Limits of Acyclic Voting and Nash Implementation

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October 15, 2012

Abstract

I prove that there is no systematic rule for aggregating individual preferences that satisfies acyclicity and the standard independence and Pareto axioms, that avoids making some voter a weak dictator, and that is minimally responsive to changes in voter preferences. The latter axiom requires that a preference reversal in the same direction by roughly one third of all voters is sufficient to break social indifference. This result substantially strengthens acyclicity theorems of Mas-Colell and Sonnenschein (1972) and Schwartz (1986). The acyclicity theorem can be applied to social choice rules via a corresponding ''revealed social preference'' rule that satisfies independence and acyclicity; when we impose monotonicity as well, additional axioms on the social choice rule translate to properties of revealed social preferences. The conclusion is that equilibrium outcome correspondences must either concentrate power in small groups or sometimes be unresponsive to substantial changes in voter preferences.

1 Introduction

Consider a set of agents with possibly heterogenous preferences who must make a collective choice from a given set of alternatives. The problem is to systematically construct a nonempty choice set --- a subset of alternatives that may represent normatively appealing choices or plausible predictions --- based on binary comparisons of alternatives. The key condition needed to construct nonempty choice sets is that the binary comparisons, or ''social preferences,'' be acyclic: there should not be a chain of social preferences beginning with one alternative and leading back to it. The main result of this paper maintains the classical assumption of three or more alternatives and assumes a large domain of possible preferences, and it demonstrates the inconsistency of a set of axioms: there is no systematic criterion for binary comparisons that satisfies the standard independence and Pareto axioms, that always produces an acyclic relation, that avoids making one agent a weak dictator, and that is minimally responsive to changes in individual preferences. The latter axiom substantially weakens Mas-Colell and Sonnenschein's (1972) positive responsiveness axiom, which requires that a tie between two alternatives is broken if a single agent reverses her preferences; in contrast, I assume that a tie is broken by a preference reversal (in the same direction) by roughly one third of all agents.

This result has implications for the properties of Nash equilibrium outcome correspondences. Specifically, given any social choice rule, we can define a ''revealed social preference'' rule that satisfies Arrow's independence axiom and generates acyclic social preferences. It is difficult to place further structure on revealed social preferences in general, but if the social choice rule further satisfies monotonicity --- if an alternative is chosen in one state of the world but not in another, then it must have moved down relative to some alternative in some agent's ranking --- then additional axioms on social choices translate to properties of revealed social preferences. Then the acyclicity theorem applies and yields immediate consequences for monotonic social choice rules. As shown by Maskin (1977, 1999), Nash equilibrium correspondences must satisfy monotonicity, so these results then have direct, and restrictive, consequences for the equilibrium outcomes generated by a given game form.

It is known that Arrow's (1963) transitivity axiom is more than sufficient for existence of maximal elements in finite sets: acyclicity is sufficient, and it is in fact necessary to ensure maximal elements in all finite subsets of alternatives. It is also known that by relaxing the transitivity axiom, it becomes possible to find non-dictatorial rules that satisfy the remaining criteria. For example, we can specify that one alternative is socially preferred to another if and only if this preference is shared by two individuals who are fixed ex ante. Or when there are fewer alternatives than agents, we can specify that one alternative is socially preferred to another if and only if this preference is contradicted by at most one agent. But the latter rule, because it demands near unanimity for a social preference, will often fail to discriminate between alternatives, and ''social indifferences'' may be plentiful and persistent; and the former rule gives two agents near dictatorial power, as each can veto a strict social preference, and together they can impose a strict preference regardless of the other agents' preferences. Broadly speaking, by relaxing Arrow's transitivity to acyclicity, there is some scope to strengthen (or add) axioms to preclude anomalies that are, if more palatable than dictatorship, still undesirable. Despite the key role of acyclicity, there are few results investigating the limits of acyclic choice under the independence axiom.

Mas-Colell and Sonnenschein (1972) impose acyclicity in their Theorem 3, and to address the anomalies presented in the above examples, they strengthen the non-dictatorship axiom to exclude weak dictatorships, and they add the axiom of positive responsiveness.[1] The content of the latter axiom is as follows. Assume each agent's preferences are represented by a weak order of the set of alternatives, and take a profile of individual weak orders as given. Suppose that two alternatives, say x and y, are socially indiffer-

[1]This axiom is also used by May (1954) and Blair, Bordes, Kelly, and Suzumura (1976).

ent. Now change individual preferences so that (i) everyone who preferred x to y still prefers x, (ii) anyone who was indifferent now weakly prefers x, and (iii) there is at least one agent for whom x strictly improves relative to y. By the latter, it is meant that either y was initially strictly preferred and x is now weakly preferred, or the agent was indifferent and now strictly prefers x. Then positive responsiveness demands that as a result of these changes, x is now strictly socially preferred to y. This axiom is restrictive: it requires social preferences to be responsive to a change in a single agent's ordering, and it is enough that a strict preference for the agent turns into an indifference or that an indifference turns into a strict preference.[2]

I follow Deb (1981) and Schwartz (1986) in weakening the responsiveness axiom so that a group of agents of some minimum size is required to break a social indifference. Obviously, the larger is this minimum size, the weaker is the axiom. Like Schwartz (but unlike Deb), I do not impose neutrality or monotonicity as axioms. Also like Schwartz (but unlike Deb), I focus on the restricted domain in which individual preferences are ''linear,'' i.e., individual preferences between distinct alternatives are strict. This dulls the responsiveness axiom, as any preference reversal must be a strict one, and has the effect of strengthening the results.[3] Letting n denote the number of agents and ignoring integer issues, Schwartz's (1986) axiom requires that a preference reversal in a group consisting of at least $\frac{n}{5}$ members break social indifference, but his result is not tight; I weaken the axiom further so that a preference reversal by roughly one third of all agents is sufficient to break social indifference. In other words, given a case of social indifference, I require that a ''vote swing'' of two thirds of the agents is enough to break indifference --- a condition that becomes increasingly weaker than positive responsiveness as the

[2] The condition used by Mas-Colell and Sonnenschein (1972) is actually stronger than this, because their condition holds whenever x is initially socially at least as good as y (not just when the alternatives are socially indifferent), so it implies a form of monotonicity.

[3] Corollary 1 shows that the results of the paper go through when all weak orderings are possible.

number of agents becomes large. In general, I refer to this condition as ‘‘r-Tie Break,’’ where r is the responsiveness threshold imposed by the axiom.

I set $r = \lfloor \frac{n-2}{3} \rfloor$ and show, assuming at least three alternatives and agents, that if a social preference rule satisfies the standard Independence and Pareto axioms, Acyclicity, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break, then some agent is a weak dictator: if that agent prefers one alternative to another, then there cannot be a strict social preference in the opposite direction; and with the agreement of $\lfloor \frac{n-2}{3} \rfloor$ other agents, that weak dictator can actually impose a strict social preference. I paraphrase the acyclicity theorem next.

> Theorem 1 Assume at least three alternatives and agents, and the possible preferences of each agent are the linear orderings of alternatives. If a social preference rule satisfies Independence, Pareto, Acyclicity, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break, then there is a weak dictator.

The result is tight: I give a three-alternative example of a social preference rule that satisfies Independence, Pareto, Acyclicity, and $(\lfloor \frac{n-2}{3} \rfloor + 1)$-Tie Break. Moreover, when there are three or at least five agents, Theorem 1 generalizes the well-known Condorcet paradox: majority rule satisfies Independence, Pareto, $\lfloor \frac{n-2}{3} \rfloor$-Tie Break (there are no ties when $n = 3$), and it does not make any agent a weak dictator, so the theorem implies that for some specification of individual preferences, majority rule generates a cycle. It does not apply to majority rule when $n = 4$, in which case $\lfloor \frac{n-2}{3} \rfloor = 0$, and this is no coincidence: when there are three alternatives and four agents, majority rule is acyclic.[4]

I then examine possibilities for monotonic social choice in light of the acyclicity theorem. First, I reformulate the r-Tie Break axiom in terms of social choices: if two alternatives, say x and y, are ranked above all others by all agents, if the social

[4]Theorem 2 of the working paper version of this paper shows that with four or more alternatives, the tie break threshold can be increased to $\lfloor \frac{n}{3} \rfloor$ to generalize majority rule with four agents.

choice set contains both, and if we change individual preferences so that x does not move down relative to any alternative in any agent's preferences and so that r agents reverse their preferences to favor x, then y does not belong to the new choice set. The results on social choice rules also assume ''Pareto Consistency,'' a weakening of the standard Pareto optimality criterion. The conclusion of Theorem 2 is not that a particular agent can unilaterally determine the social choice set, but that there is one agent such that if she prefers any x to any y, and if $\lfloor \frac{n-2}{3} \rfloor$ or more agents share that preference, then y does not belong to the social choice set; and if that x is in fact her top-ranked alternative, then it will be among those chosen.

> Theorem 2 Assume at least three alternatives and agents, and the possible preferences of each agent are the linear orderings of alternatives. If a social choice rule satisfies Monotonicity, Pareto Consistency, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break, then there is an agent who, with the agreement of $\lfloor \frac{n-2}{3} \rfloor$ others, can reject any alternative and include her top-ranked alternative among those chosen.

Under the mild Pareto Consistency axiom, we conclude that Nash equilibrium outcome correspondences either concentrate power in small groups or sometimes are unresponsive to substantial changes in individual preferences.

Background: Acyclic Preference Aggregation

The canonical problem of preference aggregation framed by Arrow (1963) is to define a social ranking of alternatives in all possible states of the world in a way that satisfies appealing or otherwise interesting axioms. If the goal is simply to construct nonempty social choice sets, however, then Arrow's requirement that social preferences form a ranking can be relaxed to acyclicity, which is sufficient for the existence of maximal elements in all finite sets of alternatives. This relaxation admits social preference rules that escape the Arrovian dictatorship result and the oligarchy results of Gibbard (1969), as the well-known example of the ''Security Council'' voting rule (Brown (1975a)) demonstrates. This example, which generates acyclic social preferences for any number of alternatives, gives each member of a small group of agents (the permanent members) veto power. The assignment of veto power can be avoided if there are m alternatives and $n > m$ agents, for we can specify that one alternative is socially preferred to another if and only if that is the preference of at least $\frac{n(m-1)}{m} + 1$ agents (ignoring integer issues), and then social preferences will be acyclic (Ferejohn and Grether (1974)), yet no agent has a veto; but this construction fails if $m \geq n$. This raises questions about the possibility of acyclic social choice, and the connections between the structure of power and the number of alternatives.

The possibility of acyclic social choice was examined in early work by Schwartz (1970), Sen (1970), and Mas-Colell and Sonnenschein (1972). The latter paper, which is the closer of the three to this one, considers the implications of positive responsiveness for acyclic social preference rules when there are three or more alternatives, and it deduces the existence of a weak dictator. By relaxing their positive responsiveness to $\lfloor \frac{n-2}{3} \rfloor$-Tie Break and working on the narrower domain of profiles of linear orders, Theorem 1 generalizes their Theorem 3.[5] Theorem 7 of Blair, Bordes, Kelly, and Suzumura (1976) is an analogue of Mas-Colell and Sonnenshein's result (using the same preference domain and notion of positive responsiveness) in the context of social choice rules. Deb (1981) considers r-Tie Break for a range of thresholds. His Theorem 3 establishes that if a social preference rule satisfies Independence,[6] Pareto, Acyclicity, and r-Tie Break, then it makes some agent a weak dictator; but the result assumes at least $r + 2$ alternatives, and he exploits the possibility of individual indifferences by allowing all profiles of weak orders.[7] Among extant

[5]In contrast, Schwartz (1970) assumes at least as many alternatives as agents and imposes a condition limiting the possibility of social indifferences; and Sen (1970) imposes the assumption that at least two agents are decisive over pairs of alternatives (his ''minimal liberalism'').

