Existence and Indeterminacy of Markovian Equilibria in Dynamic Bargaining Games

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Outline

Legislative policymaking

- A substantial literature in political economy seeks to understand legislative interaction in terms of a dynamic bargaining game.
- One-shot approach: One agent proposes an alternative to a fixed status quo, and this passes iff a majority vote to accept.
  - Romer and Rosenthal (1978)
- Infinite-horizon approach: If proposal is rejected, another agent makes a proposal; games ends once a proposal is passed.
  - Baron and Ferejohn (1989)
  - Banks and Duggan (2000, 2006)

Bargaining with an endogenous status quo

- Fully dynamic approach:
  - Each period begins with a status quo policy inherited from the previous period.
  - A legislator is chosen randomly to propose any feasible policy, which is then subject to an up or down vote.
  - If the proposal is voted up, it is implemented in that period and becomes the next period's status quo;
  - if it is voted down, the status quo is implemented and remains in place until the next period.
  - This process continues ad infinitum.

Bargaining with an endogenous status quo

- Literature: Baron, (1996), Kalandrakis (2004, 2009, 2014), Duggan and Kalandrakis (2012), Bowen and Zahran (2012), Anesi and Seidmann (2014), Nunnari (2014), Richter (2014), Baron and Bowen (2014), and Zápal (2014); among others.
- Difficulty: Existence/characterization of Markovian equilibria.
- Constructive approach: Most work involves the explicit construction of a particular Markovian equilibrium.

Pie-division environment

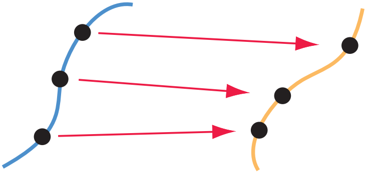
- Anesi and Seidmann (2014) analyze dynamic bargaining with an endogenous status quo in the pie-division context, allowing for:
  - any number of players,
  - concave utility over consumption,
  - heterogeneous discount factors,
  - any voting rule based on a quota.
- In the non-unanimity case, they construct a large class of Markov perfect equilibria based on simple solutions when players are sufficiently patient.
- The problem is multiplicity, not existence.

This paper

- General spacial environment.
- Extend Anesi and Seidmann's (2014) idea to semi-simple solutions.
- Three main results:
  * Every semi-simple solution is supported as the absorbing set of a Markov perfect equilibrium when players are sufficiently patient.
  * Weak gradient conditions at $x \Rightarrow$ continuum of semi-simple solutions near x.
  * Sufficient patience + weak gradient condition at $x \Rightarrow$ continuum of Markov perfect equilibria with absorbing sets near x.

This paper

- Thus, when players are sufficiently patient, Markov perfect equilibria are indeterminate, and constructions that select a single equilibrium implicitly impose a strong refinement.



- Imposing Pareto optimality of equilibrium does not dispel the indeterminacy:
    - * Alternative conditions imply a continuum of equilibria with Pareto optimal outcomes.

Outline

Bargaining framework

- Players: $N = \{1, \ldots, n\}$, $n \geq 3$.
- Alternatives: $X \subseteq \Re^d$, nonempty.
- Decisive coalitions: $\mathcal{D}$, proper, monotonic, and non-collegial, i.e., no player has a veto.
  - Ex: decisive coalitions are those satisfying $|C| \geq q$ for some quota $q \in (\frac{n}{2}, n-1]$.

Framework: protocol

- Bargaining protocol: In period t with status quo x^{t-1},
  - Some player i is selected with probability $p_i \in (0, 1)$ to propose any policy $y \in X$;
  - All players simultaneously vote to accept or to reject y;
  - If a coalition $C \in \mathcal{D}$ votes to accept y, then it is implemented as the in period t and becomes the status quo next period: $x^t = y$;
  - Otherwise, the previous status quo, x^{t-1}, is implemented again and becomes the default in period $t + 1$: $x^t = x^{t-1}$.
  - The game moves to period $t + 1$ and the protocol is repeated.

Framework: payoffs

- Path of play: In each of an infinite number of discrete periods, indexed $t = 1, 2, \ldots$, an alternative x^t is implemented.

