Subgame-Perfect Equilibrium in Games with Almost Perfect Information: Dispensing with Public Randomization

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Abstract

Harris, Reny, and Robson (*Econometrica,* 1995) add a public randomization device to dynamic games with almost perfect information to ensure existence of subgame perfect equilibria (SPE). We show that when nature's moves are atomless in the original game, SPE obtained via this channel can be "de-correlated" to produce a payoff-equivalent SPE of the original game; thus, public randomization is in a sense without loss of generality. A corollary is He and Sun's (2016) existence result for SPE that dispenses with public randomization, which in turn yields an equilibrium existence result for stochastic games with weakly continuous state transitions.

1 Introduction

A seminal result of Harris, Reny, and Robson (1995) (henceforth, HRR) ensures existence of subgame perfect equilibrium (SPE) in dynamic games with almost perfect information by augmenting such games with a public randomization device. That is, they assume that in addition to nature's moves in the original game, players observe a uniformly distributed, payoff-irrelevant public signal in every stage. This convexifies equilibrium probabilities over continuation paths in the augmented game and allows them to use limiting arguments to deduce existence of a SPE; in the original game, their construction corresponds to a generalized strategy profile in which players' actions are marked by a form of correlation. We focus on the subclass of games for which nature's moves are atomless, and we establish that in such *games with atomless moves by nature,* one can dispense with public signals in HRR's result: each SPE obtained by augmenting the original game to allow public signals can be "de-correlated" to produce a payoff-equivalent SPE of the original game involving no correlation or public signals. Therefore, for a large class of dynamic games, public randomization is without loss of generality. As a corollary, we provide a route to existence of SPE without public randomization which is relatively short in comparison to that recently taken by He and Sun (2016).

The class of games with atomless moves by nature is general enough to capture many applications of interest; in particular, it subsumes stochastic games with weakly continuous, atomless state transitions—in some respects, going well beyond the analysis in the classical literature. More formally, a stochastic game is played among n players; in each period, a state variable z is publicly observed; players then simultaneously choose actions y_i; and given the state z and action profile $y = (y_1, \ldots, y_n)$, next period's state is drawn from a transition probability $\mu(\cdot|y, z)$. This process is repeated in discrete time over an infinite horizon. In their classic analysis, Mertens and Parthasarathy (2003) do not assume that states are atomlessly distributed, but their conditions for existence of SPE impose the strong condition of norm-continuity on the state transition. Letting Z denote the set of states and $\Delta(Z)$ the set of probability measures over states, the assumption of Mertens and Parthasarathy (which is standard in the literature) is that the mapping $(y, z) \mapsto \mu(\cdot|y, z)$ is jointly measurable and continuous in y with the total variation norm on $\Delta(Z)$. This norm-continuity assumption precludes the possibility that states have a component that depends in a deterministic way on a continuous action.

In contrast, our de-correlation approach does not impose norm-continuity; rather, following HRR, we require only that state transitions are weakly continuous, i.e., as a function of action-state pairs (y, z), the probability measure $\mu(\cdot|y, z)$ over next period's state is continuous with the weak* topology on $\Delta(Z)$. Thus, a component of the state is permitted to vary in a deterministic, continuous way with respect to states and actions. Our analysis is also more general than that of Mertens and Parthasarathy in that it permits the players' payoffs and nature's moves to depend on the entire history of play, but our payoff structure is less general in one respect: this dependence has to be jointly continuous, whereas Mertens and Parthasarathy allow for payoffs and moves by nature that are continuous in the current action profile but merely measurable in the current state.

Example 1 *Strategic Growth and Autocorrelated Shocks.* In an infinite-horizon, discrete-time model of growth with two agents, let k_i represent the capital stock for agent $i = 1, 2$, and let c_i be the level of consumption of agent i. Given capital stock levels k_1 and k_2 at the end of the previous period and realized depreciation rates $r_1, r_2 \in (0, 1)$, the agents simultaneously choose consumption levels c_i subject to $0 \leq c_i \leq F_i(k_1, k_2) + (1 - r_i)k_i$, reflecting externalities in production given by F_i and a reduction in capital stocks due to depreciation. Consumption choices in the current period, in turn, leave capital stock levels at $k'_i = F_i(k_1, k_2) + (1 - r_i)k_i - c_i$. Assume that: the production functions F_i are bounded and jointly continuous; utility from consumption is bounded, continuous, and discounted over time; and that depreciation rates between periods are subject to random shocks, $r'_i = r_i + \epsilon_i$, where the shocks (ϵ_1, ϵ_2) have density $f(\epsilon_1, \epsilon_2 | c_1, c_2, k_1, k_2, r_1, r_2)$ that is jointly continuous in consumption levels, capital stocks, and shocks, thereby permitting general correlation across time.[1] This is an example of a stochastic game in which the state variable (k_1, k_2, r_1, r_2), has a deterministic component, namely (k_1, k_2), that depends on continuous actions, namely the consumption levels of the agents. As such, existence of SPE in this strategic version of the standard growth model does not follow from Mertens and Parthasarathy (2003).[2] Because the growth model is a game of almost perfect information, the result of HRR implies that the augmented game does possess a SPE, and thus the original game admits a generalized equilibrium strategy profile in which the agents' choices exhibit a form of correlation. Because depreciation rates are non-atomically distributed, however, the strategic growth model belongs to the class of games with atomless moves by nature, and our result implies that the game has a payoff-equivalent SPE that does not rely on correlation or the presence of payoff irrelevant "sunspots." □

Our analytical approach proceeds as follows. Given a game with atomless moves by nature, HRR show that there is a SPE in the game augmented with payoff-irrelevant public signals in each period. For any such SPE strategy profile, we exploit non-atomicity to "de-correlate" the SPE in each period via repeated application of Mertens' (2003) "measurable 'measurable choice' theorem." In the first period, there is no previous public signal, so the players' actions in the SPE are uncorrelated. In the second and later periods of the augmented game, however, players can condition on the public signal in the first period. We use Mertens' theorem to replace conditioning on the first-period

[1] To bound action sets of the agents, we assume there is $\underline{r} > 0$ such that $r_i \in [\underline{r}, 1)$.

[2] Dutta and Sundaram (1992) study a version of a strategic growth model allowing for weakly continuous transitions as well. They establish existence of a stationary Markov perfect equilibrium under strong concavity/differentiability assumptions.

public signal (which is payoff irrelevant) with conditioning on nature's move (which is non-atomically distributed) in the first period of the original game without public signals. The key is to do so in a way that preserves continuation payoffs in the first period, thereby maintaining equilibrium conditions on the players' first-period choices.

This step renders the first-period public signal irrelevant, but in the third and later periods of the augmented game, players SPE strategies can still condition on the public signal in the second period. We then use Mertens' theorem to replace conditioning on the second-period public signal with conditioning on nature's moves in the original game, again preserving payoffs in the first two periods and thereby maintaining equilibrium conditions. By repeating this procedure, we inductively construct a new profile of strategies such that players' actions after any history do not depend on the previous public signals, and such that the continuation payoffs from the original SPE are preserved. As a consequence, the new profile of strategies is a SPE of the original game that is payoff-equivalent to the SPE of the augmented game; in this way, we dispense with HRR's public randomization for a large class of games of interest.

Our existence corollary is consistent with an example of Luttmer and Mariotti (2003), in which there is no SPE in a game of perfect information with moves by nature that do not always have an atomless component. It is also obtained in Proposition 1 of He and Sun (2016), who establish existence of SPE without resorting to public signals under the assumption that nature's moves are atomless; their focus is on existence, and they do not provide results on "de-correlation" of equilibria in HRR's framework.[3] Methodologically, this de-correlation approach extends our previous work on dynamic games, where payoff correspondences with convex values are often needed on technical grounds but where correlation can be difficult to justify on economic grounds. Similar methods are used in Duggan (2013) to prove existence of stationary Markov perfect equilibria in noisy stochastic games: a correlated equilibrium is deduced from Nowak and Raghavan (1992), and a version of Mertens' theorem is used to construct a payoff-equivalent equilibrium in which correlation is replaced by conditioning on the noise component of the state. Barelli and Duggan (2014) use this approach to prove that for every stationary correlated equilibrium in the Nowak-Raghavan sense, there is a payoff-equivalent stationary semi-Markov equilibrium obtained by replacing correlation with

[3]In addition, He and Sun (2016) provide results that apply to a different class of dynamic games, dropping the continuity assumption on state variables in exchange for stronger continuity conditions on the moves of Nature.

conditioning on the state and actions in the previous period.[4]

The remainder of the paper is organized as follows. In Section 2, we set forth the framework of games of almost perfect information used by HRR, and we specialize this to games with atomless moves by nature. In Section 3, we present our main result, which shows that we can dispense with public randomization in any game with atomless moves by nature. In Section 4, we apply our result to stochastic games. In Section 5, we prove our de-correlation theorem, and Appendix A contains the proof of Lemma 1.