[6]Note that Deb's (1981) condition of k-monotonicity implies Independence.

[7]Blau and Deb (1977) provide a Latin square construction to show that,

results in the literature, Theorem 3.6.1 of Schwartz (1986) is closest to the acyclicity theorem of this paper: assuming at least three alternatives and all profiles of linear orders are possible, he proves that if a social preference rule satisfies Independence, Pareto, Acyclicity, and $\lfloor \frac{n}{5} \rfloor$-Tie Break, then some agent is a weak dictator. Theorem 1 generalizes this by considerably relaxing his responsiveness condition.

The interaction between the power structure of a rule and the number of alternatives arose in Schwartz (1970) and continued in early work by Brown (1973, 1975a), who showed that if the alternatives exceed the agents in number, then some agent belongs to all decisive coalitions. More complex restrictions on the collection of decisive coalitions implied by smaller numbers of alternatives were first given in Theorem 1 of Brown (1975b) and Theorem 3.1 of Nakamura (1979), both showing that acyclic social preference necessitates that the size of the smallest collection of decisive coalitions having empty intersection (a quantity now known as the ‘‘Nakamura number’’) exceed the number of alternatives. These connections were elaborated in detail in Theorem 3 of Ferejohn and Fishburn (1979), which translates acyclicity of social preferences into general conditions on the decisiveness structure of a social preference rule. Although general, these conditions are not transparent, and a number of papers have distilled them into more meaningful necessary or sufficient conditions for acyclicity, e.g., Banks (1995), Truchon (1996), and Schwartz (2001, 2007). Indeed, Theorem 1 of the current paper relies on Lemma 4, which is implied by the result of Ferejohn and Fishburn.

A number of papers give sharper characterizations of decisiveness structures for the smaller class of neutral social preference rules, e.g., Aleskerov and Vladmirov (1986), Blau and Brown (1989), and Sholomov (2000). Adding the axiom of anonymity, Ferejohn and Grether (1974) give relatively simple characterizations

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assuming at least as many alternatives as agents and imposing neutrality and monotonicity, some agent must have a veto, i.e., be a weak dictator. Kelsey (1985) formulates results of Blau and Deb (1977) and Deb (1981) with smaller sets of alternatives by focusing on implications for group vetoes.

of the acyclic social preference rules (see also Moulin (1985)). In the absence of background conditions such as neutrality and anonymity, acyclicity implies limited (but not universal) veto power on the part of some agents; Blair and Pollak (1982, 1983) and Le Breton and Truchon (1995) provide detailed analyses of the extent of veto power implied by acyclicity in this general setting.

Background: Monotonic Social Choice

Maskin (1977, 1999) proved that Nash equilibrium correspondences satisfy Monotonicity. Muller and Satterthwaite (1977) examine Monotonic social choice rules under the assumption that all profiles of linear orderings are possible. They show that if a monotonic social choice rule is ''resolute,'' in the sense of having singleton values, and has full range, in the sense that for each alternative, there is a profile at which that alternative is selected, then the rule is dictatorial. Maintaining the assumption of resoluteness and full range, Saijo (1987) illustrates the increased restrictiveness of Monotonicity when all profiles of weak orderings are possible: all such rules are in fact constant. These results are limited by the assumption that social choice sets are always singleton; Hurwicz and Schmeidler (1978) and Duggan and Schwartz (1995) provide several results allowing for choice sets of arbitrary size. Theorem 2 of Hurwicz and Schmeidler (1978) shows, essentially, that when all profiles of weak orders are possible, there is no social choice rule that satisfies Monotonicity and never chooses Pareto dominated alternatives. Closer to this paper, Theorem 7 of Duggan and Schwartz (1995) establishes that when all profiles of linear orders are possible, if a social choice rule is Monotonic, has full range, satisfies a ''minimal resoluteness'' condition, and satisfies $\lfloor \frac{n}{5} \rfloor$-Tie Break, then it is dictatorial. In contrast, the only trace of resoluteness used in the current paper lies in the $\lfloor \frac{n-2}{3} \rfloor$-Tie Break condition and Pareto Consistency (which requires that when one alternative is ranked atop all individual rankings, it is the unique choice); I explain in Section 4 how the results of Muller and Satterthwaite (1977) and Duggan and Schwartz (1995) can be obtained from Theorem 2.

2 Preliminaries

Let N be a finite set of n agents, denoted i, j, etc., who must choose from a set X of alternatives, denoted x, y, etc. Let Θ be a set of states of the world, denoted θ, which contain information about the agents' preferences over alternatives. Let $P_i(\theta)$ denote agent i's strict preference relation on X in state θ, and let $R_i(\theta)$ denote i's weak preference relation. Assume that $P_i(\theta)$ is asymmetric and negatively transitive, that $R_i(\theta)$ is complete and transitive, and that these relations are dual: for all $x,y \in X$, $xP_i(\theta)y$ if and only if not $yR_i(\theta)x$.[8] Let

$$\begin{aligned} P(x,y|\theta) &= \{i \in N \mid x\, P_i(\theta)\, y\} \\ R(x,y|\theta) &= \{i \in N \mid x\, R_i(\theta)\, y\} \end{aligned}$$

denote the set of agents who strictly and weakly, respectively, prefer x to y. Write $xI_i(\theta)y$ when neither $xP_i(\theta)y$ nor $yP_i(\theta)x$, or equivalently $xR_i(\theta)y$ and $yR_i(\theta)x$, and let $I(x,y|\theta) = \{i \in N \mid x\, I_i(\theta)\, y\}$ be the set of agents indifferent between x and y.

Letting $P(\theta) = (P_1(\theta),\ldots,P_n(\theta))$ be the profile of strict preference relations in state θ, we say *Unrestricted Domain* holds if

$$P(\Theta) = \left\{ (P_1,\ldots,P_n) \;\middle|\; \begin{array}{c} \text{for all i, } P_i(\theta) \text{ is an asymmetric and} \\ \text{negatively transitive relation on X} \end{array} \right\},$$

i.e., all profiles of weak orders are possible. We say a strict preference relation P_i is *total* if for all distinct x and y, either $x\, P_i\, y$ or $y\, P_i\, x$; this precludes indifference between two alternatives. We then say *Linear Domain* holds if

$$P(\Theta) = \left\{ (P_1,\ldots,P_n) \;\middle|\; \begin{array}{c} \text{for all i, } P_i(\theta) \text{ is an asymmetric,} \\ \text{total, and negatively transitive} \\ \text{relation on X} \end{array} \right\},$$

i.e., all profile of linear orders are possible. In this paper, I focus mainly on Linear Domain, as results proved on the smaller domain extend to the unrestricted framework.

[8] Equivalently, $yR_i(\theta)x$ if and only if not $xP_i(\theta)y$. Thus, $P_i(\theta)$ is the asymmetric part of $R_i(\theta)$. Obviously, the properties of either version of preference can be derived from the other.

A *social preference rule*, which is denoted F, is a mapping $\theta \mapsto (P_F(\theta), R_F(\theta))$ defined on the set Θ of states, where $P_F(\theta)$ is an asymmetric strict social preference relation, $R_F(\theta)$ is a complete weak social preference relation, and these relations are dual: for all $x, y \in X$, $x\, P_F(\theta)\, y$ if and only if not $y\, R_F(\theta)\, x$. I write $x\, I_F(\theta)\, y$ if neither $x\, P_F(\theta)\, y$ nor $y\, P_F(\theta)\, x$, or equivalently $x\, R_F(\theta)\, y$ and $y\, R_F(\theta)\, x$, a condition interpreted as social indifference. A special case of interest is that of a *quota rule*, where social preferences are completely specified by a single parameter $q > \frac{n}{2}$ as follows: for all x and y, $x\, P_F(\theta)\, y$ if and only if $|P(x,y|\theta)| \geq q$. Quota rules are neutral, in the sense that they treat alternatives symmetrically, but we can define a *generalized quota rule* as a social preference rule generated by a function $Q: X \times X \to \{0, \ldots, n+1\}$ as follows: for all $x, y \in X$, $x\, P_F(\theta)\, y$ if and only if $|P(x,y|\theta)| \geq Q(x,y)$. To ensure asymmetry of $P_F(\theta)$, or equivalently completeness of $R_F(\theta)$, we impose the requirement that for all $x, y \in X$, $Q(x,y) + Q(y,x) > n$. We gain *simple majority rule* as a special case of quota rule in which $q = \lceil \frac{n+1}{2} \rceil$.

I investigate the consistency of several axioms on social preference rules. The first class of axioms imposes minimal levels of responsiveness to changes in the preferences of individual agents. A classical axiom, called *Positive Responsiveness* by May (1954) and Mas-Colell and Sonnenschein (1972), is that the following holds for all states θ and θ' and all alternatives x and y:

$$\left.\begin{array}{c} P(x,y|\theta) \subseteq P(x,y|\theta'), \\ R(x,y|\theta) \subseteq R(x,y|\theta'), \\ \text{at least one inclusion strict,} \\ \text{and } x\, I_F(\theta)\, y \end{array}\right\} \Rightarrow \quad x\, P_F(\theta')\, y.$$

Intuitively, the condition says that if two alternatives are socially indifferent, and an agent changes her preferences in favor of one alternative, then this change should break the social indifference in favor of the alternative that has gained support. The condition is quite strong in two respects. First, a change in the preference of only *one* agent is sufficient to break the tie. Second, even if the agent initially was indifferent and changes to a strict preference (or initially had a strict preference and changes to indifference), this change is sufficient to break a tie; in other words, a strict preference reversal is not required to fulfill the antecedent condition of the axiom. It is well-known from May's (1954) theorem that among anonymous, neutral, and monotonic social preference rules, the Positive Responsiveness axiom is uniquely satisfied on the Unrestricted Domain by majority voting among concerned (non-indifferent) agents and on the Linear Domain by (what is the same) simple majority rule.

This paper employs a substantial weakening of the Positive Responsiveness condition. Given an integer r satisfying $0 \leq r \leq n+1$, we say F satisfies r-*Tie Break* if the following holds for all states θ and θ' and all alternatives x and y:

$$\left.\begin{array}{c} P(x,y|\theta) \subseteq P(x,y|\theta'), \\ I(x,y|\theta) = I(x,y|\theta'), \\ |P(y,x|\theta) \cap P(x,y|\theta')| \geq r, \\ \text{and } x\, I_F(\theta)\, y \end{array}\right\} \Rightarrow \; x\, P_F(\theta')\, y.$$

In words, if x and y are socially indifferent in one state, and we consider another state in which no agents have changed their preferences to favor y and in which at least r agents have reversed a strict preference for y to a strict preference for x, then x is strictly socially preferred to y at the new state. Obviously, the tie break axiom is weaker when the responsiveness threshold r is larger, the most restrictive case of the condition being 0-Tie Break, which precludes social indifference altogether. The slightly less restrictive 1-Tie Break is related to Positive Responsiveness: under Unrestricted Domain, 1-Tie Break is strictly weaker than Positive Responsiveness, and under Linear Domain, the two conditions are equivalent. Note also that under Linear Domain, the antecedent assumption $I(x,y|\theta) = I(x,y|\theta')$ in the above tie break condition is vacuous, so the condition is equivalently stated without it. Positive Responsiveness becomes most restrictive when the number n of agents is large, but in contrast the restrictiveness of r-Tie Break remains relatively constant.

