Directional Equilibria

Hun Chung and John Duggan

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Introduction

A classical approach in welfare economics is to assume utility functions $u_1, \ldots, u_n$ for each of n individuals and to prescribe an alternative that maximizes the sum of utilities:

$$\max_{x \in X} \sum_i u_i(x).$$

If each u_i is differentiable and concave, then the interior solutions are characterized by the first order condition

$$\sum_i \nabla u_i(x) \;=\; 0.$$

But obviously, the solutions are sensitive to the choice of utility representations; a positive scaling of a single u_i will in general lead to different solutions.

Intro (cont.)

A classical approach in positive political theory is to predict alternatives that are stable with respect to majority voting, i.e., the majority core.

In the multidimensional spatial model, however, the existence of such alternatives is problematic: Plott's (1967) theorem establishes that individual gradients must satisfy a restrictive radial symmetry condition at any majority core alternative.

Schofield (1983) shows that for generic preferences, the radial symmetry condition is prohibitive, and the majority core is empty.

Different solutions in the literature include the top cycle, uncovered set, structure-induced equilibria, and strong points.

Intro (cont.)

We provide a solution concept for the *m*-dimensional spatial model that provides a response to both difficulties.

For x such that no voter's gradient is zero, then we can transform the f.o.c. from the utilitarian welfare maximization problem to

$$\sum_i \frac{1}{\|\nabla u_i(x)\|} \nabla u_i(x) \quad = \quad 0.$$

This gives a system of m equations in m unknowns that is invariant with respect to positive scalings of utilities (or, for that matter, smooth transformations with positive derivatives).

A solution x is such that if all voters "pull" x in the direction of their gradients with "equal force," then x is unchanged.

Intro (cont.)

More generally, if x is a voter's ideal point, then we allow the voter to resist pulls away from x with the same force.

Such "directional equilibria" have desirable properties:

- existence
- Pareto optimality
- core extension
- uniqueness (Euclidean preferences)
- generic local uniqueness and stability
- non-cooperative foundations.

Intro (cont.)

When preferences are Euclidean, our directional equilibria reduce to the “consensus points” of Baranchuk and Dybvig (2009), who study choices of boards of directors.

In the Euclidean setting, they show existence, uniqueness, Pareto optimality, and a weak relationship to the core.

Multidimensional Spatial Model

$X \subseteq \Re^m$	nonempty set of alternatives, x, y
N	set of voters, $i, j = 1, \ldots, n$
u_i	C^1 utility function of voter i
$\hat{x}^i$	ideal point of voter i (the unique critical point of u_i)
$p^i(x)$	normalized gradient at x, e.g., if $x \neq \hat{x}^i$,

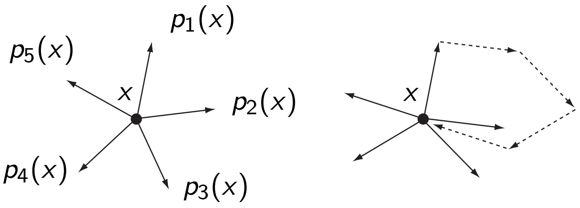
$$p^i(x) = \frac{1}{\|\nabla u_i(x)\|} \nabla u_i(x).$$

Directional Equilibrium

We say x is a **directional equilibrium** if

$$\left\| \sum_{i=1}^{n} p^i(x) \right\| \quad \leq \quad \#\{i \mid x = \hat{x}^i\}.$$

Below is an example of a "non-critical" directional equilibrium.



Directional Equilibrium (cont.)

To compare to structure-induced equilibrium, denote the normalized partial derivative (when non-zero) by

$$q_j^i(x) \;=\; \frac{\frac{\partial u_i}{\partial x_j}(x)}{|\frac{\partial u_i}{\partial x_j}(x)|}.$$

Then x is a **structure induced equilibrium** if for all $j = 1, \ldots, m$,

$$\left\| \sum_{i=1}^{n} q_j^i(x) \right\| \;\leq\; \#\{i \mid x = \hat{x}^i\}.$$

So a voter exerts unit force in the direction of each axis, even if some dimensions are less salient than others.