2 Games with Atomless Moves by Nature

We adopt the framework of HRR but for two inconsequential modifications. First, we omit the set Y_0 of starting points of the game, which was used by HRR to establish upper hemi-continuity of equilibria, whereas our focus is on de-correlation of equilibria. Second, we consider a representation of payoff functions with stage payoffs and geometric discounting, whereas HRR use continuous payoff functions defined on infinite histories. As our stage payoff functions are defined for all histories up to the current period, our representation of payoffs is equivalent to HRR's, as we show in Lemma 1, below.

Data The data of the HRR framework with our two modifications are as follows.

- There is a finite, nonempty set $N = \{1, \ldots, n\}$ of active players, indexed by i or j, and a passive player, "nature," denoted 0. Let $N_0 = N \cup \{0\}$.
- There is a countably infinite set $T = \{1, 2, \ldots\}$ of time periods, indexed by s or t. Let $T_0 = T \cup \{0\}$.
- For each $t \in T_0$ and each $i \in N$, there is a nonempty, complete, separable metric space $Y_{t,i}$ of actions denoted $y_{t,i}$. Set $Y_t = \times_{i \in N} Y_{t,i}$, with elements $y_t = (y_{t,i})_{i \in N}$, and set $Y = (Y_t)_{t \in T_0}$.
- For each $t \in T_0$, there is a nonempty, complete, separable metric space Z_t of nature's actions denoted z_t. Set $Z = (Z_t)_{t \in T_0}$.

[4]These techniques are also applied in Barelli and Duggan (2015a) to show that stationary Markov perfect equilibria in noisy stochastic games are payoff-equivalent to equilibria that select only extreme points of equilibrium payoffs in induced games, and they are applied to purification of Bayes Nash equilibria in Barelli and Duggan (2015b).

- For each $t \in T$, there is a nonempty, closed subset $X_t \subseteq \times_{s=1}^{t}(Y_s \times Z_s)$ of possible t-period histories, denoted x_t. Designate a fixed $x_0 \in Y_0 \times Z_0$ as the initial history at which the game begins, and set $X_0 = \{x_0\}$. Finally, set $X = (X_t)_{t \in T_0}$.

- For each $t \in T$ and each $i \in N$, there is continuous correspondence $A_{t,i}\colon X_{t-1} \rightrightarrows Y_{t,i}$ of feasible actions with nonempty, compact values. In addition, there is a continuous correspondence $A_{t,0}\colon X_{t-1} \rightrightarrows Z_t$ with nonempty, closed values. Set $A_t = \times_{i \in N_0} A_{t,i}$ and $A = (A_t)_{t \in T}$. Consistent with the interpretation of X_t as the set of possible t-period histories, we assume that for each $t \in T$, $X_t = \text{graph}(A_t)$.

- For each $t \in T$, there is a continuous mapping $f_{t,0}\colon X_{t-1} \to \Delta(Z_t)$, where $\Delta(\cdot)$ represents the set of Borel probability measures endowed with the weak* topology. Assume that for each $x_{t-1} \in X_{t-1}$, the support of $f_{t,0}(x_{t-1})$ is contained in $A_{t,0}(x_{t-1})$. Set $f_0 = (f_{t,0})_{t \in T}$.

- For each $t \in T$ and each $i \in N$, there is a bounded, continuous stage payoff function $u_{t,i}\colon X_t \to \mathbb{R}$. Set $u_t = (u_{t,i})_{i \in N}$ and $u = (u_t)_{t \in T}$.

- For each $i \in N$, there is a discount factor $\delta_i \in [0, 1)$. Let $\delta = (\delta_i)_{i \in N}$.

These elements describe a *game of almost perfect information* (or simply, a *game*), denoted $G = (N_0, T_0, Y, Z, X, x_0, A, f_0, u, \delta)$. Given any history x_t, we define the *subgame* at x_t, denoted $G(x_t)$, in the obvious way.

Stage payoffs Let X_∞ denote the set of infinite histories $x \in \times_{t \in T_0}(Y_t \times Z_t)$ such that for each $t \in T_0$, we have $x_t \in X_t$. We endow this with the product topology and the measurable structure generated by finite cylinder sets. Then $\Delta(X_\infty)$ is the set of probability measures ξ on infinite histories. Assume that for all $i \in N$, streams of stage payoffs are *summable,* in the sense that

$$\sup_{x \in X_\infty} \sum_{t \in T} \delta_i^{t-1} |u_{t,i}(x_t)| \quad < \quad \infty.$$

An alternative formulation, used by HRR, is to define a bounded, continuous payoff function $u_i\colon X_\infty \to \mathbb{R}$ over infinite histories for each player. Given the prevalence of geometric discounting in applications, the next lemma, which establishes that the two payoff formulations are equivalent, may be of independent interest.

Lemma 1 *For all $\delta_i \in [0,1)$ and all bounded, continuous, summable mappings $u_{t,i}\colon X_t \to \mathbb{R}$, $t \in T$, there is a bounded, continuous mapping $u_i\colon X_\infty \to \mathbb{R}$ such that for all $\xi, \xi' \in \Delta(X_\infty)$, we have*

$$\int_x u_i(x)d\xi > \int_x u_i(x)d\xi' \Leftrightarrow \int_x \sum_{t\in T} \delta_i^{t-1} u_{t,i}(x_t)d\xi > \int_x \sum_{t\in T} \delta_i^{t-1} u_{t,i}(x_t)d\xi'. \quad (1)$$

Conversely, for every bounded, continuous mapping $u_i\colon X_\infty \to \mathbb{R}$ and every $\delta_i \in (0,1)$, there are bounded, continuous, summable mappings $u_{t,i}\colon X_t \to \mathbb{R}$, $t \in T$, such that (1) holds for all $\xi, \xi' \in \Delta(X_\infty)$.

Strategies A *strategy* for player $i \in N$ is a sequence $f_i = (f_{t,i})_{t\in T}$ of Borel measurable mappings $f_{t,i}\colon X_{t-1} \to \Delta(Y_{t,i})$ such that $f_{t,i}(A_{t,i}(x_{t-1})|x_{t-1}) = 1$ for each $t \in T$ and each $x_{t-1} \in X_{t-1}$. A *strategy profile* is an ordered n-tuple $f = (f_i)_{i\in N}$. Let F_i denote the set of strategies for player i, and let $F = \times_{i\in N} F_i$ denote the set of strategy profiles.

Dynamic payoffs Given $f \in F$, $t \in T$, $i \in N$, and $x_{t-1} \in X_{t-1}$, player i's *dynamic payoff* is the expected discounted payoff defined recursively by

$$\begin{aligned} U_{t,i}(x_{t-1}, f) \;=\; & \int_y \Big[\int_z [u_{t,i}(x_{t-1}, y, z) + \\ & \delta_i U_{t+1,i}((x_{t-1}, y, z), f)] f_{t,0}(x_{t-1})(dz) \Big] (\otimes_{i\in N} f_{t,i}(x_{t-1}))(dy). \end{aligned}$$

This construction determines a mapping $U_t\colon X_{t-1} \times F \to \mathbb{R}^n$, and for each f, the dynamic payoff $U_t(x_{t-1}, f)$ is Borel measurable as a function of x_{t-1}.

Equilibrium A *subgame perfect equilibrium* is a strategy profile f such that for each $t \in T$, each $i \in N$, each $x_{t-1} \in X_{t-1}$, and each $\tilde{f}_i \in F_i$,

$$U_{t,i}(x_{t-1}, f) \;\geq\; U_{t,i}(x_{t-1}, (\tilde{f}_i, f_{-i})).$$

Clearly, a strategy profile f is a subgame perfect equilibrium of G if and only if for every history x_t, the strategies restricted to this subgame again form a subgame perfect equilibrium of the subgame $G(x_t)$.

