \end{aligned}\tag{9}$$

where β and β_F are constant variables that represent the speed of adjustment of each state variable, respectively. Following Rabanal and Friedman (2014), I assume that the speed of adjustment in contributions (which is driven by learning or imitation) is much faster compared to the evolution of distribution of types (which is driven by cultural or biological forces). Thus, $0<\beta_F\ll\beta$.

The faster adjustment of actions compared to the types implies that for a given population distribution $F(c,t)$, the players within that population are best responding and their behavior will eventually adjust to a Nash Equilibrium (NE). In this case, players with $c\leq c^*$ contribute with $P(c,t)=1$ and the rest ($c>c^*$) do not contribute ($P(c,t)=0$).

Looking at the dynamic system, this results in $\rho=\int_0^1 P(c)f(c,t)dc=F(c^*)$ and therefore the expression in brackets in equation (8) that drives the evolution of contribution can be rewritten as

$$\begin{aligned}(1-\rho)^{n-1}-c &= (1-F(c^*))^{n-1}-c\\ &= c^*-c \end{aligned}\tag{10}$$

Notice that I use equation (2) to obtain equation (10). Thus, the dynamics of contributions depend on the expression c^*-c. When $c^*-c>0$, $P(c,t)$ goes to the upper bound of one and when $c^*-c<0$, $P(c,t)$ moves to the lower bound of zero.

Next, I characterize the stationary distribution in Proposition 1, following LOW01.

Proposition 1 *Distribution F is stationary if and only if $c^* = k$ and consequently $F(k) = 1 - k^{1/(n-1)}$.*

Proof: In NE, $P = 1$ for $c \leq c^*$ and $P = 0$ for $c > c^*$. Thus, $\rho = F(c^*)$ and $c^* = (1-\rho)^{n-1}$ using equation (2). Now, equation (7) can be written as

$$\begin{aligned}\sigma(c,t) &= (c^* - k)(P(c,t) - F(c^*)) \\ &= \begin{cases} (c^* - k)(1 - F(c^*)) & \text{for } c \leq c^* \\ (k - c^*)F(c^*) & \text{for } c > c^* \end{cases}\end{aligned}$$

Distribution F is stationary if $\sigma(c,t) = 0$, which is true when $c^* = k$. At this stationary equilibrium equation (2) can be written as $F(k) = 1 - k^{1/(n-1)}\square$

2.2.1 Gradient dynamics

I employ gradient dynamics to study the evolution of types, following Friedman and Ostrov (2010, 2013). In this game, c increases (or decreases) when the fitness gradient $\frac{d\phi(c,t)}{dc}$ is positive (or negative) at a rate proportional to the steepness of the gradient. I normalize the time scale (which is infinitely slower than the time scale for the probability of contributions) to obtain $\beta_F = 1$, which then further simplifies the equation of motion to $\frac{dc}{dt} = \frac{d\phi(c,t)}{dc}$.

Using standard arguments from fluid dynamics,[6] Friedman and Ostrov conclude that when starting with an arbitrary initial density $f(c,0)$, the distribution evolves according to the partial differential equation

$$F_t(c,t) + \frac{d\phi(c,t)}{dc} \cdot f(c,t) = 0 \tag{11}$$

In steady state, the distribution is supported on isolated points (where f is a constant times a Dirac delta function) and/or on intervals where the gradient $\frac{d\phi(c,t)}{dc} = 0$. Specifically, the gradient term is computed by taking the derivative of expression (3) with respect to c

$$\frac{d\phi(c,t)}{dc} = (1 - k - \mu)P_c + (1-P)\mu_c \tag{12}$$

The adjustment of the probability of contribution is similar to the process under replicator dynamics. Combining both the evolution of probabilities of contribution and distribution of types, the system of partial differential equations with state variables $P(c,t)$ and $F(c,t)$

[6]Use of conservation of probability mass; see Friedman and Ostrov (2010, 2013).

is

$$
\begin{aligned}
P_t(c,t) &= \beta(1-P(c,t))\cdot P(c,t)\cdot\frac{\partial\phi^s(c,t)}{\partial P} \\
&= \beta(1-P(c,t))\cdot P(c,t)[(1-\rho)^{n-1}-c] && (13) \\
F_t(c,t) &= -\beta_F\frac{d\phi(c,t)}{dc}\cdot f(c,t) && (14)
\end{aligned}
$$

where $\frac{d\phi(c,t)}{dc}$ is given by equation (12) and $-\beta_F$ is the adjustment speed of the players moving their density downward (hence the minus sign) from above c to below. Thus, the distribution changes via continuous adjustment by individual players —no population mass is gained or lost— and nobody jumps.