Euclidean Special Case

Assume **Euclidean preferences,** so for each i, there is a differentiable function $f_i \colon \Re_+ \longrightarrow \Re$ with strictly negative derivative such that

$$u_i(x) \quad = \quad f_i(\|x - \hat{x}^i\|).$$

Baranchuk and Dybvig (2009) show that x is a directional equilibrium if and only if it solves

$$\min_y \sum_i \|y - \hat{x}^i\|,$$

which implies existence.

Euclidean Special Case (cont.)

Baranchuk and Dybvig (2009) also show that when ideal points are not all collinear, then the above objective function is strictly convex, which implies uniqueness.

Theorem (Baranchuk and Dybvig): Assume Euclidean preferences. If not all ideal points are collinear or if n is odd, then there is a unique directional equilibrium.

Furthermore, they show that if a majority of voters have the same ideal point, then this is the unique directional equilibrium.

Existence

Theorem Assume that X is nonempty, compact, and convex, that $\hat{x}^i \notin \text{bd}X$ for all i, and that for all $x \in \text{bd}X$, there exists $\alpha > 0$ such that

$$x + \alpha \sum_i p^i(x) \in \text{int}X.$$

Then there is a directional equilibrium.

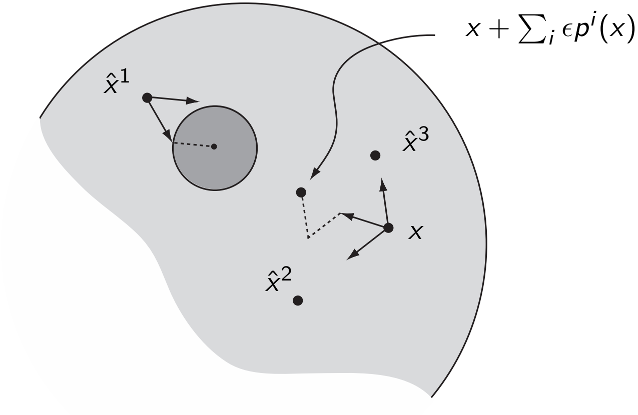
The idea of the proof is, roughly, to find a fixed point of the mapping

$$x \to x + \sum_i p^i(x).$$

There are two issues, however.

Existence (cont.)

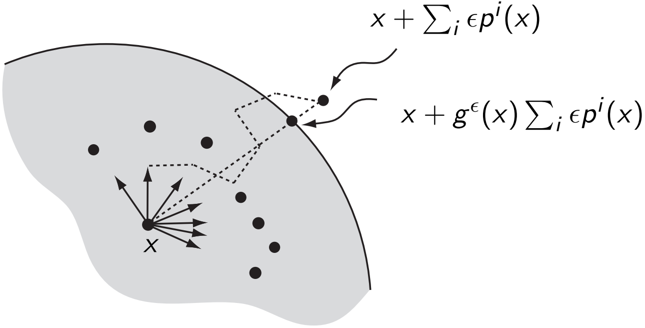
First, the above mapping is discontinuous at ideal points $x = \hat{x}^i$.



So we specify a correspondence that associates a closed disc $D_\epsilon(0)$ to each ideal point.

Existence (cont.)

Second, the mapping may take us outside X.



So we scale the values of the correspondence back toward x by a factor of $g^\epsilon(x)$. Existence follows from Kakutani's theorem.

Pareto Optimality Properties

Recall that u_i is **strictly pseudo-concave** if for all x and all $y \neq x$ such that $u_i(y) \geq u_i(x)$, we have $\nabla u_i(x) \cdot (y - x) > 0$.

Theorem Assume each u_i is strictly pseudo-concave. If x is a directional equilibrium, then it is Pareto optimal.

If x is a directional equilibrium, then $0 \in \text{conv}\{p^i(x) \mid i \in N\}$, so for every direction t, there exists i such that $p^i(x) \cdot t \leq 0$.

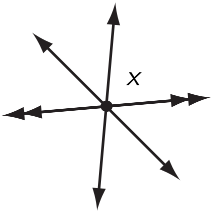
Thus, for each $y \neq x$, we can choose i s.t. $p^i(x) \cdot (y - x) \leq 0$, and it follows that $u_i(x) > u_i(y)$.

