Coalitional Bargaining Equilibria

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Outline

Motivation

- Existence of stationary subgame perfect equilibria is important in dynamic models of bargaining in political science and coalition formation in economics.
- The paper covers known environments:
  - BD 2000: convex set of alternatives, concave stage utility, "bad status quo"
  - BD 2006: convex set of alternatives, concave stage utility, common discount factor
  - RV 1999: NTU game with strict comprehensiveness*
- In all of the above, mixed proposal strategies are allowed to obtain existence.

* I cover the NTU environment, but not the rejector-becomes-proposer protocol.

Motivation (cont.)

- The paper extends existence to many environments of interest:
  - quasi-concave stage utility with bad status quo
  - heterogeneous discount factors with Pareto optimal status quo
  - exchange economies, where coalitions have property rights and consumption is non-negative
  - NTU environments without strict comprehensiveness.
- It unifies the analysis of finite and infinite models.
- And it captures the “new” class of collective dynamic programming problems with metric space of actions and finite states.

Motivation (cont.)

- Key idea: Allow mixed voting as well as mixed proposal strategies.
- Proof requires precise selection of voting equilibria, an "infinite-dimensional" problem.
- I reduce the problem to a finite-dimensional one by submerging voting strategies in the fixed point argument, backing them out at the end.
- Uses a Fatou-type lemma on upper hemicontinuity of integrals of correspondences.

Literature review

- Bargaining in political science
  - Baron and Ferejohn (1989)
  - Baron (1991)
  - Banks and Duggan (2000,2006)
  - Jackson and Moselle (2002)
  - Kalandrakis (2004)
- Coalition formation in economics
  - Bloch (1996)
  - Krishna and Serrano (1996)
  - Okada (1996)
  - Chatterjee, Dutta, Ray, and Sengupta (1993)
  - Ray and Vohra (1999)

Outline

Example 1: Mixing and delay

- Three agents, three outcomes, and stage utilities to the right.

	1	2	3
a	2	.7	0
b	0	2	.7
c	.7	0	2

- Each period, an agent is drawn at random to make a proposal or remain silent.
- A proposal passes if it is accepted by another agent, and the game ends.
- Otherwise, the process repeats.
- Payoffs are discounted by $\delta = .9$.

Example 1 (cont.)

- A pure stationary strategy consists of a proposal for each agent and a response for all possible proposals.

	1	2	3
a	2	.7	0
b	0	2	.7
c	.7	0	2

- There is no stationary equilibrium in pure strategies.
- Claim 1: $\delta v_i \geq .7$. Suppose $\delta v_1 < .7$. Then agent 3 proposes c and it passes. If agent 2 could pass b, we would have

$$\delta v_3 \;\geq\; (.3)[0 + .7 + 2] \;>\; .7 \;=\; u_3(b),$$

so agent 3 rejects b. Thus, $\delta v_2 < .7$. But then agent 1 proposes a and it passes, so $\delta v_1 \geq .7$, a contradiction.

Example 1 (cont.)

	1	2	3
a	2	.7	0
b	0	2	.7
c	.7	0	2

- Claim 2: $\delta v_i \leq .7$
  Suppose $\delta v_1 > .7$. Then agent 1 rejects c. Since the agent's payoff from remaining silent exceeds the payoff from c, agent 1 does not propose c. Then $\delta v_3 < .7$. Then agent 2 proposes b, and it passes. If agent 1 could pass a, we would have

  $$\delta v_2 \ \geq \ (.3)[.7 + 2 + 0] \ > \ .7 \ = \ u_2(a),$$

  so agent 2 rejects a. Thus, $\delta v_1 < .7$, a contradiction.
- Therefore, each agent's discounted continuation value is $\delta v_i = .7$.

Observations

- But no pure strategy profile yields these discounted continuation values, so there is no pure strategy equilibrium.
- There are stationary equilibria in pure **voting** strategies: each agent proposes her second-best outcome with probability .39 (this passes) and her best with probability .61 (this fails).
- There are a continuum of equilibria, as instead of proposing her favorite outcome, an agent can simply remain silent. . .
- . . . or she could always propose her second-favorite outcome but reject it with probability .61.

Observations (cont.)

- Voting when agent 1 proposes a has to be different when agent 2 proposes it, because agent 2 votes for a with probability at least .39 when she proposes it.
- If voting strategies were independent of proposer, then since agent 1 can propose a, we would have

$$\delta v_1 \geq (.3)[(.69)(2) + 0 + 0] > .7,$$

  and similarly for agents 2 and 3.
- But then all proposals fail, and the equilibrium breaks down.

Example 2: Mixed voting strategies

- Now assume agent 1 proposes with prob. .8, and agent 2 with probability .2.