Further, I assume that the adjustment speed of probabilities is much faster than evolution of types, $\beta_F \ll \beta$. Therefore, the choices of contribution approximate a NE for a given distribution F. Similar to the model under replicator dynamics, the probability of contribution adjusts to $P(c,t)=1$ for players with $c \leq c^*$ and $P(c,t)=0$ for $c > c^*$, see equation (10).

This pattern of contributions further simplifies the gradient term. Now, $\rho = F(c^*)$ and using equations (2) and (4), $1-\mu = c^*$. Since players are best-responding, $\mu_c = 0$. The gradient term (12) becomes

$$
\frac{d\phi(c,t)}{dc} = (c^*-k)P_c
$$

Proposition 1 also applies to gradient dynamics since the gradient term is equal to zero when $c^*=k$. However, the dynamics towards the stationary distribution are quite different compared to the replicator dynamics. Under gradient dynamics, the marginal contributor gains density as long as $c^*-k>0$ and the players that are far from the marginal contributor do not change their density since $P_c=0$. Under replicator dynamics, on the other hand, the whole population is changing.

The gradient term also shows that the initial conditions play an important role. As long as $P_c \neq 0$, the gradient term $\frac{d\phi(c,t)}{dc}$ is generally not zero for the types far away from the marginal type. Their densities change until the probability of contribution approaches the lower or upper bound. In order to illustrate these dynamics with more clarity and precision, I provide a numerical solution to the system of partial differential equations in the following section.

2.3 Solution to the dynamic models

Recall that the exogenous parameters of the model are: k, β, β_F and n, and the endogenous variables are: c^*, $P(c,t)$ and $F(c,t)$, where $c \in [0,1]$,and $t \in [t_o,T]$. The baseline parameters of the model assume that $k=0.05$, $\beta=1$, $\beta_F=0.01$ and $n=11$ under repli-

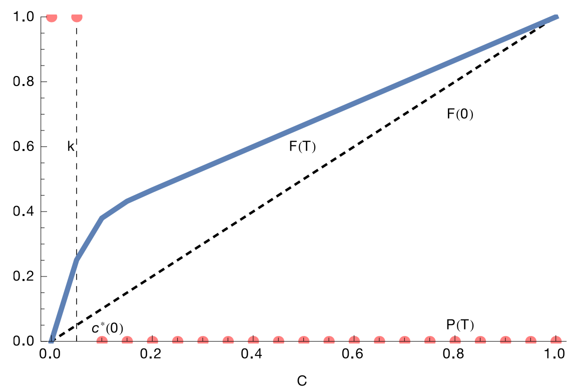


Figure 1: Initial and stationary distribution of types (F) and stationary probability of contribution (P) under replicator dynamics.

cator dynamics, and that $k = 0.05$, $\beta = 100$, $\beta_F = 1$ and $n = 11$ under gradient dynamics. I assume an initial uniform distribution of player types $F(c,0)$ and the probability of a contribution to be $P(c,0) = 0.9$ for $c \leq 0.2$ and $P(c,0) = 0.01$ for $c > 0.2$.

Prior to presenting the results, it is helpful to describe the methods used in solving each partial differential equation system. The system driven by replicator dynamics is solved using the method of lines.[7] I discretize the space of player types such that mesh grid $dc = \frac{1}{20} = .05$ and obtain 21 discrete points on the line $c \in [0,1]$. Thus, there is a system of ordinary differential equations for each point (where $T = 2000$). To solve the system, I use NDSolve in Mathematica.

Under gradient dynamics, I apply the upwind method.[8] This approach is appropriate when solving partial differential equations that may contain some discontinuity in the gradient term (or velocity according to the literature on fluid dynamics, see LeVeque, 2005). Similar to replicator dynamics, the space c is discretized, but now I assume $t \in [0,20]$ with mesh grid $dt = 0.01$. In order to satisfy the Courant-Friedrichs-Lewy condition that is necessary for stability, I allow the time mesh grid to vary in simulations. All derivatives with respect to space c in equations (8), (13) and (14) are approximated to the first-order.