Auxiliary games Given a game G, any $t \in T$, any $x_{t-1} \in X_{t-1}$, and any bounded, Borel measurable function $V\colon Y_t \times Z_t \to \mathbb{R}^n$, the *auxiliary game* induced by V at t given x_{t-1} is the strategic form game $G_t(x_{t-1}, V) = (N, (A_{t,i}(x_{t-1}))_{i\in N}, (\pi_i)_{i\in N})$ with player set N, strategy sets $A_{t,i}(x_{t-1})$ with mixed strategies $\sigma_i \in \Delta(A_{t,i}(x_{t-1}))$, and payoff functions

$$\pi_i(y) \;=\; \int_z [u_{t,i}(x_{t-1}, y, z) + \delta_i V_i(y, z)] f_{t,0}(x_{t-1})(dz).$$

Here, the values $V(y,z)$ stand in for the players' expected future payoffs given action profile (y,z); note that $U_{t+1}((x_{t-1},\cdot),f)$ in the definition of subgame perfect equilibrium plays the same role as $V(\cdot)$ in the definition of auxiliary game.

One-shot deviation principle Let $N_t(x_{t-1},V)$ denote the set of mixed strategy Nash equilibria of the auxiliary game $G_t(x_{t-1},V)$, i.e., $\sigma=(\sigma_i)_{i\in N}\in N_t(x_{t-1},V)$ if and only if for each $i\in N$ and each $y_i'\in A_{t,i}(x_{t-1})$, we have

$$\int_y \pi_i(y)(\otimes_{j\in N}\sigma_j)(dy) \quad \geq \quad \int_{y_{-i}} \pi_i(y_i', y_{-i})(\otimes_{j\neq i}\sigma_j)(dy_{-i}).$$

By the one-shot deviation principle, a strategy profile f is a subgame perfect equilibrium of G if and only if for each $t\in T$ and each $x_{t-1}\in X_{t-1}$, the mixed strategy profile $(f_{t,i}(x_{t-1}))_{i\in N}$ is a mixed strategy Nash equilibrium of the auxiliary game $G_t(x_{t-1},U_{t+1}((x_{t-1},\cdot),f))$.

Extended games Given a game $G=(N_0,T_0,Y,Z,X,x_0,A,f_0,u,\delta)$, the *extension* of G is the game $\hat{G}$ that differs from G in that for each $t\in T$, (i) for each $i\in N$, $\hat{Y}_{t,i}=Y_{t,i}$, (ii) $\hat{Z}_t=Z_t\times[0,1]$, (iii) possible t-period histories are as in the original game with the addition of a signal $\omega_s\in[0,1]$ in each period $t\geq 1$, i.e.,

$$\hat{X}_t \quad = \quad \{((y_0,z_0),(y_1,z_1,\omega_1),\ldots,(y_t,z_t,\omega_t)) \mid x_t\in X_t, (\omega_s)_{s=1}^t\in[0,1]^t\},$$

writing elements as $\hat{x}_t=(x_t,\omega_1,\ldots,\omega_t)$, (iv) for each $\hat{x}_t\in\hat{X}_{t-1}$, the marginal of $\hat{f}_{t,0}(\hat{x}_{t-1})$ on Z_t is $f_{t,0}(x_{t-1})$, and ω_t is drawn independently from the uniform distribution on $[0,1]$, that is, $\hat{f}_{t,0}(\hat{x}_{t-1})=f_{t,0}(x_{t-1})\otimes\lambda$, where λ is the uniform measure on $[0,1]$, (v) for each $i\in N_0$ and each $\hat{x}_{t-1}\in\hat{X}_{t-1}$, we have $\hat{A}_{t,i}(\hat{x}_{t-1})=A_{t,i}(x_{t-1})$, and (vi) for each $i\in N$ and each $t\in T$, we have $\hat{u}_{t,i}(\hat{x}_t)=u_{t,i}(x_t)$. Thus, ω_t is a payoff-irrelevant public signal. Without risk of confusion, given $\hat{V}\colon\hat{Y}_t\times\hat{Z}_t\to\mathbb{R}^n$, we shall use $\hat{G}_t(\hat{x}_{t-1},\hat{V})$ to denote the corresponding auxiliary game at period $t\in T$, and $\hat{U}_t(\hat{x}_{t-1},\hat{f})$ to denote dynamic payoffs at $t\in T$.

Games with atomless moves by nature A game G is a *game with atomless moves by nature* if for all $t\in T$, all $x_{t-1}\in X_{t-1}$, and all $(y_t,z_t)\in A_t(x_{t-1})$, the following holds:

$$\begin{array}{c} f_{t,0}(x_{t-1}) \text{ has} \\ \text{an atom} \end{array} \quad \Rightarrow \quad \begin{array}{c} \text{for all } i\in N,\ |A_{t+1,i}(x_t)|=1 \\ \text{and } f_{t+1,0}(x_t) \text{ is atomless.} \end{array}$$

Obviously, this condition holds if $f_{t,0}(x_{t-1})$ is atomless for all $t\in T$ and all $x_{t-1}\in X_{t-1}$, but it allows for the possibility that the distribution of nature's actions has an atom, as long as the active players have only trivial moves and nature's moves are atomless in the next period. This definition admits games in which the active players and nature move in alternate periods, a fact that allows us to apply our results to the class of stochastic games, in Section 4.

3 Dispensing with Public Randomization

Theorem 4 of HRR establishes the following equilibrium existence result:

Theorem 1 (Harris, Reny, and Robson) *For each game G, the extension $\hat{G}$ admits a subgame perfect equilibrium.*

HRR consider several approaches to establishing existence of subgame perfect equilibrium in the original game G, without public randomization. They note that their theorem implies existence of subgame perfect equilibrium in games with finite action sets and in zero-sum games. In these classes of games, nature does not play an important role.[5] Using the HRR theorem, we will take another route to dispense with public randomization that exploits the role of nature; specifically, we assume that nature's actions are realized from atomless distributions. The main result of this paper is the following theorem: for a game with atomless moves by nature, every subgame perfect equilibrium $\hat{f}$ of the extended game can be "de-correlated" to produce a subgame perfect equilibrium f of the original game that preserves the players' expected discounted payoffs from all action profiles at the initial history. Formally, we say f is *payoff-equivalent* to $\hat{f}$ if for all $i \in N$ and all $y \in \times_{i \in N} A_{1,i}(x_0)$, we have

$$\begin{aligned} &\int_z [u_{1,i}(x_0, y, z) + \delta_i U_{1,i}(x_0, y, z, f)] f_{1,0}(x_0)(dz) \\ &\quad = \int_z \Big[u_{1,i}(x_0, y, z) + \delta_i \int_\omega \hat{U}_{1,i}(x_0, y, z, \omega, \hat{f}) \lambda(d\omega) \Big] f_{1,0}(x_0)(dz), \end{aligned}$$

where $U_{1,i}$ is the dynamic payoff at time 1 in the original game and $\hat{U}_{1,i}$ is the dynamic payoff at time 1 in the extended game.

[5]HRR also claim that existence of SPE in games of perfect information is a consequence of their main result, but Luttmer and Mariotti (2003) provide a counterexample to this claim.

Theorem 2 *In a game G with atomless moves by nature, every subgame perfect equilibrium of the extended game $\hat{G}$ is payoff equivalent to a subgame perfect equilibrium of G.*

Thus, when nature's moves are already atomless, the addition of uniformly distributed payoff-irrelevant signals is without loss of generality, and can be interpreted simply as a technical device in HRR. From Theorems 1 and 2, we obtain the obvious existence result, which is also obtained in Theorem 1 of He and Sun (2016).

Corollary 1 *Every game with atomless moves by nature admits a subgame perfect equilibrium.*

Here, we present the main ideas of the proof of Theorem 2, which is located in Section 5. By the theorem of HHR, the extended game $\hat{G}$ has a subgame perfect equilibrium $\hat{f}$. The proof of Theorem 2 consists of transforming $\hat{f}$ to a subgame perfect equilibrium of G via a "triangular" sequence $(f^{t-1})_{t\in T}$ of strategy profiles of the following form.

$$\begin{aligned} f^0 &= (\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4, \hat{f}_5, \ldots) \\ f^1 &= (f_1^0, f_2^1, f_3^1, f_4^1, f_5^1, \ldots) \\ f^2 &= (f_1^0, f_2^1, f_3^2, f_4^2, f_5^2, \ldots) \\ &\vdots \end{aligned}$$

That is, the first profile f^0 is just $\hat{f}$. To define f^1, we leave period 1 strategies unchanged, and we update strategies in all periods $t = 2, 3, \ldots$. To define f^2, we leave strategies in the first two periods unchanged, and we update strategies in all periods $t = 3, 4, \ldots$, and so on. At the end of the construction, we arrive at a SPE $f^\infty = (f_t^{t-1})_{t\in T}$ of the extended game. Each component f_t^{t-1} is constructed so as to be independent of $\omega_1, \ldots, \omega_{t-1}$ and to preserve payoffs in all periods prior to t. With this profile in hand, since players' actions do not depend on the payoff-irrelevant public signals, we then define the SPE f of the game with atomless moves by nature in the obvious way by, for each period t, projecting f^∞ onto the set X_{t-1} of histories of the original game. The formal proof is somewhat complicated by the fact that a game with atomless moves by nature does allow, under specific circumstances, for atomic moves by nature; let us now convey the approach of the proof for the simpler case in which for every period t and every history $x_t \in X_t$, nature's move $f_{t,0}(x_t)$ is atomless – the complete approach follows on similar lines.