I consider the tie-break threshold $r = \lfloor \frac{n-2}{3} \rfloor$, or roughly one third of the number of agents. To parse the condition, it is helpful to write $n = 3\left\lfloor\frac{n}{3}\right\rfloor + \phi$ for $\phi \in \{0,1,2\}$, in which case

$$\left\lfloor\frac{n-2}{3}\right\rfloor = \begin{cases} \left\lfloor\frac{n}{3}\right\rfloor - 1 & \text{if } \phi = 0,1, \\ \left\lfloor\frac{n}{3}\right\rfloor & \text{if } \phi = 2. \end{cases}$$

Most closely related to the responsiveness conditions I propose is Schwartz's (1986) ''weak non-blocker,'' which is $\lfloor\frac{n}{5}\rfloor$-Tie Break.[9] As a yardstick to compare these axioms, suppose Linear Domain holds, and consider the class of generalized quota rules. Consider any distinct alternatives x and y, and for simplicity let $p = Q(x,y)$. The ''worst case'' is a state θ in which $p-1$ agents prefer x and $n-p+1$ agents prefer y, so $x\, I_F(\theta)\, y$ holds, and r members of the former group reverse their preferences; this reversal breaks the social indifference as long as $n-p+1+r \geq Q(y,x)$. Thus, a generalized quota rule satisfies r-Tie Break if and only if for all x and y, we have $r \geq Q(x,y) + Q(y,x) - n - 1$. For the special case of a quota rule with quota q, this is equivalent to $q \leq \frac{n+r+1}{2}$, so 1-Tie Break (equivalently, Positive Responsiveness) is satisfied among quota rules only by simple majority rule. For other examples, ignoring integer issues, Schwartz's $\lfloor\frac{n}{5}\rfloor$-Tie Break is satisfied for any quota up to $q = \frac{3n}{5}$, and the weaker condition of $\lfloor\frac{n-2}{3}\rfloor$-Tie Break is satisfied for quotas up to two thirds, $q = \frac{2n}{3}$.

As is standard, an agent i is a *dictator* if for all $\theta \in \Theta$ and all $x,y \in X$, $x\, P_i(\theta)\, y$ implies $x\, P_F(\theta)\, y$. We say i is a *weak dictator* if for all θ and all x and y, $x\, P_i(\theta)\, y$ implies $x\, R_F(\theta)\, y$. Thus, a weak dictator's authority is limited in the sense that a strict preference on her part precludes the opposite strict social preference, without necessitating a strict social preference in the direction of her preference. Furthermore, i is an r-*dictator* if she is a weak dictator and for all groups $G \subseteq N\setminus\{i\}$ with $|G| \geq r$, all θ, and all x and y, $\{i\} \cup G \subseteq P(x,y|\theta)$ implies $x\, P_F(\theta)\, y$. Thus, an r-dictator not only blocks a strict social preference herself but can impose a strict preference with the agreement of r other

[9] A caveat is that Schwartz's condition requires that social indifference is broken if *exactly* $\left\lfloor\frac{n}{5}\right\rfloor$ agents reverse preferences, but not if more than $\left\lfloor\frac{n}{5}\right\rfloor$ do; the extra bite of the r-Tie Break does not seem objectionable. Bordes and Salles (1978) provide a weakening of Mas-Colell and Sonnenschein's (1972) weak dictatorship axiom along these lines.

agents. A dictator is then equivalent to a 0-dictator. In obvious fashion, we say a social preference rule satisfies the axiom of *No r-Dictator* if no agent is an r-dictator. Generalized quota rules satisfy No r-Dictator as long as $n \geq 2$ and $r < n - 1$.

The other axioms used in the sequel are standard. We say a social preference rule F satisfies...

- *Independence* if for all θ and θ' and all x and y,

$$\left.\begin{array}{c} P(x,y|\theta) = P(x,y|\theta'), \\ I(x,y|\theta) = I(x,y|\theta'), \\ \text{and } x\, P_F(\theta)\, y \end{array}\right\} \Rightarrow x\, P_F(\theta')\, y.$$

- *Pareto* if for all θ and all x and y, $P(x,y|\theta) = N$ implies $x\, P_F(\theta)\, y$.

- *Acyclicity* if for all θ, all natural numbers k, and all selections $x_1,\ldots,x_k$ of k alternatives, it is not the case that

$$x_1\, P_F(\theta)\, x_2\, P_F(\theta) \cdots x_k\, P_F(\theta)\, x_1.$$

In words, respectively, the social preference between two alternatives depends only on the agents' preferences over those two alternatives; a common strict preference of the agents is inherited by the social preference relation; and the social preference rule does not produce cycles.

With Independence and Pareto, assuming Linear Domain or Unrestricted Domain, the Acyclicity axiom is necessary for the existence of socially maximal alternatives and sufficient when X is finite. Sufficiency is well-known and does not rely on other axioms or domain restrictions. To see necessity, suppose there is some state θ in which a cycle occurs, say $x_1\, P_F(\theta)\, x_2\, P_F(\theta) \cdots x_k\, P_F(\theta)\, x_1$. Even if there is a socially maximal element at state θ, we can choose another state θ' such that individual preferences restricted to $\{x_1,\ldots,x_k\}$ are as in θ, and such that these alternatives are above all others in each agent's ranking. By Independence, the cycle is maintained; and by Pareto, each alternative in the cycle is strictly socially preferred to each alternative outside it. Thus, there is no socially maximal alternative at θ'.

Generalized quota rules always satisfy Independence, and Pareto is satisfied as long as $Q(x,y) \leq n$ for all x and y. For a generalized quota rule, the Acyclicity axiom has the following implication: given any three distinct alternatives, say x, y, and z, we must have $Q(x,y) + Q(y,z) + Q(z,x) > 2n$. Indeed, if this condition is violated, then we can choose groups G, H, and I consisting, respectively, of at least $Q(x,y)$, $Q(y,z)$, and $Q(z,x)$ agents and such that $\{N \setminus G, N \setminus H, N \setminus I\}$ is a partition of N. This allows us to construct a Condorcet profile, below,

$N \setminus G$	$N \setminus H$	$N \setminus I$
y	z	x
z	x	y
x	y	z

which leads to a cycle: $x\, P_F(\theta)\, y\, P_F(\theta)\, z\, P_F(\theta)\, x$. A further implication is the well-known fact that when there are at least three alternatives (and either three or at least five agents) majority rule (or any quota rule with $q \leq \frac{2n}{3}$) violates Acyclicity.

3 Acyclic Voting

The main result on the limits of acyclic voting can now be stated. In keeping with the Arrovian tradition, I allow for any number of three or more alternatives. The theorem technically assumes at least three agents, but note that $\lfloor \frac{n-2}{3} \rfloor = 0$ for $n \in \{3,4\}$, in which case the tie break condition rules out social indifference.

Theorem 1 *Assume $|X| \geq 3$, $n \geq 3$, and Linear Domain. There does not exist a social preference rule satisfying Independence, Pareto, Acyclicity, No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break.*

The proof is relegated to the appendix. To convey the idea behind it, however, it is instructive to consider the case of a generalized quota rule. Suppose, contrary to the theorem, that there is a generalized quota rule satisfying Pareto, Acyclicity, and $\lfloor\frac{n-2}{3}\rfloor$-Tie Break. Consider any three alternatives x, y, and z, and for simplicity let $p = Q(x,y)$ and $q = Q(y,z)$. By Pareto, we have $n \geq \max\{p,q\}$. Because $Q(x,y) + Q(y,x) > n$, we can assume without loss of generality that $p \geq \frac{n}{2}$ and choose two groups G and H such that $|G| = p$, $|H| = q$, and $G \cup H = N$. Note that $Q(z,x) \geq 2n-p-q+1$, for otherwise we would have $Q(x,y)+Q(y,z)+Q(z,x) \leq 2n$, contradicting Acyclicity. And by $\lfloor\frac{n-2}{3}\rfloor$-Tie Break, we have $\lfloor\frac{n-2}{3}\rfloor \geq Q(x,z)+Q(z,x)-n-1$. Therefore,

$$Q(x,z) \ \leq \ n - Q(z,x) + 1 + \left\lfloor \frac{n-2}{3} \right\rfloor \ \leq \ p + q - n + \left\lfloor \frac{n-2}{3} \right\rfloor .$$

In addition, $\lfloor\frac{n-2}{3}\rfloor$-Tie Break implies

$$Q(z,y) \ \leq \ n - Q(y,z) + 1 + \left\lfloor \frac{n-2}{3} \right\rfloor \ = \ n - q + \left\lfloor \frac{n-2}{3} \right\rfloor + 1.$$

Setting $p' = \min\{n, p + q - n + \lfloor\frac{n-2}{3}\rfloor\}$ and $q' = \min\{n, n - q + \lfloor\frac{n-2}{3}\rfloor + 1\}$, note that $p' + q' \geq n$, since $p \geq \frac{n}{2}$. Thus, we can choose groups G' and H' such that $|G'| = p'$, $|H'| = q'$, and $G' \cup H' = N$. Note that $3\lfloor\frac{n-2}{3}\rfloor \leq n-2$. Analogous to the above argument, it then follows that

$$Q(x,y) \leq n - Q(y,x) + 1 + \left\lfloor \frac{n-2}{3} \right\rfloor \leq p' + q' - n + \left\lfloor \frac{n-2}{3} \right\rfloor \leq p - 1 < Q(x,y),$$

a contradiction. The proof in the appendix consists of a more involved analysis along these lines.

To see that Theorem 1 is tight, assume there are three alternatives ordered arbitrarily by $\succeq$, and assume for simplicity that n is divisible by three, so that $\lfloor\frac{n-2}{3}\rfloor + 1 = \frac{n}{3}$. Define a social preference rule F as follows: given any x and y with $x \succeq y$, say $x \, P_F(\theta) \, y$ holds when $|P(x,y|\theta)| \geq \frac{2n}{3}$, and $y \, P_F(\theta) \, x$ holds when $|P(x,y|\theta)| \geq \frac{2n}{3}+1$. This clearly satisfies Independence, Pareto, and No $\frac{n}{3}$-Dictator. It satisfies Acyclicity, for a cycle $x \, P_F(\theta) \, y \, P_F(\theta) \, z \, P_F(\theta) \, x$ must involve a preference that goes against the ordering $\succeq$, say $x \succ z$, and then $x \, P_F(\theta) \, y \, P_F(\theta) \, z$ implies that at least one third of the agents prefer x to z, so we cannot have $z\, P_F(\theta)\, x$. A moment's consideration shows that it also satisfies $\frac{n}{3}$-Tie Break. Indeed, assume $x \succeq y$ and $x\, I_F\, (\theta) y$, the difficult cases being $|P(x,y|\theta)| = \frac{2n}{3} - 1$ and $|P(y,x|\theta)| = \frac{n}{3} + 1$. Consider the latter, as the argument is similar in each case. If $\frac{n}{3}$ agents reverse their preference for x to now favor y in θ', then we have $|P(y,x|\theta')| = \frac{2n}{3} + 1$, so $y\, P_F(\theta')\, x$, as required.[10,11]

As mentioned above, Theorem 1 carries over when we allow arbitrary individual indifferences.

