Limits of Acyclic Voting*

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Abstract

Assuming three or more alternatives, there is no systematic rule for aggregating individual preferences that satisfies acyclicity and the standard independence and Pareto axioms, that avoids making some voter a weak dictator, and that is minimally responsive to changes in voter preferences. The latter axiom requires that a preference reversal in the same direction by roughly one third of all voters is sufficient to break social indifference. This result substantially strengthens classical acyclicity theorems of Mas-Colell and Sonnenschein (1972) and Schwartz (1986). When the set of alternatives is large, cycles become intuitively easier to construct, the acyclicity axiom has greater bite, and the responsiveness threshold can be increased to two less than the number of individuals, which yields the weakest logically possible responsiveness axiom.

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1 Introduction

Consider a set of agents with possibly heterogenous preferences who must make a collective choice from a given set of alternatives. The problem is to systematically construct a nonempty choice set (a subset of alternatives that may represent normatively appealing choices or plausible predictions) based on binary comparisons of alternatives. If the goal is choices that are maximal with respect to binary comparisons, then the key condition needed to construct nonempty choice sets is that these comparisons, or "social preferences," be acyclic: there should not be a chain of social preferences beginning with one alternative and leading back to it. The main result of this paper maintains the classical assumption of three or more alternatives and assumes a large domain of possible preferences, and it demonstrates the inconsistency of a set of axioms: there is no systematic criterion for binary comparisons that satisfies the standard independence and Pareto axioms, that always produces an acyclic relation, that avoids making one agent a weak dictator, and that is minimally responsive to changes in individual preferences. The last of these axioms substantially weakens Mas-Colell and Sonnenschein's (1972) positive responsiveness axiom, which requires that a tie between two alternatives is broken if a single agent reverses her preferences; in contrast, I assume that a tie is broken by a preference reversal (in the same direction) by roughly one third or more of all agents.

It is well known that when Arrow's (1963) transitivity axiom is relaxed, it becomes possible to find non-dictatorial rules that satisfy the remaining independence and Pareto axioms. For example, we can specify that one alternative is socially preferred to another if and only if this preference is shared by two individuals who are fixed ex ante. Or when there are fewer alternatives than agents, we can specify that one alternative is socially preferred to another if and only if all, or all but one, agent share this preference. But the latter rule, because it demands near unanimity for a social preference, will often fail to discriminate between alternatives, and "social indifferences" may be plentiful and persistent; and the former rule gives two agents near dictatorial power, as each can veto a strict social preference, and together they can impose a strict preference regardless of the other agents' preferences. Broadly speaking, by relaxing Arrow's transitivity to acyclicity, there is some scope to strengthen (or add) axioms to preclude anomalies that are, if more palatable than dictatorship, still undesirable. Despite the key role of acyclicity, however, there are few results investigating the limits of acyclic choice under the independence axiom.

The leading exception is Mas-Colell and Sonnenschein (1972), who impose acyclicity in their Theorem 3. To address the anomalies presented in the above examples, they strengthen the non-dictatorship axiom to exclude weak dictatorships, and they add the axiom of positive responsiveness.[1] The content of the latter axiom is as follows. Assume each agent's preferences are represented by a weak order of the set of alternatives, and take a profile of individual weak orders as given. Suppose that two alternatives, say x and y, are socially indifferent. Now change individual preferences so that (i) everyone who preferred x to y still prefers x, (ii) anyone who was indifferent now weakly prefers x, and (iii) there is at least one agent for whom x strictly improves relative to y. By the latter, it is meant that either y was initially strictly preferred by the agent and x is now weakly preferred, or the agent was indifferent and now strictly prefers x. Then positive responsiveness demands that as a result of these changes, x is now strictly socially preferred to y. This axiom is restrictive: it requires social preferences to be responsive to a change in a single agent's ordering, and it is enough that a strict preference for the agent turns into an indifference or that an indifference turns into a strict preference.[2]

A number of papers have investigated weakening the responsiveness axiom so that a group of agents of some minimum size can always break a social indifference; obviously, the larger is this minimum threshold, the weaker is the axiom. Among the extant results of the literature, the closest to the current paper is Theorem 3.6.1 of Schwartz (1986).[3] Letting n denote the number of agents and ignoring integer issues, Schwartz's (1986) responsiveness axiom requires that a preference reversal in a group consisting of at least $\frac{n}{5}$ members break social indifference. He shows that assuming at least three alternatives and all profiles of linear orders are possible, if a social preference rule satisfies Independence, Pareto, Acyclicity, and his responsiveness condition, then some agent is a weak dictator. Like Schwartz, I focus on the restricted domain in which individual preferences are "linear," i.e., individual preferences between distinct alternatives are strict. This dulls the responsiveness axiom, as any preference reversal must be a strict one,

[1] This axiom is also used by May (1952) and Blair, Bordes, Kelly, and Suzumura (1976).

[2] The condition used by Mas-Colell and Sonnenschein (1972) is actually stronger than this, because their condition holds whenever x is initially socially at least as good as y (not just when the alternatives are socially indifferent), so it implies a form of monotonicity.

[3] See the working paper version of this paper, Duggan (2012), for a more detailed review of the social choice literature on acyclicity.

and it has the effect of strengthening the results.[4,5] Schwartz's result is not tight, however; I weaken the axiom further so that a preference reversal by roughly one third of all agents is sufficient to break social indifference. In other words, given a case of social indifference, I require that a vote swing of one third of the agents is enough to break indifference—a condition that becomes increasingly weaker than positive responsiveness as the number of agents becomes large. In general, I refer to this condition as "r-Tie Break," where r is the responsiveness threshold imposed by the axiom.

Setting $r = \lfloor \frac{n-2}{3} \rfloor$ and assuming at least three alternatives and agents,[6] the main result of this paper shows that if a social preference rule satisfies the Independence and Pareto axioms, Acyclicity, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break, then some agent is a weak dictator: if that agent prefers one alternative to another, then there cannot be a strict social preference in the opposite direction; moreover, with the agreement of $\lfloor \frac{n-2}{3} \rfloor$ other agents, that weak dictator can actually impose a strict social preference. The result is tight: I provide a three-alternative example of a social preference rule that satisfies Independence, Pareto, Acyclicity, and $(\lfloor \frac{n-2}{3} \rfloor + 1)$-Tie Break, yet no agent is a weak dictator. Moreover, when there are three or at least five agents, Theorem 1 generalizes the well-known Condorcet paradox: majority rule satisfies Independence, Pareto, $\lfloor \frac{n-2}{3} \rfloor$-Tie Break (there are no ties when $n = 3$), and it does not make any agent a weak dictator, so the theorem implies that for some specification of individual preferences, majority rule generates a cycle. It does not apply to majority rule when $n = 4$, in which case $\lfloor \frac{n-2}{3} \rfloor = 0$, and this is no coincidence: when there are three alternatives and four agents, majority rule is acyclic.[7]

The next section introduces the formal model, and Section 3 presents

[4] Corollary 1 shows that the results of the paper go through when all weak orderings of alternatives are possible.

[5] Deb (1981) considers responsiveness with a range of thresholds. His Theorem 3 establishes that if a social preference rule satisfies Independence, Pareto, Acyclicity, and responsiveness, then it makes some agent a weak dictator; but the result assumes neutrality, allows all profiles of weak orders and counts a change in individual preferences toward the threshold, as do Mas-Colell and Sonnenschein (1972), even when an indifference is made or broken. Deb, Kelsey, and Schimmelpfennig (2002) generalize Deb (1981) but also exploit the possibility of individual indifferences and require responsiveness to preference reversals that are not strict.

[6] Given a real number x, the notation $\lfloor x \rfloor$ refers to the greatest integer less than or equal to x. Similarly, $\lceil x \rceil$ denotes the least integer greater than or equal to x.

[7] Theorem 2 of the working paper version of this paper shows that with four or more alternatives, the tie break threshold can be increased to $\lfloor \frac{n}{3} \rfloor$ to generalize majority rule with four agents.

the main results: Theorem 1 is discussed above, and Theorem 2 shows that when there are more than three alternatives, the Acyclity axiom has greater bite, and the r-Tie Break axiom can be weakened considerably. When there are sufficiently many alternatives, the responsiveness threshold can in fact be increased to its largest logically possible level, $r = n - 2$. Section 4 contains several extensions of Theorem 1, and the appendix contains proofs of Theorems 1 and 2 and extensions.

2 Preliminaries

Let N be a finite set of n agents, denoted i, j, etc., who must choose from a set X of alternatives, denoted x, y, etc. Let Θ be a set of states of the world, denoted θ, which contain information about the agents' preferences over alternatives. Let $P_i(\theta)$ denote agent i's strict preference relation on X in state θ, and let $R_i(\theta)$ denote i's weak preference relation. Assume that $P_i(\theta)$ is asymmetric and negatively transitive, that $R_i(\theta)$ is complete and transitive, and that these relations are dual: for all x,$y \in X$, $x \; P_i(\theta) \; y$ if and only if not $y \; R_i(\theta) \; x$. Let

$$\begin{aligned} P(x,y|\theta) &= \{i \in N \mid x \; P_i(\theta) \; y\} \\ R(x,y|\theta) &= \{i \in N \mid x \; R_i(\theta) \; y\} \end{aligned}$$

denote the set of agents who strictly and weakly, respectively, prefer x to y. Write $x \; I_i(\theta) \; y$ when $x \; R_i(\theta) \; y$ and $y \; R_i(\theta) \; x$, and let $I(x,y|\theta) = \{i \in N \mid x \; I_i(\theta) \; y\}$ be the set of agents indifferent between x and y.

Letting $P(\theta) = (P_1(\theta),...,P_n(\theta))$ be the profile of strict preference relations in state θ, we say *Unrestricted Domain* holds if

$$P(\Theta) = \left\{ (P_1,...,P_n) \mid \begin{array}{c} \text{for all } i, \, P_i \text{ is an asymmetric and} \\ \text{negatively transitive relation on } X \end{array} \right\},$$

i.e., all profiles of weak orders are possible. We say a strict preference relation P_i is *total* if for all distinct x and y, either $x \; P_i \; y$ or $y \; P_i \; x$; this precludes indifference between two alternatives. We then say *Linear Domain* holds if

$$P(\Theta) = \left\{ (P_1,...,P_n) \mid \begin{array}{c} \text{for all } i, \, P_i \text{ is an asymmetric,} \\ \text{total, and negatively transitive} \\ \text{relation on } X \end{array} \right\},$$

i.e., all profiles of linear orders are possible. This paper focusses mainly on Linear Domain, as results proved on the smaller domain extend to the unrestricted framework.

A *social preference rule*, denoted F, is a mapping $\theta \mapsto (P_F(\theta)\ ,\, R_F(\theta))$ defined on the set Θ of states, where $P_F(\theta)$ is an asymmetric strict social preference relation, $R_F(\theta)$ is a complete weak social preference relation, and these relations are dual: for all $x,y \in X$, $x\ P_F(\theta)\ y$ if and only if not $y\ R_F(\theta)\ x$. We write $x\ I_F(\theta)\ y$ if $x\ R_F(\theta)\ y$ and $y\ R_F(\theta)\ x$, a condition interpreted as social indifference. A special case of interest is that of a *quota rule*, where social preferences are completely specified by a single parameter $q > \frac{n}{2}$ as follows: for all x and y, $x\ P_F(\theta)\ y$ if and only if $|P(x,y|\theta)| \geq q$. Quota rules are neutral, in the sense that they treat alternatives symmetrically, but we can define a *generalized quota rule* as a social preference rule that allows the quota to depend on the pair of alternatives under consideration. Such a rule is parameterized by a function $Q\colon X \times X \to \{0,...,n+1\}$ and defined as follows: for all $x,y \in X$, $x\ P_F(\theta)\ y$ if and only if $|P(x,y|\theta)| \geq Q(x,y)$. To ensure asymmetry of $P_F(\theta)$, we impose the requirement that for all $x,y \in X$, $Q(x,y) + Q(y,x) > n$. Of course, *simple majority rule* is the special case of quota rule for which $q = \lceil \frac{n+1}{2} \rceil$.

We investigate the consistency of several axioms on social preference rules. The first class of axioms imposes degrees of responsiveness to changes in the preferences of individual agents. A classical axiom defined by May (1952) and Mas-Colell and Sonnenschein (1972) has two components. First, it requires that if a social preference holds for one alternative over another, and then individual preferences change so that the first has greater support, then the social preference is maintained. Formally, a social preference rule is *monotonic* if for all states θ and θ' and all alternatives x and y, if $P(x,y|\theta) \subseteq P(x,y|\theta')$, $R(x,y|\theta) \subseteq R(x,y|\theta')$, and $x\ P_F(\theta)\ y$, then $x\ P_F(\theta')\ y$. Second, it requires that if two alternatives are socially indifferent, and an agent changes her preferences in favor of one alternative, then this change should break the social indifference in favor of that alternative. The next axiom separates the responsiveness component from the first part of the original axiom.

