Sequential Median Location

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Abstract

This note analyzes a sequential location game in one dimension with single-peaked preferences. If the order of location alternates between agents on one side of the median alternative and the other, then the unique pure strategy subgame perfect equilibrium outcome is the median. The result extends to sequential play of any Condorcet consistent game, and adding risk aversion, it extends to mixed strategies and games with stochastic outcomes.

1 Introduction

This note shows that strategic incentives in one-dimensional location games can lead to the median ideal point as the unique subgame perfect equilibrium outcome in pure strategies. The key is that the order of location alternates, with someone to one side of the median alternative moving first, then someone (after observing the earlier agent) from the other side moving next, and back again in this fashion. Beyond that restriction, the result is quite general. Initially, I assume that once a vector of locations $(x_1, \ldots, x_n)$ is determined for all agents, the final outcome is the "plurality winner," i.e., the location to which most agents move. This basic specification of the sequential location game is reminiscent of Feddersen's (1992) model of elections, in which voters simultaneously locate, with small cost, at positions in a policy space with the outcome being the plurality winner. When the policy space is one-dimensional, Feddersen finds that in Nash equilibria of this game, there are at most two locations that receive a positive number of votes, and in general there may be an infinite number of equilibria. The sequential median voter theorem of this note provides conditions under which this multiplicity is avoided in the sequential version of the game, yielding the median as the unique prediction.

The sequential median voter theorem extends to any Condorcet consistent game, in which there is an action profile $(a_1^*, \ldots, a_n^*)$ such that if a majority of agents choose their designated actions a_i^*, then the median is the outcome regardless of the actions of other agents. This captures Condorcet consistent

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voting rules, such as the top cycle, uncovered set, Copeland set, or sequential Borda elimination. To obtain these rules as special cases, we specify that the agents sequentially announce ballots, and that the outcome is any selection from the voting rule of interest; and we designate a_i^* as any ballot in which the median alternative is top ranked. Thus, regardless of which of the above voting rules is used, if the agents alternate from one side of the median to the other, then the unique pure strategy subgame perfect equilibrium outcome is the median. Adding risk aversion, the result extends to mixed strategy equilibria and games with stochastic outcomes.

2 Basic Sequential Location Game

An odd number of agents, $N = \{1, \ldots, n\}$, choose an alternative belonging to a set $X \subseteq \Re$ of alternatives by playing the following perfect information game. Agent 1 locates at any alternative x_1; after observing the previous location, agent 2 locates at any x_2; in general, after observing the previous locations $x_1, \ldots, x_{i-1}$, agent i locates at any x_i. Given all locations $x_1, \ldots, x_n$, the final outcome is the plurality winner, with ties being broken in favor of the greatest plurality winning location:

$$g(x_1, \ldots, x_n) \quad = \quad \max\{x_i \mid \text{for all } j,\ v_i(x_1, \ldots, x_n) \geq v_j(x_1, \ldots, x_n)\},$$

where

$$v_i(x_1, \ldots, x_n) \quad = \quad \#\{j \mid x_j = x_i\}$$

is the number of agents who locate at x_i. We will see in Section 4 that this specification of the tie-breaking rule is without consequence, as all results hold for rules that are arbitrary functions of histories.

Each agent i has preferences over X represented by a utility function $u_i \colon \Re \to \Re$ assumed to be single-peaked on X:[1] agent i's utility is uniquely maximized at ideal point $\hat{x}^i \in X$, and for all $x, y \in X$, $0 < \frac{x - \hat{x}^i}{y - \hat{x}^i} < 1$ implies $u_i(x) > u_i(y)$. Since the number of agents is odd, the distribution of ideal points $\hat{x}^1, \ldots, \hat{x}^n$ has a unique median, say x^*. Thus, a majority of agents have ideal points equal to or less than the median x^*, and a majority have ideal points equal to or greater than the median.

