Coalitional Bargaining Equilibria

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Motivation

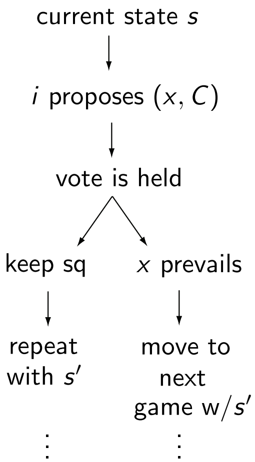
- The paper covers bargaining models from political science and related models of coalition formation from economics.
- Existence of stationary equilibria is not known in a wide class of bargaining models.
  - heterogeneous discount factors with Pareto optimal status quo, even in one dimension
  - non-concave stage utilities, even with strict quasi-concavity
  - exchange economies, where coalitions have "property rights" and consumption is non-negative
  - NTU environments where transfers are not too large
- I generalize known existence results to cover these cases, and I unify analysis of finite and infinite models.
- Key idea: Allow mixed voting as well as mixed proposal strategies.

Literature review

- Bargaining in political science
  - Baron and Ferejohn (1989)
  - Baron (1991)
  - Banks and Duggan (2000,2006)
  - Jackson and Moselle (2002)
  - Kalandrakis (2004)
- Coalition formation in economics
  - Bloch (1996)
  - Krishna and Serrano (1996)
  - Okada (1996)
  - Chatterjee, Dutta, Ray, and Sengupta (1993)
  - Ray and Vohra (1999)

Literature review (cont.)

- Basic protocol
- Variations
  - proposer selection is random vs. deterministic
  - environment is policy space with given winning coalitions vs. TU game vs. NTU game
  - voting is simultaneous vs. sequential (and next state depends on first rejector)
  - status quo payoff is bad vs. not
  - end when x prevails vs. not
- We do not capture general models with infinite state space.



Model

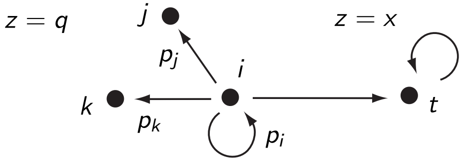
- State s in finite S is publicly observed.
- Legislator $i(s)$ proposes (y, C), where y lies in metric space X.
- Legislator vote simultaneously to accept or reject y.
- If $y \in X_C(s)$ and all members of C accept, then period t outcome is $z = y$. Otherwise, it is $z = q$.
- Stage utility $\delta_i^{t-1} u_i(z, s)$ accrues to i.
- New state is drawn from $p(s'|z, s)$, and we repeat.

Assumptions

- Sets X and $X_C(s)$ are compact.
- Implicit coalitions: $X_C(s) \cap X_{C'}(s) = \emptyset$.
- Proposers can pass: $q \in X_{\{i(s)\}}(s)$.
- Functions $u_i(x, s)$ and $p(s'|x, s)$ are continuous.
- No perfect patience: $\delta_i < 1$.

Special cases

- Baron-Ferejohn
  - $S = N \cup \{t\}$, $i \in N$ proposer, t terminal,
  - X a simplex in $\Re^n$ plus default q,
  - $X_C(s)$ is X if $|C| > \frac{n}{2}$, and is empty otherwise.
  - transition probability, $p(s'|z,s)$:



  - stage utilities:

$$
\begin{aligned}
u_i(z,j) &= \begin{cases} \frac{z}{1-\delta_i} & \text{if } z \text{ in simplex,} \\ 0 & \text{else,} \end{cases} \\
u_i(z,t) &= 0.
\end{aligned}
$$

Special cases (cont.)

- Spatial Baron-Ferejohn:
  - X is a subset of $\Re^d$ plus q,
  - u_i need not possess any convexity properties,
  - status quo utility $u_i(q)$ need not be low,
  - discount factors may be heterogenous.

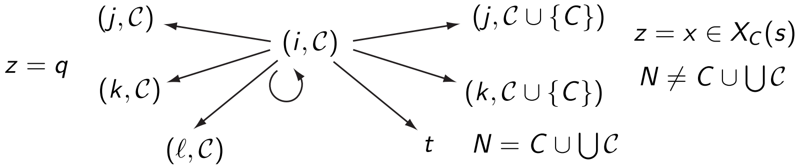
Special cases (cont.)