The idea of the construction of f^1 is to select, in a measurable way, a SPE of the subgame $\hat{G}(x_1, \omega_1)$ for each $(x_1, \omega_1) \in X_1 \times [0, 1]$, and to do so in a way that is independent of ω_1, yet preserves the active players' expected discounted payoffs from every action profile y_1 in period 1. This is possible for two reasons. First, the public signal ω_1 is payoff irrelevant, so that the set of subgame perfect equilibria in subgame $\hat{G}(x_1, \omega_1)$ is in fact independent of ω_1. Second, the assumption that nature's move $f_1(x_0)$ is atomless allows us to use Lyapunov's theorem to replace conditioning on the period one realization ω_1 of the public signal with conditioning on the realization of nature's action z_1 in period one. Several technical challenges must be addressed in the process. In the "de-correlation" step, we select subgame perfect equilibrium payoffs as a function of z_1 to preserve payoffs from a given action profile y_1, but we must then "sew together" these payoff selections as a measurable function of y_1; for this, we use the "measurable 'measurable choice' theorem of Mertens (2003). Once this is done, we must translate the selection of equilibrium payoffs in each subgame $\hat{G}(x_1, \omega_1)$ into a measurable strategy profile in the extended game $\hat{G}$, and for this we rely on Proposition 10 of HRR. As a result, we dispense with dependence on the public signal ω_1 of moves in periods $t \geq 2$ in a way that preserves payoffs from every action profile y_1 in period 1, and thus the originally specified moves $\hat{f}_1(x_0) = f_1^0(x_0)$ form a Nash equilibrium of the induced game $\hat{G}_1(x_0, \hat{U}_2((x_0, \cdot), f^1))$ in the first period; that is, f^1 is a SPE of the extended game in which payoffs in the first period are maintained and actions in future periods are not conditioned on the public signal ω_1.

We then construct f^2 by selecting, in a measurable way, subgame perfect equilibrium payoffs of the subgames $\hat{G}(x_2, \omega_1, \omega_2)$ for each $(x_2, \omega_1, \omega_2) \in X_2 \times [0, 1]^2$, substituting conditioning on z_2 for conditioning on ω_2 in periods $t \geq 3$ and preserving expected payoffs in period 2. We leave moves in the first two periods unchanged, so that $f_1^2 = f_1^1 = \hat{f}_1$ and $f_2^2 = f_2^1$, and we construct measurable strategies $f_3^2, f_4^2, \ldots$ such that f^2 is a SPE of the extended game in which payoffs in the first two periods are maintained and actions in periods $t \geq 3$ are not conditioned on the public signals ω_1 and ω_2. The construction continues in this way, generating a sequence $f^0, f^1, f^2, \ldots$, and we construct a SPE $f^\infty = (f_t^{t-1})_{t \in T}$ of the extended game by selecting moves along the "diagonal" of this sequence. Since f_t^{t-1} is independent of public signals prior to period t, we project each f_t^{t-1} onto X_{t-1} to produce a SPE f of the original game with atomless moves by nature that is payoff-equivalent to $\hat{f}$.

More formally, the proof of Theorem 2 consists of five steps, taking as given a SPE f^{t-1} that it is independent of public signals in the first $t-1$ periods, and that preserves the payoffs of f_s^{t-2} in all periods $s < t$. In Step

1, for each $x_{t-1} \in X_{t-1}$ and $y \in \times_{i \in N} A_{t,i}(x_{t-1})$, we construct a selection of SPE payoffs, as a function of $z \in A_{t,0}(x_{t-1})$, from the subgames starting at (x_{t-1}, y, z). This selection does not depend on public signals in period t or earlier, and it preserves expected discounted payoffs in period t (and thus in earlier periods as well), where the expectation is taken with respect to nature's move in period t. Hence, the "correlation" captured by the dependency on the period t public signal is substituted for conditioning on nature's move in that period.[6] In Step 2, we sew together the selection found in Step 1 to obtain a selection that is measurable in all three variables, (x_{t-1}, y, z). In Step 3, we use Proposition 10 of HRR to construct a SPE f^t of $\hat{G}$ that generates the payoffs from Step 2. In Step 4, we verify that f^t is independent of public signals and preserves payoffs in the first t periods. Iterating this argument, we arrive at the sequence $f^0, f^1, f^2, \ldots$ of SPE profiles in $\hat{G}$. Finally, in Step 5, we construct the SPE $f^\infty = (f_t^{t-1})_{t \in T}$ of the extended game, and we project each f_t^{t-1} onto X_{t-1} to produce the desired SPE f of G.

4 Application to Stochastic Games

A classical (discounted) *stochastic game* is a dynamic game played in discrete time among n players such that: in each period, a state $z \in Z$ is publicly observed; each player i chooses an action y_i from a feasible set $A_i(z) \subseteq Y_i$; payoffs $u_i(y, z)$ are realized; a new state is drawn from the transition probability $\mu(\cdot|y, z) \in \Delta(Z)$; and the process is repeated. The standard assumptions have that each Y_i is a compact metric space with its Borel structure, Z is a measurable space, the feasible action correspondences A_i are lower measurable with nonempty, compact values, and each $u_i(y, z)$ is jointly measurable and continuous in y. Typically, the transition probability $\mu\colon Y \times Z \to \Delta(Z)$ is assumed to be measurable and, with an appropriate topology on $\Delta(Z)$, continuous in y; general existence results for stochastic games have imposed the total variation norm topology on $\Delta(Z)$, which makes the continuity assumption quite restrictive. We refer to this condition as *norm continuity of transitions.* Note that feasible action correspondences depend only on the current state, and payoff functions depend only on the current state and profile of actions. Finally, payoffs are computed using the discounted sum of stage

[6] In the formal proof, Step 1 is broken into two parts to address the possibility that nature's move in period $t-1$ has an atom, in which case the active players' moves in period t are exogenously fixed in a game with atomless moves by nature. The current discussion describes Step 1.1, which takes up the atomless case.

payoffs using discount factors $\delta_i \in [0, 1)$ for each player i.

We subsume this classical stochastic game framework within the class of games of almost perfect information as follows. First, we specify that for all $t \in T$, we have $Z_t = Z$ and $Y_{t,i} = Y_i$. Second, to capture the timing of a stochastic game, we have active players move in *odd periods* and nature in *even periods* by adding dummy moves: in even periods, active players' moves are fixed at some given $\bar{y} \in Y = \times_{i \in N} Y_i$, and in odd periods, nature's move is fixed at some given $\bar{z} \in Z$. Formally, we specify that for each odd period t and each history x_{t-1}, player i's feasible set $A_{t,i}(x_{t-1}) = A_i(z_{t-1})$ is given by the action correspondence from the stochastic game; and in even periods, the action set is the singleton $A_{t,i}(z_{t-1}) = \{\bar{y}_i\}$. For nature's moves, for each even period $t \geq 2$ and each history x_{t-1}, we set $f_{0,t}(x_{t-1}) = \mu(\cdot | y_{t-1}, z_{t-2})$, consistent with the transition probability μ using the players' actions in the previous period and the state in the period preceding that; and in odd periods, we set $f_{t,0}(x_{t-1})$ to be equal to the point-mass at $\bar{z}$. Thus, nature's move z_t in an even period determines the state for the following period, in which players move. Finally, we define stage payoffs so that for each odd period t and each history x_t, $u_{t,i}(x_t) = u_i(y_t, z_{t-1})$, so that player i's payoff is determined by the current action profile and state; and in each even period t, we set $u_{t,i} = 0$. This alternation of moves implies that stage payoffs from odd periods are subject to double discounting, so to preserve dynamic preferences in the stochastic game, we specify that player i's discount factor is $\sqrt{\delta_i}$ in the corresponding game of almost perfect information.