Figure 1 shows the numerical solution using replicator dynamics. The stationary distribution (represented at time T) is achieved when the marginal contributor has a subjective cost equal to k. Players with cost lower than or equal to k contribute to the public good and $P(T) = 1$ while players with cost higher than k do not contribute and $P(T) = 0$.

The evolution of types under replicator dynamics shows that players with subjective cost below the marginal contributor gain mass as opposed to players with cost above the marginal contributor. I present the distribution F and the associated value for the marginal

[7]This method is also known as semidiscretization, since only one dimension (space) is discretized. For example, see Iserles (2006).

[8]This scheme is called upwind, because it follows the direction that gradient term flows. For a description of the scheme applied to a continuous action space, see Appendix B of Friedman and Ostrov (2013).

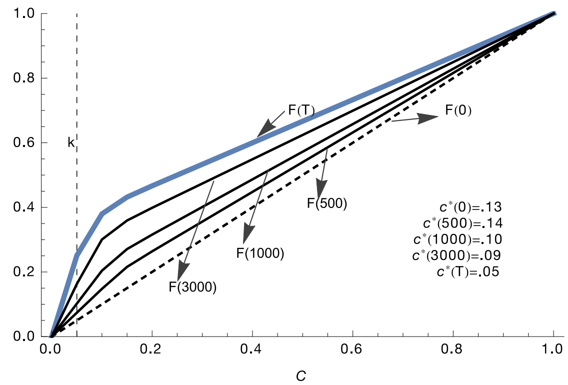


Figure 2: Evolution of distribution of types (F) and marginal type c^* under replicator dynamics for iteration 0, 500, 1000, 3000 and T.

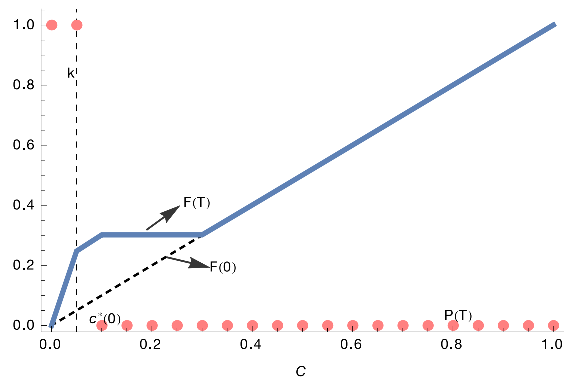


Figure 3: Initial and stationary distribution of types (F) and stationary probability of contribution (P) under gradient dynamics.

contributor c^* at different points in time in Figure 2. This figure illustrates that the marginal contributor evolves towards the material cost $k = 0.05$. The stationary distribution clearly shows that the types which increase in mass are the ones located at or below the value of k, followed by the types located between the values of k and $c^*(0)$. This adjustment occurs at the expense of types associated with greater subjective cost. Consequently, the types with cost greater than $c^*(0)$ lose mass.

When using gradient dynamics, I assume the same initial conditions as under replicator dynamics. The stationary distribution depicted in Figure 3 shows a significant mass around k, which corresponds to the value of the marginal contributor. Players with cost lower than or equal to the marginal type contribute with probability equal to one, while players with cost higher than the marginal type do not contribute.

Figure 4 shows that under gradient dynamics the distribution of types adjusts smoothly towards k. In the steady state, the density of types higher than k is either zero or unchanged from the initial distribution. The stationary distribution also depends on the initial conditions of the probability of contribution. For example, Figure 5 assumes that the initial probability of contribution is inversely related (at a constant rate) to c. Comparing dynam-

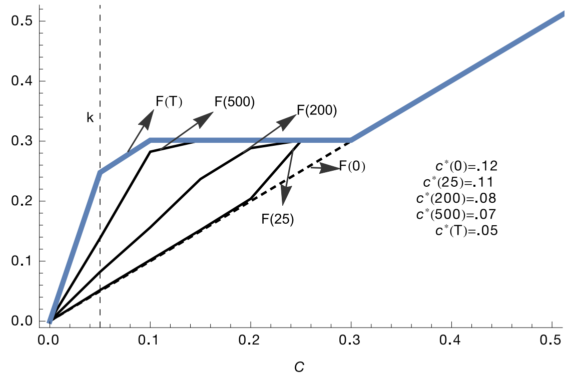


Figure 4: Evolution of distribution of types (F) and marginal type c^* under gradient dynamics for iteration 0, 25, 200, 500 and T.