- Payoffs: Player i's payoff from alternative sequence $\{x^t\}$ is
$$(1 - \delta_i) \sum_{t=1}^{\infty} \delta_i^{t-1} u_i \left(x^t\right) \ ,$$
where $\delta_i \in (0, 1)$ and u_i is C^2 and bounded above.

Example: private goods

- In a mixed economy, $x = (x_1, \ldots, x_n, g)$ consists of private component $(x_1, \ldots, x_n) \in \Re^n_+$ and other, possibly public goods $g \in \Re^{d-n}_+$.
- Then alternatives are given by a resource constraint

$$X = \left\{ x \in \Re^d_+ : f(\sum_{i \in N} x_i, g) \leq 0 \right\},$$

  where $f : \Re^d_+ \to \Re$ is continuous, weakly monotonic.
- Assume $\frac{\partial u_i}{\partial x_i}(x) > 0$ and u_i is constant in x_{-i}.

Example: pie division

- A special case of mixed economy is pie-division with disposal, in which

$$X = \left\{ (x_1, \ldots, x_n) \in [0,1]^n : \sum_{i \in N} x_i \leq 1 \right\}$$

and $u_i(x) = x_i$ for all i.

Example: pie division

- We model pie-division with no disposal as

$$X = \left\{ (x_1, \ldots, x_{n-1}) \in [0,1]^{n-1} : \sum_{i=1}^{n-1} x_i \leq 1 \right\},$$

  $u_i(x) = x_i$ for all $i < n$, and $u_n(x) = 1 - \sum_{i<n} x_i$.

- Here, X is $(n-1)$-dimensional.

Pareto optimality

- Given coalition C, let $PO(C)$ denote the Pareto optimal alternatives for C, i.e., x such that there is no y satisfying $u_i(y) \geq u_i(x)$ for all $i \in C$, with strict inequality for at least one member.

Strategies and equilibrium

- Stationary Markov perfect equilibrium: Focus on pure-strategy subgame perfect equilibria σ in which each player uses a stationary Markov strategy.
- Stage-undominated voting: Voting strategies are not weakly dominated, given continuation play of σ.

$\Rightarrow$ "stationary bargaining equilibrium" (SBE) .

- Note: Since we seek to show indeterminacy, our results strengthened by using a simple class of equilibria.

Strategies and equilibrium

- Say x is an absorbing point for σ if it remains in place once implemented, i.e., given status quo x, the alternative implemented is x again with probability one.
- Say σ is no-delay if at every status quo alternative an absorbing point is implemented with probability one.
- Note: No-delay implies that we can't "cheat" by encoding histories in the current status quo.

Outline

Semi-simple solutions

- Definition: A set of alternatives $S \subset X$ is a semi-simple solution if (i) there is a one-to-one mapping $\rho \colon S \to N$ such that for all $x \in S$,

$$\rho(x) \in C(x) \equiv \left\{ i \in N : u_i(x) = \max_{z \in S} u_i(z) \right\} \in \mathcal{D},$$

  and (ii) for all $i \in N$, u_i is not constant on S.

- Here, $C(x)$ is the supporters of x, and part (i) requires $C(x)$ is decisive and a 1-1 selection ρ of a supporter for each $x \in S$.

Semi-simple solutions

- Remarks:
  - For each i, $\max_{z \in S} u_i(z)$ is the reward payoff...
  - and $\min_{z \in S} u_i(z)$ is the punishment payoff.
  - The selection ρ may not be onto, players can have multiple punishment payoffs.
  - The 1-1 requirement follows from our focus on pure proposal strategies.

Example: semi-simple solution

- Example: Assume $n = 5$ and $\mathcal{D}$ is majority rule. Consider four alternatives with the payoffs below,

	1	2	3	4
$\bar{x}^1$	1	1	1	0
$\bar{x}^2$	1	1	0	1
$\bar{x}^3$	1	0	1	1
$\bar{x}^4$	0	1	1	1

while player 5 is not indifferent over all four alternatives. So, $C(\bar{x}^1) = \{1, 2, 3, 5?\}$, $C(\bar{x}^2) = \{1, 2, 4, 5?\}$, $C(\bar{x}^3) = \{1, 3, 4, 5?\}$, and $C(\bar{x}^4) = \{2, 3, 4, 5?\}$, and we define $\rho(\bar{x}^k) = k$ for all k.