The analysis uses the following restriction on the order of location.

Alternating protocol The order of location alternates between agents with ideal points to one side of the median and the other, i.e.,

$$\max\{\hat{x}^i \mid i \text{ is odd}\} \ \leq \ x^* \ \leq \ \min\{\hat{x}^i \mid i \text{ is even}\}.$$

[1] More precisely, agent i's preferences over alternatives are represented by the restriction of u_i to X.

As usual, a pure strategy for agent i is a mapping $s_i : X^{i-1} \to X$ from all player i histories to actions. A pure strategy profile $s = (s_1, \ldots, s_n)$ determines a unique path of locations from the initial node, $(x_1(s), \ldots, x_n(s))$, and a unique outcome, $x(s) = g(x_1(s), \ldots, x_n(s))$. More generally, given any history $h = (x_1, \ldots, x_{i-1})$, a strategy profile s determines a unique path $(x_i(h|s), \ldots, x_n(h|s))$ from that history and a unique outcome $x(h|s) = g(h, x_i(h|s), \ldots, x_n(h|s))$ from that history. A pure strategy subgame perfect equilibrium is a pure strategy profile s such that for all agents i and all histories $h = (x_1, \ldots, x_{i-1})$, agent i is choosing optimally given the strategies of the agents restricted to the continuation game: for all $x \in X$,

$$u_i(x(h, s_i(h)|s)) \quad \geq \quad u_i(x(h, x|s)).$$

This completes the description of the *basic sequential location game.*

When X is allowed to be infinite, the players' payoffs will generally be discontinuous in histories. Specifically, because the outcome function $g(x_1, \ldots, x_n)$ is discontinuous in $(x_1, \ldots, x_n)$, the composition $u_i(g(x_1, \ldots, x_n))$ will also be discontinuous. This makes existence of equilibrium (pure or mixed) a difficult issue. Assuming X is finite, however, the basic sequential location game is a finite game of perfect information, and thus it automatically admits at least one subgame perfect equilibrium in pure strategies.

Theorem 1 *Assume X is finite. The basic sequential location game possesses a pure strategy subgame perfect equilibrium.*

The sequential location game may actually admit multiple pure strategy subgame perfect equilibria, but the following sequential median voter theorem establishes that they all determine the same outcome: the median ideal point of the agents. Note that the result does not assume X finite, and so the statement of uniqueness does not imply existence. That is, the result establishes that if there is a pure strategy subgame perfect equilibrium, then the outcome determined by that equilibrium is the median.

Theorem 2 *The unique pure strategy subgame perfect equilibrium outcome of the basic sequential location game is the median, x^*.*

Let s be any subgame perfect equilibrium profile. Given arbitrary history $h = (x_1, \ldots, x_{i-1})$, let $\mu(h) = \#\{j < i \mid x_j = x^*\}$ denote the number of agents who have already located at the median, and define

$$\begin{aligned} \lambda(h) &= \#\{j \geq i \mid \hat{x}^j < x^*\} \\ \rho(h) &= \#\{j \geq i \mid x^* < \hat{x}^j\} \end{aligned}$$

as the numbers of agents who have not yet located and have ideal points, respectively, strictly less than and strictly greater than the median.

The proof is by induction on k using the following hypothesis:

> For every history h satisfying $k + \mu(h) \geq m$, if $\lambda(h) \geq k - 1$ and $\rho(h) \geq k - 1$, then $x(h|s) = x^*$.

Consider $k = 0$. Given a history h satisfying $\mu(h) \geq m$, a majority of agents have located at the x^*, and the median is obviously the outcome: $x(h|s) = x^*$.