Positive Responsiveness For all states θ, θ' and all alternatives x and y:

$$\left.\begin{array}{c} P(x,y|\theta) \subseteq P(x,y|\theta'), \\ R(x,y|\theta) \subseteq R(x,y|\theta'), \\ \text{at least one inclusion strict,} \\ \text{and } x\ I_F(\theta)\ y \end{array}\right\} \quad \Rightarrow \quad x\ P_F(\theta')\ y.$$

This responsiveness axiom is quite strong in two respects. First, a change in the preference of only *one* agent is sufficient to break the tie. Second, even if the agent initially was indifferent and changes to a strict preference (or initially had a strict preference and changes to indifference), this change is sufficient to break a tie; in other words, a strict preference reversal is not required to fulfill the antecedent condition of the axiom. It is well-known from May's (1952) theorem that among anonymous, neutral, and monotonic social preference rules, the Positive Responsiveness axiom is uniquely satisfied on the Unrestricted Domain by majority voting among concerned (non-indifferent) agents and on the Linear Domain by (what is the same) simple majority rule.[8]

This paper employs a substantial weakening of the Positive Responsiveness condition. In the following definition, we take an integer r satisfying $0 \leq r \leq n+1$ as given.

r-Tie Break For all states θ and θ' and all alternatives x and y:

$$\left.\begin{array}{c} P(x,y|\theta) \subseteq P(x,y|\theta'), \\ I(x,y|\theta) = I(x,y|\theta'), \\ |P(y,x|\theta) \cap P(x,y|\theta')| \geq r, \\ \text{and } x \; I_F(\theta) \; y \end{array}\right\} \quad \Rightarrow \quad x \; P_F(\theta') \; y.$$

In words, if x and y are socially indifferent in one state, and we consider another state in which no agents have changed their preferences to favor y and in which at least r agents have reversed a strict preference for y to a strict preference for x, then x is strictly socially preferred to y at the new state. Obviously, the tie break axiom is weaker when the responsiveness threshold r is larger, the most restrictive case of the condition being 0-Tie Break, which precludes social indifference altogether. The slightly less restrictive 1-Tie Break is related to Positive Responsiveness: under Unrestricted Domain, 1-Tie Break is strictly weaker than Positive Responsiveness, and under Linear Domain, the two conditions are equivalent. Note also that under Linear Domain, the antecedent assumption $I(x,y|\theta) = I(x,y|\theta')$ in the above tie break condition is vacuous, so the condition is equivalently stated without it.

[8] Although May (1952) restricted his analysis to the case of two alternatives, $|X| = 2$, his anonymity and neutrality axioms can be formulated in such a way that the characterization holds for any number of alternatives; cf. Theorem 3.10 of Austen-Smith and Banks (1999).

We consider the tie-break threshold $r = \lfloor \frac{n-2}{3} \rfloor$, or roughly one third of the number of agents. Most closely related to the responsiveness conditions I propose is Schwartz's (1986) "weak positive responsiveness," which is $\lfloor \frac{n}{5} \rfloor$-Tie Break. For future use, consider the class of generalized quota rules under Linear Domain. Let x and y be distinct alternatives, and for simplicity let $p = Q(x,y)$. The "worst case" scenario for r-Tie Break is a state θ in which $p-1$ agents prefer x, $n-p+1$ agents prefer y, and $x \; I_F(\theta) \; y$ holds. If r members of the former group reverse their preferences, then this reversal breaks the social indifference as long as $n-p+1+r \geq Q(y,x)$. Thus, a generalized quota rule satisfies r-Tie Break if and only if for all x and y, we have

$$r \quad \geq \quad Q(x,y) + Q(y,x) - n - 1, \tag{1}$$

an inequality that is useful for later the exposition. For the special case of a quota rule with quota q, it is equivalent to $q \leq \frac{n+r+1}{2}$, so 1-Tie Break (equivalently, Positive Responsiveness) is satisfied among quota rules only by simple majority rule. For other examples (ignoring integer issues), Schwartz's $\lfloor \frac{n}{5} \rfloor$-Tie Break is satisfied for any quota up to $q = \frac{3n}{5}$, and the weaker condition of $\lfloor \frac{n-2}{3} \rfloor$-Tie Break is satisfied for quotas up to two thirds, $q = \frac{2n}{3}$.

As is standard, an agent i is a *dictator* if for all $\theta \in \Theta$ and all $x,y \in X$, $x \; P_i(\theta) \; y$ implies $x \; P_F(\theta) \; y$. We say i is a *weak dictator* if for all θ and all x and y, $x \; P_i(\theta) \; y$ implies $x \; R_F(\theta) \; y$. Thus, a weak dictator's authority is limited in the sense that a strict preference on her part precludes the opposite strict social preference, without necessitating a strict social preference in the direction of her preference. Furthermore, i is an *r-dictator* if she is a weak dictator and for all groups $G \subseteq N \setminus \{i\}$ with $|G| \geq r$, all θ, and all x and y, $\{i\} \cup G \subseteq P(x,y|\theta)$ implies $x \; P_F(\theta) \; y$. Thus, an r-dictator not only blocks a strict social preference herself but can impose a strict preference with the agreement of r other agents. In obvious fashion, we say a social preference rule satisfies the axiom of No r-dictator if no agent is an r-dictator.

No r-Dictator No agent is an r-dictator.

This definition provides a parameterized family of dictatorship axioms. At one extreme, a dictator is equivalent to a 0-dictator, and the standard Arrovian axiom is No 0-Dictator. Of course, the No r-Dictator axiom becomes more restrictive as we increase the threshold r, for the axiom with a higher threshold rules out agents who need more help in imposing a strict social preference.[9] Setting $r = n+1$, an $(n+1)$-dictator is just a weak dictator, and the weak dictatorship axiom of Mas-Colell and Sonnenschein is No $(n+1)$-Dictator, the most restrictive axiom in this family.[10] Generalized quota rules satisfy No r-Dictator as long as $n \geq 2$ and $r < n-1$.

The other axioms used in the sequel are standard.

Pareto For all θ and all x and y, $P(x,y|\theta) = N$ implies $x\ P_F(\theta)\ y$.

Acyclicity For all θ, all natural numbers m, and all selections $x_1,\ldots,x_m$ of m alternatives, it is not the case that

$$x_1\ P_F(\theta)\ x_2\ P_F(\theta) \cdots x_m\ P_F(\theta)\ x_1.$$

Independence For all θ and θ' and all x and y,

$$\left.\begin{array}{c} P(x,y|\theta) = P(x,y|\theta'), \\ I(x,y|\theta) = I(x,y|\theta'), \\ \text{and } x\ P_F(\theta)\ y \end{array}\right\} \Rightarrow \quad x\ P_F(\theta')\ y.$$

In words, respectively, a common strict preference of the agents is passed to the social preference relation; social preferences do not admit cycles; and the social preference between two alternatives depends only on the agents' preferences over those alternatives. A noteworthy implication of Independence is that social preferences can be elicited from binary votes between pairs of alternatives: to determine a social preference for one alternative x over another y, it is enough to know which individuals prefer x and which prefer y.

With Independence and Pareto, assuming Linear Domain or Unrestricted Domain, the Acyclicity axiom is necessary for the existence of socially maximal alternatives and sufficient when X is finite. Sufficiency is well-known and does not rely on other axioms or domain restrictions. To see necessity, suppose there is some state θ in which a cycle occurs, say $x_1\ P_F(\theta)\ x_2\ P_F(\theta) \cdots x_m\ P_F(\theta)\ x_1$. Even if there is a socially maximal element at state θ, we can choose another state θ' such that individual preferences restricted to $\{x_1,\ldots,x_m\}$ are as in θ, and such that these alternatives are above all others in each agent's ranking. By Independence, the cycle is maintained; and by Pareto, each alternative in the cycle is strictly socially preferred to each alternative outside it. Thus, there is no socially maximal alternative at θ'.

[9] Bordes and Salles (1978) provide a weakening of Mas-Colell and Sonnenschein's (1972) weak dictatorship axiom along these lines.

[10] Schwartz's (1986) Weak Non-Blocker axiom is equivalent to No $(n+1)$-Dictator.

Generalized quota rules always satisfy Independence, and Pareto is satisfied as long as $Q(x,y) \leq n$ for all x and y. For future use, if a generalized quota rule satisfies the Pareto and Acyclicity axioms, then given any three distinct alternatives, say x, y, and z, we must have

$$Q(x,y) + Q(y,z) + Q(z,x) \quad > \quad 2n. \tag{2}$$

Indeed, if this condition is violated, then $(n - Q(x,y)) + (n - Q(y,z)) + (n - Q(z,x)) \geq n$, and we can choose groups G, H, and I consisting, respectively, of at least $Q(x,y)$, $Q(y,z)$, and $Q(z,x)$ agents and such that $\{N \setminus G, N \setminus H, N \setminus I\}$ is a partition of N. This allows us to construct a Condorcet profile, below,

$N \setminus G$	$N \setminus H$	$N \setminus I$
y	z	x
z	x	y
x	y	z

which leads to a cycle: $x \ P_F(\theta) \ y \ P_F(\theta) \ z \ P_F(\theta) \ x$. A further implication is the well-known fact that when there are at least three alternatives (and either three or at least five agents) majority rule (or any quota rule with $q \leq \frac{2n}{3}$) violates Acyclicity.

3 Acyclicity Theorems

The main result on the limits of acyclic voting can now be stated. In keeping with the Arrovian tradition, any number of three or more alternatives is allowed.[11]

Theorem 1 *Assume $|X| \geq 3$, $n \geq 3$, and Linear Domain. There does not exist a social preference rule satisfying Independence, Pareto, Acyclicity, No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break.*

The general proof is relegated to Appendix A. To convey the idea behind the proof, however, it is instructive to consider generalized quota rules. Specialized to this particular case, the result of Theorem 1 is not especially difficult; indeed, the anonymity of generalized quota rules renders the No $\lfloor \frac{n-2}{3} \rfloor$-Dictator axiom redundant. The properties of generalized quota rules vastly

[11]The theorem technically assumes at least three agents, but note that $\lfloor \frac{n-2}{3} \rfloor = 0$ for $n \in \{3,4\}$, in which case the tie break condition rules out social indifference.

reduce the complexity of social preference rules allowed, and under Pareto, they allow us to express the $\lfloor \frac{n-2}{3} \rfloor$-Tie Break and Acyclicity axioms in terms of inequalities (1) and (2). In fact, it is straightforward to see that applied to any triple of alternatives, the inequalities are inconsistent.[12] In contrast, the general proof of Theorem 1 in Section A.1 does not assume anonymity or monotonicity, and it must address the possibility that a particular agent has the authority to block a strict social preference for one alternative over another, while others may not have this authority. The purpose of Lemmas 5–7 is to extract implications of the axioms following just such an instance, and Lemmas 8 and 9 set the stage for the final steps of the proof. There, I work directly with groups G, H, G^*, and H^* that are carefully chosen to satisfy certain properties, whereas here, in the simplified case of generalized quota rules, we can work with the sizes of those groups; in each step below, a natural number s will correspond to a group of size s in the general argument.

The argument below for the special case of generalized quota rules mirrors the key steps of the general proof. Although not the most efficient proof for this special case, because it does not leverage the full force of anonymity or monotonicity, the four steps presented here are close analogues of Steps 1–4 in Section A.1. To set up the argument, suppose, contrary to the theorem, that there is a generalized quota rule satisfying Pareto, Acyclicity, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break. Set $r = \lfloor \frac{n-2}{3} \rfloor$ for the discussion, and consider any three alternatives a, b, and c. In Figure 1 and accompanying analysis, the notation $a\ D^s\ b$ means that groups of size s meet or exceed the quota $Q(a, b)$, so that their members can impose a common strict preference for a over b (i.e., groups of size s are "decisive for a over b"), with similar conventions for other pairs of alternatives.

First, I provide a brief overview. Step 1 begins with a chain $a\ D^p\ b\ D^q\ c$ from a to c, where $p = Q(a, b)$ is the quota needed for a strict social preference for a over b, and $q = Q(b, c)$ is the quota for a strict preference for b over c. Step 2 employs a "doubling back" argument, using r-Tie Break to find an arc $c\ D^{n-q+1+r}\ b$ from c back to b. Step 3 applies a weak transitivity argument to the chain from Step 1 to deduce an arc $a\ D^{p+q-n+r}\ c$ directly from a to c. Finally, Step 4 applies a similar transitivity argument to the chain

$$a\ D^{p+q-n+r}\ c\ D^{n-q+1+r}\ b$$

[12] Applying (1) to the three pairs of alternatives and summing, we obtain $\lfloor \frac{n-2}{3} \rfloor \geq (Q(x, y) + Q(y, z) + Q(z, x)) + (Q(y, x) + Q(x, z) + Q(z, y)) - 3(n + 1)$, and we can apply (2) to obtain $3\lfloor \frac{n-2}{3} \rfloor \geq n - 1$, which is false. I thank an anonymous referee for this observation.

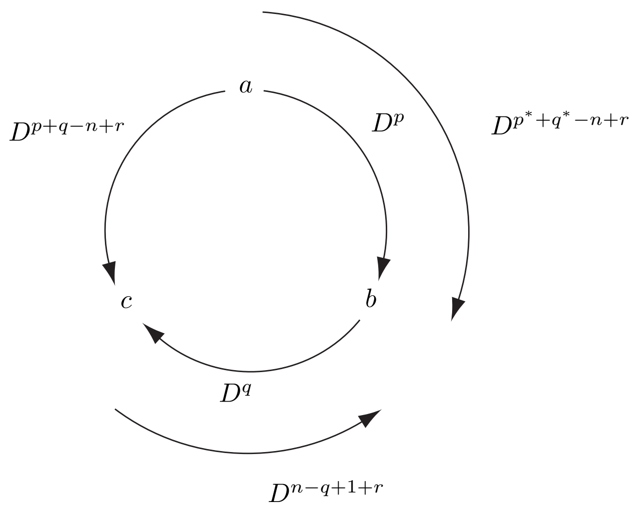

Figure 1: Proof for generalized quota rules

to deduce an arc $a\ D^{p^*+q^*-n+r}\ b$. It is shown, however, that $p^*+q^*-n+r < p$, so that a strict social preference for a over b is obtained by a group that fails to meet the quota, a contradiction.