By the above argument, any stochastic game can be represented as a game G of almost perfect information as long as: (i) Z is a complete, separable metric space, (ii) each A_i is continuous, (iii) each u_i is continuous, and (iv) the transition probability $\mu: Y \times Z \to \Delta(Z)$ is weak* continuous. It should now be apparent why we formulated games with atomless moves by nature as we have. In a general game of almost perfect information, our definition allows for atoms in moves of nature in periods preceding a "no play" period; in stochastic games, this corresponds to having atoms (in fact, nature's move is completely atomic) in odd periods and no atoms in even periods. Thus, a stochastic game can be viewed as a game with atomless moves by nature if, in addition to (i)–(iv), we have: (v) the distribution over states, $\mu(\cdot | y, z)$, is atomless for all $(y, z) \in Y \times Z$. In this case, we refer to G as a *stochastic game with atomless moves by nature.*

Our results for games with atomless moves by nature have immediate application to stochastic games. Clearly, Theorem 1 implies that given any stochastic game G satisfying (i)–(iv), the extension $\hat{G}$ admits a SPE. In turn, Theorem 2 shows that if the transition probability is atomless, then every SPE of the extended game corresponds to a SPE of the original stochastic game.

Corollary 2 *In a stochastic game G with atomless moves by nature, every subgame perfect equilibrium of the extended game $\hat{G}$ is payoff equivalent to a subgame perfect equilibrium of G. In particular, every stochastic game satisfying (i)–(v) admits a subgame perfect equilibrium.*

The existence of subgame perfect equilibrium in stochastic games satisfying (i)–(v) is also obtained in Corollary 2 of He and Sun (2016). This existence result is not logically nested with Theorem 1 of Mertens and Parthasarathy (2003), as they allow feasible action sets, payoffs, and the state transition to depend on the state in a measurable way, whereas we assume continuous dependence.[7] However, as we mentioned above, Mertens and Parthasarathy require norm continuity of transitions, precluding the possibility that the state has a component that varies deterministically as a function of players' continuous actions. Such deterministic dependence is natural in many applications, such as the strategic growth model in Example 1, and thus Corollary 2 has comparatively broad applicability.

The de-correlation result of the preceding corollary relies on the application of Theorem 1 to a stochastic game G satisfying (i)–(v), but upon close inspection, HRR's theorem applies in a somewhat unexpected way. In particular, the extended game $\hat{G}$ is defined by adding public signals in every period, including even periods in which nature moves, but also odd periods in which the active players move. Thus, $\hat{G}$ requires that the active players in an odd period observe *two* public signals, ω_{t-2} and ω_{t-1}, between moves—in effect, duplicating the public signal. The standard interpretation of a correlated equilibrium (e.g., Nowak and Raghavan (1992)) has players observing only one signal for a given play of the stage game, complicating the interpretation of equilibria in $\hat{G}$ as correlated equilibria of the original game. It is possible to recover the standard interpretation by having both public signals be drawn in the same period, so that players in effect observe one one, albeit two-dimensional, public signal between moves; then the signal can be reduced to one dimension via

[7]See also Proposition 4 of He and Sun (2016), which allows measurable dependence on the state and weak continuity with respect to actions. On the other hand, their result assumes that nature's moves are absolutely continuous with respect to a fixed, atomless measure, precluding deterministic transition probabilities.

Skorokhod's theorem. Instead, we follow a more direct approach using Theorem 2 to establish existence of a SPE in correlated strategies, a result recently alluded to by Jaśkiewicz and Nowak (2017).

Given a stochastic game G satisfying (i)–(iv), we define the *associated game with public randomization*, denoted $\tilde{G}$, by adding a payoff-irrelevant public signal, drawn from the uniform distribution λ on $[0, 1]$, in every *even period* t. Denoting histories in the associated game by $\tilde{x}_t$, nature's move in every even period t is then $\tilde{f}_{t,0}(\tilde{x}_{t-1}) = f_{t,0}(x_{t-1}) \otimes \lambda$. Regardless of the structure of the original game, $\tilde{f}_{t,0}$ is atomless, and thus the associated game $\tilde{G}$ with public randomization is a game with atomless moves by nature. Thus, we have the following corollary of Theorem 2.

Corollary 3 *For every stochastic game G satisfying (i)–(iv), the associated game $\tilde{G}$ with public randomization admits a subgame perfect equilibrium.*

By Corollary 3, any stochastic game satisfying (i)–(iv) admits a form of correlated subgame perfect equilibrium, complementing the well-known result of Nowak and Raghavan (1992), which establishes existence of a *stationary* correlated equilibrium under norm continuity of transitions.[8] Importantly, by allowing for subgame perfect equilibria in which strategies are history-dependent, we extend this existence result to stochastic games G in which the transition probability may have atoms and is merely weak* continuous. For example, it may be that the transition probability in G is a deterministic, continuous function of the previous state and actions. Returning to Example 1, we can remove noise from the depreciation rates, simply fixing them at $\bar{r}_1$ and $\bar{r}_2$, so that in any period, the capital stock levels, k_1 and k_2, and consumption levels, c_1 and c_2, determine new capital stock levels $k'_i = F_i(k_1, k_2) + (1 - \bar{r}_i)k_i - c_i$ in a deterministic way. Such deterministic transitions obviously violate norm continuity, but Corollary 3 delivers a SPE in the associated game, where the agents observe a single, one-dimensional public signal between periods.

5 Proof of Theorem 2

Before proceeding to the proof, we extend the concept of payoff equivalence to any period $t \in T$ and history $\hat{x}_{t-1} \in \hat{X}_{t-1}$ as follows. Given strategy profiles

[8] They also assume that the transitions are absolutely continuous with respect to a fixed measure.

$\hat{f}$ and $\hat{f}'$ in the extended game $\hat{G}$, we say that $\hat{f}'$ is *payoff-equivalent to* $\hat{f}$ *at* $\hat{x}_{t-1}$ if for all $i \in N$ and all $y \in \times_{i\in N} A_{t,i}(x_{t-1})$, we have

$$\int_z \Big[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t,i}(x_{t-1}, y, z, \omega, \hat{f}')\lambda(d\omega)\Big] f_{t,0}(x_{t-1})(dz)$$
$$= \int_z \Big[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t,i}(x_{t-1}, y, z, \omega, \hat{f})\lambda(d\omega)\Big] f_{t,0}(x_{t-1})(dz),$$

so that the expected discounted payoffs from $\hat{f}'$ and $\hat{f}$, calculated at $\hat{x}_{t-1}$, from every action profile are the same for every active player.

Let G be any game with atomless moves by nature, and fix a SPE $\hat{f}$ of $\hat{G}$ and set $f^0 = \hat{f}$. Recursively, for all $t \in T$ with $t \geq 2$, we take f^{t-1} as a given SPE of $\hat{G}$ satisfying the following conditions:[9]

(C1$_t$) for all $s \in T$ and all $\hat{x}_{s-1} \in \hat{X}_{s-1}$, $f_s^{t-1}(\hat{x}_{s-1})$ is independent of $\omega_1, \ldots, \omega_{t-2}$,

(C2$_t$) for all $s \geq t$ and all $\hat{x}_{s-1} \in \hat{X}_{s-1}$ such that $f_{t-1,0}(x_{t-2})$ is atomless, $f_s^{t-1}(\hat{x}_{s-1})$ is independent of $\omega_1, \ldots, \omega_{t-1}$,

(C3$_t$) for all $\hat{x}_{t-1} \in \hat{X}_{t-1}$ such that $f_{t-1,0}(x_{t-2})$ has an atom, $f_t^{t-1}(\hat{x}_{t-1})$ is independent of $\omega_1, \ldots, \omega_{t-1}$,

(C4$_t$) for all $\hat{x}_{t-1} \in \hat{X}_{t-1}$, f^{t-1} is payoff-equivalent to f^{t-2} at $\hat{x}_{t-1}$.

That is, the strategy profile f^{t-1} does not depend on public signals in periods up to and including $t-2$, and unless nature's move has an atom at history x_{t-2}, strategies are also independent of ω_{t-1}; moreover, expected discounted payoffs calculated in period $t-1$ are the same for f^{t-1} and f^{t-2}. Note that (C3$_t$) is in fact redundant: if $f_{t-1,0}(x_{t-2})$ has an atom, then by the definition of a game with atomless moves by nature, action sets at history $\hat{x}_{t-1}$ are singleton and independent of ω_{t-1}. In particular, note that actions in period t, namely $f_t^{t-1}(\hat{x}_{t-1})$, are independent of $\omega_1, \ldots, \omega_{t-1}$.