	1	2	3
a	2	.7	0
b	0	2	.7
c	.7	0	2

- There is no stationary equilibrium in pure voting strategies.
- Claim 1: When agent 1 proposes a, it cannot fail with probability one. Suppose it does. Then $\delta v_2 \geq .7$. If agent 1 proposes a with positive probability in equilibrium, then since c will pass, we have $\delta v_1 \geq .7$, so agent 2 proposes a with positive probability, but then agent 2's expected payoff from proposing is only .7, so $\delta v_2 < .7$. So agent 1 never proposes a. Likewise, agent 1 does not remain silent.

Example 2 (cont.)

Thus, agent 1 proposes c in equilibrium, and $\delta v_2 < .7$, a contradiction.

	1	2	3
a	2	.7	0
b	0	2	.7
c	.7	0	2

- Claim 2: When agent 1 proposes a, it cannot pass with probability one. Suppose it does. Then $\delta v_2 \leq .7$. Then

$$\delta v_3 \ \leq\ (.9)[(.8)(0) + (.2)(2)] \ <\ .7 \ =\ u_3(b),$$

so agent 2 proposes b, and it passes. But then

$$\delta v_2 \ \geq\ (.9)[(.8)(.7) + (.2)(2)] \ >\ .7,$$

so agent 2 rejects a, a contradiction.

Outline

Model

- State s in S is publicly observed.
- Agent $i(s)$ proposes x, where x belongs to metric space X.
- Agents vote simultaneously to accept or reject x.
- If $x \in X_C(s)$ and all members of C accept, then period t outcome is $z = x$. Otherwise, it is $z = q$.
- Stage utility $\delta_i^{t-1} u_i(z, s)$ accrues to i.
- New state is drawn from $p(s'|z, s)$, and we repeat.

Assumptions

- Finite set S of states.
- Sets $X_C(s)$ are compact.
- Monotonicity: $C \subseteq C'$ implies $X_C(s) \subseteq X_{C'}(s)$.
- Proposers can remain silent: $q \in X_{\{i(s)\}}(s)$.
- Functions $u_i(x, s)$ and $p(s'|x, s)$ are continuous.
- Imperfect patience: $\delta_i < 1$.

- No convexity conditions are imposed.

Strategies and payoffs

- Proposal strategies: $\pi_i \colon S_i \to \mathcal{P}(X)$
- Voting strategies: $\alpha_i \colon X \times S \to [0,1]$
- Continuation value to agent i beginning in state s is:

$$\begin{aligned} v_i(s) &= \int_X \Big[\alpha(x,s)[u_i(x,s) + \delta_i \sum_{s'} p(s'|x,s) v_i(s')] \\ &\quad + \rho(x,s)[u_i(q,s) + \delta_i \sum_{s'} p(s'|q,s) v_i(s')]\Big] \pi_{i(s)}(dx). \end{aligned}$$

Strategies and payoffs (cont.)

- Dynamic payoff from outcome z in state s is:

$$U_i(z,s) \quad = \quad u_i(z,s) + \delta_i \sum_{s' \in S} p(s'|z,s) v_i(s').$$

Note that U_i is continuous.

Coalitional bargaining equilibrium

- Optimal proposals: $\pi_i(s)$ puts probability one on solutions to

$$\max_{x \in X} \alpha(x,s)U_i(x,s) + \rho(x,s)U_i(q,s)$$

- Optimal voting:

$$\alpha_i(x,s) \quad = \quad \begin{cases} 1 & \text{if } U_i(x,s) > U_i(q,s) \\ 0 & \text{if } U_i(x,s) < U_i(q,s), \end{cases}$$

  with no restriction when indifferent.
- Refines stationary Markov perfect equilibrium by eliminating stage-dominated voting strategies.

Outline

Existence

Theorem

There exists a coalitional bargaining equilibrium.

Special case: Legislative bargaining

- States are the proposers plus a terminal state.
- In every period, each agent j is selected with probability p_j to make a proposal or remain silent.
- Feasible sets are determined by a fixed, monotonic collection $\mathcal{D}$ of decisive coalitions.
- If a proposal x is accepted by a decisive coalition $C \in \mathcal{D}$, then the game transitions to the terminal state, and agents receive $u_i(x)$ in each subsequent period.
- Otherwise, agents receive $u_i(q)$, a new proposer is selected, and the game is played again.

Legislative bargaining (cont.)

- Typically, $X \subseteq \Re^d$ is convex, and stage utilities u_i are concave.
- Say the status quo is "bad" if $u_i(q) = 0$ and for all $x \neq q$, $u_i(x) > 0$.
- It is customary to normalize payoffs by $(1 - \delta_i)$, so an agent i's vote is determined by the comparison:

$$u_i(x) \quad \text{vs.} \quad (1 - \delta_i)u_i(q) + \delta_i v_i.$$

- Coalitional bargaining equilibria correspond to the usual concept, except we allow voting strategies to depend on the proposer as well as the proposal — see Example 1.