Step 1: To simplify notation, let $p = Q(a,b)$ and $q = Q(b,c)$, which gives us $a\ D^p\ b\ D^q\ c$, as depicted in Figure 1.[13]

Step 2: Recall from (1) that r-Tie Break implies $r \geq Q(c,b) + Q(b,c) - n - 1$, which yields

$$Q(c,b) \;\; \leq \;\; n - Q(b,c) + 1 + r \;\; = \;\; n - q + r + 1,$$

so each group of size $n-q+r+1$ can impose a common strict preference of its members for c over b, indicated by the arc $c\ D^{n-q+1+r}\ b$ in Figure 1.[14]

Step 3: By Pareto and (2), we have $Q(a,b)+Q(b,c)+Q(c,a) \leq 2n$, which yields $p+q-n \geq n - Q(c,a)+1$. And again from (1), r-Tie Break yields $r \geq Q(a,c) + Q(c,a) - n - 1$. Therefore,

$$Q(a,c) \;\; \leq \;\; n - Q(c,a) + 1 + r \;\; \leq \;\; p + q - n + r.$$

This is indicated by the $a\ D^{p+q-n+r}\ c$ arc in Figure 1.[15]

[13]This corresponds to the chain $a\ D^G\ b\ D^H\ c$ derived in Step 1 of Section A.1.

[14]This corresponds to the the derivation of $c\ D^{(G\setminus H)\cup\{j\}\cup I}\ b$ in Step 2 of Section A.1.

[15]This corresponds to the the derivation of $c\ D^{(G\cap H)\cup J}\ b$ in Step 3 of Section A.1.

Step 4: Set $p^* = p+q-n+r$ and $q^* = n-q+r$. By arguments analogous to Step 3, it then follows that

$$Q(a,b) \;\leq\; n - Q(b,a) + 1 + r \;\leq\; p^* + q^* - n + r,$$

as indicated in Figure 1 by the arc $a\ D^{p^*+q^*-n+r}\ b$.[16] But $3r \leq n-2$ implies

$$Q(a,b) \;\leq\; p^* + q^* - n + r \;\leq\; p - 1,$$

a contradiction. The proof in Section A.1 consists of a more involved analysis along these lines.

To see that Theorem 1 is tight, assume there are three alternatives ordered arbitrarily by $\succeq$, and assume for simplicity that n is divisible by three, so that $\lfloor \frac{n-2}{3} \rfloor + 1 = \frac{n}{3}$. Define a social preference rule F as follows: given any x and y with $x \succeq y$, say $x\ P_F(\theta)\ y$ holds when $|P(x,y|\theta)| \geq \frac{2n}{3}$, and $y\ P_F(\theta)\ x$ holds when $|P(y,x|\theta)| \geq \frac{2n}{3} + 1$. This clearly satisfies Independence, Pareto, and No $\frac{n}{3}$-Dictator. It satisfies Acyclicity, for a cycle $x\ P_F(\theta)\ y\ P_F(\theta)\ z\ P_F(\theta)\ x$ must involve a preference that goes against the ordering $\succeq$, say $x \succ z$, and then $x\ P_F(\theta)\ y\ P_F(\theta)\ z$ implies that at least one third of the agents prefer x to z, so we cannot have $z\ P_F(\theta)\ x$. A moment's consideration shows that it also satisfies $\frac{n}{3}$-Tie Break. Indeed, assume $x \succeq y$ and $x\ I_F(\theta)\ y$, the two difficult cases being when the thresholds for strict preference are missed by a single voter, i.e., $|P(x,y|\theta)| = \frac{2n}{3} - 1$ or $|P(y,x|\theta)| = \frac{2n}{3}$. Consider the former, as the argument is similar in each case. Then $|P(y,x|\theta)| = \frac{n}{3} + 1$, and if $\frac{n}{3}$ agents reverse their preference for x to now favor y in θ', then we have $|P(y,x|\theta')| = \frac{2n}{3} + 1$, so $y\ P_F(\theta')\ x$, as required.[17,18]

We conclude from Theorem 1 that in the binary framework, in which social choices are modeled via a social preference rule, either a single agent must have near dictatorial power, or social preferences must be unresponsive to substantial changes in individual preferences. Although Theorem 1 holds even if there are just three alternatives, it requires that any preference reversal (in the same direction) of roughly one third of all agents breaks social indifference; to the extent that higher thresholds could be considered, the responsiveness condition could be viewed as demanding. But when there are

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[16] This corresponds to the derivation of $a\ D^{(G^* \cap H^*) \cup K}\ b$ in Step 4 of Section A.1.

[17] When $n = 3\lfloor \frac{n}{3} \rfloor + 1$, we use the quota rule with $q = 2\lfloor \frac{n}{3} \rfloor + 1$, and when $n = 3\lfloor \frac{n}{3} \rfloor + 2$, we use the quota rule with $q = 2\lfloor \frac{n}{3} \rfloor + 2$.

[18] The construction used in the above example relies on the assumption of three alternatives; this is demonstrated in Theorem 2 of the working paper version of this paper, which assumes at least four alternatives and uses $\lfloor \frac{n}{3} \rfloor$-Tie Break and No $\lfloor \frac{n}{3} \rfloor$-Dictator.

more than three alternatives, it becomes easier to construct cycles, and the Acyclicity axiom has greater bite, allowing us to relax the responsiveness axiom. We end with a result that draws out the consequences of this logic and establishes that the limits of acyclic voting become arbitrarily constrictive as the set of alternatives becomes large: in this case, we can impose the No r-Dictator and r-Tie Break axioms with threshold up to $r = n - 2$. This threshold is the highest that is logically possible, as $(n-1)$-Tie Break is implied by Pareto, and then the quota rule with $q = n$ satisfies all of the axioms, regardless of the number of alternatives.

Theorem 2 *Assume $n \geq 3$ and Linear Domain. Let $r \leq n-2$. Then there exists an integer m that is large relative to n such that when $|X| \geq m$, there does not exist a social preference rule satisfying Independence, Pareto, Acyclicity, No r-Dictator, and r-Tie Break.*

Setting $r = n-2$, it follows that for a large enough set of alternatives and any social preference rule satisfying Independence, Pareto, Acyclicity, and No $(n-2)$-Dictator, there is a profile of preferences and a pair of alternatives, say x and y, such that one agent prefers x to y, all other agents prefer y to x, and the two alternatives are socially indifferent; moreover, when $n-2$ of the agents in the second group reverse their preference to x over y (so now $n-1$ agents share this preference), the social indifference is maintained. The theorem holds for general $n \geq 3$, and so the quantity $n-1$ could be made arbitrarily large (at the cost of increasing the number of alternatives) and $(n-2)$-Tie Break correspondingly weak.

Theorem 2 highlights an interesting trade off that can be formalized by a threshold function ρ defined follows. Assuming $n \geq 3$ and Linear Domain, for each natural number $m \geq 3$, let $\rho(m)$ be the highest threshold such that when $|X| = m$ holds, if a social preference rule satisfies Independence, Pareto, Acyclicity, and r-Tie Break, then some agent is a weak dictator. Theorem 1 and the subsequent tightness example show that $\rho(3) = \lfloor \frac{n-2}{3} \rfloor$. It is straightforward to show that the threshold function is weakly increasing, as it becomes easier to construct cycles when the number of alternatives is larger. Theorem 2 establishes that the threshold function achieves the logical upper bound $\rho(\overline{m}) = n-2$ for some $\overline{m}$. The nature of this trade off, i.e., the structure of the threshold function for arbitrary $m \geq 4$, is an interesting—but challenging—issue that is outside the scope of this paper.

4 Extensions

4.1 Unrestricted domain

As mentioned above, Theorem 1 carries over when we allow arbitrary individual indifferences.

Corollary 1 *Assume $|X| \geq 3$, $n \geq 3$, and Unrestricted Domain. There does not exist a social preference rule satisfying Independence, Pareto, Acyclicity, No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break.*

The example following Theorem 1 can be extended to Unrestricted Domain with some additional argument. We again assume for simplicity that n is divisible by three, and we order three alternatives by $\succeq$. Given a state θ and alternatives x and y, let $n_{x,y}(\theta) = n - |P(x, y|\theta)| - |P(y, x|\theta)|$ be the number of agents who are not indifferent between the alternatives. We then define the social preference relation $P_F(\theta)$ by applying the earlier example to the group $P(x, y|\theta) \cup P(y, x|\theta)$ of individuals who are not indifferent. That is, given any x and y with $x \succeq y$, we specify that $x \; P_F(\theta) \; y$ holds when

$$|P(x, y|\theta)| \quad \geq \quad \frac{2[|P(x, y|\theta)| + |P(y, x|\theta)|]}{3},$$

and $y \; P_F(\theta) \; x$ holds when

$$|P(y, x|\theta)| \quad \geq \quad \frac{2[|P(x, y|\theta)| + |P(y, x|\theta)|]}{3} + 1.$$

This rule satisfies Independence, Pareto, and No $\frac{n}{3}$-dictator. It satisfies $\frac{n}{3}$-Tie Break by the above arguments, as a preference reversal of one third of the agents will, when some agents are indifferent, have an even a larger impact on the proportions of voters with a strict preference in one direction or the other. Acyclicity follows from some algebra. Suppose we have a cycle, say $x \; P_F(\theta) \; y \; P_F(\theta) \; z \; P_F(\theta) \; x$, and without loss of generality assume $x \succeq z$. There are 12 possible weak orders on $\{x, y, z\}$ that are non-trivial, in the sense of not being indifferent across all three alternatives. These are listed below, with α_k being the number of agents with the corresponding ranking.

α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
z	xz	x	x	x	xy	y	y	y	yz	z	z
x	y	z	yz	y	z	x	xz	z	x	y	xy
y		y		z		z		x		x	

The cycle implies the three inequalities below,

$$\begin{aligned}\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 &\geq 2[\alpha_7+\alpha_8+\alpha_9+\alpha_{10}+\alpha_{11}]\\ \alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9 &\geq 2[\alpha_1+\alpha_2+\alpha_3+\alpha_{11}+\alpha_{12}]\\ \alpha_1+\alpha_9+\alpha_{10}+\alpha_{11}+\alpha_{12} &> 2[\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7]\end{aligned}$$

but adding the inequalities and canceling terms, it is straightforward to show that they cannot simultaneously hold. Thus, Corollary 1 is also tight, in the sense that the result does not hold for thresholds greater than $\lfloor\frac{n-2}{3}\rfloor$.

4.2 Saturated Domains

Following Le Breton and Weymark (2011), a pair $\{x, y\} \subseteq X$ is *trivial* if for all θ, either $P(x, y|\theta) = N$ or $P(y, x|\theta) = N$. Otherwise, it is *non-trivial.* A triple $Y \subseteq X$ is *free* if for every profile of weak orders on Y, there exists $\theta \in \Theta$ such that individual preferences restricted to Y are as in the profile. Two triples Y and Y' are *connected* if there is a sequence $Z_1, ..., Z_m$ of free triples such that $Y = Z_1$, $Y' = Z_m$, and for all $k = 1, ..., m-1$, $|Z_k \cap Z_{k+1}| = 2$. The domain $P(\Theta)$ is *saturated* if (i) every non-trivial pair is contained in a free triple, and (ii) every two free triples are connected. Le Breton and Weymark (2011) show that many common preference domains (including public good domains and the multidimensional spatial voting model) are saturated, and thus the following corollary extends Theorem 1 to many environments of interest.

Corollary 2 *Assume $|X| \geq 3$, $n \geq 3$, and Saturated Domain. There does not exist a social preference rule satisfying Independence, Pareto, Acyclicity, No $\lfloor\frac{n-2}{3}\rfloor$-Dictator, and $\lfloor\frac{n-2}{3}\rfloor$-Tie Break.*

4.3 Pareto Dominant m-Cycles

Theorem 1 can be stated using a weaker acyclicity axiom that proscribes cycles of any particular length, and in fact only such cycles involving alternatives that Pareto dominate all others.

Top m-Acyclicity for all θ and all $Z = \{z_1, ..., z_m\}$ with $|Z| = m$, if $P(z_j, x|\theta) = N$ for all $j = 1, ..., m$ and all $x \in X \setminus Z$, then it is not the case that

$$z_1 \; P_F(\theta) \; z_2 \; P_F(\theta) \cdots z_m \; P_F(\theta) \; z_1.$$

In the above, the set Z is a *cycle of length* m that runs through m distinct alternatives, and these alternatives are strictly preferred by every agent to the alternatives outside the cycle. Of special interest are two extremes: Top 3-Acyclicity rules out only cycles among triples of alternatives that occur at the tops of the agents' rankings, and assuming $|X| < \infty$, we can set $m = |X|$ to rule out only cycles that run through the entire set of alternatives with no repetitions. A necessary condition for the existence of socially maximal elements in every state is that for all $m \geq 3$, Top m-Acyclicity holds, and thus Top m-Acyclicity may be viewed as a minimal requirement. Nevertheless, the incompatibility presented in Theorem 1 is maintained.