Now assume the hypothesis is true for an arbitrary natural number $k < m$. To show the hypothesis must hold for $k + 1$, consider a history h satisfying $k + 1 + \mu(h) \geq m$, and suppose $\lambda(h) \geq k$ and $\rho(h) \geq k$. Assume without loss of generality that at h, agent i makes her location decision and i is odd. If i locates at $x_i = x^*$, then $k + \mu(h, x_i) \geq m$. Furthermore, since i is odd, we have $\hat{x}^i \leq x^*$, and this implies $\lambda(h, x_i) \geq k - 1$ and $\rho(h, x_i) \geq k$. By hypothesis, this implies $x(h, x_i|s) = x^*$. Since s_i is optimal at h, it follows that agent i's payoff from locating at $s_i(h)$ is at least as high as that from locating at x^*. Noting that $x(h|s) = x(h, s_i(h)|s)$, we therefore have

$$u_i(x(h|s)) \quad \geq \quad u_i(x^*),$$

and by single peakedness, $x(h|s) \leq x^*$.

I claim that $x(h|s) = x^*$. Since i is odd, $i + 1$ is even. Note that $k + 1 + \mu(h, s_i(h)) \geq m$, $\lambda(h, s_i(h)) \geq k - 1$, and $\rho(h, s_i(h)) \geq k$. If $i + 1$ locates at $x_{i+1} = x^*$, then $k+\mu(h, s_i(h), x_{i+1}) \geq m$. Furthermore, $\lambda(h, s_i(h), x_{i+1}) \geq k-1$ and $\rho(h, s_i(h), x_{i+1}) \geq k-1$. By hypothesis, this implies $x(h, s_i(h), x_{i+1}|s) = x^*$. Since s_{i+1} is optimal at $(h, s_i(h))$, it follows that agents $i + 1$'s payoff from locating at $s_{i+1}(h, s_i(h))$ is at least as high as that from locating at x^*, and thus we have

$$u_{i+1}(x(h|s)) \quad \geq \quad u_i(x^*),$$

and by single peakedness, $x(h|s) \geq x^*$. And then $x^* \leq x(h|s) \leq x^*$ yields the claim.

Finally, to prove the theorem, note that the antecedent of the induction hypothesis holds at the initial history by setting $k = m$. Therefore, the unique pure strategy subgame perfect equilibrium outcome from the initial history is the median location x^*, as desired.

3 Counterexample for Non-alternating Protocols

The prominence of the median expressed in Theorem 2 relies on the assumption of alternating protocol. To see this, assume $n = 5$, $X = \Re$, and assume that agents have Euclidean preferences with ideal points as follows:

$$\hat{x}^1 = \hat{x}^2 = -1, \;\; \hat{x}^3 = 0, \;\; \hat{x}^4 = \hat{x}^5 = 2.$$

More precisely, define utilities by $u_i(x) = -|x - \hat{x}^i|$, with ideal points as above. Finally, let the agents locate in the order of their indexing. In this example, the only possible subgame perfect equilibrium outcome is $x = -1$. To see this, consider history $(-1, -1)$, where the first two agents have located at their ideal point. If agent 3, the "median voter," locates at $x_3 = -1$, then this is clearly the outcome. If the agent does not locate at $x_3 = -1$, then the unique equilibrium in the subgame $(-1, -1, x_3)$ is for agents 4 and 5 to locate at their ideal point,

$x_4 = x_5 = 2$, and by the tie-breaking rule assumed, this is the outcome.[2] Thus, agent 3 optimally locates at $x_3 = -1$. Since this is the ideal point of agents 1 and 2, their locations are optimal for them as well.

The logic of this example is instructive. Agents 1 and 2 could ensure the median outcome by locating there, but by locating at $x_1 = x_2 = -1$, agents 4 and 5 become a threat to the median voter, and she is forced to side with agents 1 and 2, the lesser of two evils.

This example does not rely critically on agents 1 and 2 sharing the same ideal point, nor on the assumption that agents 4 and 5 do as well. If ties are broken in favor of the minimum plurality winning alternative, then the example would be modified by a reflection about zero. Note that the example is robust even if ties are broken in favor of the median (if among the plurality winners). The argument for this example has not specified strategies at all histories, which would be a complicated task. But the example carries over if the set X of alternatives consists of just ideal points of the agents—the simplest possible case—in which case there is a subgame perfect equilibrium, and the unique equilibrium outcome is $x = -1$.