- NTU environments:
  - V is a coalition function, monotonic, each $V(C)$ compact
  - S consists of $(i, \mathcal{C})$ pairs and terminal state t, where
    - i is a proposer,
    - $\mathcal{C}$ is a pairwise disjoint collections of coalitions that have "formed,"
  - X is $V(N)$ plus q,
  - feasible outcomes:

$$X_C(s) = \begin{cases} V(C) & \text{if } C \cap \bigcup \mathcal{C} = \emptyset, \\ \emptyset & \text{else.} \end{cases}$$

Special cases (cont.)

- more on NTU environments:
  - transition probability:



  - stage utilities:

$$u_i(z, s) = \begin{cases} x_i & \text{if } z = x \in X_C(s) \text{ and } i \in C, \\ V(\{i\}) & \text{else.} \end{cases}$$

Mixed stationary strategies

- Proposal strategies: $\pi_i \colon S_i \to \mathcal{P}(X)$
- Voting strategies: $\alpha_i \colon X \times S \to [0,1]$
- Continuation values: $v_i \colon S \to \Re$
- Dynamic payoff from outcome x in state s is:

$$U_i(x,s) \quad = \quad u_i(x,s) + \delta_i \sum_{s' \in S} p(s'|x,s) v_i(s').$$

Note that U_i is continuous.

Coalitional bargaining equilibrium

- Optimal proposals: $\pi_i(s)$ puts probability one on solutions to

$$\max_{x \in X(s)} \alpha(x,s) U_i(x,s) + (1 - \alpha(x,s)) U_i(q,s),$$

  where $X(s) = \bigcup_{C \subseteq N} X_C(s)$.
- Optimal voting:

$$\alpha_i(x,s) \quad = \quad \begin{cases} 1 & \text{if } U_i(x,s) > U_i(q,s) \\ 0 & \text{if } U_i(x,s) < U_i(q,s), \end{cases}$$

  with no restriction when indifferent.
- Refines stationary Markov perfect equilibrium.

Existence

Theorem

There exists a coalitional bargaining equilibrium.

Spatial Baron-Ferejohn model

- Assume $u_i(x) > 0$ concave and status quo payoff zero.
- Pure voting strategies correspond to acceptance sets:

$$
\begin{aligned}
A_i &= \{y \in X \mid u_i(x) \geq \delta_i v_i\} \\
A_C &= \bigcap_{i \in C} A_i \\
A &= \bigcup_{C : |C| > n/2} A_C .
\end{aligned}
$$

- Proposer's problem: $\max_{y \in A} u_i(y)$.

Spatial Baron-Ferejohn model (cont.)

- Stationary equilibrium is fixed point of correspondence from $\pi = (\pi_1, \ldots, \pi_n)$ to π',

$$\pi \longrightarrow \{v_i(\pi)\}_{i \in N} \longrightarrow A(\pi) \longrightarrow \left\{ \mathcal{P}\left(\arg\max_{y \in A(\pi)} u_i(y) \right) \right\}_{i \in N}$$

  where $v_i(\pi) = \sum_j p_j u_i(x) \pi_j(dx)$ is continuous in π.
- Proof hinges on continuity of $A(\cdot)$ correspondence.
- It suffices to show that either $|A_C(\pi)| = 1$ or $A_C(\pi)$ has nonempty interior.
- By concavity,

$$u_i(E_\pi[x]) \geq v_i(\pi) > \delta_i v_i(\pi),$$

  so $E_\pi[x] \in \text{int} A_C(\pi)$, and $A(\cdot)$ is continuous.

Spatial Baron-Ferejohn model (cont.)

- Assume u_i concave, common discount factor, and a voter constraint qualification (e.g., strict quasi-concavity).
- Acceptance sets:

$$A_i \;=\; \{y \in X \mid u_i(x) \geq (1-\delta)u_i(q) + \delta v_i\},$$

  and define A_C and A as before.
- We use a similar fixed point approach to existence, and again we must show that either $|A_C(\pi)| = 1$ or $A_C(\pi)$ has nonempty interior.
- By concavity and common discount factor,

$$u_i((1-\delta)q + \delta E_\pi[x]) \;\geq\; (1-\delta)u_i(q) + \delta v_i(\pi),$$

  so $|A_C(\pi)| \geq 1$.

Spatial Baron-Ferejohn model (cont.)