Core Extension Properties

Recall that the **majority core** consists of alternatives x such that for all y,

$$\#\{i \mid u_i(y) > u_i(x)\} \quad \leq \quad \frac{n}{2}.$$

Theorem Assume n is odd, let $x \in \text{int}X$, and assume at most one voter k such that $\hat{x}^k = x$. If x is a majority core alternative, then it is a directional equilibrium.

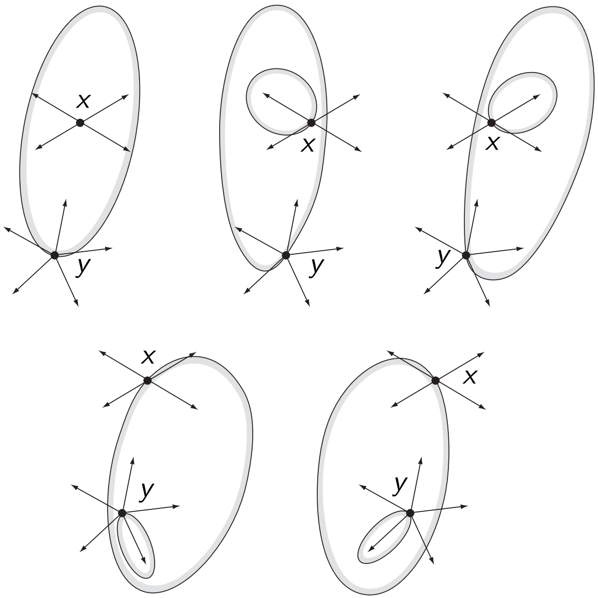


Core Extension Properties (cont.)

Corollary: Assume n is odd, Euclidean preferences, and letting $x \in \text{int}X$, there is at most one voter k such that $\hat{x}^k = x$. If x is a majority core alternative, then x is the unique directional equilibrium.

The result does not hold for general convex, continuous preferences. . . .

Core Extension Properties (cont.)



Local Uniqueness

Assume X is open, and parameterize preferences by

$$u_i(x) + \theta^i \cdot x,$$

where θ^i is a perturbation that lives in an open subset of $\Re^m$.

Let $\hat{x}^i(\theta^i)$ be the unique ideal point of voter i at θ^i, and again assume it is the unique critical point for all θ^i.

Let $p^i(x, \theta^i)$ be the normalized gradient parameterized by θ^i.

Local Uniqueness (cont.)

Say x is a **directional equilibrium at** θ if

$$\left\| \sum_i p^i(x, \theta^i) \right\| \quad \leq \quad \#\{i \mid x = \hat{x}^i(\theta^i)\}.$$

A directional equilibrium x at θ is **locally unique** if some neighborhood of x contains no other directional equilibria.

Theorem Assume n is odd and each u_i is C^2. Then for almost all θ, the directional equilibria at θ are locally unique.

We say a critical directional equilibrium at θ is locally unique if some neighborhood contains no other such equilibria, with a similar convention for non-critical equilibria.

Local Uniqueness (cont.)

A key to the proof is the mapping $f\colon X \times \mathbb{R}^n_{++} \times \Theta^n \to \mathbb{R}^{m+n}$ defined by

$$f(x,\alpha,\theta) \quad = \quad \begin{bmatrix} \sum_{i=1}^{n-1} \alpha_i (D_1 u_i(x) + \theta^i_1) \\ \vdots \\ \sum_{i=1}^{n-1} \alpha_i (D_m u_i(x) + \theta^i_m) \\ \|\nabla u_1(x) + \theta^1\|^2 - \frac{1}{\alpha_1^2} \\ \vdots \\ \|\nabla u_n(x) + \theta^n\|^2 - \frac{1}{\alpha_n^2} \end{bmatrix}.$$

Note that x satisfies $f(x,\alpha,\theta) = 0$ iff it is a non-critical directional equilibrium at θ.

Local Uniqueness (cont.)

We show that 0 is a regular value of f, i.e., the Jacobian $Df(x, \alpha, \theta)$ at every solution to $f = 0$ has full row rank, and thus for almost all θ, the set of solutions to $f(\cdot, \theta) = 0$ is a manifold of dimension

$$m + n - m - n \ = \ 0.$$

It follows that all non-critical directional equilibria are locally unique.