Example: semi-simple solution

- Example: Assume $n = 7$, majority rule, i.e., $q = 4$, and preferences over six alternatives as below,

	1	2	3	4	5	6
$\bar{x}^1$	1	1	1	1	0	2
$\bar{x}^2$	1	1	1	0	0	2
$\bar{x}^3$	1	1	0	0	1	2
$\bar{x}^4$	1	0	0	1	1	2
$\bar{x}^5$	0	0	1	1	1	2
$\bar{x}^6$	0	1	1	1	1	0

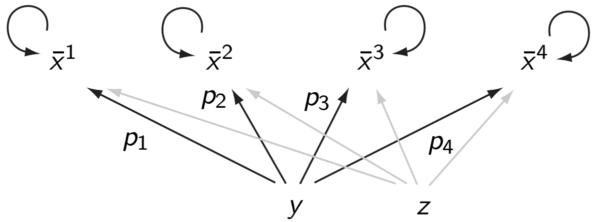
while player 7 is not indifferent over all six alternatives. So, $C(\bar{x}^k) \setminus \{7\} = \{1, \ldots, 6\} \setminus \{6 - (k \bmod 6)\}$, and we define $\rho(\bar{x}^1) = 1$, $\rho(\bar{x}^2) = 2$, $\rho(\bar{x}^3) = 5$, $\rho(\bar{x}^4) = 6$, $\rho(\bar{x}^5) = 3$, and $\rho(\bar{x}^6) = 4$.

Existence of equilibrium

- Theorem 1: Let S be a semi-simple solution. There exists $\bar{\delta} \in (0,1)$ such that if $\min_{i \in N} \delta_i > \bar{\delta}$, then there exists a no-delay SBE with absorbing points equal to S.
- Thus, if we can show that a semi-simple solution exists, then necessarily the spatial bargaining game with patient players will possess an SBE.

Proof of Theorem 1

- Assume $n = 5$, majority rule, $S = \{\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4\}$, and $\rho(\bar{x}^i) = i$.



- Note: It is important that no player is indifferent over S, so reward payoff $>$ punishment payoff.

Outline

Sufficient conditions (easy)

(C1′) $\{\nabla u_i(x) : i \in N\}$ is linearly independent.

- Remarks:
  - Trivially satisfied at every x in a mixed economy, including pie division with disposal.
  - If $d \geq n$, then the condition generically holds outside a closed set of measure zero of alternatives (Smale, 1974).
  - Cannot be satisfied when $d < n$.
  - Not satisfied at any alternatives in pie division model with no disposal.

Continuum of semi-simple solutions (easy)

- Theorem $2'$: Let x be any interior point of X at which ($C1'$) is satisfied. Every open neighborhood U of x contains a continuum of semi-simple solutions.
- Remarks:
  - Applies to every $x \in \text{int}X$ in a mixed economy.
  - Gradient restriction is independent of voting rule, but...
  - we generalize ($C1'$) to weaken the degree of linear independence required.
  - The construction uses Pareto inefficient alternatives.

Proof of Theorem 2′

- Assume $x \in \text{int}X$ satisfies (C1′), and let U be an open neighborhood of x.
- For example, assume $n = 3$.
- Define $f : X^3 \to \Re^9$ be defined as

$$f\left(x^1, x^2, x^3\right) \equiv \left(u_1(x^i), u_2(x^i), u_3(x^i)\right)_{i \in N} \ .$$

- Condition (C1′) implies $Df(x, x, x)$ has full row rank.

Proof of Theorem 2′

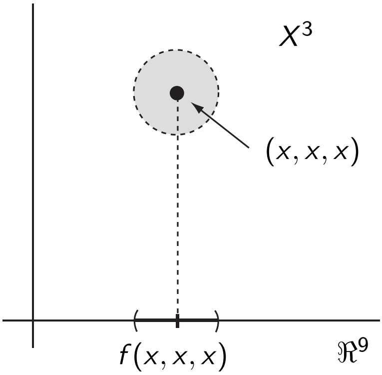
- To see this, note that

$$Df(x,x,x) = \begin{bmatrix} Du_1(x) & 0 & 0 \\ Du_2(x) & 0 & 0 \\ Du_3(x) & 0 & 0 \\ 0 & Du_1(x) & 0 \\ 0 & Du_2(x) & 0 \\ 0 & Du_3(x) & 0 \\ 0 & 0 & Du_1(x) \\ 0 & 0 & Du_2(x) \\ 0 & 0 & Du_3(x) \end{bmatrix}$$

has full row rank.