- And if $|A_C(\pi)| \geq 2$, then the voter constraint qualification implies $\text{int}A_C(\pi) \neq \emptyset$.
- For example, under strict quasi-concavity,

$$u_i(x), u_i(y) \quad \geq \quad (1-\delta)u_i(q) + \delta v_i(\pi)$$

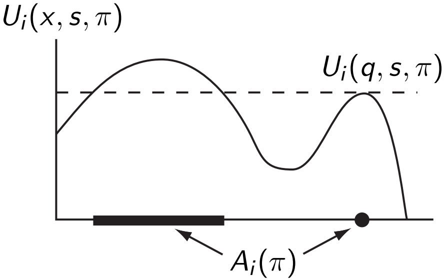
for $x \neq y$ implies

$$u_i\left(\frac{1}{2}x + \frac{1}{2}y\right) \quad > \quad (1-\delta)u_i(q) + \delta v_i(\pi),$$

so $\frac{1}{2}x + \frac{1}{2}y \in \text{int}A_C(\pi)$.

Challenges

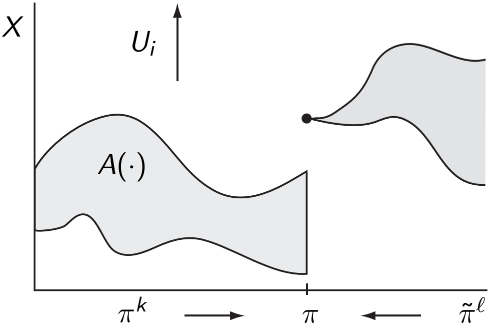
- Now drop bad status quo, common discount factor, voter constraint qualification. The correspondences $A_C(\pi)$ need not be continuous...



- ...and the previous fixed point correspondence does not work.

Challenges (cont.)

- One possible solution: indifferent voters reject proposals.



- But this creates problems for consistency across sequences of mixed proposal strategies.

Challenges (cont.)

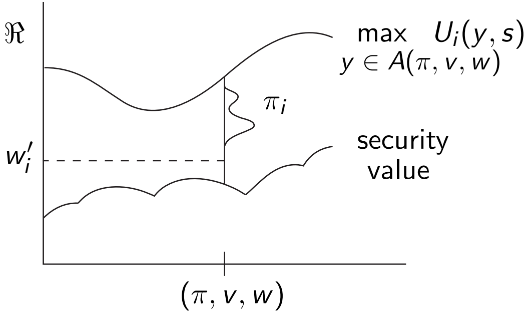
- Another possible solution: include voting strategies in domain of fixed point correspondence...

$$(\pi, \alpha) \longrightarrow \{v_i(\pi, \alpha)\}_{i \in N} \longrightarrow A(\pi, \alpha) \longrightarrow (\pi', \alpha')$$

- ... but the space of voting strategies $\alpha_i\colon X \times S \to [0, 1]$ is infinite-dimensional and does not have a nice topology.

Solution

- Include continuation values and proposer's payoff in domain: $(\pi, v, w) \rightarrow (\pi', v', w')$.



- Collect all new proposer payoffs between security value and the worst alternative in the support of π_i.

Solution (cont.)

- The proposer's payoff, however, must be the same for each x in the support of his mixing probability.
- We can have indifferent voters mix to bring payoffs from all $x \in \operatorname{supp} \pi_i'$ down to w_i'. Specifically, the probability, $\alpha(x, s)$, that x passes should satisfy

$$w_i' \;=\; \alpha(x, s) U_i(x, s) + (1 - \alpha(x, s)) U_i(q, s).$$

- This pins down acceptance probabilities, except where $U_i(x, s) = U_i(q, s)$.
- Each selection of acceptance probabilities gives us new continuation values, v_i', for the agents.
- The correspondence from (π, v, w) to (π', v', w') satisfies the conditions of Glicksberg's thm.

Extensions

- Existence theorems of Banks and Duggan (2000,2006) for pure voting strategies can be obtained as corollaries.
- Arguments can be extended to allow for sequential voting and for distribution of next period's s' to depend on first rejector. In particular, first rejector may be next proposer.
- Limited externalities can be accommodated: period t stage utility $u_i(x, s, h_t)$ can depend on history in a limited way. This allows us to capture bargaining in partition function environments.
- More externalities appears to require correlation for existence.