Standard proof approach

- Consider legislative bargaining with concave utilities and bad status quo.
- We can restrict attention to no-delay equilibria in pure voting strategies: agents accept when indifferent.
- Continuation values can then be calculated from proposal strategies:

$$v_i(\pi) \quad = \quad \sum_j p_j \int_X u_i(x)\pi_j(dx).$$

Standard approach (cont.)

- Each agent's voting strategy is characterized by an acceptance set:

$$A_i(\pi) \;\; = \;\; \{x \mid u_i(x) \geq \delta_i v_i(\pi)\}.$$

- The set of proposals that will pass is:

$$A(\pi) \;\; = \;\; \bigcup_{\mathcal{D}} \bigcap_{i \in C} A_i(\pi).$$

Standard approach (cont.)

- The optimization problem of a proposer is to maximize over the set $A(\pi)$ of proposals that pass.
- Existence follows from a fixed point argument in the space of proposal strategy profiles:

$$\pi \longrightarrow \{v_i(\pi)\} \longrightarrow A(\pi) \longrightarrow \left\{ \mathcal{P}\left(\arg\max_{x \in A(\pi)} u_i(x) \right) \right\}$$

- Here, to find "best response" proposals, it is sufficient to begin with proposal strategies.

General legislative bargaining

- Keeping the legislative bargaining framework, drop convexity and concavity, and allow Pareto optimal status quo and heterogeneous discount factors.
- Then we cannot assume indifferent agents always accept... or always reject.
- Example 2 shows that we must exploit the possibility that indifferent agents can use mixed voting strategies.
- Then a proposer's optimal proposals are not determined by proposal strategies alone; we need information about voting outcomes as well.

Idea of proof

- One possibility is to include voting strategies in the domain of the correspondence, as in $(\pi, \alpha) \to (\pi', \alpha')$, but the space of voting strategies (measurable functions in $[0,1]^{X \times S}$) does not appear to have a useful topology.
- I circumvent the difficulty by reducing the problem to a finite-dimensional one...
- ...by expanding the domain to include continuation values and the expected payoff, denoted w_i, to each agent i when selected as proposer.
- We want a correspondence $(\pi, w, v) \to (\pi', w', v')$ such that: (i) a fixed point exists, and (ii) fixed points correspond to coalitional bargaining equilibria.

Backing out acceptance probabilities

- To update (π, w, v) to vectors (π', w', v'), we need to know acceptance probabilities of proposed outcomes in different states. We back them out.
- Suppose agent i proposes x in state s such that

$$u_i(x) \;\geq\; w_i \;\geq\; (1-\delta_i)u_i(q) + \delta_i v_i.$$

- Then the acceptance probability satisfies

$$w_i \;\;=\;\; \alpha(x,i)\big[u_i(x)\big] + \rho(x,i)\big[(1-\delta_i)u_i(q) + \delta_i v_i\big].$$

Backing out (cont.)

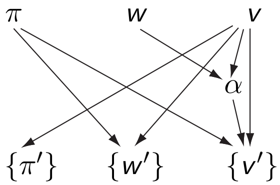
- That is,

$$\alpha(x,i) \quad = \quad \frac{w_i - (1-\delta_i)u_i(q) - \delta_i v_i}{u_i(x) - (1-\delta_i)u_i(q) - \delta_i v_i}.$$

- We can bound this between zero and one, so it is well-defined as long as $u_i(x) > (1-\delta_i)u_i(q) + \delta_i v_i$.
- When equality holds, the acceptance probability is indeterminate, so we can take any **selection** of acceptance probabilities.
- We can then update proposer payoffs to w' and v' for each such selection.

Fixed point correspondence

- So the correspondence has the form to the right.
- By a Fatou's lemma argument, the correspondence has closed graph, etc., so it has a fixed point, satisfying (i).
- For (ii), more details about the correspondence are needed.



Equilibrium strategies

- The correspondence is defined so that π_i is optimal given the selection α of acceptance probabilities, and so that the indeterminacy in acceptance probabilities is only critical when the indifferent agents are pivotal.
- If they all reject x, then it fails, and if they all accept x, then it passes.
- By the intermediate value theorem, they can mix so that x indeed passes with probability $\alpha(x, i)$ when proposed by i.
- After minor adjustments, we have a coalitional bargaining equilibrium.

Outline

Conclusion

- Coalitional bargaining equilibria exist when stage utilities are non-concave, or the status quo is Pareto optimal and discount factors are heterogeneous, or consumption is bounded below.
- The game need not end when the first agreement is reached, so we cover collective dynamic programming problems with metric space of actions and finite states.
- I do not cover the rejector-becomes-proposer protocol: equilibrium outcomes in such voting games are potentially non-convex, creating problems for our argument.
- I conjecture that existence is obtained if we allow for correlated voting equilibria.