Proof of Theorem 2′

- The local submersion theorem implies that f is locally diffeomorphic to the projection mapping at (x, x, x).



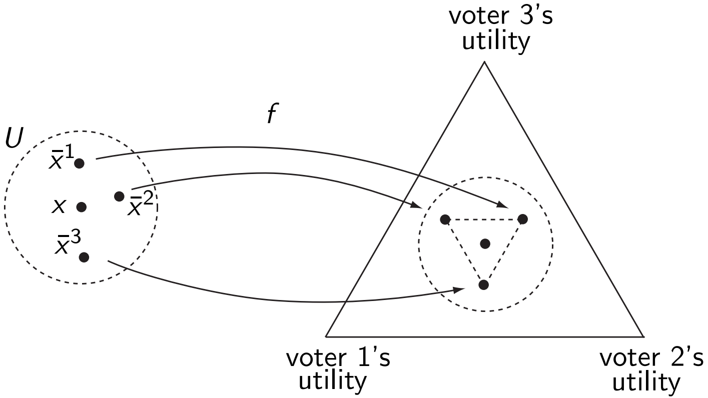
Proof of Theorem 2′

- Thus, restricted to some open subset of U^3 containing (x, x, x), f is onto an open set around $f(x, x, x)$.
- So for small enough $\epsilon > 0$, we can perturb x to alternatives $\bar{x}^1, \bar{x}^2, \bar{x}^3 \in U$ such that for all i,

$$\begin{aligned} u_i\left(\bar{x}^i\right) &= u_i(x) - \epsilon \\ u_j\left(\bar{x}^i\right) &= u_j(x) + \epsilon\ ,\ \forall j \neq i\ . \end{aligned} \tag{1}$$

- Set $S_\epsilon = \{\bar{x}^1, \bar{x}^2, \bar{x}^3\}$.

Proof of Theorem 2′



Proof of Theorem 2′

- Let $\rho(\bar{x}^i) = i + 1 \mod 3$, and note that $\bar{x}^{i+1}$ maximizes the utility of player $\rho(\bar{x}^i)$ over the set S_ϵ.
- Since $\mathcal{D}$ is non-collegial, the coalitions

$$C(\bar{x}^1) = \{2, 3\},\ C(\bar{x}^2) = \{1, 3\},\ C(\bar{x}^3) = \{1, 2\}$$

  are decisive. Clearly, u_i is not constant on S_ϵ.
- Thus, S_ϵ is a semi-simple solution.
- Repeating for $\epsilon' \in (0, \epsilon)$ yields a continuum $\{S_{\epsilon'} \mid 0 < \epsilon' < \epsilon\}$ of semi-simple solutions.

Pareto optimal semi-simple solutions (easy)

(C2$'$) (i) x is Pareto optimal,

(ii) for all i, the gradients $\{\nabla u_j(x) : j \in N \setminus \{i\}\}$ are linearly independent,

(iii) for all i, $D^2 u_i(x)$ is negative definite.

- Remarks:
  - Typically, satisfied in spatial model at Pareto optimal alternatives when $d = n - 1$.
  - In the paper, we use a weaker concavity condition that is commonly satisfied in mixed economies.
  - Part (iii) is unneeded in pie-division with no disposal.

Pareto optimal semi-simple solutions (easy)

- Theorem $3'$: Let x be any interior point of X at which (C$2'$) is satisfied. Every open neighborhood U of x is such that $U \cap PO(N)$ contains a continuum of simple solutions.
- Remark: We generalize (C$2'$) to weaken the degree of linear independence required.