Corollary 3 *Assume $|X| \geq 3$, $n \geq 3$, and Linear or Unrestricted Domain. Let m satisfy $3 \leq m \leq |X|$. There does not exist a social preference rule satisfying Independence, Pareto, Top m-Acyclicity, No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break.*

A Proofs of Acyclicity Theorems

This appendix begins with supporting results that will lay the groundwork for the proof of Theorem 1. Throughout Appendix A, assume $|X| \geq 3$, $n \geq 3$, Linear Domain, and $r \leq n-2$. Consider a social preference rule F satisfying Independence, Pareto, Acyclicity, and r-Tie Break. Given a group G and distinct alternatives x and y, we say G is *semi-decisive for x over y*, written $x\ D^G\ y$, if for all θ such that $P(x,y|\theta) = G$, we have $x\ P_F(\theta)\ y$. In parallel fashion, given distinct x and y, we say G is *semi-blocking for x over y*, written $x\ B^G\ y$, if for all θ such that $P(x,y|\theta) = G$, we have $x\ R_F(\theta)\ y$. Thus, a group G that is semi-decisive decisive for x over y can impose a strict social preference for x over y in any state, as long as the set of individuals who prefer x to y is exactly G; in contrast, if G is semi-blocking for x over y, then it can impose a weak social preference under the same conditions.[19] Note that if i is a weak dictator, then for all distinct x and y, we have $x\ B^{\{i\}}\ y$.

The first lemma, a direct implication of the Independence axiom, estab-

[19]The definition of semi-decisiveness does not imply that a group G that is semi-decisive for x over y can impose a strict social preference for x over y in a state where the set of individuals who prefer x to y contains individuals in addition to G. If $x\ P_F(\theta)\ y$ for all θ such that $G \subseteq P(x, y|\theta)$, then we would say G is "decisive" for x over y. Similar remarks hold for semi-blocking and blocking groups.

lishes that if an alternative x is strictly socially preferred to another y at some state, then the group of agents who strictly prefer x to y in that state is semi-decisive for x over y. A similar implication holds for weak social preference, in which case the group of agents who prefer x to y is semi-blocking for x over y. Finally, for every group G all x and y, either G is semi-decisive for x over y, or the complement $N \setminus G$ is semi-blocking in the opposite direction.

Lemma 1 *For all distinct x and y and all θ,*

(i) $x\ P_F(\theta)\ y$ if and only if $x\ D^{P(x,y|\theta)}\ y$

(ii) $x\ R_F(\theta)\ y$ if and only if $x\ B^{P(x,y|\theta)}\ y$

(iii) for all G, either $x\ D^G\ y$ or $y\ B^{N \setminus G}\ x$, but not both.

Proof Parts (i) and (ii) follow immediately from Linear Domain and Independence. For part (iii), consider any G. By Linear Domain, there exists $\theta \in \Theta$ such that $P(x,y|\theta) = G$, and therefore $P(y,x|\theta) = N \setminus G$. By duality, either $x\ P_F(\theta)\ y$ or $y\ R_F(\theta)\ x$, but not both. The first case is equivalent to $x\ D^G\ y$ by part (i), and the second is equivalent to $y\ B^{N \setminus G}\ x$ by part (ii). ■

The next lemma is implied by Theorem 3 of Ferejohn and Fishburn (1979).[20] Setting $k = 3$, the lemma implies that there do not exist distinct alternatives x, y, and z and groups G, H, and I such that $x\ D^G\ y\ D^H\ z\ D^I\ x$, $G \cup H \cup I = N$, and $G \cap H \cap I = \emptyset$. Of course, Pareto implies

[20] The statement of Ferejohn and Fishburn's result requires the concept of a "constitution" induced by a social preference rule, F, assumed to satisfy Independence. The idea is complicated in their framework by the possibility of individual indifferences, but under Linear Domain, it is a mapping C that assigns to each pair (x, y) of distinct alternatives a collection $C(x, y)$ of groups as follows: $G \in C(x, y)$ if and only if there is a state θ such that $x\ P_F(\theta)\ y$ and $P(x,y|\theta) = G$. Then the authors' Theorem 3, simplified to Linear Domain, states that F satisfies Acyclicity if and only if for all k, all distinct alternatives $x_1, ..., x_k$, and all groups $G_1, ..., G_k$ such that $G_j \in C(x_j, x_{j+1})$ for all $j = 1, ..., k-1$ and $G_k \in C(x_k, x_1)$, the following condition holds: there exists j such that either

$$G_j \not\subseteq \bigcup \{N \setminus G_h \mid h \in \{1, ..., k\} \setminus \{j\}\}$$

or

$$N \setminus G_j \not\subseteq \bigcup \{G_h \mid h \in \{1, ..., k\} \setminus \{j\}\}.$$

The latter condition simplifies further to $\bigcap_{j=1}^k G_j \neq \emptyset$ or $\bigcup_{j=1}^k G_k \neq N$.

that for all x and z, we have $z\ D^N\ x$, and thus the lemma has the following direct implication: there do not exist distinct alternatives x, y, and z and disjoint groups G and H such that $x\ D^G\ y\ D^H\ z$. This observation is used in the application of Lemma 2 in the proofs of Lemmas 4, 6, and 8, below. Moreover, note that the relation $x\ D^G\ y$ implies that G is non-empty. Lastly, a careful reading of the Appendix A will reveal that Lemma 2 distills all of the content of the Acyclicity axiom that is used in the analysis; beyond the application of the lemma, the axiom is not invoked in the remainder of the appendix.

Lemma 2 *For all natural numbers k, all distinct alternatives $x_1,\ldots,x_k$, and all groups $G_1,\ldots,G_k$, at least one of the following is violated:*

(i) $x_h\ D^{G_h}\ x_{h+1}$ *for all* $h = 1,\ldots,k-1$ *and* $x_k\ D^{G_k}\ x_1$,

(ii) $\bigcup_{h=1}^{k} G_h = N$,

(iii) $\bigcap_{h=1}^{k} G_h = \emptyset$.

An implication of the previous lemma is a weak transitivity property of the semi-decisiveness relation.

Lemma 3 *For all distinct alternatives $x_1,\ldots,x_k$ and all groups $G_1,\ldots,G_k$, if $x_1\ D^{G_1}\ x_2\ D^{G_2}\ x_3 \cdots x_k\ D^{G_k}\ x_{k+1}$, then $x_1\ B^{\bigcap_{j=1}^{k} G_j}\ x_{k+1}$.*

Proof Consider alternatives $x_1,...,x_{k+1}$ and groups $G_1,...,G_k$ as in the statement of the lemma, and let $G^* = \bigcap_{j=1}^{k} G_j$. Suppose that $x_1\ B^{G^*}\ x_{k+1}$ fails. By Lemma 1, we have $x_{k+1}\ D^{N\setminus G^*}\ x_1$, but then we have

$$x_1\ D^{G_1}\ x_2\ D^{G_3}\ x_3 \cdots x_k\ D^{G_k}\ x_{k+1}\ D^{N\setminus G^*}\ x_1,$$

and we have

$$\left(\bigcup_{j=1}^{k} G_j\right) \cup (N \setminus G^*) = N \quad \text{and} \quad \left(\bigcap_{j=1}^{k} G_j\right) \cap (N \setminus G^*) = \emptyset,$$

contradicting Lemma 2. ■

An implication of the following lemma is that if an agent i is a weak dictator, then she is an r-dictator. Of note, the proof of this result is the single instance in which r-Tie Break is invoked in Appendix A; it fully encapsulates all content of the axiom that is needed for the results in the remainder of the appendix.

Lemma 4 *For all G, all H with $|H| \geq r$ and $G \cap H = \emptyset$, and all x and y, if $x\ B^G\ y$, then $x\ D^{G \cup H}\ y$.*

Proof Consider any $G \subseteq N$, any H with $|H| \geq r$ and $G \cap H = \emptyset$, any x and y such that $x\ B^G\ y$, and any θ such that $G \cup H = P(x,y|\theta)$. By Lemma 1, to establish $x\ D^{G \cup H}\ y$, it suffices to show $x\ P_F(\theta)\ y$. Let θ' be such that $G = P(x,y|\theta')$. Since $x\ B^G\ y$, we have either $x\ P_F(\theta')\ y$ or $x\ I_F(\theta')\ y$. In the latter case, r-Tie Break immediately implies $x\ P_F(\theta)\ y$. Consider the former case. If not $x\ P_F(\theta)\ y$, then either we have $x\ I_F(\theta)\ y$, so r-Tie Break implies $y\ P_F(\theta')\ x$ and contradicts $x\ P_F(\theta')\ y$, or we have $y\ P_F(\theta)\ x$. Thus, we must preclude the possibility of a social preference reversal: $x\ P_F(\theta')\ y$ and $y\ P_F(\theta)\ x$. If so, Lemma 1 implies $x\ D^G\ y$ and $y\ D^{N \setminus (G \cup H)}\ x$. Letting $z \in X \setminus \{x,y\}$, we will show that $x\ B^G\ z$. Otherwise, by Lemma 1, we have $z\ D^{N \setminus G}\ x$, but then we have $z\ D^{N \setminus G}\ x\ D^G\ y$, we have $(N \setminus G) \cup G = N$, and we have $(N \setminus G) \cap G) = \emptyset$, contradicting Lemma 2. Thus, $x\ B^G\ z$, as claimed. Consider θ'' such that $G = P(x,z|\theta'')$. Since $x\ B^G\ z$, there are two cases. Case 1: $x\ P_F(\theta'')\ z$. By Lemma 1, $x\ D^G\ z$. Then we have $y\ D^{N \setminus (G \cup H)}\ x\ D^G\ z\ D^N\ y$, we have $(N \setminus (G \cup H)) \cup G \cup N = N$, and we have $(N \setminus (G \cup H)) \cap G \cap N = \emptyset$, contradicting Lemma 2. Case 2: $x\ I_F(\theta'')\ z$. Letting θ''' be such that $G \cup H = P(x,z|\theta''')$, r-Tie Break implies $x\ P_F(\theta''')\ z$, and then $x\ D^{G \cup H}\ z$ by Lemma 1. Then we have $y\ D^{N \setminus (G \cup H)}\ x\ D^{G \cup H}\ z$, we have $(N \setminus (G \cup H)) \cup (G \cup H) = N$, and we have $(N \setminus (G \cup H)) \cap (G \cup H) = \emptyset$, again contradicting Lemma 2. ■

A.1 Proof of Theorem 1

Now set $r = \lfloor \frac{n-2}{3} \rfloor$, an assumption that is used in the proofs of Lemmas 5, 6, and 9, and in Step 2 at the end of this section.[21] For future use, it is

[21] Specifically, the assumption that r is small is used in the construction of the groups H, I, and J in the proof of Lemma 6; it is used in the construction of G_1 and G_2 and in Cases 3 and 4 of the proof of Lemma 9; and it is used in the construction of groups I and J in Step 2.

helpful to write $n = 3\left\lfloor \frac{n}{3} \right\rfloor + \phi$ for $\phi \in \{0,1,2\}$, in which case

$$\left\lfloor \frac{n-2}{3} \right\rfloor = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor - 1 & \text{if } \phi = 0,1, \\ \left\lfloor \frac{n}{3} \right\rfloor & \text{if } \phi = 2. \end{cases}$$

Assuming F satisfies Independence, Pareto, Acyclicity, and r-Tie Break, the goal is to deduce that F violates No r-Dictator. The proof of Theorem 1 consists of an initial series of lemmas, Lemmas 5–7, that are used to extract implications following an instance in which some agent blocks a social preference: if $x\ B^{\{i\}}\ y$ for some agent and some alternatives, then the agent is an r-dictator. Lemma 8 builds on this by establishing that if there exist alternatives and groups such that $x\ D^G\ y\ D^H\ z$, then one of two cases holds: either some agent is an r-dictator, or there are at least two agents in the intersection of these groups, i.e., $|G \cap H| \geq 2$. Lemma 9 establishes the existence of such a configuration. Finally, Steps 1–4 of the proof rule out the second case, leaving an r-dictator.

We begin by showing that if one agent is semi-blocking for each alternative over every other, then she is an r-dictator.