4 Sequential Location with Condorcet Consistent Outcomes

In Section 2, we examined the sequential location problem with a plurality rule, where the alternative at which the most agents located was the outcome. Again assuming an odd number of agents with single-peaked preferences over a subset $X \subseteq \Re$ of alternatives, we now allow the agents to play an arbitrary *Condorcet consistent* game: each player i has a set A_i of actions, the outcome from action profile $(a_1, \ldots, a_n) \in A = \prod_i A_i$ is $G(a_1, \ldots, a_n) \in X$, and there is an action profile $(a_1^*, \ldots, a_n^*)$ such that for all $a \in A$,

$$|\{i \in N \mid a_i = a_i^*\}| \geq m \quad \Rightarrow \quad G(a) = x^*.$$

That is, if a majority of agents choose their designated action a_i^*, then the outcome is the median, regardless of the actions of others.

The *sequential location game with Condorcet consistent outcomes* is defined analogous to the basic game: agent 1 chooses any action a_1; after observing this, agent 2 chooses any action a_2; in general, after observing the previous actions $a_1, \ldots, a_{i-1}$, agent i chooses any a_i. Given action profile a, the final outcome is given by $G(a)$, and the payoff of agent i is $u_i(G(a))$. We carry forward notational conventions from the previous section: a strategy for i is a mapping $s_i \colon \prod_{j<i} A_j \to A_i$, and $x(h|s)$ is the outcome dictated by strategy profile s conditional on starting at history h. Note that this captures the basic sequential location game with action sets $A_i = X$ equal to the set of alternatives,

[2] If $x_3 = 2$, then there may be multiple subgame equilibria in the subgame isolated by $(-1, -1, 2)$, but at least one of agents 4 and 5 will choose their ideal point, and the unique outcome remains $\hat{x}^4 = \hat{x}^5 = 2$.

outcome function $G = g$, and a_i^* equal to any weak order with x^* ranked above all other alternatives; indeed, if a majority of agents report a_i^*, then x^* will be the plurality winner, and thus it will be the outcome regardless of the actions of others. We initially assumed that ties were broken in favor of the greatest plurality winning alternative, but the analysis of this section allows for arbitrary tie-breaking rules.

Beyond that, the model now captures general Condorcet consistent voting rules. For example, assuming X is finite, it may be that for each agent i, A_i is the set of weak orders on X, so an action a_i is viewed as a ballot expressing the preferences of the agent. We could define $G(a)$ as a selection from the top cycle, uncovered set, Copeland set, the survivors of sequential Borda elimination, or winners of Black's rule. Another possibility is that agents play an approval voting game, in which each reports a subset of alternatives, with the choice being the alternative belonging to the most ballots. For any such voting rule, note that we can define a_i^* as any weak order in which x^* is ranked above all other alternatives; then if a majority of agents report a_i^*, then x^* will be a Condorcet winner, and it is the outcome regardless of actions of others, as required.

It is a simple matter to extend Theorems 1 and 2 to this more general framework. Assuming action sets are finite, we again have a finite game of perfect information, so existence of pure strategy subgame perfect equilibrium follows immediately.

Theorem 3 *Assume each A_i is finite. The sequential location game with Condorcet consistent outcomes possesses a pure strategy subgame perfect equilibrium.*

The equilibrium characterization also extends to the general case. The proof is a minor modification of the original.