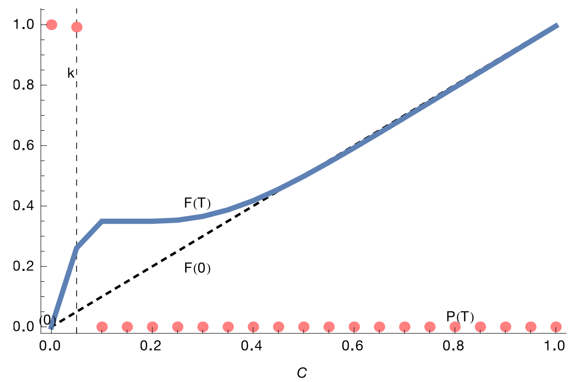


Figure 5: Stationary distribution of types (F) and probability of contribution (P) under gradient dynamics for decreasing initial probabilities of contribution (at a constant rate).

ics under two sets of initial conditions, we can see that the stationary distribution under the inversely related probability shows a larger mass around k and therefore more types, with cost higher than k, will have no density.

Both families of dynamics presented in this section achieve the predictions specified in Proposition 1. The stationary distribution shows that the marginal contributor has a subjective cost equal to k, the material cost. Players with subjective cost lower than k contribute with probability equal to one, while players with subjective cost higher than k do not contribute at all. The adjustment of the distribution as well as the stationary distribution, however, are quite different under alternative dynamics.

Gradient dynamics assumes that evolution is costly and the stationary distribution indeed shows that types who are far from the marginal type do not change their density. Replicator dynamics assumes that evolution is driven by the difference in rates of birth and death within a population. Thus, under replicator dynamics, prevalence of some types increases at the expense of other types. This result is different compared to gradient dynamics, in which the gain in mass around k decreases the density (to zero) of the types that are locally bigger than k. The density of other types with cost higher than k remains unchanged.

Based on the numerical and graphic analysis of the public goods game, it appears that the initial conditions of the probability of contribution only affect gradient and not replicator dynamics. Next, I present a Cournot duopoly model, based on the work of KLS14.

3 A Cournot duopoly model

KLS14 analyze the evolutionary stability of two firms (Cournot duopoly) who are randomly matched at the time. The firms can be either standard profit maximizers (labeled P) or socially responsible (labeled S). S firms seek to maximize the sum of profits as well as the consumer surplus (CS). Each firm i weighs the CS with a fixed parameter $\theta_i \in [0,1]$ in its profit function. θ_i can be interpreted as the level of awareness of firm i to consumer well-being. Thus, P firms set $\theta_i = 0$ and S firms choose any $\theta_i \in (0,1]$. KLS14 work with two possible firm types $\theta_i \in \{0,\theta\}$. For this analysis, I assume that the types belong to a continuum and that all products are perfect substitutes.[9]

The utility of a consumer who chooses products $q_i(\theta_i,t)$ is

$$U = \sum_{i=1}^{2} a(\theta_i) \cdot q_i - \frac{1}{2}\sum_{i=1}^{2} q_i^2 - q_i q_{-i} \tag{15}$$

where $a(\theta_i) = 1+\theta_i, \in [1,2]$. This implies that consumers are willing to pay more for products when firms place a higher weight on CS. Note when $a = 1$, the firm is a standard profit maximizer, P, and when $a > 1$, the firm is type S. The consumers, facing a typical budget constraint with prices p_i, behave rationally, giving rise to a standard inverse demand $p_i = a(\theta_i) - q_i - q_{-i}$, where the subscript $-i$ refers to the other firm. All firms face a constant variable cost c_i and their subjective profits are given by

$$\pi_i^s(\theta_i,t) = (p_i - c_i)q_i + \theta_i \cdot CS \tag{16}$$

where $CS = U - p_i q_i - p_{-i}q_{-i} = \frac{1}{2}(\sum_{i=1}^{2} q_i^2) + q_i q_{-i}$. The profits in equation (16) are subjective because type S firms give weight to CS. This awareness of CS does not directly affect the material payoff. The fact that type S firms react to subjective payoffs is standard in indirect evolutionary approach to preferences (e.g. Güth and Yaari, 1992). Type P firms do not consider CS and therefore the subjective profits of these firms are equivalent to their material profits. The subjective profit is relevant when determining the best-response for

[9] This assumption simplifies the presentation of the model. KLS14 analyze exhaustively both complements and substitutes in their static analysis, but make similar assumptions to ease the exposition of their dynamics.

each firm, while the material profit helps determine the evolution of types. Similar to my previous example, I use both replicator and gradient dynamics to explain the evolution of θ_i.