By assumption, there are at most n critical directional equilibria, so they are locally unique and cannot "accumulate."

Local Uniqueness (cont.)

The remainder of the proof consists of showing that generically, non-critical directional equilibria cannot accumulate around a critical equilibrium.

If $\{x^k\}$ is a sequence of non-critical directional equilibria at θ that converge to a critical equilibrium x, then either x is the ideal point of at least two voters, or x is a **tight** critical equilibrium, i.e., there is one voter i such that $x = \hat{x}^i(\theta^i)$ and

$$\sum_j p^j(x, \theta^j) \;=\; 1.$$

We show that generically, neither situation can occur.

Stability Properties

Let Π be a metric space of model parameters, which enter utility as in

$$u_i(x, \pi) + \theta^i \cdot x.$$

A directional equilibrium x at θ given π is **stable** if there exist an open set $U \subseteq \Pi$ with $\pi \in U$, and open set $V \subseteq \Re^m$ with $x \in V$, and a continuous mapping $F\colon U \to V$ such that for all $\pi' \in U$ and all $y \in V$, y is a directional equilibrium at θ given π' iff $y = F(\pi')$.

Theorem Assume n is odd and each u_i is C^2. Then for all π and almost all θ, every directional equilibrium at θ given π is stable.

The result extends as expected to a smooth definition of stability.

Non-cooperative Foundations

Given $x \in \text{int}X$ and $\epsilon > 0$ with $D_{n\epsilon}(x) \subseteq X$, the ϵ**-local game at** x is the strategic form game with strategy sets $S_i = D_\epsilon(0)$ and payoffs

$$U_i(s_1, \ldots, s_n) \quad = \quad u_i(x + \sum_j s_j).$$

We say x is **non-cooperatively stable** if there exist $\epsilon > 0$ and a Nash equilibrium $s = (s_1, \ldots, s_n)$ of the ϵ-local game at x such that $\sum_j s_j = 0$.

Non-cooperative Foundations (cont.)

Theorem Given $x \in \text{int}X$, if x is non-cooperatively stable, then it is a directional equilibrium; conversely, if x is a directional equilibrium such that each $D^2 u_i(x)$ is negative definite, then it is non-cooperatively stable.

Suppose x is non-cooperatively stable, and let s be a Nash equilibrium with $\sum_j s_j = 0$. Then for each voter i, s_i solves

$$\max_{y \in \Re^m} u_i\left(x + (\textstyle\sum_{j \neq i} s_j) + y\right)$$
$$\text{s.t. } \|y\|^2 \ \leq \ \epsilon^2.$$

Non-cooperative Foundations (cont.)

The f.o.c. is

$$
\begin{aligned}
\nabla u_i(x) &= 2\lambda_i s_i \\
\lambda_i(\epsilon^2 - \|s_i\|^2) &= 0 \\
\lambda_i &\geq 0.
\end{aligned}
$$

Letting G be the set of voters with $\lambda_i > 0$, we have

$$
\sum_i p^i(x) = \sum_{i \in G} p^i(x) = \sum_{i \in G} \frac{1}{\|s_i\|} s_i = \frac{1}{\epsilon} \sum_{i \in G} s_i = -\frac{1}{\epsilon} \sum_{i \notin G} s_i,
$$

which yields the result...

The converse direction is similar, relying on sufficient second order conditions for a local maximizer.

Conclusion

The directional equilibria provide an ordinal version of utilitarianism that is based on a normalization of the usual first order condition.

This solution concept extends the majority core in a way that maintains existence when radial symmetry is violated.

In addition to existence, directional equilibria are Pareto optimal, extend the core, and have desirable uniqueness and stability properties.

Conclusion (cont.)

Comparison to structure-induced equilibria.

	D.E.	S.I.E.
existence	y	y
Pareto optimality	y	n
core extension	y	y
unique (Euclidean)	y	y
locally unique	y	y
stable	y	y
rotation inv.	y	n
stretch inv.	n	y

Interestingly, the strong points do well by these criteria, but Euclidean uniqueness is known only in two dimensions, and general local uniqueness and stability are not known.