Corollary 1 *Assume* $|X| \geq 3$*,* $n \geq 3$*, and Unrestricted Domain. There does not exist a social preference rule satisfying Independence, Pareto, Acyclicity, No* $\lfloor \frac{n-2}{3} \rfloor$*-Dictator, and* $\lfloor \frac{n-2}{3} \rfloor$*-Tie Break.*

Proof Consider a social preference rule F defined on the domain of all profiles of weak orders and satisfying Independence, Pareto, Acyclicity, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break. Let F' denote the restriction of F to the Linear Domain, and note that F' satisfies the same four axioms. Then Theorem 1 implies that some agent i is a $\lfloor \frac{n-2}{3} \rfloor$-dictator for F'. I claim that F violates No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, which proves the corollary. I first show that i is a weak dictator for F. Consider any θ and any x and y such that $x\, P_i(\theta)\, y$, and suppose $y\, P_F(\theta)\, x$. Now let z be distinct from x and y, and consider θ' with individual preferences over these three alternatives as follows, where $N \setminus \{i\}$ is partitioned into groups G and H such that $\min\{|G|,|H|\} \geq \lfloor \frac{n-2}{3} \rfloor$.

i	G	H
x	x,y?	z
z	z	x,y?
y		

[10] When $n = 3\lfloor \frac{n}{3} \rfloor + 1$, we use the quota rule with $q = 2\lfloor \frac{n}{3} \rfloor + 1$, and when $n = 3\lfloor \frac{n}{3} \rfloor + 2$, we use the quota rule with $q = 2\lfloor \frac{n}{3} \rfloor + 2$.

[11] The construction used in the above example relies on the assumption of three alternatives, as demonstrated in Theorem 2 of the working paper version of this paper, which assumes at least four alternatives and uses $\lfloor \frac{n}{3} \rfloor$-Tie Break and No $\lfloor \frac{n}{3} \rfloor$-Dictator.

Here, ``x,y?'' indicates that individual preferences between x and y are as in θ. By Independence we have $y\, P_F(\theta')\, x$. Of course, there exists θ'' such that indifferences between x and y are broken while maintaining preferences between x and z and between y and z, i.e., $P_j(\theta'')$ is a linear order for all j and $P(x,z|\theta'') = P(x,z|\theta')$ and $P(z,y|\theta'') = P(z,y|\theta')$. Since i is a $\lfloor\frac{n-2}{3}\rfloor$-dictator for F′, we then have $x\, P_{F'}(\theta'')\, z$ and $z\, P_{F'}(\theta'')\, y$. Since F′ is the restriction of F, this implies $x\, P_F(\theta'')\, z$ and $z\, P_F(\theta'')\, y$. By Independence, we have $x\, P_F(\theta')\, z\, P_F(\theta')\, y\, P_F(\theta')\, x$, contradicting Acyclicity. We conclude that i is a weak dictator for F.

Now consider θ, x, and y such that $x\, P_i(\theta)\, y$ and, letting $G = P(x,y|\theta) \setminus \{i\}$, such that $|G| \geq \lfloor\frac{n-2}{3}\rfloor$. Suppose that $y\, R_F(\theta)\, x$. I claim that $y\, I_F(\theta)\, x$, for suppose $y\, P_F(\theta)\, x$. Let z be distinct from x and y, and consider θ' with individual preferences over these three alternatives as follows, where $H = N \setminus (G \cup \{i\})$.

i	G	H
x	x	z
z	z	x,y?
y	y	

Again, there exists θ'' such that $P_j(\theta'')$ is a linear order for all j and $P(x,z|\theta'') = P(x,z|\theta')$ and $P(z,y|\theta'') = P(z,y|\theta')$. Since i is a $\lfloor\frac{n-2}{3}\rfloor$-dictator for F′, we then have $x\, P_{F'}(\theta'')\, z$ and $z\, P_{F'}(\theta'')\, y$, and this leads to a violation of Acyclicity of F in θ', as above. We conclude that $y\, I_F(\theta)\, x$, as claimed. Now consider $\hat{\theta}$ such that $P(x,y|\hat{\theta}) = \{i\}$ and $P(y,x|\hat{\theta}) = G \cup P(y,x|\theta)$. Note that the antecedent of $\lfloor\frac{n-2}{3}\rfloor$-Tie Break is satisfied, and in particular that $|P(y,x|\hat{\theta}) \cap P(x,y|\theta)| \geq |G| \geq \lfloor\frac{n-2}{3}\rfloor$. Therefore, we have $y\, P_F(\hat{\theta})\, x$, contradicting the fact that i is a weak dictator. We conclude that $x\, P_F(\theta)\, y$, as required. Q.E.D.

Theorem 1 establishes the incompatibility of several axioms with Acyclicity, which precludes the possibility of any cycle. The theorem can be stated using weaker acyclicity axioms, and such statements can actually be obtained as corollaries of the theorem. The axiom of *Atricyclicity,* which precludes only social preference cycles of length three, e.g., $x_1\, P_F(\theta)\, x_2\, P_F(\theta)\, x_3\, P_F(\theta)\, x_1$, is weaker than the conventional axiom. A further weakening precludes only cycles of length three that involve alternatives that are preferred to all others by all agents: say F satisfies

- *Top Atricyclicity* if for all θ and all selections x_1, x_2, x_3 of alternatives such that $x_j P_i(\theta) y$ for all agents i, all $j \in \{1,2,3\}$, and all $y \in X \setminus \{x_1, x_2, x_3\}$, it is not the case that $x_1 \, P_F(\theta) \, x_2 \, P_F(\theta) \, x_3 \, P_F(\theta) \, x_1$.

Under Pareto, the axiom of Top Atricyclicity is necessary if social preferences admit maximal elements in every state, and thus it may be viewed as a minimal requirement. Nevertheless, the incompatibility presented in Theorem 1 is maintained.

Corollary 2 *Assume $|X| \geq 3$, $n \geq 3$, and Linear or Unrestricted Domain. There does not exist a social preference rule satisfying Independence, Pareto, Top Atricyclicity, No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break.*

Proof Consider a social preference rule F satisfying Independence, Pareto, No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break. I claim that there is some selection of three distinct alternatives a, b, and c, such that no agent i is a $\lfloor \frac{n-2}{3} \rfloor$-dictator on $\{a,b,c\}$, in the sense that (i) for all θ and all $x, y \in \{a,b,c\}$, $x P_i(\theta) y$ implies $x \, R_F(\theta) \, y$, and (ii) for all groups $G \subseteq N \setminus \{i\}$ with $|G| \geq \lfloor \frac{n-2}{3} \rfloor$, all θ, and all $x, y \in \{a,b,c\}$, $\{i\} \cup G \subseteq P(x,y|\theta)$ implies $x \, P_F(\theta) \, y$. To establish the claim, suppose otherwise. By No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, there must exist triples $\{a,b,c\}$ and $\{d,e,f\}$ and distinct agents i and j such that i is a $\lfloor \frac{n-2}{3} \rfloor$-dictator on $\{a,b,c\}$ and j is a $\lfloor \frac{n-2}{3} \rfloor$-dictator on $\{d,e,f\}$. In case the triples have nonempty intersection, let a belong to both, and let b and d be distinct. Choose disjoint groups G and H with $|G| = \lfloor \frac{n-2}{3} \rfloor$ and $|H| = \lfloor \frac{n-2}{3} \rfloor$. Let θ be a state such that $P(a,d|\theta) = \{j\} \cup G$, $P(d,b|\theta) = N$, and $P(b,a|\theta) = \{i\} \cup H$, and further specify that for all agents k, all $x \in \{a,b,d\}$, and all $y \in X \setminus \{a,b,d\}$, we have $x P_k(\theta) y$. Using Pareto, we then have $a \, P_F(\theta) \, d \, P_F(\theta) \, b \, P_F(\theta) \, a$, contradicting Top Atricyclicity. In the remaining case that the triples are disjoint, our supposition implies that some agent k is a $\lfloor \frac{n-2}{3} \rfloor$-dictator on $\{a,d,e\}$. If $k \neq i$, then we apply the preceding argument to the triples {a,b,c} and {a,b,d}; and if k = i, then we apply the preceding argument to {a,d,e} and {d,e,f}. This establishes the claim.

We may therefore choose distinct alternatives a, b, and c such that no agent is a $\lfloor\frac{n-2}{3}\rfloor$-dictator on {a,b,c}. Now define the rule F' for the restricted environment in which the set of alternatives is $X' = \{a,b,c\}$ as follows: for all θ and all $x,y \in X'$, $x \, P_{F'}(\theta) \, y$ if and only if $x \, P_F(\theta) \, y$, with $R_{F'}(\theta)$ defined as the dual of $P_{F'}(\theta)$. Then F' satisfies Independence, Pareto, No $\lfloor\frac{n-2}{3}\rfloor$-dictator, and $\lfloor\frac{n-2}{3}\rfloor$-Tie Break. By Theorem 1 or Corollary 1, it violates Acyclicity, so there is a state θ such that $a \, P_{F'}(\theta) \, b \, P_{F'}(\theta) \, c \, P_{F'}(\theta) \, a$ (or the reverse) holds. By construction, $a \, P_F(\theta) \, b \, P_F(\theta) \, c \, P_F(\theta) \, a$ (or the reverse) holds. Then choose θ' in the unrestricted environment so that individual preferences over {a,b,c} are as in θ, and for all $x \in \{a,b,c\}$ and all $y \in X \setminus \{a,b,c\}$, $xP_i(\theta')y$. By Independence, $a \, P_F(\theta') \, b \, P_F(\theta') \, c \, P_F(\theta') \, a$ (or the reverse) still holds, violating Top Atricyclicity, as required. Q.E.D.

We conclude that in the binary framework, in which social choices are modeled via a social preference rule, either a single agent must have near dictatorial power, or social preferences must be unresponsive to substantial changes in individual preferences. In the next section, we will see that the analysis of acyclic voting has restrictive implications for equilibrium outcome correspondences.

4 Implementation

A *social choice rule,* denoted f, is a mapping $\theta \mapsto f(\theta)$, where $f(\theta)$ is a non-empty subset of X. A common interpretation is that $f(\theta)$ is the set of collective choices that satisfy some normative criterion, but an alternative interpretation is that $f(\theta)$ is the set of alternatives that are plausible predictions under some positive solution, such as Nash equilibrium. Consistent with the Arrovian usage, an agent i is a *dictator* if for all θ, all $x \in f(\theta)$, and all y, we have $xR_i(\theta)y$. We say f satisfies...

- *Resoluteness* if for all θ, $|f(\theta)| = 1$.
- *Full Range* if $\bigcup_{\theta \in \Theta} f(\theta) = X$.
- *No Dictator* if no i is a dictator.

We define a *game form* as a pair (S,g), where $S = S_1 \times \cdots \times S_n$ is an n-fold product of sets, each factor S_i is a set of strategies available to agent i, and $g \colon S \to X$ is an outcome function mapping strategy profiles to alternatives. As is standard, we write $s = (s_1, \ldots, s_n)$ for a strategy profile and s_{-i} for a profile of strategies for all agents but i. Then s is a (pure strategy) *Nash equilibrium* in state θ if for all $i \in N$ and all $s'_i \in S_i$, we have $g(s) R_i(\theta) g(s'_i, s_{-i})$; and we write $N_{(S,g)}(\theta)$ for the set of Nash equilibria in θ. Finally, the game form (S,g) *Nash implements* f if for all $\theta \in \Theta$, we have $f(\theta) = g(N_{(S,g)}(\theta))$, i.e., if the social choice set matches the set of Nash equilibrium outcomes of the game form (S,g) in every state, and f is *Nash implementable* if there is some game form that Nash implements it. Put differently, a social choice rule is Nash implementable if it is the equilibrium outcome correspondence for some game form.