Lastly, KLS14 assume that the distribution of firms is drawn from a single population and that firms are not able to perfectly observe the competitor's type. They develop a complete system of ordinary differential equations that explains the evolution of firm types. I enrich their discrete two type model by developing a dynamic system that allows for a continuum of types.

3.1 Dynamics of adjustment

The model assumes a continuum of types $\theta_i \in [0,1]$ distributed with cdf $F(\theta,t)$. Assuming zero marginal cost ($c_i = 0$), without the loss of generality, the subjective profits, π_i^s, of firm i are then

$$\pi_i^s(\theta_i,t) = [a_i(\theta_i) - q_i - Eq]q_i + \theta_i[\frac{1}{2}q_i^2 + \frac{1}{2}(Eq)^2 + (q_i)(Eq)] \tag{17}$$

where $Eq = \int_0^1 q(\theta)f(\theta,t)d\theta$ is the expectation of the other firm's behavior. Similarly, the fitness for each firm is given by

$$\pi_i(\theta_i,t) = [a(\theta_i) - q_i - Eq]q_i \tag{18}$$

I assume that a firm's production decision is driven by sign-preserving dynamics. That is, the firm will increase (decrease) production as long as the gradient $d\pi_i^s/dq_i > 0\,(< 0)$. The gradient with respect to production is[10]

$$\begin{aligned} d\pi_i^s/dq_i &= a(\theta_i) - 2q_i - Eq + \theta_i q_i + \theta_i Eq \\ &= a(\theta_i) - (2-\theta_i)q_i - (1-\theta_i)Eq \end{aligned} \tag{19}$$

Assuming that firms behave rationally and that the expected quantity is the same as the actual quantity chosen by the other firm, the optimal production for firm i is $q_i^* = \frac{a(\theta_i)\cdot(2-\theta_{\_i})-a(\theta_{\_i})\cdot(1-\theta_i)}{3-\theta_i-\theta_{\_i}}$, where $a(\theta_i)$ is the intercept of the (inverse) demand for the products of firms with awareness θ_i.

The evolution of the distribution of types can be described by two systems: replicator and gradient dynamics. Under gradient dynamics (Friedman and Ostrov 2010, 2013), θ_i increases (or decreases) with the value of the fitness gradient $\frac{d\pi_i(\theta_i,t)}{d\theta_i}$. Specifically, they

[10] As mentioned earlier, I adopt gradient dynamics since they are a familiar specification that are a reasonable approximation for dynamics arising from a wide variety of evolutionary processes. KLS14 work with best-replies, which are equivalent to gradient dynamics at $d\pi_i/dq_i = 0$.

conclude that beginning from an arbitrary initial density $f(\theta_i,0)$, the distribution evolves according to the partial differential equation

$$F_{it}(\theta_i,t)+\frac{d\pi_i(\theta_i,t)}{d\theta_i}\cdot f_i(\theta_i,t)=0 \tag{20}$$

The gradient term can be computed as

$$\frac{d\pi_i(\theta_i,t)}{d\theta_i}=(a(\theta_i)-c-2q_i-Eq)\frac{dq_i}{d\theta_i}+q_i\frac{da(\theta_i)}{d\theta_i}-q_i\frac{dEq}{d\theta_i} \tag{21}$$

In steady state, the distribution is supported on isolated points (where f_i is a constant times a Dirac delta function) and/or on intervals where the gradient $\frac{d\pi_i(\theta_i,t)}{d\theta_i}=0$.

In the discrete case, KLS14 find three families of equilibria, which are dependent on the values of θ_i (see their Proposition 4, pp.12).[11] When $a>1$ and θ_i is low, the stationary equilibrium requires that all firms are type S; for intermediate values of θ_i, both types survive and when θ_i is high enough, only type P firms survive. The following proposition proves that the first scenario (only type S firms survive) is a stationary equilibrium in the continuous case of the Cournot duopoly game with $a(\theta_i)=1+\theta_i$ and $c_i=0$.