Theorem 4 *The unique pure strategy subgame perfect equilibrium outcome of the sequential location game is the median, x^*.*

Let s be any subgame perfect equilibrium profile. Given arbitrary history $h = (a_1, \ldots, a_{i-1})$, let $\mu(h) = \#\{j < i \mid a_j = x^*\}$ denote the number of agents j who have already chosen a_j^*, and define

$$\begin{aligned} \lambda(h) &= \#\{j \geq i \mid \hat{x}^j < x^*\} \\ \rho(h) &= \#\{j \geq i \mid x^* < \hat{x}^j\} \end{aligned}$$

as the numbers of agents who have not yet located and have ideal points, respectively, strictly less than and strictly greater than the median.

The proof is by induction on k using the same hypothesis as in the proof of Theorem 2:

> For every history h satisfying $k + \mu(h) \geq m$, if $\lambda(h) \geq k - 1$ and $\rho(h) \geq k - 1$, then $x(h|s) = x^*$.

Consider $k = 0$. Given a history h satisfying $\mu(h) \geq m$, a majority of agents j have chosen a_j^*, and the median is the outcome by assumption of Condorcet consistent outcomes: $x(h|s) = x^*$.

Now assume the hypothesis is true for an arbitrary natural number $k < m$. To show the hypothesis must hold for $k+1$, consider a history h satisfying $k + 1 + \mu(h) \geq m$, and suppose $\lambda(h) \geq k$ and $\rho(h) \geq k$. Assume without loss of generality that at h, agent i makes her action choice and i is odd. If i chooses $a_i = a_i^*$, then $k + \mu(h, a_i) \geq m$. Furthermore, since i is odd, we have $\hat{x}^i \leq x^*$, and this implies $\lambda(h, a_i) \geq k - 1$ and $\rho(h, a_i) \geq k$. By hypothesis, this implies $x(h, a_i|s) = x^*$. Since s_i is optimal at h, it follows that agent i's payoff from choosing $s_i(h)$ is at least as high as that from choosing a_i^*. Noting that $x(h|s) = x(h, s_i(h)|s)$, we therefore have

$$u_i(x(h|s)) \quad \geq \quad u_i(x^*),$$

and by single peakedness, $x(h|s) \leq x^*$.

I claim that $x(h|s) = x^*$. Since i is odd, $i+1$ is even. Note that $k + 1 + \mu(h, s_i(h)) \geq m$, $\lambda(h, s_i(h)) \geq k - 1$, and $\rho(h, s_i(h)) \geq k$. If $i+1$ chooses $a_{i+1} = a_i^*$, then $k + \mu(h, s_i(h), a_{i+1}) \geq m$. Furthermore, $\lambda(h, s_i(h), a_{i+1}) \geq k-1$ and $\rho(h, s_i(h), a_{i+1}) \geq k-1$. By hypothesis, this implies $x(h, s_i(h), a_{i+1}|s) = x^*$. Since s_{i+1} is optimal at $(h, s_i(h))$, it follows that agents $i+1$'s payoff from choosing $s_{i+1}(h, s_i(h))$ is at least as high as that from choosing a_{i+1}^*, and thus we have

$$u_{i+1}(x(h|s)) \quad \geq \quad u_i(x^*),$$

and by single peakedness, $x(h|s) \geq x^*$. And then $x^* \leq x(h|s) \leq x^*$ yields the claim.

Finally, to prove the theorem, note that the antecedent of the induction hypothesis holds at the initial history by setting $k = m$. Therefore, the unique pure strategy subgame perfect equilibrium outcome from the initial history is the median location x^*, as desired.