Lemma 5 *For all i, if for all distinct x and y, we have $x\ B^{\{i\}}\ y$, then i is an r-dictator.*

Proof Consider i as in the statement of the lemma, any distinct x and y, and any θ such that $x\ P_i(\theta)\ y$. Let $G = P(x,y|\theta) \setminus \{i\}$. Suppose not $x\ R_F(\theta)$ y, so that $y\ P_F(\theta)\ x$. By Lemma 1, $y\ D^{N\setminus(G\cup\{i\})}\ x$. Let $\{H,I\}$ be a partition of $N \setminus \{i\}$ such that $\min\{|H|,|I|\} \geq r$, and let $z \notin X \setminus \{x,y\}$. Using Lemma 4, we have $y\ D^{N\setminus(G\cup\{i\})}\ x\ D^{\{i\}\cup H}\ z\ D^{\{i\}\cup I}\ y$, we have $(N \setminus (G \cup \{i\})) \cup (\{i\} \cup H) \cup (\{i\} \cup I) = N$, and we have $(N \setminus (G \cup \{i\})) \cap (\{i\} \cup H) \cap (\{i\} \cup I) = \emptyset$, contradicting Lemma 2. Therefore, $x\ R_F(\theta)\ y$, and it follows that i is a weak dictator. Then Lemma 4 implies that i is in fact an r-dictator. ■

The next lemma shows, essentially, that if an agent is semi-blocking for some x over some y, then she is semi-blocking for every alternative over y and for x over every alternative. To convey the structure of the first half of the proof, suppose toward a contradiction that we have $x\ B^{\{i\}}\ y\ D^{N\setminus\{i\}}\ z$. Here, $N \setminus \{i\}$ is semi-decisive for y over z, but it may properly contain a smaller such group. Letting G be a minimal subset of $N \setminus \{i\}$ such that $y\ D^G\ z$, this gives us the chain $x\ B^{\{i\}}\ y\ D^G\ z$, but that is not the end goal. Rather, the goal is to find partition of N, say $\{K, L\}$, such that

$x\ D^K\ z\ D^L\ y$. To this end, we use the weak transitivity result from Lemma 3 to obtain $x\ B^{G \cap H}\ z$, and we use minimality of G; specifically, if we remove an agent j from G, the smaller group is no longer semi-decisive for y over z, and we have $z\ B^{(N \setminus G) \cup \{j\}}\ y$. This gives us,

$$x\ B^{G \cap H}\ z\ B^{(N \setminus G) \cup \{j\}}\ y.$$

To create the desired semi-decisive groups, it is shown that there are at least $2r$ agents "left over," allowing us to add a set I of at least r agents to $(N \setminus G) \cup \{j\}$ and a set J of at least r agents to $G \cap H$. Setting $L = (N \setminus G) \cup \{j\} \cup I$ and $K = (G \cap H) \cup J$, we obtain $x\ D^K\ z\ D^L\ y$, which contradicts Lemma 2. In the remainder of Appendix A, we use the fact that given two groups G and H, we have $|G \cap H| = |G| + |H| - |G \cup H|$; in particular, when $G \cup H = N$, we have $|G \cap H| = |G| + |H| - n$.

Lemma 6 *For all i and all x, y, and z, it is not the case that $x\ B^{\{i\}}\ y\ D^{N \setminus \{i\}}\ z$, and it is not the case that $x\ D^{N \setminus \{i\}}\ y\ B^{\{i\}}\ z$.*

Proof Consider any i, x, y, and z, and first suppose toward a contradiction that $x\ B^{\{i\}}\ y\ D^{N \setminus \{i\}}\ z$. Note that $x \neq z$, by Lemma 1. Since $y\ D^{N \setminus \{i\}}\ z$, we can choose a minimal group G such that $y\ D^G\ z$ and $i \notin G$. I claim that $|G| \geq n - r$. Otherwise $|G| \leq n - r - 1$, and we have $|N \setminus (G \cup \{i\})| = n - |G| - 1 \geq r$, and using $x\ B^{\{i\}}\ y$ and $N \setminus G = \{i\} \cup (N \setminus (G \cup \{i\}))$, Lemma 4 implies $x\ D^{N \setminus G}\ y$. But then we have $x\ D^{N \setminus G}\ y\ D^G\ z$, we have $(N \setminus G) \cup G = N$, and we have $(N \setminus G) \cap G = \emptyset$, contradicting Lemma 2. This establishes the claim. Of course, we then have $|N \setminus G| \leq r$.

Now let H be a group with $i \notin H$ and $|H| = r$ and such that $N \setminus (G \cup \{i\}) \subseteq H$. By Lemma 4, we have $x\ D^{\{i\} \cup H}\ y$, and then by Lemma 3, $x\ D^{\{i\} \cup H}\ y\ D^G\ z$ implies $x\ B^{G \cap (\{i\} \cup H)}\ z$. Since $i \notin G$, this reduces to $x\ B^{G \cap H}\ z$. Note that since $G \cup H = N \setminus \{i\}$, we have $|G \cap H| = |G| + r - n + 1$.

Choose $j \in G \setminus H$. Since $|G| \geq n - r$ and $|G \cap H| = |G| + r - n + 1$, we can partition $G \setminus (H \cup \{j\})$ into two groups I and J such that $\min\{|I|,|J|\} \geq r$. Indeed, this is possible because

$$\begin{aligned} |G \setminus (H \cup \{j\})| &= |G| - |\{j\}| - |G \cap H| \\ &= |G| - 1 - |G| - r + n - 1 \\ &= n - r - 2 \\ &\geq 2r, \end{aligned}$$

where the inequality follows from $r \leq \frac{n-2}{3}$. From minimality of G, with Lemma 1, it follows that $z\ B^{(N \setminus G) \cup \{j\}}\ y$. Since $I \cap ((N \setminus G) \cup \{j\}) = \emptyset$ and $|I| \geq r$, Lemma 4 implies $z\ D^{(N \setminus G) \cup \{j\} \cup I}\ y$. Recall that $x\ B^{G \cap H}\ z$. Since $J \cap (G \cap H) = \emptyset$ and $|J| \geq r$, Lemma 4 implies $x\ D^{(G \cap H) \cup J}\ z$. But then we have $x\ D^{(G \cap H) \cup J}\ z\ D^{(N \setminus G) \cup \{j\} \cup I}\ y$, we have $((N \setminus G) \cup \{j\} \cup I) \cup ((G \cap H) \cup J) = N$, and we have $((N \setminus G) \cup \{j\} \cup I) \cap ((G \cap H) \cup J) = \emptyset$, contradicting Lemma 2.

When $x\ D^{N \setminus \{i\}}\ y\ B^{\{i\}}\ z$, we deduce a contradiction by a similar argument. Since $x\ D^{N \setminus \{i\}}\ y$, we can choose a minimal group G such that $x\ D^G\ y$ and $i \notin G$. If $|G| < n - r$, then we have $x\ D^G\ y\ D^{N \setminus G}\ z$, a contradiction, so $|G| \geq n - r$. Let H be a group with $i \notin H$ and $|H| = r$ such that $N \setminus (G \cup \{i\}) \subseteq H$. Since $y\ B^{\{i\}}\ z$, we have $y\ D^{\{i\} \cup H}\ z$, and then $x\ D^G\ y\ D^{\{i\} \cup H}\ z$ implies $x\ B^{G \cap H}\ z$. Letting $j \in G$, partition $G \setminus (\{j\} \cup H)$ into groups I and J with $\min\{|I|,|J|\} \geq r$. By minimality of G, we have $y\ B^{(N \setminus G) \cup \{j\}}\ x$, and therefore $y\ D^{(N \setminus G) \cup \{j\} \cup I}\ x\ D^{(G \cap H) \cup J}\ z$, a contradiction. ■

The next lemma strengthens Lemma 5 by establishing that an agent is an r-dictator if she is semi-blocking for one alternative over one other.

Lemma 7 *For all i and all x and y, if $x\ B^{\{i\}}\ y$, then i is an r-dictator.*

Proof Assume that for some i and some x and y, we have $x\ B^{\{i\}}\ y$. Consider any $z \in X \setminus \{x,y\}$. If either $x\ B^{\{i\}}\ z$ or $z\ B^{\{i\}}\ y$ fails, then using Lemma 1, either $z\ D^{N \setminus \{i\}}\ x\ B^{\{i\}}\ y$ or $x\ B^{\{i\}}\ y\ D^{N \setminus \{i\}}\ z$, in both cases contradicting Lemma 6. Thus, we have $x\ B^{\{i\}}\ z$ and $z\ B^{\{i\}}\ y$. Now consider any a and b, and choose $c \in X \setminus \{y,b\}$. By the above argument, $x\ B^{\{i\}}\ c$. Now choose $d \in X \setminus \{b,c\}$, and note that the same argument applied sequentially yields $d\ B^{\{i\}}\ c$, $d\ B^{\{i\}}\ b$, and $a\ B^{\{i\}}\ b$. Since a and b are arbitrary, it follows that i is semi-blocking for every alternative over every other, and Lemma 5 implies that i is an r-dictator. ■

The next lemma distills the key implication of Lemmas 5–7 for the remainder of the proof: either No r-Dictator is violated, or groups that are decisive for adjacent pairs of alternatives must contain at least two agents in their intersection.

Lemma 8 *For all G and H and all distinct x, y, and z, if $x\ D^G\ y\ D^H\ z$, then either $G \cap H = \{i\}$ and i is an r-dictator for some i, or $|G \cap H| \geq 2$.*

Proof Given G, H, and distinct x, y, and z such that $x\ D^G\ y\ D^H\ z$, Lemma 2 implies $G \cap H \neq \emptyset$, so either $|G \cap H| \geq 2$ or for some i, $G \cap H = \{i\}$

In the latter case, Lemma 3 implies $x\ B^{\{i\}}\ z$, and Lemma 7 implies that i is an r-dictator. ■

The next lemma exhibits groups that are not too large, that exhaust the agents, and that are decisive for adjacent pairs of alternatives. To convey the idea of the proof, fix distinct alternatives x, y, and z. The proof begins by finding a group G^1 consisting of roughly two thirds of the agents that is semi-decisive for one alternative over another. This is possible by Lemmas 1 and 4: start with a group A_1 consisting of one third of the agents, and if this group is blocking for x over y, then we can add another third of the agents to obtain $x\ D^{G_1}\ y$; if not, then we set G_1 equal to the complement of A_1, which contains two thirds of the agents, to obtain $y\ D^{G_1}\ x$. We perform a similar operation, starting with a group A_2 consisting of half of the members of G_1, to find a group G_2 consisting of two thirds of the agents such that either $y\ D^{G_2}\ z$ or $z\ D^{G_2}\ y$. The conclusion of the lemma is then verified in each of the four corresponding cases. The formal proof relies on a careful accounting of inequalities, and it uses the fact that $r = \lfloor \frac{n-2}{3} \rfloor \leq \lfloor \frac{n}{3} \rfloor$.

Lemma 9 *There exist groups G and H and distinct alternatives a,b,c such that max*$\{|G|,|H|\} \leq \lceil \frac{2n}{3} \rceil$, $G \cup H = N$, *and* $a\ D^G\ b\ D^H\ c$.

Proof Consider any distinct alternatives x, y, and z, and let A_1 be any group consisting of $\lfloor \frac{n}{3} \rfloor$ members. If $x\ B^{A_1}\ y$, then let B_1 be any group of $\lceil \frac{2n}{3} \rceil - \lfloor \frac{n}{3} \rfloor \geq \lfloor \frac{n}{3} \rfloor$ members disjoint from A_1, and set $G_1 = A_1 \cup B_1$. By Lemma 4, we have $x\ D^{G_1}\ y$. Otherwise, Lemma 1 implies $y\ D^{N \setminus A_1}\ x$, and we set $G_1 = N \setminus A_1$ to obtain $y\ D^{G_1}\ x$. Note that in either case, we have $|G_1| = \lceil \frac{2n}{3} \rceil$.

Next, let A_2 be a group consisting of $\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{3} \rceil$ members of G_1, which is possible since $|G_1| = \lceil \frac{2n}{3} \rceil \geq \lceil \frac{n}{3} \rceil$. If $y\ B^{A_2}\ z$, then let $B_2 = N \setminus G_1$, and set $G_2 = A_2 \cup B_2$. Since $|B_2| = n - |G_1| = \lfloor \frac{n}{3} \rfloor$, Lemma 4 implies $y\ D^{G_2}\ z$. Otherwise, $z\ D^{N \setminus A_2}\ y$, and we set $G_2 = N \setminus A_2$, so that $z\ D^{G_2}\ y$. Note that in the latter case, $|G_2| = n - \lceil \frac{2n}{3} \rceil + \lceil \frac{n}{3} \rceil = \lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{3} \rceil \in \{\lfloor \frac{2n}{3} \rfloor, \lceil \frac{2n}{3} \rceil\}$, and in the former case, $|G_2| = \lceil \frac{2n}{3} \rceil - \lceil \frac{n}{3} \rceil + n - |G_1| = \lfloor \frac{2n}{3} \rfloor$. Thus, we have $|G_2| \in \{\lfloor \frac{2n}{3} \rfloor, \lceil \frac{2n}{3} \rceil\}$ in either case. In addition, $G_1 \cup G_2 = N$ holds.

To complete the proof, we consider four possible cases.

Case 1: $x\ D^{G_1}\ y\ D^{G_2}\ z$. Set $a = x$, $b = y$, $c = z$, $G = G_1$, and $H = G_2$. Then $a\ D^{G_1}\ b\ D^{G_2}\ c$, and as above, $\max\{|G|,|H|\} \leq \lceil \frac{2n}{3} \rceil$, and $G \cup H = N$.

Case 2: $z\ D^{G_2}\ y\ D^{G_1}\ x$. Set $a = z$, $b = y$, $c = x$, $G = G_2$, and $H = G_1$. Then $a\ D^{G_1}\ b\ D^{G_2}\ c$, and again, $\max\{|G|,|H|\} \leq \lceil \frac{2n}{3} \rceil$, and $G \cup H = N$.