Proposition 2 *Distribution F is stationary when* $\frac{d\pi_i(\theta_i,t)}{d\theta_i}=0$. *Therefore, parameter values that satisfy this condition are* $\theta^*=0.31$ *and* $q^*=0.56$.
Proof: I assume two possible equilibria, following the work of KLS14: (i) both types survive or (ii) all surviving firms are of the same type. I begin by analyzing the first case.

In the case that both P and S firms survive, the distribution of types is bimodal, in which $\theta_i=0$ and $\theta_{-i}=\theta$. Recall that this is an incomplete information game and therefore each firm takes into consideration the expected production and distribution of types. The equilibrium depends on three endogenous variables: firm i production, firm $-i$ production and the share of firm types. These three variables can be determined using the best response of each firm, equation (19), and the equivalent payoff condition, which states

[11] Defining the parameters $c=0$, $\theta_A=\frac{3-a}{2}-a^2+a\sqrt{a^2+a-2}$, and $\theta_B=\frac{1}{2}(\sqrt{9a^2-6a-3}-3a+3)$, they find the following

- When $a>1$ and $0<\theta<\theta_A$, the model admits two equilibria: (i) all firms are type S, producing $a/(3-2\theta)$, which is locally stable, and (ii) all firms are type P, producing $1/3$, which is a saddle point.
- When $a>1$ and $\theta_A<\theta<\theta_B$, the model admits two corner equilibria, and an inner equilibrium. Both corners equilibria are saddle points, and the inner equilibrium is defined as coexistence of both types. The type S produces $(a(\theta(2\theta-3)+2)-(\theta-3)\theta-2-2\sqrt{\psi})/\theta^2$, where $\psi=(a-1)(\theta-1)^2(a((\theta-1)\theta+1)+\theta-1)$. The type P responds as $h(q)=(2-\theta)q/2(1-\theta)+(1-a-\theta)/(2(1-\theta))$.
- When $a>1$ and $\theta_B<\theta\leq 1$, or when $a=1$ and $0<\theta\leq 1$, the model admits two corner equilibria. All firms are type P, which is locally stable; or all firms are type S, which is a saddle point.

that the profit of being a type S or type P firm is the same. If the profits are not the same, then one type will have an evolutionary advantage over the other type and only one type of firm will prevail.[12] The next step is to corroborate that this equilibrium is stationary. To do so, I need to evaluate whether the gradient term $\frac{d\pi_{-i}}{d\theta_{-i}}$ is zero. I replace the production decision in the profit function (18) by firm's best response and use the share of firm types that satisfies the equal payoff condition. The profit function now depends on θ_i and θ_{-i}. I compute $\frac{d\pi_{-i}}{d\theta_{-i}}$ and find that it is negative for $\theta_{-i} \in (0,1]$.[13] Hence, a bimodal distribution is not part of the stationary equilibrium.

Next, I analyze the second case, where in equilibrium all surviving firms are of the same type. The optimal firm production is $q^* = \frac{1+\theta^*}{3-2\theta^*}$. To find the value of $\theta_i \in [0,1]$ that satisfies $\frac{d\pi_i}{d\theta_i} = 0$, consider a firm thinking about deviating from this equilibrium (choosing a different θ). This firm will encounter, with equal probability, a firm of the same type and a firm of a different type, where the production of the other firm is taken as given. Using the optimal production decision q^* in the profit function (18), I compute the value of θ_i that satisfies $\frac{d\pi_i}{d\theta_i} = 0$. I find that $\theta^* = 0.31$ and consequently $q^* = 0.56$ $\square$

The stationary distribution suggested by Proposition 2 is consistent with the results of KLS14.[14] Prior to verifying the static analysis with a simulation of the dynamic system, I present how θ_i evolves under replicator dynamics. According to Oechssler and Riedel (2001, 2002), the distribution of types varies according to the difference in individual and population payoffs. The population payoff is computed as $\bar{\pi} = \int_0^1 \pi_i(\theta,t)\, f_i(\theta,t)d\theta$ and therefore the dynamic system that governs the evolution of production and types (under both replicator and gradient dynamics) consists of the following partial differential equations, which depend on time t and space θ (I omit the subscript i for convenience),

$$q_t(\theta,t) = \beta q(a-q)\frac{d\pi^S(\theta,t)}{dq} \tag{22}$$

$$F_t(\theta,t) = -\frac{d\pi(\theta,t)}{d\theta} f(\theta,t) \qquad \textbf{[gradient]} \tag{23}$$

$$F_t(\theta,t) = \int_0^\theta (\pi(\theta,t)-\bar{\pi})dF(\theta,t) \qquad \textbf{[replicator]} \tag{24}$$

where the evolution of production $q(\theta,t)$ in equation (22) includes the term $q(a-q)$, to help restrict quantity to the $[0,a]$ range, and where the gradient of the subjective profits with respect to production is given by equation (19). Lastly, the speed of adjustment of