We will construct a SPE f^t of $\hat{G}$ satisfying (C1$_{t+1}$)–(C4$_{t+1}$) such that f^t shares the first t moves with f^{t-1}, i.e.,

$$(f_1^t, \ldots, f_t^t) \;=\; (f_1^{t-1}, \ldots, f_t^{t-1}).$$

The remaining moves will be constructed to be independent of $\omega_1, \ldots, \omega_{t-1}$, and when $f_{t,0}(x_{t-1})$ is atomless, to also be independent of ω_t. Importantly,

[9] Note that (C1$_t$) is vacuously satisfied for $t = 2$.

they must preserve dynamic payoffs in period t, so that f_t^t will indeed be a Nash equilibrium of the auxiliary game induced by f^t at $\hat{x}_{t-1}$. Because f^{t-1} does not depend on public signals prior to $t-1$, by ($C1_t$), we suppress the realizations of public signals prior to period $t-1$ and write $f_t^{t-1}(\hat{x}_{t-1})$ as $f_t^{t-1}(x_{t-1}, \omega_{t-1})$. When $f_{t-1,0}(x_{t-2})$ is atomless, ($C2_t$) allows us to write this simply as $f_t^{t-1}(x_{t-1})$.

As described before, the proof consists of five steps. For each $s \in T$, let $\tilde{X}_s$ consist of histories x_s of the game G such that $f_{s+1,0}(x_s)$ is atomless. Note that the set of atomless probability measures is open, and since $f_{s+1,0}$ is Borel measurable, the set $\tilde{X}_s$ is itself Borel measurable. Given $x_{t-1} \notin \tilde{X}_{t-1}$, since $f_t(x_{t-1})$ has an atom, the definition of game with atomless moves by nature implies that nature's move $f_{t-1}(x_{t-2})$ is atomless, and by ($C2_t$), that $f^{t-1}(\hat{x}_{t-1})$ is independent of $\omega_1, \ldots, \omega_{t-1}$, as required for ($C1_{t+1}$). Thus, our arguments focus on histories $\hat{x}_{t-1} \in \tilde{X}_{t-1} \times [0,1]^{t-1}$ at which nature's moves are atomless. The de-correlation step, next, is broken into two parts, depending on whether nature's move $f_{t-1,0}(x_{t-2})$ in period $t-2$ is atomless.

Step 1.1: De-correlation after atomless moves. Fix a history $\hat{x}_{t-1} \in \tilde{X}_{t-1} \times [0,1]^{t-1}$ such that nature's move $f_{t,0}(x_{t-1})$ is atomless and such that $\hat{x}_{t-2} \in \tilde{X}_{t-2} \times [0,1]^{t-2}$, so that $f_{t-1,0}(x_{t-2})$ is atomless as well. Since f^{t-1} is a SPE of the extended game $\hat{G}$, it follows that f_t^{t-1} is a Nash equilibrium of the auxiliary game $\hat{G}_t(\hat{x}_{t-1}, \hat{U}_{t+1}((\hat{x}_{t-1}, \cdot), f^{t-1}))$. Note that player i's expected dynamic payoff at $\hat{x}_{t-1}$ from an arbitrary action profile $y \in \times_{i \in N} A_{t,i}(x_{t-1})$ is

$$\int_z \Big[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t+1,i}((\hat{x}_{t-1}, y, z, \omega), f^{t-1}) \lambda(d\omega) \Big] f_{t,0}(x_{t-1})(dz).$$

In particular, the equilibrium payoff of player $i \in N$ is

$$\begin{aligned} &\hat{U}_{t,i}(\hat{x}_{t-1}, f^{t-1}) = \\ &\quad \int_y \Big[\int_z [u_{t,i}(x_{t-1}, y, z) + \\ &\quad \delta_i \int_\omega \hat{U}_{t+1,i}((\hat{x}_{t-1}, y, z, \omega), f^{t-1}) \lambda(d\omega)] f_{t,0}(x_{t-1})(dz) \Big] (\otimes_{i \in N} f_{t,i}^{t-1}(\hat{x}_{t-1}))(dy). \end{aligned}$$

By ($C2_t$), strategies f_s^{t-1} in periods $s \geq t+1$ are independent of public signals $\omega_1, \ldots, \omega_{t-1}$, so we can write $\hat{U}_{t,i}(\hat{x}_{t-1}, f^{t-1})$ as a function $\hat{U}_{t,i}(x_{t-1}, f^{t-1})$ of past actions alone, and similarly we can write $\hat{U}_{t+1,i}((\hat{x}_{t-1}, y_t, z_t, \omega_t), f^{t-1})$ as $\hat{U}_{t+1,i}((x_{t-1}, y_t, z_t, \omega_t), f^{t-1})$. Then the expected payoff to action profile y

simplifies to

$$\int_z \Big[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t+1,i}((x_{t-1}, y, z, \omega), f^{t-1}) \lambda(d\omega) \Big] f_{t,0}(\hat{x}_{t-1})(dz).$$

In addition, as in HRR, let $E_{t+1}(x_t)$ be the set of subgame perfect equilibrium payoffs in any subgame $\hat{G}(\hat{x}_t)$ of $\hat{G}$ such that $\hat{x}_t$ embeds actions x_t. Because the signals $\omega_1, \ldots, \omega_t$ are payoff irrelevant, this set is well-defined, and by Theorem 5 of HRR, the mapping $E_{t+1}\colon X_t \rightrightarrows \mathbb{R}^n$ so-defined has closed graph and, thus, is lower measurable (Theorem 18.20, Aliprantis and Border (2006)).

Now, fix a pair $(y, z) \in A_t(x_{t-1})$. Since the public signal is payoff irrelevant, it follows that for all ω, the dynamic payoff $\hat{U}_{t+1}((x_{t-1}, y, z, \omega), f^{t-1})$ belongs to $E_{t+1}(x_{t-1}, y, z)$. Therefore, the integral

$$\int_\omega \hat{U}_{t+1}((x_{t-1}, y, z, \omega), f^{t-1}) \lambda(d\omega)$$

of dynamic payoffs belongs to

$$\int_\omega E_{t+1}(x_{t-1}, y, z) \lambda(d\omega) \;\; = \;\; \mathrm{co} E_{t+1}(x_{t-1}, y, z),$$

where the equality follows from Theorem 4, p.64, of Hindenbrand (1974).

Define the ex ante vector of continuation payoffs $\hat{V}_{t+1}(\cdot|x_{t-1})\colon A_t(x_{t-1}) \to \mathbb{R}^n$ by

$$\hat{V}_{t+1,i}(y, z|x_{t-1}) \;\; = \;\; \int_\omega \hat{U}_{t+1,i}((x_{t-1}, y, z, \omega), f^{t-1}) \lambda(d\omega), \tag{2}$$

which gives the expected discounted payoffs from (y, z), integrating over the public signal in period t. Integrating over z, we obtain a function only of y, which by the observation above, satisfies

$$\int_z \hat{V}_{t+1}(y, z|x_{t-1}) f_{t,0}(x_{t-1})(dz) \;\; \in \;\; \int_z \mathrm{co} E_{t+1}(x_{t-1}, y, z) f_{t,0}(x_{t-1})(dz).$$

Since $f_{t,0}(x_{t-1})$ is atomless and $E_{t+1}(x_{t-1}, y, z)$ is closed, Lyapunov's theorem (Theorem 8.6.3, Aubin and Frankowska (1990)), implies

$$\int_z \hat{V}_{t+1}(y, z|x_{t-1}) f_{t,0}(x_{t-1})(dz) \;\; \in \;\; \int_z E_{t+1}(x_{t-1}, y, z) f_{t,0}(x_{t-1})(dz).$$

Thus, for all $(x_{t-1}, y) \in \tilde{X}_{t-1} \times Y_t$ with $x_{t-2} \in \tilde{X}_{t-2}$ and $y \in \times_{i \in N} A_{t,i}(x_{t-1})$, there is a Borel measurable selection $\psi_{t+1}(\,\cdot\,|x_{t-1}, y)\colon A_{t,0}(x_{t-1}) \to \mathbb{R}^n$ from the correspondence $E_{t+1}(x_{t-1}, y, \cdot)$ of SPE payoffs such that the expected continuation payoff is maintained, i.e.,

$$\int_z \psi_{t+1}(z|x_{t-1}, y) f_{t,0}(x_{t-1})(dz) \quad = \quad \int_z \hat{V}_{t+1}(y, z|x_{t-1}) f_{t,0}(x_{t-1})(dz). \quad (3)$$

Note that this selection is independent of signals $\omega_1, \ldots, \omega_t$.