Case 3: $x\ D^{G_1}\ y$ and $z\ D^{G_2}\ y$. Let A_3 be a group with $\lceil \frac{n}{3} \rceil$ members that contains $N \setminus G_2$ and is contained in G_1, i.e., $N \setminus G_2 \subseteq A_3 \subseteq G_1$. This is possible because $\lfloor \frac{2n}{3} \rfloor \leq |G_2|$, $|G_1| = \lceil \frac{2n}{3} \rceil$, and $G_1 \cup G_2 = N$. If $x\ B^{A_3}\ z$, then let $B_3 = N \setminus G_1$, so that $|B_3| = \lfloor \frac{n}{3} \rfloor$, and set $a = x$, $b = z$, $c = y$, $G = A_3 \cup B_3$, and $H = G_2$. Otherwise, we have $z\ D^{N \setminus A_3}\ x$, and we set $a = z$, $b = x$, $c = y$, $G = N \setminus A_3$, and $H = G_1$. In the latter case, $|G| = n - |A_3| = \lfloor \frac{2n}{3} \rfloor$ and $|H| = |G_1| = \lceil \frac{2n}{3} \rceil$, and in the former case, $|G| = |A_3| + |B_3| = \lceil \frac{n}{3} \rceil + \lfloor \frac{n}{3} \rfloor \leq \lceil \frac{2n}{3} \rceil$ and $|H| = |G_2| \leq \lceil \frac{2n}{3} \rceil$. Thus, $\max\{|G|,|H|\} \leq \lceil \frac{2n}{3} \rceil$ in either case. In addition, $G \cup H = N$ holds, and using Lemma 4, we have $a\ D^G\ b\ D^H\ c$, as required.

Case 4: $y\ D^{G_1}\ x$ and $y\ D^{G_2}\ z$. Let A_4 be a group with $\lfloor \frac{n}{3} \rfloor$ members that contains $N \setminus G_1$ and is contained in G_2, i.e., $N \setminus G_1 \subseteq A_4 \subseteq G_2$. This is possible because $|G_1| = \lceil \frac{2n}{3} \rceil$ and $\lfloor \frac{2n}{3} \rfloor \leq |G_2|$. If $x\ B^{A_4}\ z$, then let $B_4 = N \setminus G_2$, so that $|B_4| \in \{\lfloor \frac{n}{3} \rfloor, \lceil \frac{n}{3} \rceil\}$, and set $a = y$, $b = x$, $c = z$, $G = G_1$, and $H = A_4 \cup B_4$. Otherwise, we have $z\ D^{N \setminus A_4}\ x$, and we set $a = y$, $b = z$, $c = x$, $G = G_2$, and $H = N \setminus A_4$. In the latter case, $|G| = |G_2| \leq \lceil \frac{2n}{3} \rceil$ and $|H| = n - |A_4| = \lceil \frac{2n}{3} \rceil$, and in the former case, $|G| = |G_1| = \lceil \frac{2n}{3} \rceil$ and $|H| = |A_4| + |B_4| \leq \lceil \frac{2n}{3} \rceil$. Thus, $\max\{|G|,|H|\} \leq \lceil \frac{2n}{3} \rceil$ in either case. In addition, $G \cup H = N$ holds, and using Lemma 4, we have $a\ D^G\ b\ D^H\ c$, as required. ■

Finally, using Lemmas 8 and 9, we can complete the proof of Theorem 1. The argument is decomposed into four steps paralleling the steps described following the statement of the theorem. In contrast to the argument for generalized quota rules, where we can find any groups G and H with $|G| = Q(a,b)$ and $|H| = Q(b,c)$ to conclude $a\ D^G\ b\ D^H\ c$, Step 1 now establishes a chain $a\ D^G\ b\ D^H\ c$ such that G and H are the smallest groups with this property. By Lemma 8, either the intersection $G \cap H$ consists of an r-dictator, or it contains at least two agents. Steps 2–4 deduce a contradiction from the latter hypothesis. Step 2 then "doubles back" using minimality of H and r-Tie Break to conclude $c\ D^{(G \setminus H) \cup \{j\} \cup I}\ b$, where $|I| = r$. Step 3 uses the weak transitivity result from Lemma 3 and carefully chooses a group J such that $a\ D^{(G \cap H) \cup J}\ c$, where $|J| = r$. Step 4 applies the transitivity argument to obtain $a\ D^{(G^* \cap H^*) \cup K}\ b$, where K is a carefully chosen group with $|K| = r$, but it is shown that $|(G^* \cap H^*) \cup K| < |G|$, contradicting minimality of G.

Step 1: Choose distinct alternatives a, b, and c as in Lemma 9, and define the collection Π of pairs of groups by

$$\Pi \;=\; \left\{ (G,H) \;\middle|\; \begin{array}{c} a\ D^G\ b\ D^H\ c,\ |G| \le \lceil \frac{2n}{3} \rceil, \\ |H| \le \lceil \frac{2n}{3} \rceil, \text{ and } G \cup H = N \end{array} \right\}.$$

By Lemma 9, Π is nonempty, so we can choose a pair $(G,H) \in \Pi$ that is minimal, in the sense that there is no $(G',H') \in \Pi$ such that $|G'| \le |G|$ and $|H'| \le |H|$ with at least one inequality strict, i.e., $\max\{|G'|-|G|,|H'|-H|\} \le 0 < \max\{|G|-|G'|,|H|-|H'|\}$. Note that $|G \cap H| = |G| + |H| - n$, and because $a\ D^G\ b\ D^H\ c$, Lemma 8 establishes that one of two cases holds: either there is an r-dictator, or $|G \cap H| \ge 2$. The remainder of the proof shows that the supposition $|G \cap H| \ge 2$ leads to a contradiction.

Step 2: Choose any $j \in G \cap H$. If $b\ D^{H \setminus \{j\}}\ c$, then we can set $G' = G$ and $H' = H \setminus \{j\}$ and obtain $(G',H') \in \Pi$, contradicting minimality. Using Lemma 1, we then have $c\ B^{(N \setminus H) \cup \{j\}}\ b$, and since $G \cup H = N$, we can write $c\ B^{(G \setminus H) \cup \{j\}}\ b$. Setting $\alpha = \lceil \frac{2n}{3} \rceil - 2 - r$, it is verified at the end of Appendix A that $\lfloor \frac{n}{3} \rfloor \ge \alpha \ge r$. Thus, since $|N \setminus G| \ge n - \lceil \frac{2n}{3} \rceil = \lfloor \frac{n}{3} \rfloor$, we can choose a group I disjoint from G with $|I| = \alpha$. Since $c\ B^{(G \setminus H) \cup \{j\}}\ b$ and $((G \setminus H) \cup \{j\}) \cap I = \emptyset$, Lemma 4 yields $c\ D^{(G \setminus H) \cup \{j\} \cup I}\ b$.

Step 3: Since $a\ D^G\ b\ D^H\ c$, Lemma 3 implies $a\ B^{G \cap H}\ c$. Because $G \cup H = N$, we have $I \subseteq H \setminus G$, and with $G \cap I = \emptyset$, we can deduce

$$\begin{aligned} |N \setminus (G \cup I)| &= |H \setminus ((G \cap H) \cup I)| \\ &= |H| - |G \cap H| - |I| \\ &\le \left\lceil \frac{2n}{3} \right\rceil - 2 - \alpha \\ &= r, \end{aligned}$$

where the inequality uses $|H| \le \lceil \frac{2n}{3} \rceil$ and $|G \cap H| \ge 2$. In addition,

$$|N \setminus ((G \cap H) \cup I)| \;=\; n - (|G| + |H| - n) - \alpha \;\ge\; 2n - 2\left\lceil \frac{2n}{3} \right\rceil - \alpha \;\ge\; r,$$

where the first inequality uses $\max\{|G|,|H|\} \le \lceil \frac{2n}{3} \rceil$, and the second is verified at the end of the appendix. Thus, we can choose a group J such that $N \setminus (G \cup I) \subseteq J \subseteq N \setminus ((G \cap H) \cup I)$ with $|J| = r$. Since $a\ B^{G \cap H}\ c$ and $(G \cap H) \cap J = \emptyset$, Lemma 4 yields $a\ D^{(G \cap H) \cup J}\ c$.

Step 4: Since $N \setminus G \subseteq I \cup J$, it follows that $N = ((G \setminus H) \cup \{j\} \cup I) \cup ((G \cap H) \cup J)$. Setting $G^* = (G \cap H) \cup J$ and $H^* = (G \setminus H) \cup \{j\} \cup I$, note that

$G^* \cap (H^* \cap H) = G^* \cap (\{j\} \cup I) = \{j\}$. Moreover, we have $a\ D^{G^*}\ c\ D^{H^*}\ b$, which implies $a\ B^{G^* \cap H^*}\ b$ by Lemma 3. Since $G^* \cup H^* = N$, we have

$$\begin{aligned} |G^* \cap H^*| &= |G^*| + |H^*| - n \\ &= |(G \cap H) \cup J| + |(G \setminus H) \cup \{j\} \cup I| - n \\ &= |G| + |H| - n + r + n - |H| + 1 + \alpha - n \\ &= |G| + 1 - n + r + \alpha. \end{aligned}$$

Setting $\beta = |G| - 1 - |G^* \cap H^*|$, we have

$$\begin{aligned} |N \setminus ((G^* \cap H^*) \cup H)| &= n - (|G^* \cap H^*| + |H| - |(G^* \cap H^*) \cap H|) \\ &= n - |G^* \cap H^*| - |H| + 1 \\ &= n + \beta - |G| + 2 - |H| \\ &\leq \beta, \end{aligned}$$

where the second equality uses $(G^* \cap H^*) \cap H = \{j\}$, and the inequality follows from $|G| + |H| - n = |G \cap H| \geq 2$. In addition,

$$|N \setminus (G^* \cap H^*)| = n - |G^* \cap H^*| = n - (|G| - 1 - \beta) = n - |G| + 1 + \beta \geq \beta.$$

Thus, we can choose a group K such that $N \setminus ((G^* \cap H^*) \cup H) \subseteq K \subseteq N \setminus (G^* \cap H^*)$ with $|K| = \beta$. In particular, $(G^* \cap H^*) \cup K \cup H = N$. Finally, note that

$$\beta = |G| - 1 - |G^* \cap H^*| = |G| - 1 - |G| - 1 + n - r - \alpha = \left\lfloor \frac{n}{3} \right\rfloor \geq r.$$

Therefore, Lemma 4 yields $a\ D^{(G^* \cap H^*) \cup K}\ b\ D^H\ c$. But defining $G' = (G^* \cap H^*) \cup K$ and $H' = H$, we then have $(G', H') \in \Pi$ and

$$|G'| = |G^* \cap H^*| + \beta = |G| - 1 < |G|,$$

contradicting minimality of (G,H).

Remainders: Writing $n = 3\left\lfloor \frac{n}{3} \right\rfloor + \phi$ for $\phi \in \{0,1,2\}$, we have

$$\left\lfloor \frac{n-2}{3} \right\rfloor = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor & \text{if } \phi = 2 \\ \left\lfloor \frac{n}{3} \right\rfloor - 1 & \text{if } \phi = 0,1, \end{cases}$$

and

$$\left\lceil \frac{2n}{3} \right\rceil = \begin{cases} 2\left\lfloor \frac{n}{3} \right\rfloor & \text{if } \phi = 0 \\ 2\left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{if } \phi = 1 \\ 2\left\lfloor \frac{n}{3} \right\rfloor + 2 & \text{if } \phi = 2. \end{cases}$$

Therefore, $\alpha = \lfloor \frac{n}{3} \rfloor - 1$ when $\phi = 0$, and $\alpha = \lfloor \frac{n}{3} \rfloor$ when $\phi = 1$,2. In each case, we have $\lfloor \frac{n}{3} \rfloor \geq \alpha \geq r$. Moreover, since $\lceil \frac{2n}{3} \rceil \leq 2 \lfloor \frac{n}{3} \rfloor + 2$, we have $2n - 2\lceil \frac{2n}{3} \rceil - \alpha = 2 \lfloor \frac{n}{3} \rfloor + 2 - \lceil \frac{2n}{3} \rceil + r \geq r$, as required.

A.2 Proof of Theorem 2

We now return to the general restriction $r \leq n-2$ on the responsiveness threshold, still assuming F satisfies Independence, Pareto, Acyclicity, and r-Tie Break. In addition, we add the assumption that F satisfies No r-Dictator and show that there is a natural number m such that if $|X| \geq m$, then a contradiction arises. First, we show that if one agent is semi-blocking for each alternative over every other, then she is an r-dictator.