5 Stochastic Sequential Location

Finally, we extend the preceding analysis to allow for mixed strategies and stochastic outcomes. Again assume an odd number of agents, and furthermore assume that each agent's preferences over lotteries on the Borel set $X \subseteq \Re$ are represented by a strictly concave von Neumann-Morgenstern utility function $u_i \colon \Re \to \Re$.[3] The action set of each agent i is the measurable space $(A_i, \mathcal{A}_i)$, and an action profile $a = (a_1, \ldots, a_n)$ now determines a Borel probability measure $\Gamma(\cdot|a)$ on X. To ensure that expected payoffs from mixed strategies are well-defined, assume that for all Borel $Y \subseteq X$, the function $\Gamma(Y|a)$ is measurable with

[3]More precisely, agent i's preferences over lotteries on X are represented, via expected utility, by the restriction of a strictly concave function u_i to X.

respect to the product measure $\mathcal{A} = \otimes_i \mathcal{A}_i$. The game is *Condorcet consistent* if there is an action profile $(a_1^*, \ldots, a_n^*)$ such that for all $a \in A$,

$$|\{i \in N \mid a_i = a_i^*\}| \geq m \quad \Rightarrow \quad \Gamma(\{x^*\}|a) = 1.$$

That is, if a majority of agents choose their designated action a_i^*, then the outcome is the median with probability one, regardless of others' actions.

The *sequential location game with Condorcet consistent stochastic outcomes* is defined in the expected way, the only difference from the previous section being that actions $a_1, \ldots, a_n$ determines a lottery $\Gamma(\cdot|a)$, so that the payoff of agent i is the expected utility $\int_x u_i(x)\Gamma(dx|a)$. We now modify notational conventions to allow for stochastic outcomes and mixed strategies. In particular, a behavioral strategy for agent i is a mapping $\sigma_i\colon \prod_{j<i} A_j \to \Delta(A_i)$, with codomain equal to the set $\Delta(A_i)$ of probability measures on $(A_i, \mathcal{A}_i)$. Starting from any history h, a mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ determines a lottery $\xi(\cdot|h, \sigma)$ over alternatives.

It is a simple matter to extend Theorems 3 and 4 to this more general framework. Assuming action sets are finite, we no longer have a game of perfect information, but existence of pure strategy subgame perfect equilibrium follows from a simple backward induction argument.

Theorem 5 *Assume each A_i is finite. The sequential location game with Condorcet consistent stochastic outcomes possesses a pure strategy subgame perfect equilibrium.*

The equilibrium characterization also extends to the general case. The proof is a minor modification of the original.

Theorem 6 *The unique mixed strategy subgame perfect equilibrium outcome of the sequential location game with Condorcet consistent stochastic outcomes is the median, x^*, with probability one.*

Let σ be any mixed strategy subgame perfect equilibrium profile. Given arbitrary history $h = (a_1, \ldots, a_{i-1})$, let $\mu(h) = \#\{j < i \mid a_j = x^*\}$ denote the number of agents j who have already chosen a_j^*, and define

$$\begin{aligned} \lambda(h) &= \#\{j \geq i \mid \hat{x}^j < x^*\} \\ \rho(h) &= \#\{j \geq i \mid x^* < \hat{x}^j\} \end{aligned}$$

as the numbers of agents who have not yet located and have ideal points, respectively, strictly less than and strictly greater than the median.

The proof is by induction on k after a slight modification of the hypothesis in the proof of Theorem 2:

> For every history h satisfying $k + \mu(h) \geq m$, if $\lambda(h) \geq k - 1$ and $\rho(h) \geq k - 1$, then $\xi(\{x^*\}|h, s) = 1$.

Consider $k = 0$. Given a history h satisfying $\mu(h) \geq m$, a majority of agents j have chosen a_j^*, and the median is the outcome with probability one by assumption of Condorcet consistent outcomes: $\xi(\{x^*\}|h, s) = 1$.