Step 1.2: De-correlation after moves with atoms. Fix a history $\hat{x}_{t-2} \notin \tilde{X}_{t-2} \times [0,1]^{t-2}$, so that $f_{t-1,0}(x_{t-2})$ has an atom. By definition of a game with atomless moves by nature, it follows that the active players' moves are trivial in period t at history $\hat{x}_{t-1} = (\hat{x}_{t-2}, y_{t-1}, z_{t-1}, \omega_{t-1})$ for all $(y_{t-1}, z_{t-1}, \omega_{t-1}) \in \hat{A}_{t-1}(\hat{x}_{t-2})$, so their actions are predetermined in period t; moreover, nature's move $f_{t,0}(x_{t-1})$ in period t at $\hat{x}_{t-1}$ is atomless, i.e., $\hat{x}_{t-1} \in \tilde{X}_{t-1} \times [0,1]^{t-1}$. By (C1$_t$), for all $s \geq t-1$ and all histories $\hat{x}_s$ extending $\hat{x}_{t-2}$, $f_s^{t-1}(\hat{x}_s)$ is independent of $\omega_1, \ldots, \omega_{t-2}$, but future actions may depend on the public signal ω_{t-1} in period $t-1$. Since $f_{t-1,0}(x_{t-2})$ has an atom, the arguments from Step 1.1 cannot be used to eliminate dependence of f_s^{t-1} on ω_{t-1}, but we can use a similar argument to replace dependence on (ω_{t-1}, ω_t) with an appropriate selection of equilibrium payoffs as a function of z_t, the distribution of which is atomless. Because action sets are singleton at $\hat{x}_{t-1}$, this selection may be constructed without concern for equilibrium incentives in period t, but we must now maintain expected continuation payoffs in period $t-1$.

Player i's expected dynamic payoff at $\hat{x}_{t-2}$ from the pair $(y, z) \in A_{t-1}(x_{t-2})$ is

$$u_{t-1,i}(x_{t-2}, y, z) + \delta_i \int_{z'} \Big[u_{t,i}(x_{t-2}, y, z, y', z') + \delta_i \int_{(\omega,\omega')} \hat{U}_{t+1,i}((x_{t-2}, y, z, \omega, y', z', \omega'), f^{t-1}) \lambda^2(d(\omega, \omega')) \Big] f_{t,0}(x_{t-2}, y, z)(dz'),$$

where y' is the unique feasible action profile at (x_{t-2}, y, z), and λ^2 is Lebesgue measure on the unit square $[0,1]^2$. We can write y' explicitly as a function $\alpha_t(x_{t-2}, y, z)$ of history; and because the feasible action correspondences $A_{t,i}$ are continuous, the mapping $\alpha_t \colon \{x_{t-1} \in X_{t-1} \mid x_{t-2} \notin \tilde{X}_{t-2}\} \to Y_t$ so-defined is continuous.

Since the public signal is payoff irrelevant, it follows that for all (ω, ω'), we have

$$\hat{U}_{t+1,i}((x_{t-2}, y, z, \omega, y', z', \omega'), f^{t-1}) \in E_{t+1}(x_{t-1}, y, z, y', z'),$$

and thus the integral

$$\int_{(\omega,\omega')} \hat{U}_{t+1,i}((x_{t-2}, y, z, \omega, y', z', \omega'), f^{t-1})\lambda^2(d(\omega, \omega'))$$

belongs to co$E_{t+1}(x_{t-2}, y, z, y', z')$, by Theorem 4, p.64, of Hindenbrand (1974).

Denote the feasible action choices by players and nature in periods $t-1$ and t, given x_{t-2}, by

$$A^2_{t-1}(x_{t-2}) = \{(y, z, y', z') \mid (x_{t-2}, y, z, y', z') \in X_t\},$$

and note that for every quadruple $(y, z, y', z') \in A^2_{t-1}(x_{t-2})$, we have $y' = \alpha_t(x_{t-2}, y, z)$, since feasible actions are uniquely pinned down at (x_{t-2}, y, z). Define the ex ante vector of continuation payoffs $\hat{V}_{t+1}(\cdot|x_{t-2})\colon A^2_{t-1}(x_{t-2}) \to \mathbb{R}^n$ by

$$\hat{V}_{t+1}(y, z, y', z'|x_{t-2}) = \int_{(\omega,\omega')} \hat{U}_{t+1,i}((x_{t-2}, y, z, \omega, y', z', \omega'), f^{t-1})\lambda^2(d(\omega, \omega')),$$

and note that

$$\int_{z'} \hat{V}_{t+1}(y, z, y', z'|x_{t-2}) f_{t,0}(x_{t-2}, y, z)(dz')$$

belongs to

$$\begin{aligned} &\int_z \text{co}E_{t+1}(x_{t-2}, y, z, y', z') f_{t,0}(x_{t-2}, y, z)(dz') \\ &= \int_z E_{t+1}(x_{t-2}, y, z, y', z') f_{t,0}(x_{t-2}, y, z)(dz'), \end{aligned}$$

where the equality follows from the assumption that $f_{t,0}(x_{t-2}, y, z)$ is atomless and Lyapunov's theorem.

Thus, for all $(x_{t-1}, y') \in \tilde{X}_{t-1} \times Y_t$ with $x_{t-2} \notin \tilde{X}_{t-2}$ and $y' \in \times_{i \in N} A_{t,i}(x_{t-1})$, i.e., $y' = \alpha_t(x_{t-1})$, we conclude that there is a Borel measurable selection $\psi_{t+1}(\cdot|x_{t-1}, y')\colon A_{t,0}(x_{t-1}) \to \mathbb{R}^n$ from the correspondence $E_{t+1}(x_{t-1}, y', \cdot)$ of SPE payoffs such that the expected continuation payoff is maintained, i.e.,

$$\int_{z'} \psi_{t+1}(z'|x_{t-1}, y') f_{t,0}(x_{t-1})(dz') = \int_{z'} \hat{V}_{t+1}(y, z, y', z'|x_{t-2}) f_{t,0}(x_{t-1})(dz').$$

Note that this selection is independent of signals $\omega_1, \ldots, \omega_t$.

Step 2: Sewing payoff selections together. Now we invoke the "measurable 'measurable choice' theorem" of Mertens (2003), which establishes the existence of a Borel measurable mapping $\Psi_{t+1}: \tilde{X}_t \to \mathbb{R}^n$ such that for all $(x_{t-1}, y) \in \tilde{X}_{t-1} \times Y_t$ with $y \in \times_{i\in N} A_{t,i}(x_{t-1})$, the mapping $\Psi_{t+1}(x_{t-1}, y, \cdot)$ is a Borel measurable selection from $E_{t+1}(x_{t-1}, y, \cdot)$, and

$$\int_z \Psi_{t+1}(x_{t-1}, y, z) f_{t,0}(x_{t-1})(dz) \;=\; \int_z \psi_{t+1}(z|x_{t-1}, y) f_{t,0}(x_{t-1})(dz). \quad (4)$$

This gives us a selection of subgame perfect equilibrium payoffs that is independent of $\omega_1, \ldots, \omega_t$ and such that after every history $\hat{x}_t$, player i's expected dynamic payoff from each action profile $y \in \times_{i\in N} A_{t,i}(x_{t-1})$ is

$$\int_z \Big[u_{t,i}(x_{t-1}, y, z) + \delta_i \Psi_{t+1,i}(x_{t-1}, y, z) f_{t,0}(x_{t-1})(dz)$$

$$= \int_z \Big[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t+1,i}((x_{t-1}, y, z, \omega), f^{t-1}) \lambda(d\omega) \Big] f_{t,0}(\hat{x}_{t-1})(dz),$$

so that the selection Ψ_{t+1} preserves payoffs from each action profile y under f^{t-1}.