Lemma 10 *Assume $|X| \geq \frac{n-2}{n-r-1} + 1$. For all i, if for all distinct x and y, $x\ B^{\{i\}}\ y$, then i is an r-dictator.*

Proof Consider i as in the statement of the lemma, any distinct x and y, and any θ such that $x\ P_i(\theta)\ y$. Let $G = P(x,y|\theta) \setminus \{i\}$. Suppose not $x\ R_F(\theta)\ y$, so that $y\ P_F(\theta)\ x$. By Lemma 1, $y\ D^{N\setminus(G\cup\{i\})}\ x$. Let $|N \setminus (G \cup \{i\})| = m$, and note that since $G \neq \emptyset$, we have $m \leq n-2$. Because $|N \setminus (G\cup\{i\})| = m \leq (n-r-1)\lceil \frac{m}{n-r-1} \rceil$, we can partition $N \setminus (G \cup \{i\})$ into groups $G_1, ..., G_k$ such that $k \leq \lceil \frac{m}{n-r-1} \rceil$ and for all $j = 1, ..., k$, we have $|G_j| \leq n-r-1$. For all $j = 1, ..., k$, let $H_j = (N \setminus (G \cup \{i\})) \setminus G_j$, so that

$$|H_j| \ = \ m - |G_j| \ \geq \ n - |G| - 1 - (n-r-1) \ = \ r - |G|$$

and $\bigcap_{j=1}^{k} H_j = \emptyset$. In particular, $|G| + |H_j| \geq r$. Using $|X| \geq \lceil \frac{n-2}{n-r-1} \rceil + 1 \geq \lceil \frac{m}{n-r-1} \rceil + 1$, select distinct alternatives $x_1, ..., x_{k+1} \in X$ with $x = x_1$ and $y = x_{k+1}$. For all $j = 1, ..., k$, we have $x_j\ B^{\{i\}}\ x_{j+1}$ by assumption, and therefore $x_j\ D^{\{i\}\cup G \cup H_j}\ x_{j+1}$ by Lemma 4. Then we have

$$y\ D^{N\setminus(G\cup\{i\})}\ x = x_1\ D^{\{i\}\cup G\cup H_1}\ x_2\ D^{\{i\}\cup G\cup H_2}\ x_3 \cdots x_k\ D^{\{i\}\cup G\cup H_k}\ x_{k+1} = y,$$

and we have

$$\begin{aligned} (N \setminus (G \cup \{i\})) \cup \left(\bigcup_{j=1}^{k} (\{i\} \cup G \cup H_j) \right) &= N \\ (N \setminus (G \cup \{i\})) \cap \left(\bigcap_{j=1}^{k} (\{i\} \cup G \cup H_j) \right) &= \emptyset, \end{aligned}$$

contradicting Lemma 2. Therefore, $x\ R_F(\theta)\ y$, and it follows that i is a weak dictator. Then Lemma 4 implies that i is in fact an r-dictator. ■

The next lemma, which is implied by Theorem 1 of Blair and Pollack (1982),[22] establishes, in sufficiently large subsets of alternatives, existence of two alternatives belonging to the subset and an agent who is blocking for one alternative over the other.

Lemma 11 *If $Y \subseteq X$ satisfies $|Y| > n+1$, then there exist i and $x, y \in Y$ such that $x\ B^{\{i\}}\ y$.*

To complete the proof of Theorem 2, note that No r-Dictator, with Lemma 10, establishes that for each i, there exist distinct alternatives z_i and w_i such that it is not the case that $z_i\ B^{\{i\}}\ w_i$. By Lemma 1, this implies $w_i\ D^{N\setminus\{i\}}\ z_i$. Assume that $|X| \geq 2n + (n-1)n(n+2)$. Let $X^0 = \{z_i, w_i \mid i \in N\}$, and let $X^1, ..., X^{(n-1)n} \subseteq X \setminus X^0$ be pairwise disjoint sets of alternatives such that for all $j = 1, ..., (n-1)n$, we have $|X^j| \geq n+2$. By Lemma 11, for all $j = 1, ..., n(n-1)$, there exist alternatives $x_j, y_j \in X^j$ and an agent $i(j)$ such that $x_j\ B^{\{i(j)\}}\ y_j$. Then there is an agent i such that $|\{j \mid i = i(j)\}| \geq n-1$. Without loss of generality, assume $i = 1$, and assume $i = i(1) = i(2) = \cdots = i(k)$, where $k \geq n-1$. For all $j = 1, ..., k$ and all agents $h \geq 2$, since $x_j\ B^{\{i\}}\ y_j$ and $r \leq n-2$, Lemma 4 implies that $x_j\ D^{N\setminus\{h\}}\ y_j$. But then we have

$$w_1\ D^{N\setminus\{1\}}\ z_1\ D^N\ x_1\ D^{N\setminus\{2\}}\ y_1\ D^N\ x_2 \cdots x_{n-1}\ D^{N\setminus\{n\}}\ y_{n-1}\ D^N\ w_1,$$

and we have

$$\begin{aligned}(N\setminus\{1\}) \cup N \cup (N\setminus\{2\}) \cup N \cdots \cup (N\setminus\{n\}) \cup N &= N \\ (N\setminus\{1\}) \cap N \cap (N\setminus\{2\}) \cap N \cdots \cap (N\setminus\{n\}) \cap N &= \emptyset,\end{aligned}$$

contradicting Lemma 2.

[22]Say agent i is *semi-decisive on* the pair $\{x, y\}$ of alternatives if $x\ B^{\{i\}}\ y$ and $y\ B^{\{i\}}\ x$. Theorem 1 of Blair and Pollack states that if $|X| \geq \max\{n+1, 4\}$, then there is an individual who is semi-decisive on at least $(|X| - n + 1)(|X| - 1)$ pairs of alternatives. See also Theorem 3 of Le Breton and Truchon (1995).

B Proofs of Corollaries

B.1 Proof of Corollary 1

Consider a social preference rule F defined on the domain of all profiles of weak orders and satisfying Independence, Pareto, Acyclicity, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break. Let F' denote the restriction of F to the Linear Domain, and note that F' satisfies the same four axioms. Then Theorem 1 implies that some agent i is a $\lfloor \frac{n-2}{3} \rfloor$-dictator for F'. To see that F violates No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, proving the corollary, we first show that i is a weak dictator for F. Consider any θ and any x and y such that $x\ P_i(\theta)\ y$, and suppose $y\ P_F(\theta)\ x$. Now let z be distinct from x and y, and consider θ' with individual preferences over these three alternatives as follows, where $N \setminus \{i\}$ is partitioned into groups G and H such that $\min\{|G|,|H|\} \geq \lfloor \frac{n-2}{3} \rfloor$.

i	G	H
x	$x,y?$	z
z	z	$x,y?$
y		

Here, "$x,y?$" indicates that individual preferences between x and y are as in θ. By Independence we have $y\ P_F(\theta')\ x$. Of course, there exists θ'' such that indifferences between x and y are broken while maintaining preferences between x and z and between y and z, i.e., $P_j(\theta'')$ is a linear order for all j and $P(x,z|\theta'') = P(x,z|\theta')$ and $P(z,y|\theta'') = P(z,y|\theta')$. Since i is a $\lfloor \frac{n-2}{3} \rfloor$-dictator for F', we then have $x\ P_{F'}(\theta'')\ z$ and $z\ P_{F'}(\theta'')\ y$. Since F' is the restriction of F, this implies $x\ P_F(\theta'')\ z$ and $z\ P_F(\theta'')\ y$. By Independence, we have $x\ P_F(\theta')\ z\ P_F(\theta')\ y\ P_F(\theta')\ x$, contradicting Acyclicity. We conclude that i is a weak dictator for F.

Now consider θ, x, and y such that $x\ P_i(\theta)\ y$ and, letting $G = P(x,y|\theta) \setminus \{i\}$, such that $|G| \geq \lfloor \frac{n-2}{3} \rfloor$. Suppose toward a contradiction that $y\ R_F(\theta)\ x$. With the fact that i is a weak dictator, this implies $y\ I_F(\theta)\ x$. Now consider $\hat{\theta}$ such that $P(x,y|\hat{\theta}) = \{i\}$ and $P(y,x|\hat{\theta}) = G \cup P(y,x|\theta)$. Note that the antecedent of $\lfloor \frac{n-2}{3} \rfloor$-Tie Break is satisfied, and in particular that $|P(y,x|\hat{\theta}) \cap P(x,y|\theta)| \geq |G| \geq \lfloor \frac{n-2}{3} \rfloor$. Therefore, we have $y\ P_F(\hat{\theta})\ x$, contradicting the fact that i is a weak dictator. We conclude that $x\ P_F(\theta)\ y$, as required.

B.2 Proof of Corollary 2

Assume F satisfies Pareto, Independence, and Acyclicity, and consider any free triple Y. Define an associated collective choice environment denoted (Θ', N', X', PR') as follows: $\Theta' = \Theta$, $N' = N$, $X' = Y$, $PR'(\theta) = PR(\theta)|_Y$. Define a restricted SPR F', $\theta \mapsto (P_{F'}(\theta), R_{F'}(\theta))$, by $P_{F'}(\theta) = P_F(\theta)|_Y$ and $R_{F'}(\theta) = R_F(\theta)|_Y$. Note that the SPR F' inherits Pareto, Independence, Acyclicity, and $\lfloor\frac{n-2}{3}\rfloor$-Tie Break from F, and Corollary 1 implies F' admits a $\lfloor\frac{n-2}{3}\rfloor$-dictator. Let $i(Y)$ denote the $\lfloor\frac{n-2}{3}\rfloor$-dictator for F restricted to Y. Next, it is shown that if Y and Y' are free triples, then $i(Y) = i(Y')$. Indeed, let $Z_1, ..., Z_m$ be as in the definition of connected triples, consider any $k = 1, ..., m-1$, and let $Z_k \cap Z_{k+1} = \{x, y\}$. Suppose in order to show a contradiction that $i(Z_k) \neq i(Z_{k+1})$. Since Z_k is free, there is a state θ such that $x \ P_{i(Z_k)}(\theta) \ y$ and $y \ P_{i(Z_{k+1})}(\theta) \ x$. Since $i(Z_k)$ is the $\lfloor\frac{n-2}{3}\rfloor$-dictator for F restricted to Z_k, we have $x \ P_F(\theta)|_{Z_k} \ y$, and likewise since $i(Z_{k+1})$ is the $\lfloor\frac{n-2}{3}\rfloor$-dictator for F restricted to Z_{k+1}, we have $y \ P_F(\theta)|_{Z_k} \ x$, a contradiction. It follows that $i(Z_k) = i(Z_{k+1})$, and we conclude that $i(Y) = i(Y')$. We may then denote this agent by i, independent of the free triple. Finally, we argue that i is a $\lfloor\frac{n-2}{3}\rfloor$-dictator for F. Indeed, consider any two alternatives x and y and any state such that $x \ P_i(\theta) \ y$. If $\{x, y\}$ is trivial, then $P(x, y|\theta) = N$, and Pareto implies $x \ P_F(\theta) \ y$. Otherwise, $\{x, y\}$ is contained in a free triple Y, and $x \ P_F(\theta)|_Y \ y$, as required.

B.3 Proof of Corollary 3

Let m be such that $3 \leq m \leq |X|$. To deduce a contradiction, suppose there is a social preference rule F satisfying Independence, Pareto, No $\lfloor\frac{n-2}{3}\rfloor$-Dictator, $\lfloor\frac{n-2}{3}\rfloor$-Tie Break, and Top m-Acyclicity. An overview of the proof is as follows. Given a subset $Z \subseteq X$, we say an agent i is a *weak dictator on* Z if for all θ and all $x, y \in Z$, (i) $x \ P_i(\theta) \ y$ implies $x \ R_F(\theta) \ y$. Say i is an $\lfloor\frac{n-2}{3}\rfloor$-*dictator on* Z if she is a weak dictator on Z and for all θ and all $x, y \in Z$, (ii) $x \ P_i(\theta) \ y$ and $|P(x, y|\theta) \setminus \{i\}| \geq \lfloor\frac{n-2}{3}\rfloor$ imply $x \ P_F(\theta) \ y$. The first step is to show that if an agent i is a weak dictator on Z, then she is a $\lfloor\frac{n-2}{3}\rfloor$-dictator on Z. In particular, this implies that no agent is a weak dictator on X. It is still conceivable that for every subset $Z \subseteq X$ of m alternatives, there is an agent i who is a weak dictator on Z, as the weak dictator i could vary with the subset. The next step is to show that this is not actually possible: in fact, there are distinct alternatives a and b such that no agent i is a weak dictator on $\{a, b\}$. Once this pair is obtained, the proof proceeds by selecting $m-2$ other alternatives from X and defining a reduced environment with three alternatives, a, b, and c, where c is a "composite" alternative that corresponds in a particular way, vis-a-vis F, to the $m-2$ other alternatives in the original environment. I show that F induces an aggregation rule $\tilde{F}$ that satisfies Independence, Pareto, No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break. By Theorem 1, $\tilde{F}$ violates the Acyclicity axiom in the smaller environment, and finally this leads to a Pareto dominant m-cycle for F in the original environment.

To proceed, we argue that if some agent i is a weak dictator on Z, then i is an $\lfloor \frac{n-2}{3} \rfloor$-dictator on Z. Assume i is a weak dictator on Z, and consider any state θ and alternatives $x, y \in Z$ such that $x \ P_i(\theta) \ y$ and $|P(x,y|\theta) \setminus \{i\}| \geq \lfloor \frac{n-2}{3} \rfloor$. Since i is a weak dictator on Z, we have $x \ R_F(\theta) \ y$, and we must show that $x \ P_F(\theta) \ y$, i.e., we must exclude $x \ I_F(\theta) \ y$. Choose state θ' such that $P(x,y|\theta') = \{i\}$. If $x \ I_F(\theta) \ y$ did hold, then $\lfloor \frac{n-2}{3} \rfloor$-Tie Break would imply $y \ P_F(\theta') \ x$, but because i is a weak dictator on Z, we have $x \ R_F(\theta') \ y$, a contradiction. Thus, we have $x \ P_F(\theta) \ y$, and we conclude that i is an $\lfloor \frac{n-2}{3} \rfloor$-dictator on Z, as required. Since F satisfies No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, an immediate implication of the above argument is that no agent is a weak dictator on X.