By studying implementable social choice rules, we may therefore derive restrictions on equilibrium behavior independent of the particular game form under consideration. Maskin (1977, 1999) has shown, for example, that every Nash implementable social choice rule satisfies *Monotonicity:* for all $\theta, \theta' \in \Theta$ and all $x \in f(\theta) \setminus f(\theta')$, there exists $i \in N$ and $y \in X$ such that $x R_i(\theta) y$ and $y P_i(\theta') x$. In words, if x is a viable social choice in state θ but not in θ', then there must be a preference reversal involving x and another alternative, where x is weakly preferred in θ but not in θ'. This opens an avenue for the general analysis of equilibrium behavior by deducing implications of Monotonicity, and in fact, results of Muller and Satterthwaite (1977) and Saijo (1987) show, via this logic, that when the domain of preferences is large, a game form admits a unique equilibrium outcome in each state only under very restrictive conditions.[12]

[12]Assuming at least three alternatives and Linear Domain, Muller and Satterthwaite (1977) show that there is no social choice rule satisfying Res-

The latter results apply to Resolute social choice rules, but less is known about the restrictiveness of Monotonicity for social choice rules that are multi-valued. To apply the analysis of acyclic social preference to this problem, I define conditions on social choice rules paralleling No r-Dictator and r-Tie Break, as well as a condition intermediate between Full Range and the usual Pareto optimality condition. Given a set Y of alternatives, let Θ^Y denote the set of states in which all agents prefer every alternative in Y to every alternative outside Y:

$$\Theta^Y = \{\theta \mid \forall i \in N : \forall y \in Y : \forall z \in X \setminus Y : y\, P_i(\theta)\, z\}.$$

We say f satisfies r-*Tie Break* if the following holds for all x and y and all $\theta, \theta' \in \Theta^{\{x,y\}}$:

$$\left.\begin{array}{c} P(x,y|\theta) \subseteq P(x,y|\theta'), \\ I(x,y|\theta) = I(x,y|\theta'), \\ |P(y,x|\theta) \cap P(x,y|\theta')| \geq r, \\ \{x,y\} \subseteq f(\theta) \end{array}\right\} \Rightarrow y \notin f(\theta').$$

In words, if x and y are ranked above all other alternatives by all agents, if the social choice set contains both, and if we consider another state in which x and y are still ranked above all other alternatives by all agents, in which no agents have changed their preferences to prefer x over y, and in which at least r agents have reversed their preferences to favor x, then y is no longer socially viable. Obviously, under Linear Domain the condition can be simplified by omitting the antecedent requirement that $I(x,y|\theta) = I(x,y|\theta')$.

An agent i is an r-*dictator* if for all θ, all x and y, and all groups $G \subseteq N \setminus \{i\}$ with $|G| \geq r$ such that $\{i\} \cup G \subseteq P(x,y|\theta)$, (i) $y \notin f(\theta)$, and (ii) if $xR_i(\theta)z$ for all z, then $x \in f(\theta)$. Thus, an r-dictator can reject any alternative and include her favorite alternatives among the choices with the support of at least r other agents. We say *No* r-*Dictator* holds if no agent is an r-dictator.

oluteness, Monotonicity, Full Range, and No Dictator; when the preference domain is expanded to Unrestricted Domain, Saijo (1987) shows that if a social choice rule is Resolute and Monotonic, then it is constant.

A social choice rule f satisfies *Pareto Consistency* if (i) for all finite $Y \subseteq X$ and all $\theta \in \Theta^{Y}$, we have $f(\theta) \cap Y \neq \emptyset$, and (ii) for all x and all $\theta \in \Theta^{\{x\}}$, we have $f(\theta) = \{x\}$. This is clearly much weaker than the usual Pareto optimality condition, which would require that whenever x is strictly preferred to y by all agents, the social choice set does not contain y: Pareto Consistency excludes y only when x is ranked above all other alternatives by all agents, so it applies in a strictly smaller set of cases than the usual condition.

The analysis of social choices is connected to the analysis of social preference by the following version of revealed preference: given a state θ, we define $x\,P_f(\theta)\,y$ if and only if for all θ',

$$\begin{array}{l} P(x,y|\theta) = P(x,y|\theta') \\ I(y,x|\theta) = I(y,x|\theta') \end{array} \quad \Rightarrow \quad y \notin f(\theta').$$

I claim that as long as f is Pareto consistent, the revealed preference relation $P_f(\theta)$ is asymmetric: given alternatives x and y and state θ with $x\,P_f(\theta)\,y$, we may choose a state $\theta' \in \Theta^{\{x,y\}}$ such that individual preferences between x and y are unchanged; then part (i) of Pareto Consistency requires either $x \in f(\theta')$, in which case not $y\,P_f(\theta)\,x$, or $y \in f(\theta')$, in which case not $x\,P_f(\theta)\,y$. Defining $x\,R_F(\theta)\,y$ to hold if there exists θ' such that $P(x,y|\theta) = P(x,y|\theta')$, $I(x,y|\theta) = I(x,y|\theta')$, and $x \in f(\theta')$, we have defined a social preference rule $\theta \mapsto (P_f(\theta), R_f(\theta))$. As usual, we define $x\,I_f(\theta)\,y$ if neither $x\,P_f(\theta)\,y$ nor $y\,P_f(\theta)\,x$, or equivalently, $x\,R_f(\theta)\,y$ and $y\,R_f(\theta)\,x$.

The next lemma shows that revealed social preference rules, under Pareto Consistency, always satisfy Independence and Acyclicity; note that these properties hold without Monotonicity. Adding the latter condition, we deduce Pareto as well.

Lemma 1 *Assume Linear or Unrestricted Domain. Given any social choice rule* f*, (i) the revealed social preference rule* $\theta \mapsto (P_f(\theta), R_f(\theta))$ *satisfies Independence; (ii) if* f *satisfies Pareto Consistency, then* $\theta \mapsto (P_f(\theta), R_f(\theta))$ *satisfies Acyclicity; and (iii) if* f *satisfies Monotonicity and Pareto Consistency, then* $\theta \mapsto (P_f(\theta), R_f(\theta))$ *satisfies Pareto.*

Proof The revealed preference rule satisfies Independence by construction. To establish Acyclicity, consider any θ and any finite set Y of alternatives, and choose $\theta' \in \Theta^Y$ so that individual preferences restricted to Y are as in θ. By part (i) of Pareto Consistency, there exists $x \in f(\theta') \cap Y$, in which case $y\ P_f(\theta)\ x$ for no $y \in Y$, and we conclude that there is no $P_f(\theta)$-cycle through Y. Since Y was arbitrary, Acyclicity follows. To prove Pareto, consider θ, x, and y such that $P(x,y|\theta) = N$. If not $x\ P_f(\theta)\ y$, then there exists θ' such that $P(x,y|\theta') = P(x,y|\theta)$, $I(x,y|\theta') = I(x,y|\theta)$, and $y \in f(\theta')$. Let $\theta'' \in \Theta^{\{x,y\}}$ be a state such that individual preferences between x and y are unchanged, with those alternatives above all others in each agent's ranking. Then Monotonicity implies that $y \in f(\theta'')$, but note that $\theta'' \in \Theta^{\{x\}}$, so that part (ii) of Pareto Consistency implies $y \notin f(\theta'')$, a contradiction. Q.E.D.

The goal is to show that Pareto Consistency and Monotonicity are incompatible with our r-Tie Break and No r-Dictator conditions. The strategy of proof is to show that when f is Monotonic, the revealed social preference rule inherits key properties from f, allowing us to apply Theorem 1 on acyclic social preferences.

Lemma 2 *Assume Linear or Unrestricted Domain. If a social choice rule* f *satisfies Monotonicity, No* r*-Dictator, and* r*-Tie Break, where* $r < \left\lfloor \frac{n}{2} \right\rfloor$*, then the revealed preference rule* $\theta \mapsto (P_f(\theta), R_f(\theta))$ *satisfies No* r*-Dictator, and* r*-Tie Break.*

Proof Assume f satisfies Monotonicity, Pareto Consistency, No r-Dictator, and r-Tie Break. To deduce a contradiction, suppose the revealed social preference rule admits an r-dictator i; in particular, for all groups $G \subseteq N \setminus \{i\}$ with $|G| \geq r$, all θ, and all x and y, $\{i\} \cup G \subseteq P(x,y|\theta)$ implies $x\ P_f(\theta)\ y$. I claim that i is an r-Dictator for f. For part (i), consider any θ, any x and y, and any $G \subseteq N \setminus \{i\}$ with $|G| \geq r$ such that $\{i\} \cup G \subseteq P(x,y|\theta)$. By supposition, $x\ P_f(\theta)\ y$, which implies $y \notin f(\theta)$. For (ii), suppose in addition that $x R_i(\theta) z$ for all z, let $G' \subseteq G$ satisfy $|G'| = r$ and $H = N \setminus (G' \cup \{i\})$, so $|H| \geq r$, and let $Z = X \setminus \{x,y\}$. Consider a state θ' with individual rankings below, where preferences over Z are arbitrary.

i	G'	H
x	Z	y
y	x	Z
Z	y	x

From part (i), shown above, $\{i\} \cup G' \subseteq P(x,y|\theta')$ implies $y \notin f(\theta')$, and $\{i\} \cup H \subseteq P(y,z|\theta')$ for all $z \in Z$ implies $Z \cap f(\theta') = \emptyset$; therefore, $f(\theta') = \{x\}$. Then Monotonicity implies that $x \in f(\theta)$, as required.

To prove r-Tie Break, consider any θ and θ' and x and y such that $P(x,y|\theta) \subseteq P(x,y|\theta')$, $I(x,y|\theta) = I(x,y|\theta')$, $|P(y,x|\theta) \cap P(x,y|\theta')| \geq r$, and $x\ I_f(\theta)\ y$. Revealed indifference implies that there exist θ_x and θ_y such that $P(x,y|\theta_x) = P(x,y|\theta_y) = P(x,y|\theta)$, $I(x,y|\theta_x) = I(x,y|\theta_y) = I(x,y|\theta)$, $x \in f(\theta_x)$, and $y \in f(\theta_y)$. Let $\hat{\theta} \in \Theta^{\{x,y\}}$ be a state such that individual preferences between x and y are unchanged from θ (and from θ_x and θ_y) with those alternatives above all others in the agents' rankings; then Monotonicity implies $\{x,y\} \subseteq f(\hat{\theta})$. Let $\hat{\theta}' \in \Theta^{\{x,y\}}$ be a state in which individual preferences between x and y are unchanged from θ' with those alternatives above all others, and note that $P(x,y|\hat{\theta}) \subseteq P(x,y|\hat{\theta}')$, $I(x,y|\hat{\theta}) = I(x,y|\hat{\theta}')$, and $|P(y,x|\hat{\theta}) \cap P(x,y|\hat{\theta}')| \geq \lfloor \frac{n-2}{3} \rfloor$; then r-Tie Break implies $y \notin f(\hat{\theta}')$. To complete the proof, suppose that $y\ R_f(\theta')\ x$, so there exists θ'' such that $P(x,y|\theta'') = P(x,y|\theta') = P(x,y|\hat{\theta}')$, $I(x,y|\theta'') = I(x,y|\theta') = I(x,y|\hat{\theta}')$, and $y \in f(\theta'')$; but then Monotonicity implies $y \in f(\hat{\theta}')$, a contradiction. Q.E.D.