Structure of Pareto optimals

- Lemma: Assume
  (i) x is Pareto optimal for C,
  (ii) for all $i \in C$, the gradients $\{\nabla u_j(x) \mid j \in C \setminus \{i\}\}$ are linearly independent,
  (iii) $\sum_{i \in C} \alpha_i D^2 u_i(x)$ is negative definite for some positive coefficients $\alpha_i > 0$.

  Then $PO(C)$ is locally a manifold of dimension $|C| - 1$.
- Extends a result of Smale (1976) from exchange economies to general spatial model.

Proof of Theorem 3′

- Now we can only control utilities of two players at a time.
- Define $f: PO(N)^3 \to \Re^6$ as

$$f(x^1, x^2, x^3) = \begin{cases} f^1(x^1) = (u_2(x^1), u_3(x^1)) \\ f^2(x^2) = (u_1(x^2), u_3(x^2)) \\ f^3(x^3) = (u_1(x^3), u_2(x^3)) . \end{cases}$$

- Condition (C2′) implies $Df^i(x)$ has full row rank.
- Thus, $Df(x, x, x)$ has full row rank as well.

Proof of Theorem 3′

- Local submersion theorem applied to f implies that for small enough $\epsilon > 0$, we can perturb x to alternatives $\bar{x}^i \in U$ such that for all i,

$$u_j\left(\bar{x}^i\right) = u_j(x) + \epsilon \text{ , } \forall j \neq i.$$

- Since x is Pareto optimal, we have $u_i(\bar{x}^i) < u_i(x)$.
- Set $S_\epsilon = \{\bar{x}^1, \bar{x}^2, \bar{x}^3\}$, which is a semi-simple solution.

Outline

General sufficient conditions

- A coalition C is oversized if for all $i \in C$, $C \setminus \{i\} \in \mathcal{D}$.

(C1) There is an oversized coalition C^* such that

(i) the gradients $\{\nabla u_i(x) : i \in C^*\}$ are linearly independent,

(ii) there exists $j \in C^*$ such that for all $k \in N \setminus C^*$, the gradients

$$\{\nabla u_i(x) : i \in (C^* \setminus \{j\}) \cup \{k\}\}$$

are linearly independent.

General sufficient conditions

- Theorem 2: Let x be any interior point of X at which (C1) is satisfied. Every open neighborhood U of x contains a continuum of simple solutions.
- Remarks:
  - For a quota rule, we now need only $d \geq q$.
  - The proof follows the lines of the proof of Theorem $2'$ using C^* instead of N.
  - But when $C^* \subsetneq N$, a subtle perturbation argument is needed to ensure u_i is not constant for players $i \notin C^*$.
  - The construction uses Pareto inefficient alternatives.

Pareto optimal semi-simple solutions

(C2) There is an oversized coalition C^* such that

(i) x is Pareto optimal for C^*,

(ii) for all $i \in C^*$, the gradients $\{\nabla u_j(x) : j \in C^* \setminus \{i\}\}$ are linearly independent,

(iii) $\sum_{i \in C^*} \alpha_i D^2 u_i(x)$ is negative definite,

(iv) there exist $i, j \in C^*$ such that for all $k \in N \setminus C^*$, $L_{i,j,k}(x) \overline{\pitchfork} M_{C^*}(x)$.

- This condition is parsed on the following slides...

Parsing (C2)

- Given (i) and (ii), there are positive coefficients α_i for each $i \in C^*$ such that

$$\sum_{i \in C^*} \alpha_i \nabla u_i(x) \;\; = \;\; 0,$$

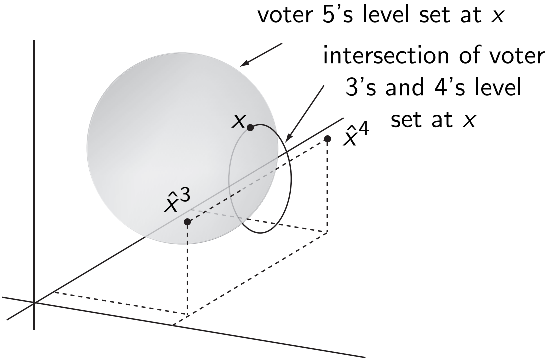
  and these coefficients are unique up to a common scaling.