[12] To ease the exposition of the paper, I do not re-write most of the equations presented by KLS14 that describe firm production and firm types share.

[13] The gradient term for the type S firm evaluated at the optimal share and firm's best reply is $\frac{4\theta(15\theta^3-7\theta^4-48\sqrt{\theta^2+\theta^4}+\theta(48+2\sqrt{\theta^2+\theta^4})+\theta^2(-6+7\sqrt{\theta^2+\theta^4}))}{5(\theta-\theta^2+\sqrt{\theta^2+\theta^4})^3}$.

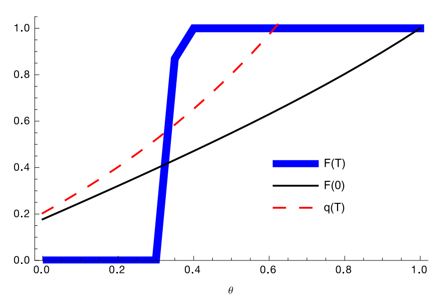
[14] I confirm their results by replacing θ^* and $a = 1+\theta^*$ in the expressions that prove their Proposition 4. Out of the three families of equilibria described in footnote 11, only the first family survives in the long run.

production and distribution of types may be different ($\beta \geq 1$). The evolution of types, represented by the cdf F, can be driven by two alternative specifications: equation (23) assumes gradient dynamics, while equation (24) follows replicator dynamics. The main reason for analyzing two different specifications for the evolution of θ is to compare whether they lead to the same stationary distribution under reasonable initial conditions.

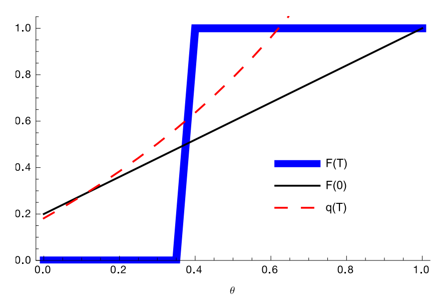
3.2 Numerical solution

The baseline parameters used in the numerical solution are $c = 0$, $\beta = 1$, and $a = 1 + \theta, \in [1,2]$. Space θ is discretized into 21 points. The method for solving the system of equations differs according to the underlying dynamics. Under replicator dynamics, I apply the method of lines. That is, each discrete point in space θ contains a system of ordinary differential equations (that depends on time). I use Mathematica (NDSolve) to solve this system and work with 80,000 iterations. Under gradient dynamics, I solve the system using the upwind scheme.[15] Although I use $dt = 0.01$ in my analysis, dt can vary to satisfy the Courant-Friedrichs-Lewy condition, which is necessary for stability (LeVeque, 2005). The number of iterations under gradient dynamics is 8,000.[16] The initial distribution of types is 0.2 when $\theta = 0$, and increases at a constant rate when $\theta > 0$. The initial quantity is 0.05 for all types.

Figure 6 depicts the results of the dynamic system under replicator (left panel) and gradient (right panel) dynamics. Both systems achieve the same stationary distribution. However, the distribution under gradient dynamics is slightly to the right of the replicator dynamics by one discrete point due to difference in the numerical schemes used. All type S firms choose $\theta^* = 0.3$ and produce $q^* = 0.56$. These results are in line with the predictions stated in Proposition 2.