With the above lemmata, the following result on Nash implementable social choice rules is immediate. The analysis has direct implications for equilibrium outcome correspondences: if the equilibrium outcomes of a game form satisfy Pareto Consistency, then the game form must either concentrate power in small groups containing a particular agent (i.e., violate No $\lfloor \frac{n-2}{3} \rfloor$-Dictator) or be unresponsive to significant changes in individual preferences (i.e., violate $\lfloor \frac{n-2}{3} \rfloor$-Tie Break). Note that, using Corollary 1, we can state the result for Linear or Unrestricted Domain.

Theorem 2 *Assume* $|X| \geq 3$, $n \geq 3$, *and Linear or Unrestricted Domain.*

There does not exist a social choice rule satisfying Monotonicity, Pareto Consistency, No $\lfloor \frac{n-2}{3} \rfloor$-Dictator, and $\lfloor \frac{n-2}{3} \rfloor$-Tie Break.

The No r-Dictator axiom may be slightly stronger than expected, because it is not necessarily the case that the top-ranked alternative of an r-dictator, say i, will be included in the social choice set. It will if r or more agents prefer agent i's top-ranked alternative to some other given alternative; but if, for example, i's top-ranked alternative is bottom-ranked by all other agents, then it may not be chosen. In fact, if we were to weaken the axiom to only preclude an r-dictator in the more restrictive sense (whose top-ranked alternative is always among those chosen), then Theorem 2 would fail even for r = 1. To see this, assume X finite and Linear Domain, and construct a social choice rule f as follows. Given state θ, we specify that $f(\theta)$ include the top-ranked alternative, say x, of agent 1 if and only if it is not bottom-ranked by all other agents; and if y Pareto dominates x among agents 2,...,n, i.e., $\{2,\ldots,n\} \subseteq P(y,x|\theta)$, and is itself not Pareto dominated among agents 2,...,n, then y also belongs to $f(\theta)$. Evidently, this social choice rule satisfies Monotonicity, Pareto Consistency, and 1-Tie Break. Indeed, to fulfill the antecedent of 1-Tie Break, it must be that agent 1's top-ranked alternative is x, followed by y, while all other agents rank y first, followed by x; then a preference reversal by agent 1 leads to y as the unique choice, and a preference reversal by $i \neq 1$ leads to x as the unique choice, because y no longer Pareto dominates x among 2,...,n. Because agent 1's top-ranked alternative is not always chosen, however, the social choice rule satisfies the weaker version of No 1-Dictator.

Duggan and Schwartz (1995) prove a result related to Theorem 2 assuming a stronger tie break condition (which requires singleton social choice sets in some situations) and imposing a *Minimal Resoluteness* axiom: if two alternatives, say x and y, are preferred to all others by all agents, and if all or all but one agent prefers x to y, then the social choice set is a singleton; formally, for all x and y and all $\theta \in \Theta^{\{x,y\}}$ with $|P(x,y|\theta)| \geq n-1$, we have

$|f(\theta)| = 1$.[13] With the latter condition, they weaken Pareto Consistency to Full Range, and they weaken No r-Dictator to No Dictator. When there are at least three alternatives, it is straightforward to see that under Monotonicity and Minimal Resoluteness, the latter axioms actually imply Pareto Consistency and No r-Dictator. For example, suppose agent i is an r-dictator. Consider a state θ in which x is top-ranked for agent i, and suppose some other alternative y is among those chosen. Now consider a state θ' in which x and y are moved to the tops of all agents' rankings, with agent i preferring x and all other agents preferring y. By Monotonicity, y is still among those chosen; by Minimal Resoluteness, it is the unique choice. But now all agents strictly prefer x (agent i's top-ranked alternative) to z (a third alternative), and then x must be chosen by definition of r-dictator, a contradiction. Thus, i is in fact a dictator for f. In contrast, the only trace of resoluteness in the conditions of Theorem 2 is in part (ii) of Pareto Consistency, which only applies when one alternative is preferred to all others by all agents, and in r-Tie Break.

It is apparent that Resoluteness implies not only Minimal Resoluteness but r-Tie Break as well: regardless of the threshold, the antecedent of r-Tie Break (which only applies when two distinct alternatives are chosen) is false, so the axiom is vacuously satisfied. For a resolute social choice rule, Full Range and Monotonicity together imply Pareto Consistency, and an r-dictator is a dictator. Thus, Theorem 2 generalizes the theorem of Muller and Satterthwaite (1977), which establishes that every social choice rule satisfying Resoluteness, Full Range, and Monotonicity is dictatorial.

A Proof of Acyclicity Theorem

This appendix begins with supporting results that will lay the groundwork for the proof of Theorem 1. Throughout the appendix,

[13] See also Duggan and Schwartz (2000) for an application of minimal resoluteness to strategy-proof social choice.

assume $|X| \geq 3$, $n \geq 3$, and Linear Domain; set $r = \lfloor \frac{n-2}{3} \rfloor$; and consider a social preference rule F satisfying Independence, Pareto, Acyclicity, and r-Tie Break. Ultimately, the goal is to deduce that F violates the No r-Dictator axiom. Given a group G and distinct alternatives x and y, we say G is *semi-decisive for* x *over* y*,* written $x\, D^G\, y$, if for all θ such that $P(x,y|\theta) = G$, we have $x\, P_F(\theta)\, y$. In parallel fashion, given distinct x and y, we say G is *semi-blocking for* x *over* y*,* written $x\, B^G\, y$, if for all θ such that $P(x,y|\theta) = G$, we have $x\, R_F(\theta)\, y$. Note that if i is a weak dictator, then for all distinct x and y, we have $x\, B^{\{i\}}\, y$.

Lemma 3 *For all distinct* x *and* y *and all* θ*,*

(i) $x\, P_F(\theta)\, y$ *if and only if* $x\, D^{P(x,y|\theta)}\, y$

(ii) $x\, R_F(\theta)\, y$ *if and only if* $x\, B^{P(x,y|\theta)}\, y$

(iii) for all G*, either* $x\, D^G\, y$ *or* $y\, B^{N\setminus G}\, x$*, but not both.*

Proof   Parts (i) and (ii) follow immediately from Linear Domain and Independence. For part (iii), consider any G. By Linear Domain, there exists $\theta \in \Theta$ such that $P(x,y|\theta) = G$, and therefore $P(y,x|\theta) = N \setminus G$. By duality, either $x\, P_F(\theta)\, y$ or $y\, R_F(\theta)\, x$, but not both. The first case is equivalent to $x\, D^G\, y$ by part (i), and the second is equivalent to $y\, B^{N\setminus G}\, x$ by part (ii). Q.E.D.

The next lemma is implied by Theorem 3 of Ferejohn and Fishburn (1979). For $k = 3$, it implies that there do not exist alternatives x, y, and z and groups G, H, and I such that $x\, D^G\, y\, D^H\, z\, D^I\, x$, $G \cup H \cup I = N$, and $G \cap H \cap I = \emptyset$. And because $z\, D^N\, x$, by Pareto, it implies that there do not exist x, y, and z and a group G such that $x\, D^G\, y\, D^{N\setminus G}\, z$. In particular, if $x\, D^G\, y$, then G is non-empty.

Lemma 4 *There do not exist natural number* k*, distinct alternatives* $x_1,\ldots,x_k$*, and groups* $G_1,\ldots,G_k$ *such that*

(i) $x_h\, D^{G_h}\, x_{h+1}$ *for all* $h = 1,\ldots,k-1$ *and* $x_k\, D^{G_k}\, x_1$*,*

(ii) $\bigcup_{h=1}^{k} G_h = N$,

(iii) $\bigcap_{h=1}^{k} G_h = \emptyset$.

An implication of the previous lemma is a weak transitivity property of the semi-decisiveness relation.

Lemma 5 *For all* G *and* H *and all distinct* x, y, *and* z, *if* $x\, D^G\, y\, D^H\, z$, *then* $x\, B^{G\cap H}\, z$.

Proof Consider groups G and H and alternatives x, y, and z such that $x\ D^G\ y\ D^G\ z$, and suppose that $x\ B^{G\cap H}\ z$ fails. By Lemma 3, we have $z\ D^{N\setminus(G\cap H)}\ x$, but then we have $x\ D^G\ y\ D^H\ z\ D^{N\setminus(G\cap H)}\ x$, we have $G\cup H\cup(N\setminus(G\cap H)) = N$, and we have $G\cap H\cap(N\setminus(G\cap H)) = \emptyset$, contradicting Lemma 4. Q.E.D.

An implication of the following lemma is that if an agent i is a weak dictator, then she is a r-dictator.

Lemma 6 *For all* G, *all* H *with* $|H| \geq r$ *and* $G\cap H = \emptyset$, *and all* x *and* y, *if* $x\, B^G\, y$, *then* $x\, D^{G\cup H}\, y$.

Proof Consider any $G \subseteq N$, any H with $|H| \geq r$ and $G\cap H = \emptyset$, any x and y such that $x\, B^G\, y$, and any θ such that $G\cup H = P(x,y|\theta)$. By Lemma 3, to establish $x\, D^{G\cup H}\, y$, it suffices to show $x\, P_F(\theta)\, y$. Let θ' be such that $G = P(x,y|\theta')$. Since $x\ B^G\ y$, we have either $x\ P_F(\theta')\ y$ or $x\ I_F(\theta')\ y$. In the latter case, r-Tie Break immediately implies $x\, P_F(\theta)\, y$. Consider the former case. If not $x\, P_F(\theta)\, y$, then either $x\ I_F(\theta)\ y$, so r-Tie Break implies $y\, P_F(\theta')\, x$ and contradicts $x\, P_F(\theta')\, y$, or $y\, P_F(\theta)\, x$. Thus, we must preclude the possibility of a social preference reversal: $x\ P_F(\theta')\ y$ and $y\ P_F(\theta)\ x$. If so, Lemma 3 implies $x\ D^G\ y$ and $y\, D^{N\setminus(G\cup H)}\ x$. Letting $z \in X\setminus\{x,y\}$, I claim that $x\, B^G\, z$. Otherwise, by Lemma 3, we have $z\, D^{N\setminus G}\, x$, but then $z\, D^{N\setminus G}\, x\, D^G\, y$ and $(N\setminus G)\cap G) = \emptyset$, contradicting Lemma 4. Thus, $x\, B^G\, z$, as claimed. Consider θ'' such that $G = P(x,z|\theta'')$. Since $x\, B^G\, z$, there are two cases. Case 1: $x\ P_F(\theta'')\ z$. By Lemma 3, $x\, D^G\, z$. Then we have $y\, D^{N\setminus(G\cup H)}\ x\, D^G\, z$ and $(N\setminus(G\cup H))\cap G = \emptyset$, contradicting Lemma 4. Case 2: $x\ I_F(\theta'')\ z$. Letting θ''' be such that

$G \cup H = P(x,z|\theta''')$, r-Tie Break implies $x\ P_F(\theta''')\ z$, and then $x\ D^{G\cup H}\ z$ by Lemma 3. Then we have $y\ D^{N\setminus(G\cup H)}\ x\ D^{G\cup H}\ z$ and $(N\setminus(G\cup H))\cap(G\cup H) = \emptyset$, again contradicting Lemma 4. Q.E.D.