Now assume the hypothesis is true for an arbitrary natural number $k < m$. To show the hypothesis must hold for $k + 1$, consider a history h satisfying $k + 1 + \mu(h) \geq m$, and suppose $\lambda(h) \geq k$ and $\rho(h) \geq k$. Assume without loss of generality that at h, agent i makes her action choice and i is odd. If i chooses $a_i = a_i^*$, then $k + \mu(h, a_i) \geq m$. Furthermore, since i is odd, we have $\hat{x}^i \leq x^*$, and this implies $\lambda(h, a_i) \geq k - 1$ and $\rho(h, a_i) \geq k$. By hypothesis, this implies $\xi(\{x^*\}|(h, a_i), \sigma) = 1$. Since σ_i is optimal at h, it follows that agent i's payoff from mixing according to $\sigma_i(\cdot|h)$, which determines the distribution over outcomes $\xi(\cdot|h, \sigma)$, is at least as high as that from choosing a_i^*. Letting $\overline{x}$ denote the mean of this distribution, we have

$$u_i(\overline{x}) \;\geq\; \int_x u_i(x)\xi(dx|h, \sigma) \;\geq\; u_i(x^*), \tag{1}$$

where the first inequality follows from concavity of u_i. By single peakedness, $\overline{x} \leq x^*$.

I claim that $\xi(\{x^*\}|h, s) = 1$. Since i is odd, $i + 1$ is even. Note that for every $a_i \in A_i$, we have $k + 1 + \mu(h, a_i) \geq m$, $\lambda(h, a_i) \geq k - 1$, and $\rho(h, a_i) \geq k$. If $i + 1$ chooses $a_{i+1} = a_{i+1}^*$, then $k + \mu(h, a_i, a_{i+1}) \geq m$. Furthermore, $\lambda(h, a_i, a_{i+1}) \geq k - 1$ and $\rho(h, a_i, a_{i+1}) \geq k - 1$. By hypothesis, this implies $\xi(\{x^*\}|h, a_i, a_{i+1}, \sigma) = 1$. Since σ_{i+1} is optimal at (h, a_i), it follows that agent $i+1$'s payoff from choosing $\sigma_{i+1}(\cdot|h, a_i)$ is at least as high as that from choosing a_{i+1}^*. Note that $\xi(\cdot|(h, a_i), \sigma)$ is the lottery over outcomes at history (h, a_i) when agent $i + 1$ mixes according to $\sigma_{i+1}(\cdot|h, a_i)$, and let $\overline{x}(a_i)$ denote the mean of $\xi(\cdot|(h, a_i), \sigma)$. Then we have

$$u_{i+1}(\overline{x}(a_i)) \;\geq\; \int_x u_{i+1}(x)\xi(dx|(h, a_i), \sigma) \;\geq\; u_{i+1}(x^*),$$

where the first inequality follows from concavity of u_{i+1}. By single peakedness, $x^* \leq \overline{x}(a_i)$. Writing $\overline{x}$ as

$$\begin{aligned}\overline{x} &= \int_{a_i}\int_x x\xi(dx|(h, a_i), \sigma)\sigma_i(da_i|h) \\ &= \int_{a_i} \overline{x}(a_i)\sigma_i(da_i|h),\end{aligned}$$

we conclude that $x^* \leq \overline{x}$.

Combining the inequalities $x^* \leq \overline{x} \leq x^*$, we conclude that $x^* = \overline{x}$. That is, the mean of the lottery $\xi(\cdot|h, \sigma)$ is equal to the median. To see that the lottery is actually degenerate on the median, note that equation (1) becomes

$$u_i(\overline{x}) \;=\; \int_x u_i(x)\xi(dx|h, \sigma).$$

But if $\xi(\cdot|h,\sigma)$ were not degenerate on x^*, then strict concavity of u_i would dictate

$$u_i(\overline{x}) \quad > \quad \int_x u_i(x)\xi(dx|h,\sigma),$$

a contradiction. This establishes the claim.

Finally, to prove the theorem, note that the antecedent of the induction hypothesis holds at the initial history by setting $k = m$. Therefore, the unique mixed strategy subgame perfect equilibrium outcome from the initial history is the median location x^* with probability one, as desired.

References

[1] T. Feddersen (1992) "A Voting Model Implying Duverger's Law and Positive Turnout," *American Journal of Political Science,* 36: 938–962.