(a)



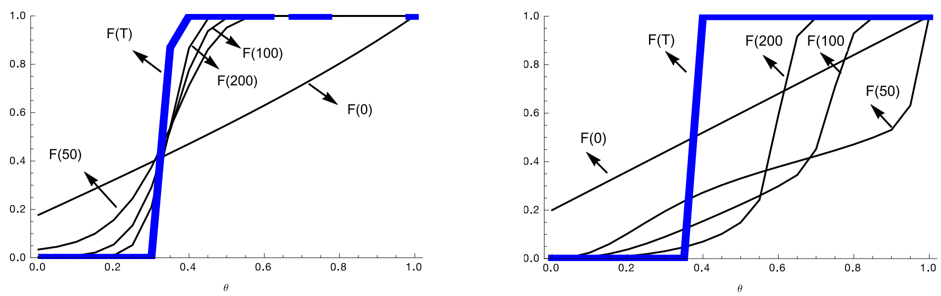
(b)

Figure 6: Production and distribution of firm types under (a) replicator and (b) gradient dynamics.

[15] See footnote 8.

[16] In this numerical solution, I use a larger number of iterations compared to the public goods game since the initial distribution of types (uniform) is quite different compared to the stationary distribution.

Interestingly, the adjustment towards the stationary distribution is different in each system (see Figure 7). Under replicator dynamics, the types below θ^* lose density in favor of the types greater than θ^*. Under gradient dynamics, the distribution F shifts down for all types except those with the highest θ. Eventually, these high θ types are outperformed by intermediate θ types and the distribution shifts towards θ^*.



(a) (b)

Figure 7: Distribution of types at iteration 0, 50, 100, 200 and T under (a) replicator and (b) gradient dynamics.

The dynamic model can also be used to analyze the parametric assumptions of $a(\theta)$ made by KLS14, or a number of other specifications. For example, in the case that a is constant and equal to one for all types, the numerical solution shows that only type P firms survive. Moreover, if a is a binary parameter that increases to same constant value when $\theta_i > 0$, then the stationary distribution has a unique mass at the minimum value of θ_i greater than zero.

4 Discussion

I have analyzed the dynamic stability of two different games, in which a continuum of types is present, using two families of dynamics: replicator and gradient. In the first example, the underlying game relies on the individual decision to contribute to the public good. Each player has a subjective cost and her fitness depends on the contribution decisions of self and others, and a fixed material cost. The equilibrium is characterized by players with cost lower than or equal to some (endogenous) threshold who contribute to the public good, while types with higher costs do not contribute. The stationary equilibrium is achieved when the subjective cost of a marginal contributor is equal to the fixed material cost. Although, both replicator and gradient dynamics achieve the equilibrium predictions, the resulting stationary distributions and their evolutionary dynamics are quite different.

Under replicator dynamics, the adjustment arises from changing birth and death rates of different types. The types with costs lower than the marginal contributor exhibit higher growth rates, compared to the types with costs higher than the marginal contributor. Under gradient dynamics, the topology of types is key to determining evolution in the long-run. Simulations show that adjustment towards the fixed material cost happens smoothly and locally. These smooth dynamics do not impede us from observing some discontinuities in the stationary distribution. The resulting distribution shows that the types locally above the marginal contributor have no density.

The second example complements the recent work of KLS14. The authors study the evolution of two possible types of firms: the standard profit maximizer firms and the socially responsible firms who consider consumer surplus as part of their subjective profits. I enrich the work of KLS14 by endogenizing the (continuous) firm types, using replicator and gradient dynamics. I assume that the willingness to pay is a linear function that incorporates firm awareness of consumer welfare (θ). I show that the same stationary distribution is achieved under both dynamics, and that in equilibrium only socially responsible firms are present. The stationary equilibrium found in this analysis is part of a plethora of equilibria described in KLS14. The advantage of introducing gradient and replicator dynamics is that I am able to identify which type of equilibrium survives.

Perhaps, the main question that arises when using alternative family of dynamics is which specification is more appropriate? Although I illustrate that in both examples the underlying dynamics are not trivial, in the second example, gradient dynamics might be more appropriate. In the Cournot duopoly example, gradient dynamics can represent an internal firm process that selects a level of awareness regarding consumer surplus. The distribution of types (and the level of production) captures the beliefs of the firms, and it is only when the process converges to an invariant belief distribution that the firm commits to production. Additional field or laboratory data can provide further insight in selecting appropriate dynamics. The techniques developed in this paper center on two families, but can easily be extended to a wide variety of dynamics, and more importantly to different types of games.

5 Acknowledgements

I am grateful to two anonymous referees of this journal, Mark Cheung, Dan Friedman, Michael Kopel and Olya Rabanal for their comments and suggestions.

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