We can now show that if one agent is semi-blocking for each alternative over every other, then she is an r-dictator.

Lemma 7 *For all* i*, if for all distinct* x *and* y*,* $x\ B^{\{i\}}\ y$*, then* i *is an* r*-dictator.*

Proof Consider i as in the statement of the lemma, any distinct x and y, and any θ such that $x\ P_i(\theta)\ y$. Let $G = P(x,y|\theta) \setminus \{i\}$. Suppose not $x\ R_F(\theta)\ y$, so that $y\ P_F(\theta)\ x$. By Lemma 3, $y\ D^{N\setminus(G\cup\{i\})}\ x$. Let $\{H,I\}$ be a partition of $N \setminus \{i\}$ such that $\min\{|H|,|I|\} \geq r$, and let $z \notin X \setminus \{x,y\}$. Using Lemma 6, we have $y\ D^{N\setminus(G\cup\{i\})}\ x\ D^{\{i\}\cup H}\ y\ D^{\{i\}\cup I}\ z$, we have $(N \setminus (G \cup \{i\})) \cup (\{i\} \cup H) \cup (\{i\} \cup I) = N$, and we have $(N \setminus (G \cup \{i\})) \cap (\{i\} \cup H) \cap (\{i\} \cup I) = \emptyset$, contradicting Lemma 4. Therefore, $x\ R_F(\theta)\ y$, and it follows that i is a weak dictator. Then Lemma 6 implies that i is in fact an r-dictator. Q.E.D.

The next lemma shows, essentially, that if an agent is semi-blocking for some x over some y, then she is semi-blocking for every alternative over y and for x over every alternative. In the remainder of the appendix, I use the fact that given two groups G and H, we have $|G\cap H| = |G| + |H| - |G\cup H|$; in particular, when $G\cup H = N$, we have $|G \cap H| = |G| + |H| - n$.

Lemma 8 *For all* i *and all* x*,* y*, and* z*, it is not the case that* $x\ B^{\{i\}}\ y\ D^{N\setminus\{i\}}\ z$*, and it is not the case that* $x\ D^{N\setminus\{i\}}\ y\ B^{\{i\}}\ z$*.*

Proof Consider any i, x, y, and z, and first suppose $x\ B^{\{i\}}\ y\ D^{N\setminus\{i\}}\ z$. Note that $x \neq z$, by Lemma 3. Since $y\ D^{N\setminus\{i\}}\ z$, we can choose a minimal group G such that $y\ D^G\ z$ and $i \notin G$. I claim that $|G| \geq n - r$, for otherwise $|G| \leq n-r-1$, and we have $|N\setminus(G\cup\{i\})| = n-|G|-1 \geq r$, and using $x\ B^{\{i\}}\ y$ and $N \setminus G = \{i\} \cup (N \setminus (G \cup \{i\}))$, Lemma 6 implies $x\ D^{N\setminus G}\ y$. But then $x\ D^{N\setminus G}\ y\ D^G\ z$ and $(N\setminus G)\cap G = \emptyset$, contradicting Lemma 4. This establishes the claim. Of course, $|N \setminus G| \leq r$. Now let H be a group with $i \notin H$ and $|H| = r$ and such that $N\setminus(G\cup\{i\}) \subseteq H$. By Lemma

6, we have x $D^{\{i\}\cup H}$ y, and then by Lemma 5, x $D^{\{i\}\cup H}$ y D^G z implies x $B^{G\cap(\{i\}\cup H)}$ z. Since $i \notin G$, this reduces to x $B^{G\cap H}$ z. Note that since $G \cup H = N \setminus \{i\}$, we have $|G \cap H| = |G| + r - n + 1$.

Choose $j \in G \setminus H$. Since $|G| \geq n - r$ and $|G \cap H| = |G| + r - n + 1$, we can partition $G \setminus (H \cup \{j\})$ into two groups I and J such that $\min\{|I|,|J|\} \geq r$. Indeed, this is possible because

$$\begin{aligned} |G \setminus (H \cup \{j\})| &= |G| - |\{j\}| - |G \cap H| \\ &= |G| - 1 - |G| - r + n - 1 \\ &= n - r - 2 \\ &\geq 2r, \end{aligned}$$

where the inequality follows from $r \leq \frac{n-2}{3}$. From minimality of G, with Lemma 3, it follows that z $B^{(N\setminus G)\cup\{j\}}$ y. Since $I\cap((N\setminus G)\cup\{j\}) = \emptyset$ and $|I| \geq r$, Lemma 6 implies z $D^{(N\setminus G)\cup\{j\}\cup I}$ y. Recall that x $B^{G\cap H}$ z. Since $J\cap(G\cap H) = \emptyset$ and $|J| \geq r$, Lemma 6 implies x $D^{(G\cap H)\cup J}$ z. But then we have x $D^{(G\cap H)\cup J}$ z $D^{(N\setminus G)\cup\{j\}\cup I}$ y, we have $((N\setminus G)\cup\{j\}\cup I)\cap((G\cap H)\cup J) = N$, and $((N \setminus G) \cup \{j\} \cup I) \cap ((G \cap H) \cup J) = \emptyset$, contradicting Lemma 4.

When x $D^{N\setminus\{i\}}$ y $B^{\{i\}}$ z, we deduce a contradiction by a similar argument. Since x $D^{N\setminus\{i\}}$ y, we can choose a minimal group G such that x D^G y and $i \notin G$. If $|G| < n - r$, then we have x D^G y $D^{N\setminus G}$ z, a contradiction, so $|G| \geq n - r$. Let H be a group with $i \notin H$ and $|H| = r$ such that $N \setminus (G \cup \{i\}) \subseteq H$. Since y $B^{\{i\}}$ z, we have y $D^{\{i\}\cup H}$ z, and then x D^G y $D^{\{i\}\cup H}$ z implies x $B^{G\cap H}$ z. Letting $j \in G$, partition $G\setminus(\{j\}\cup H)$ into groups I and J with $\min\{|I|,|J|\} \geq r$. By minimality of G, we have y $B^{(N\setminus G)\cup\{j\}}$ x, and therefore y $D^{(N\setminus G)\cup\{j\}\cup I}$ x $D^{(G\cap H)\cup J}$ z, a contradiction. Q.E.D.

The next lemma strengthens Lemma 7 by establishing that an agent is an r-dictator if she is semi-blocking for one alternative over one other.

Lemma 9 *For all* i *and all* x *and* y*, if* x $B^{\{i\}}$ y*, then* i *is an* r-*dictator.*

Proof Suppose that for some i and some x and y, we have y $B^{\{i\}}$ x. Consider any $z \in X \setminus \{x,y\}$. If either y $B^{\{i\}}$ z or z $B^{\{i\}}$ x fails, then using Lemma 3, either $z\ D^{N\setminus\{i\}}\ y\ B^{\{i\}}\ x$ or $y\ B^{\{i\}}\ x\ D^{N\setminus\{i\}}\ z$, in both cases contradicting Lemma 8. Now consider any a and b, and choose $c \in X \setminus \{y,b\}$. By the above argument, $y\ B^{\{i\}}\ c$. Now choose $d \in X \setminus \{b,c\}$, and note that by the same argument, $d\ B^{\{i\}}\ c$, $d\ B^{\{i\}}\ b$, and $a\ B^{\{i\}}\ b$. Since a and b are arbitrary, it follows that i is semi-blocking for every alternative over every other, and Lemma 7 implies that i is an r-dictator. Q.E.D.

An implication of the preceding lemma is that either No r-Dictator is violated, or groups that are decisive over adjacent pairs of alternatives must contain at least two agents in their intersection.

Lemma 10 *For all* G *and* H *and all distinct* x, y, *and* z, *if* $x\ D^G\ y\ D^H\ z$, *then either* $G \cap H = \{i\}$ *and* i *is an* r*-dictator for some* i, *or* $|G \cap H| \geq 2$.

Proof Given G, H, and distinct x, y, and z such that $x\ D^G\ y\ D^H\ z$, Lemma 4 implies $G \cap H \neq \emptyset$, so either $|G \cap H| \geq 2$ or for some i, $G \cap H = \{i\}$ In the latter case, Lemma 5 implies $x\ B^{\{i\}}\ z$, and Lemma 9 implies that i is an r-dictator. Q.E.D.

The next lemma exhibits groups that are not too large, that exhaust the agents, and that are decisive for adjacent pairs of alternatives. The proof uses the fact that $r = \lfloor \frac{n-2}{3} \rfloor \leq \lfloor \frac{n}{3} \rfloor$.

Lemma 11 *There exist groups* G *and* H *and distinct alternatives* a,b,c *such that* $\max\{|G|,|H|\} \leq \lceil \frac{2n}{3} \rceil$, $G \cup H = N$, *and* $a\ D^G\ b\ D^H\ c$.

Proof Consider any distinct alternatives x, y, and z, and let A_1 be any group consisting of $\lfloor \frac{n}{3} \rfloor$ members. If $x\ B^{A_1}\ y$, then let B_1 be any group of $\lceil \frac{2n}{3} \rceil - \lfloor \frac{n}{3} \rfloor \geq \lfloor \frac{n}{3} \rfloor$ members disjoint from A_1, and set $G_1 = A_1 \cup B_1$. By Lemma 6, we have $x\ D^{G_1}\ y$. Otherwise, Lemma 3 implies $y\ D^{N\setminus A_1}\ x$, and we set $G_1 = N \setminus A_1$ to obtain $y\ D^{G_1}\ x$. Note that in either case, we have $|G_1| = \lceil \frac{2n}{3} \rceil$.

Next, let A_2 be a group consisting of $\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{3} \rceil$ members of G_1, which is possible since $|G_1| = \lceil \frac{2n}{3} \rceil \geq \lceil \frac{n}{3} \rceil$. If $y\ B^{A_2}\ z$, then let

$B_2 = N \setminus G_1$, and set $G_2 = A_2 \cup B_2$. Since $|B_2| = n - |G_1| = \lfloor \frac{n}{3} \rfloor$, Lemma 6 implies $y\, D^{G_2}\, z$. Otherwise, $z\, D^{N \setminus A_2}\, y$, and we set $G_2 = N \setminus A_2$, so that $z\, D^{G_2}\, y$. Note that in the latter case, $|G_2| = n - \lceil \frac{2n}{3} \rceil + \lceil \frac{n}{3} \rceil = \lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{3} \rceil \in \{\lfloor \frac{2n}{3} \rfloor, \lceil \frac{2n}{3} \rceil\}$, and in the former case, $|G_2| = \lceil \frac{2n}{3} \rceil - \lceil \frac{n}{3} \rceil + n - |G_1| = \lfloor \frac{2n}{3} \rfloor$. Thus, we have $|G_2| \in \{\lfloor \frac{2n}{3} \rfloor, \lceil \frac{2n}{3} \rceil\}$ in either case. In addition, $G_1 \cup G_2 = N$ holds.

To complete the proof, I consider four possible cases.

Case 1: $x\, D^{G_1}\, y\, D^{G_2}\, z$. Set $a = x$, $b = y$, $c = z$, $G = G_1$, and $H = G_2$. Then $a\, D^{G_1}\, b\, D^{G_2}\, c$, and as above, $\max\{|G|, |H|\} \leq \lceil \frac{2n}{3} \rceil$, and $G \cup H = N$.