- Given (i)–(iii), the Pareto set $PO(C^*)$ has a manifold structure in an open set G around x; let $M_{C^*}(x) = G \cap PO(C^*)$.

- $L_{i,j,k}(x)$ is the level set of $(u_h)_{h \in (C^* \setminus \{i,j\}) \cup \{k\}}$ through x.

Parsing (C2)

- Then part (iv) entails that $L_{i,j,k}(x) \cap PO(C^*)$ is a zero-dimensional manifold and is robust to perturbations of x.
- Ex: $n = 5$, $q = 3$, $C^* = \{1, 2, 3, 4\}$



Pareto optimal semi-simple solutions

- Theorem 3: Let x be any interior point of X at which (C2) is satisfied using coalition C^*. Every open neighborhood U of x is such that $U \cap PO(C^*)$ contains a continuum of simple solutions.
- Remarks:
  - The proof roughly follows the lines of the proof of Theorem $3'$ using C^* instead of N.
  - And when $C^* \subsetneq N$, a subtle perturbation argument is needed to ensure u_i is not constant for players $i \notin C^*$.

Outline

Indeterminacy of SBE

- Theorems 1 and $2'$ almost immediately imply equilibrium indeterminacy:
- Theorem 4: Let x be any interior point of X at which (C1$'$) is satisfied. For every open neighborhood U of x, there exists $\hat{\delta} \in (0, 1)$ such that if $\min_{i \in N} \delta_i > \hat{\delta}$, then there is a continuum of semi-simple solutions in U corresponding to absorbing sets of no-delay stationary bargaining equilibria with discount factors $\delta_1, \ldots, \delta_n$.
- Remark: Because we use Theorem $2'$, indeterminacy may rely on Pareto inefficient alternatives.

Indeterminacy of SBE

- Say x is dynamically stable if it is an equilibrium absorbing point when players are sufficiently patient: there exists $\hat{\delta} \in (0,1)$ such that if $\min_{i \in N} \delta_i > \hat{\delta}$, then there is an SBE σ such that x is absorbing for σ.
- Corollary 1: If the set of alternatives satisfying (C1$'$) is dense in int$X$, then the dynamically stable alternatives are dense in int$X$.
- Remark: In a mixed economy, (C1$'$) is satisfied at every x, so the corollary applies.

Proof of Theorem 4

- Let $U \subseteq X$ be an arbitrary open neighborhood of x.
- Let $\mathcal{S}$ be the continuum of semi-simple solutions $S \subseteq U$ from Theorem 2′.
- For all $S \in \mathcal{S}$, let $\bar{\delta}_S < 1$ be the threshold associated with S in Theorem 1.
- For each natural number k, set

$$\mathcal{S}^k = \left\{ S \in \mathcal{S} : \bar{\delta}_S < 1 - \frac{1}{k} \right\}.$$

- Note that $\mathcal{S} = \bigcup_{k=1}^{\infty} \mathcal{S}^k$.
- Then $\mathcal{S}$ is the union of countably many sets, and by the axiom of choice, some $\mathcal{S}^k$ is a continuum: set $\hat{\delta} = 1 - \frac{1}{k}$.

Indeterminacy of Pareto optimal SBE

- Theorem 5: Let x be any interior point of X at which (C2$'$) is satisfied. For every open neighborhood U of x, there exists $\hat{\delta} \in (0,1)$ such that if $\min_{i \in N} \delta_i > \hat{\delta}$, then there is a continuum of semi-simple solutions in $U \cap PO(N)$ corresponding to absorbing sets of no-delay stationary bargaining equilibria with discount factors $\delta_1, \ldots, \delta_n$.

- Corollary 2: If the set of alternatives satisfying (C2$'$) is dense in $PO(N) \cap \text{int}X$, then the dynamically stable alternatives are dense in $PO(N) \cap \text{int}X$.

Conclusion

- When players are patient and a weak gradient condition holds, stationary bargaining equilibria are indeterminate.
- For example, in mixed economies, the dynamically stable alternatives are dense in $\text{int}X$.
- Constructive approach used in the literature implicitly relies on a restrictive refinement.
- Where do we go from here?
  - low-dimensional spaces, impatient players...
  - additional refinements
  - add noise to model, à la Duggan and Kalandrakis (2012)?