We now claim that there is a subset $Y \subseteq X$ with $|Y| = 2$ such that no agent i is a weak dictator on Y. To deduce a contradiction, suppose otherwise. Note that if there is one agent i such that i is a weak dictator on every doubleton $Y \subseteq X$ with $|Y| = 2$, then i is a weak dictator for F, which is impossible. Thus, there must exist subsets Y and Y' of alternatives with $|Y| = |Y'| = 2$ and distinct agents i and j such that i is a weak dictator on Y and j is a weak dictator on Y'. By the above argument, it follows that in fact i is a $\lfloor \frac{n-2}{3} \rfloor$-dictator on Y, and j is a $\lfloor \frac{n-2}{3} \rfloor$-dictator on Y'. In case the subsets have nonempty intersection, let $Y = \{x, y\}$, $Y' = \{y, z\}$, and choose $m-3$ alternatives $Z = \{z_1, ..., z_{m-3}\} \subseteq X \setminus (Y \cup Y')$. (If $m = 3$, then these alternatives are excluded from the argument.) Let $\{G, H\}$ be a partition of $N \setminus \{i, j\}$ with $\min\{|G|, |H|\} \geq \lfloor \frac{n-2}{3} \rfloor$. Choose a state θ such that preferences over $Y \cup Y' \cup Z$ are as below,

$\{i\} \cup G$	$\{j\} \cup H$
z	y
z_1	z
$\vdots$	z_1
z_{m-3}	$\vdots$
x	z_{m-3}
y	x

and such that for all $s \in Y \cup Y' \cup Z$ and all $t \in X \setminus (Y \cup Y' \cup Z)$, we have $P(s, t|\theta) = N$. By Independence and Pareto, we have

$$x \; P_F(\theta) \; y \; P_F(\theta) \; z \; P_F(\theta) \; z_1 \; P_F(\theta) \; z_2 \cdots P_F(\theta) \; z_{m-3} \; P_F(\theta) \; x,$$

which is a Pareto dominant cycle of length m, contradicting Top m-Acyclicity of F. In the remaining case that the subsets are disjoint, choose alternatives $y \in Y$ and $y' \in Y'$. Then our supposition implies that some agent k is a $\lfloor \frac{n-2}{3} \rfloor$-dictator on $(Y \setminus \{y\}) \cup \{y'\}$. If $k \neq i$, then we apply the preceding argument to the subsets Y and $(Y \setminus \{y\}) \cup \{y'\}$; and if $k = i$, then we apply the preceding argument to $(Y \setminus \{y\}) \cup \{y'\}$ and Y'. This establishes the claim.

Therefore, we may choose a pair $\{a, b\}$ such that no agent is a dictator on $\{a, b\}$. Let $Z \subseteq X$ be a subset of m alternatives containing this pair, i.e., $\{a, b\} \subseteq Z$ and $|Z| = m$. Now consider a three-alternative environment with set of alternatives $\tilde{X} = \{a,b,c\}$, in which the third alternative c proxies for the additional set $Z \setminus \{a, b\}$ of $m - 2$ alternatives. Let $\tilde{\Theta}$ denote the set of states in the smaller environment, where each $\tilde{\theta} \in \tilde{\Theta}$ corresponds to a profile of linear orders of $\tilde{X}$ and Linear Domain holds. Define a social preference rule $\tilde{F}$ for the smaller environment as follows: for all $\tilde{\theta}$ and all $x,y \in \tilde{X}$, $x \; P_{\tilde{F}}(\tilde{\theta}) \; y$ if and only if any one of three cases obtains.

- $\{x, y\} = \{a, b\}$ and $x \; P_F(\theta) \; y$ for some state θ with individual preferences restricted to $\{a, b\}$ as in $\tilde{\theta}$, i.e., $P(a, b|\theta) = P(a, b|\tilde{\theta})$,

- $y = c$ and for some state θ and alternative $z \in Z \setminus \{a, b\}$ such that $P(x, y|\tilde{\theta}) = P(x, z|\theta)$, we have $x \; P_F(\theta) \; z$.

- $x = c$ and for some state θ and alternative $z \in Z \setminus \{a, b\}$ such that $P(x, y|\tilde{\theta}) = P(z, y|\theta)$, we have $z \; P_F(\theta) \; y$.

Thus, for example, a is strictly socially preferred to the composite alternative c in state $\tilde{\theta}$ of the reduced model when there is a state θ in the original model such that individual preferences between a and c in the reduced model are as they are between a and an element $z \in Z \setminus \{a, b\}$ in the original model, and $a \; P_F(\theta) \; z$ holds.

We must check that $\tilde{F}$ is a valid social preference function, in the sense that $P_{\tilde{F}}(\tilde{\theta})$ is asymmetric. Consider any $\tilde{\theta}$, and suppose toward a contradiction that $x \; P_{\tilde{F}}(\tilde{\theta}) \; y$ and $y \; P_{\tilde{F}}(\tilde{\theta}) \; x$. It must then be that either $x = c$ or $y = c$; without loss of generality, we consider the latter case. Then there exist states θ', θ'' and alternatives $z', z'' \in Z \setminus \{a, b\}$ such that $P(x, y|\tilde{\theta}) = P(x, z'|\theta')$, $P(y, x|\tilde{\theta}) = P(z'', x|\theta'')$, $x \; P_F(\theta') \; z'$, and $z'' \; P_F(\theta'') \; x$. Note that $z' \neq z''$, for otherwise Independence implies $x \; P_F(\theta') \; z'$ and $z' \; P_F(\theta') \; x$, contradicting asymmetry of $P_F(\theta)$. Now index $Z \setminus \{x\}$ as $\{z_1, ..., z_{m-1}\}$ with $z_1 = z'$ and $z_{m-1} = z''$, and choose a state θ such that for all i,

$$x \; P_i(\tilde{\theta}) \; y \quad \text{implies} \quad x \; P_i(\theta) \; z_1 \; P_i(\theta) \; z_2 \cdots P_i(\theta) \; z_{m-1}$$

and

$$y \; P_i(\tilde{\theta}) \; x \quad \text{implies} \quad z_1 \; P_i(\theta) \; z_2 \cdots P_i(\theta) \; z_{m-1} \; P_i(\theta) \; x,$$

and for all $z \in Z$ and all $w \in X \setminus Z$, we have $P(z, w|\theta) = N$. By Independence and Pareto, we have

$$x \; P_F(\theta) \; z_1 \; P_F(\theta) \; z_2 \cdots P_F(\theta) \; z_{m-1} \; P_F(\theta) \; x,$$

which is a Pareto dominant cycle of length m, contradicting Top m-Acyclicity. Thus, social preferences are asymmetric, as required.

The rule $\tilde{F}$ directly inherits Independence and Pareto from F. We now verify that it satisfies $\lfloor \frac{n-2}{3} \rfloor$-Tie Break. Consider states $\tilde{\theta}, \tilde{\theta}'$ and alternatives $x, y \in \tilde{X}$ such that $P(x, y|\tilde{\theta}) \subseteq P(x, y|\tilde{\theta}')$, $|P(y, x|\tilde{\theta}) \cap P(x, y|\tilde{\theta}')| \geq \lfloor \frac{n-2}{3} \rfloor$, and $x \; I_{\tilde{F}}(\tilde{\theta}) \; y$. In case $\{x, y\} = \{a, b\}$, then clearly $x \; P_{\tilde{F}}(\tilde{\theta}') \; y$, so consider the remaining cases that $x = c$ or $y = c$. The cases are symmetric, so we focus on the second. Choose any $z \in Z \setminus \{a, b\}$ and states θ, θ' in the original model such that $P(x, z|\theta) = P(x, y|\tilde{\theta})$ and $P(x, z|\theta') = P(x, y|\tilde{\theta}')$. Since $x \; I_{\tilde{F}}(\tilde{\theta}) \; y$, we have $x \; I_F(\theta) \; z$, and then the assumption that F satisfies $\lfloor \frac{n-2}{3} \rfloor$-Tie Break implies $x \; P_F(\theta') \; z$, which implies $x \; P_{\tilde{F}}(\tilde{\theta}') \; y$.

Next, we establish that $\tilde{F}$ satisfies No $\lfloor \frac{n-2}{3} \rfloor$-Dictator. Suppose toward a contradiction that some agent i is an $\lfloor \frac{n-2}{3} \rfloor$-Dictator for $\tilde{F}$, i.e., for all $\tilde{\theta}$

and all $x, y \in \tilde{X}$ such that $x \; P_i(\tilde{\theta}) \; y$, we have $x \; R_{\tilde{F}}(\tilde{\theta}) \; y$ and, if in addition $|P(x,y|\tilde{\theta}) \setminus \{i\}| \geq \lfloor \frac{n-2}{3} \rfloor$, then $x \; P_{\tilde{F}}(\tilde{\theta}) \; y$. By choice of $\{a, b\}$, i is not a weak dictator on $\{a, b\}$, so without loss of generality there is a state $\theta^* \in \Theta$ such that $b \; P_i(\theta^*) \; a$ but $a \; P_F(\theta^*) \; b$. Let $G = P(a,b|\theta^*)$, and let $\{H, I\}$ be a partition of $N \setminus \{i\}$ such that $\min\{|H|, |I|\} \geq \lfloor \frac{n-2}{3} \rfloor$. Let $\tilde{\theta}$ be such that $P(b,c|\tilde{\theta}) = \{i\} \cup H$ and $P(c,a|\tilde{\theta}) = \{i\} \cup I$. Since i is an $\lfloor \frac{n-2}{3} \rfloor$-dictator for $\tilde{F}$, we have $b \; P_{\tilde{F}}(\tilde{\theta}) \; c \; P_{\tilde{F}}(\tilde{\theta}) \; a$. Then there exist states θ', θ'' and alternatives $z', z'' \in Z \setminus \{a, b\}$ such that $P(b,c|\tilde{\theta}) = P(b,z'|\theta')$, $P(c,a|\tilde{\theta}) = P(z'',a|\theta'')$, $b \; P_F(\theta') \; z'$, and $z'' \; P_F(\theta'') \; a$. Now index $Z \setminus \{a, b\}$ as $\{z_1, ..., z_{m-2}\}$ with $z_1 = z'$ and $z_{m-2} = z''$. Choose a state θ such that preferences over Z are as below,

i	$H \setminus G$	$I \setminus G$	$G \cap H$	$G \cap I$
b	b	z_1	a	z_1
z_1	a	$\vdots$	b	$\vdots$
$\vdots$	z_1	z_{m-2}	z_1	z_{m-2}
z_{m-2}	$\vdots$	b	$\vdots$	a
a	z_{m-2}	a	z_{m-2}	b

and such that for all $s \in Z$ and all $t \in X \setminus Z$, we have $P(s,t|\theta) = N$. By Independence and Pareto, we have

$$a \; P_F(\theta) \; b \; P_F(\theta) \; z_1 \; P_F(\theta) \; z_2 \cdots P_F(\theta) \; z_{m-2} \; P_F(\theta) \; a,$$

which is a Pareto dominant cycle of length m, contradicting Top m-Acyclicity of F. We conclude that the rule $\tilde{F}$ in the reduced environment satisfies No $\lfloor \frac{n-2}{3} \rfloor$-Dictator.

Finally, Theorem 1 then implies that $\tilde{F}$ must violate the Acyclicity axiom, and we will argue that this, in turn, contradicts Top m-Acyclicity of F. Let $\tilde{\theta}$ be such that $P_{\tilde{F}}(\tilde{\theta})$ admits a cycle, and assume without loss of generality that $a \; P_{\tilde{F}}(\tilde{\theta}) \; b \; P_{\tilde{F}}(\tilde{\theta}) \; c \; P_{\tilde{F}}(\tilde{\theta}) \; a$. Let $G_1 = P(a,b|\tilde{\theta})$, $G_2 = P(b,c|\tilde{\theta})$, and $G_3 = P(c,a|\tilde{\theta})$, and note that $G_1 \cap G_2 \cap G_3 = \emptyset$ and $G_1 \cup G_2 \cup G_3 = N$. Then there exist states θ', θ'' and alternatives $z', z'' \in Z \setminus \{a, b\}$ such that $G_2 = P(b,z'|\theta')$, $G_3 = P(z'',a|\theta'')$, $b \; P_F(\theta') \; z'$, and $z'' \; P_F(\theta'') \; a$. Now index the alternatives in $Z \setminus \{a, b\}$ as $\{z_1, ..., z_{m-2}\}$ with $z_1 = z'$ and $z_{m-2} = z''$.

Choose a state θ such that preferences over Z are as below,

$G_1 \setminus (G_2 \cup G_3)$	$G_2 \setminus (G_1 \cup G_3)$	$G_3 \setminus (G_1 \cup G_2)$	$G_1 \cap G_2$	$G_2 \cap G_3$	$G_1 \cap G_3$
a	b	z_1	a	b	z_1
z_1	a	$\vdots$	b	z_1	$\vdots$
$\vdots$	z_1	z_{m-2}	z_1	$\vdots$	z_{m-2}
z_{m-2}	$\vdots$	b	$\vdots$	z_{m-2}	a
b	z_{m-2}	a	z_{m-2}	a	b

and such that for all $z \in Z$ and all $w \in X \setminus Z$, we have $P(z, w|\theta) = N$. By Independence and Pareto, we have

$$a\ P_F(\theta)\ b\ P_F(\theta)\ z_1\ P_F(\theta)\ z_2 \cdots P_F(\theta)\ z_{m-2}\ P_F(\theta)\ a,$$

which is a Pareto dominant cycle of length m, contradicting Top m-Acyclicity of F.

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