Case 2: $z\, D^{G_2}\, y\, D^{G_1}\, x$. Set $a = z$, $b = y$, $c = x$, $G = G_2$, and $H = G_1$. Then $a\, D^{G_1}\, b\, D^{G_2}\, c$, and again, $\max\{|G|, |H|\} \leq \lceil \frac{2n}{3} \rceil$, and $G \cup H = N$.

Case 3: $x\, D^{G_1}\, y$ and $z\, D^{G_2}\, y$. Let A_3 be a group with $\lceil \frac{n}{3} \rceil$ members that contains $N \setminus G_2$ and is contained in G_1, i.e., $N \setminus G_2 \subseteq A_3 \subseteq G_1$. This is possible because $\lfloor \frac{2n}{3} \rfloor \leq |G_2|$, $|G_1| = \lceil \frac{2n}{3} \rceil$, and $G_1 \cup G_2 = N$. If $x\, B^{A_3}\, z$, then let $B_3 = N \setminus G_1$, so that $|B_3| = \lfloor \frac{n}{3} \rfloor$, and set $a = x$, $b = z$, $c = y$, $G = A_3 \cup B_3$, and $H = G_2$. Otherwise, we have $z\, D^{N \setminus A_3}\, x$, and we set $a = z$, $b = x$, $c = y$, $G = N \setminus A_3$, and $H = G_1$. In the latter case, $|G| = n - |A_3| = \lfloor \frac{2n}{3} \rfloor$ and $|H| = |G_1| = \lceil \frac{2n}{3} \rceil$, and in the former case, $|G| = |A_3| + |B_3| = \lceil \frac{n}{3} \rceil + \lfloor \frac{n}{3} \rfloor \leq \lceil \frac{2n}{3} \rceil$ and $|H| = |G_2| \leq \lceil \frac{2n}{3} \rceil$. Thus, $\max\{|G|, |H|\} \leq \lceil \frac{2n}{3} \rceil$ in either case. In addition, $G \cup H = N$ holds, and using Lemma 6, we have $a\, D^G\, b\, D^H\, c$, as required.

Case 4: $y\, D^{G_1}\, x$ and $y\, D^{G_2}\, z$. Let A_4 be a group with $\lfloor \frac{n}{3} \rfloor$ members that contains $N \setminus G_1$ and is contained in G_2, i.e., $N \setminus G_1 \subseteq A_4 \subseteq G_2$. This is possible because $|G_1| = \lceil \frac{2n}{3} \rceil$ and $\lfloor \frac{2n}{3} \rfloor \leq |G_2|$. If $x\, B^{A_4}\, z$, then let $B_4 = N \setminus G_2$, so that $|B_4| \in \{\lfloor \frac{n}{3} \rfloor, \lceil \frac{n}{3} \rceil\}$, and set $a = y$, $b = x$, $c = z$, $G = G_1$, and $H = A_4 \cup B_4$. Otherwise, we have $z\, D^{N \setminus A_4}\, x$, and we set $a = y$, $b = z$, $c = x$, $G = G_2$, and $H = N \setminus A_4$. In the latter case, $|G| = |G_2| \leq \lceil \frac{2n}{3} \rceil$ and $|H| = n - |A_4| = \lceil \frac{2n}{3} \rceil$, and in the former case, $|G| = |G_1| = \lceil \frac{2n}{3} \rceil$ and $|H| = |A_4| + |B_4| \leq \lceil \frac{2n}{3} \rceil$. Thus, $\max\{|G|, |H|\} \leq \lceil \frac{2n}{3} \rceil$ in either case. In addition, $G \cup H = N$ holds, and using Lemma 6, we have $a\, D^G\, b\, D^H\, c$, as required. Q.E.D.

To complete the proof of Theorem 1, choose distinct alternatives a, b, and c as in Lemma 11, and define the collection Π of pairs of groups by

$$\Pi \;=\; \left\{ (G,H) \;\middle|\; \begin{array}{c} a\,D^{G}\,b\,D^{H}\,c,\ |G| \leq \lceil \frac{2n}{3} \rceil, \\ |H| \leq \lceil \frac{2n}{3} \rceil,\text{ and } G \cup H = N \end{array} \right\}.$$

By Lemma 11, Π is nonempty, so we can choose a pair $(G,H) \in \Pi$ that is minimal, in the sense that there is no $(G',H') \in \Pi$ such that $|G'| \leq |G|$ and $|H'| \leq |H|$ with at least one inequality strict, i.e., $\max\{|G'| - |G|, |H'| - H|\} \leq 0 < \max\{|G| - |G'|, |H| - |H'|\}$. Note that $|G \cap H| = |G| + |H| - n$, and because $a\,D^{G}\,b\,D^{H}\,c$, Lemma 5 implies $a\,B^{G\cap H}\,c$. By Lemma 9, either $|G \cap H| \geq 2$ or there is an r-dictator. The remainder of the proof shows that the supposition $|G \cap H| \geq 2$ leads to a contradiction.

To proceed, choose any $j \in G \cap H$. If $b\,D^{H\setminus\{j\}}\,c$, then we can set $G' = G$ and $H' = H \setminus \{j\}$ and obtain $(G',H') \in \Pi$, contradicting minimality. Using Lemma 3, we then have $c\,B^{(N\setminus H)\cup\{j\}}\,b$, and since $G \cup H = N$, we can write $c\,B^{(G\setminus H)\cup\{j\}}\,b$. Setting $\alpha = \lceil \frac{2n}{3} \rceil - 2 - r$, it is verified at the end of the appendix that $\lfloor \frac{n}{3} \rfloor \geq \alpha \geq r$. Thus, since $|N \setminus G| \geq n - \lceil \frac{2n}{3} \rceil = \lfloor \frac{n}{3} \rfloor$, we can choose a group I disjoint from G with $|I| = \alpha$ Since $c\,B^{(G\setminus H)\cup\{j\}}\,b$ and $((G \setminus H) \cup \{j\}) \cap I = \emptyset$, Lemma 6 implies $c\,D^{(G\setminus H)\cup\{j\}\cup I}\,b$. Because $G \cup H = N$, we have $I \subseteq H \setminus G$, and with $G \cap I = \emptyset$, we can deduce

$$\begin{aligned} |N \setminus (G \cup I)| &= |H \setminus ((G \cap H) \cup I)| \\ &= |H| - |G \cap H| - |I| \\ &\leq \left\lceil \frac{2n}{3} \right\rceil - 2 - \alpha \\ &= r, \end{aligned}$$

where the inequality uses $|H| \leq \lceil \frac{2n}{3} \rceil$ and $|G \cap H| \geq 2$. In addition,

$$|N \setminus ((G \cap H) \cup I)| \;=\; n - (|G| + |H| - n) - \alpha \;\geq\; 2n - 2\left\lceil \frac{2n}{3} \right\rceil - \alpha \;\geq\; r,$$

where the first inequality uses $\max\{|G|, |H|\} \leq \lceil \frac{2n}{3} \rceil$, and the second is verified at the end of the appendix. Thus, we can choose a group J such that $N \setminus (G \cup I) \subseteq J \subseteq N \setminus ((G \cap H) \cup I)$ with $|J| = r$. Since $a\,B^{G\cap H}\,c$ and $(G \cap H) \cap J = \emptyset$, Lemma 6 implies $a\,D^{(G\cap H)\cup J}\,c$.

Since $N \setminus G \subseteq I \cup J$, it follows that $N = ((G \setminus H) \cup \{j\} \cup I) \cup ((G \cap H) \cup J)$. Setting $G^{*} = (G \cap H) \cup J$ and $H^{*} = (G \setminus H) \cup \{j\} \cup I$, note that $G^{*} \cap (H^{*} \cap H) =$

$G^* \cap (\{j\} \cup I) = \{j\}$. Moreover, we have a D^{G^*} c D^{H^*} b, which implies a $B^{G^* \cap H^*}$ b by Lemma 5. Since $G^* \cup H^* = N$, we have

$$\begin{aligned} |G^* \cap H^*| &= |G^*| + |H^*| - n \\ &= |(G \cap H) \cup J| + |(G \setminus H) \cup \{j\} \cup I| - n \\ &= |G| + |H| - n + r + n - |H| + 1 + \alpha - n \\ &= |G| + 1 - n + r + \alpha. \end{aligned}$$

Setting $\beta = |G| - 1 - |G^* \cap H^*|$, we have

$$\begin{aligned} |N \setminus ((G^* \cap H^*) \cup H)| &= n - (|G^* \cap H^*| + |H| - |(G^* \cap H^*) \cap H|) \\ &= n - |G^* \cap H^*| - |H| + 1 \\ &= n + \beta - |G| + 2 - |H| \\ &\leq \beta, \end{aligned}$$

where the second equality uses $(G^* \cap H^*) \cap H = \{j\}$, and the inequality follows from $|G| + |H| - n = |G \cap H| \geq 2$. In addition,

$$|N \setminus (G^* \cap H^*)| = n - |G^* \cap H^*| = n - (|G| - 1 - \beta) = n - |G| + 1 + \beta \geq \beta.$$

Thus, we can choose a group K such that $N \setminus ((G^* \cap H^*) \cup H) \subseteq K \subseteq N \setminus (G^* \cap H^*)$ with $|K| = \beta$. In particular, $(G^* \cap H^*) \cup K \cup H = N$.

Finally, note that

$$\beta = |G| - 1 - |G^* \cap H^*| = |G| - 1 - |G| - 1 + n - r - \alpha = \left\lfloor \frac{n}{3} \right\rfloor \geq r.$$

Therefore, using Lemma 6, we have a $D^{(G^* \cap H^*) \cup K}$ b D^H c. But defining $G' = (G^* \cap H^*) \cup K$ and $H' = H$, we then have $(G', H') \in \Pi$ and

$$|G'| = |G^* \cap H^*| + \beta = |G| - 1 < |G|,$$

contradicting minimality of (G,H).

Remainders: Writing $n = 3\left\lfloor \frac{n}{3} \right\rfloor + \phi$ for $\phi \in \{0,1,2\}$, we have

$$\left\lfloor \frac{n-2}{3} \right\rfloor = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor & \text{if } \phi = 2 \\ \left\lfloor \frac{n}{3} \right\rfloor - 1 & \text{if } \phi = 0,1, \end{cases}$$

and

$$\left\lceil \frac{2n}{3} \right\rceil = \begin{cases} 2\left\lfloor \frac{n}{3} \right\rfloor & \text{if } \phi = 0 \\ 2\left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{if } \phi = 1 \\ 2\left\lfloor \frac{n}{3} \right\rfloor + 2 & \text{if } \phi = 2. \end{cases}$$

Therefore, $\alpha = \left\lfloor \frac{n}{3} \right\rfloor - 1$ when $\phi = 0$, and $\alpha = \left\lfloor \frac{n}{3} \right\rfloor$ when $\phi = 1,2$. In each case, we have $\left\lfloor \frac{n}{3} \right\rfloor \geq \alpha \geq r$. Moreover, since $\lceil \frac{2n}{3} \rceil \leq 2\left\lfloor \frac{n}{3} \right\rfloor + 2$, we have $2n - 2\lceil \frac{2n}{3} \rceil - \alpha = 2\left\lfloor \frac{n}{3} \right\rfloor + 2 - \lceil \frac{2n}{3} \rceil + r \geq r$, as required.

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