Next, we must assign, in a measurable way, a SPE of every subgame $\hat{G}(\hat{x}_t)$ with $x_t \in \tilde{X}_t$ to generate payoffs $\Psi_{t+1}(x_t)$. We apply Proposition 10 of HRR to construct the desired equilibrium selection, say $\tilde{f}^t$. Actions are uniquely determined at any history $\hat{x}_t$ with $x_t \notin \tilde{X}_t$, and we then splice these together with $\tilde{f}^t$ to arrive at the desired strategy profile f^t.

Step 3: From payoffs to actions. To apply Proposition 10 of HRR, we set $t = 1$ in their result. We identify their set of starting points, $\hat{X}_0 = Y_0$, with our $\tilde{X}_t$, their correspondence ΨC_2 with our E_{t+1}, and their mapping c_1 with our Ψ_{t+1}. Hence, $c_1 = \Psi_{t+1}$ maps $x_t \in \tilde{X}_t$ to equilibrium payoff vectors in $(\Psi C_2)(x_t) = E_{t+1}(x_t)$. By HRR's Proposition 10, there exist Borel measurable mappings $\tilde{f}^t_{t+1}: \tilde{X}_t \to \Delta(Y_{t,i})$ for each $i \in N$ and a Borel measurable selection $\tilde{\Psi}_{t+2}: \tilde{X}_t \times (Y_{t+1} \times Z_{t+1} \times [0,1]) \to \mathbb{R}^n$ from E_{t+2} such that for all $x_t \in \tilde{X}_t$, (i) $\tilde{f}^t_{t+1} = (\tilde{f}^t_{t+1,i}(x_t))_{i\in N}$ is a Nash equilibrium of the induced game $G_t(x_{t-1}, \tilde{\Psi}_{t+2})$, and (ii) equilibrium payoffs from $\tilde{f}^t_{t+1}$ in the induced game are $\Psi_{t+1}(x_t)$. We then extend $\tilde{f}^t_{t+1}$ to $\hat{X}_t$ so that for all $\hat{x}_t \in \hat{X}_t$, $\tilde{f}^t_{t+1}(\hat{x}_t)$ is independent of $\omega_1, \ldots, \omega_t$.

Applying HRR's Proposition 10 recursively (as in the proof of Lemma 18 of HRR), we obtain a sequence $\tilde{f}^t_{t+1}, \tilde{f}^t_{t+2}, \ldots$ such that for all $s \in T$ with $s > t$ and all $\hat{x}_{s-1} \in \hat{X}_{s-1}$, the profile $\tilde{f}^t_s(\hat{x}_{s-1})$ is a Nash equilibrium of the induced game with continuation payoffs generated by $\tilde{f}_{s+1}, \tilde{f}_{s+2}, \ldots$. We then define the strategy profile $f^t = (f^t_s)_{s \in T}$ as follows. For all $s \in T$ and all $\hat{x}_{s-1} \in \hat{X}_{s-1}$, if $s \leq t$, then set $f^t_s(\hat{x}_{s-1}) = f^{t-1}_s(\hat{x}_{s-1})$; if $s > t$ and $x_{t-1} \in \tilde{X}_{t-1}$, then set $f^t_s(\hat{x}_{s-1}) = \tilde{f}^t_s(\hat{x}_{s-1})$; and if $s > t$ and $x_{t-1} \notin \tilde{X}_{t-1}$, then set $f^t_s(\hat{x}_{s-1}) = f^{t-1}_s(\hat{x}_{s-1})$. In words, we use the strategies specified by f^{t-1} in all periods up to and including t. In later periods, we use $\tilde{f}^t_s$ at any history such that nature's move in period t is atomless; and at other histories, we use f^{t-1}_s. Because $\tilde{X}_{t-1}$ is Borel measurable, the mappings f^t_s so-defined are Borel measurable for all $s \in T$.

Step 4: Completing the induction. In periods $s \leq t-1$, (C1$_t$) implies that the strategies $f^t_s = f^{t-1}_s$ are independent of $\omega_1, \ldots, \omega_{t-2}$, and they are trivially independent of ω_{t-1}. In period $s = t$, we have $f^t_s = f^{t-1}_t$, and by (C2$_t$) and (C3$_t$), these strategies are again independent of $\omega_1, \ldots, \omega_t$. Now consider any period $s \geq t+1$ and history $\hat{x}_{s-1} \in \hat{X}_{s-1}$ such that $x_{t-1} \in \tilde{X}_{t-1}$, so that nature's move in period t, namely $f_{t,0}(x_{t-1})$, is atomless. Then $f^t_s = \tilde{f}^t_s$, and $\tilde{f}^t_s(\hat{x}_{s-1})$ is independent of $\omega_1, \ldots, \omega_t$, satisfying (C2$_{t+1}$). At a history $\hat{x}_{s-1} \in \hat{X}_{s-1}$ with $x_{t-1} \notin \tilde{X}_{t-1}$, nature's move in period t, namely $f_{t,0}(x_{t-1})$, has an atom, and we have specified that $f^t_s = f^{t-1}_s$. The definition of a game with atomless moves by nature requires that nature's move at $\hat{x}_{t-2}$, namely $f_{t-1,0}(x_{t-2})$ is atomless. Then (C2$_t$) implies that $f^{t-1}_s(\hat{x}_{s-1})$ is independent of $\omega_1, \ldots, \omega_{t-1}$. We conclude that (C1$_{t+1}$) is satisfied, and as discussed at the beginning of the proof, (C1$_{t+1}$) and (C2$_{t+1}$) imply (C3$_{t+1}$). Finally, note that for all $s \in T$ with $s > t$ and all $\hat{x}_s \in \hat{X}_s$ such that $x_{t-1} \in \tilde{X}_{t-1}$, and all $i \in N$, we have

$$\begin{aligned} & \int_z \int_\omega \hat{U}_{t+1,i}((x_{t-1}, y, z, \omega), f^t)\lambda(d\omega) f_{t,0}(x_{t-1})(dz) \\ = & \int_z \Psi_{t+1,i}(x_{t-1}, y, z) f_{t,0}(x_{t-1})(dz) \\ = & \int_z \int_\omega \hat{U}_{t+1,i}((x_{t-1}, y, z, \omega), f^{t-1})\lambda(d\omega), \end{aligned}$$

where the first equality follows from (ii) of Step 3 and the second equality from (2)–(4). This yields (C4$_{t+1}$), as required.

Finally, we construct a SPE of the game with atomless moves of nature, G, that is payoff-equivalent to $\hat{f}$.

Step 5: Construction of f. Let $f^\infty = (f^{t-1}_t)_{t \in T}$ be the strategy profile in $\hat{G}$ such that play in every period t is determined by f^{t-1}. For each $t \in T$, (C2$_t$) and (C3$_t$) imply that for all $\hat{x}_{t-1} \in \hat{X}_{t-1}$, $f^\infty_t(\hat{x}_{t-1}) = f^{t-1}_t(\hat{x}_{t-1})$ is independent of $\omega_1, \ldots, \omega_{t-1}$. Thus, for all $i \in N$, we can define $f_{t,i}: X_{t-1} \to \Delta(Y_{t,i})$ as follows: for all $x_{t-1} \in X_{t-1}$, choose an arbitrary $\hat{x}_{t-1} \in \hat{X}_{t-1}$ that embeds x_{t-1}, and set $f_{t,i}(x_{t-1}) = f^{\infty}_{t,i}(\hat{x}_{t-1})$. Then define $f_t = (f_{t,i})_{i \in N}$, and define $f = (f_t)_{t \in T}$. That is, we remove the nominal dependence of f^{∞}_t on $\omega_1, \ldots, \omega_{t-1}$, and we specify that at any history x_{t-1} of G, players move as in f^{t-1}_t.

To see that f is a SPE of G, note that for each $t \in T$, f^{t-1}_t is a SPE of $\hat{G}$. Thus, for each $x_{t-1} \in X_{t-1}$ and each $\hat{x}_{t-1} \in \hat{X}_{t-1}$ that embeds x_{t-1}, $f^{t-1}_t(\hat{x}_{t-1})$ is a Nash equilibrium of the auxiliary game $\hat{G}_t(\hat{x}_{t-1}, \hat{U}_{t+1}((\hat{x}_{t-1}, \cdot), f^{t-1}))$. By (C4$_{t+1}$), f^t is payoff-equivalent to f^{t-1} at $\hat{x}_{t-1}$, and thus $f^{t-1}_t(\hat{x}_{t-1})$ is also a Nash equilibrium of the auxiliary game $\hat{G}_t(\hat{x}_{t-1}, \hat{U}_{t+1}((\hat{x}_{t-1}, \cdot), f^t))$. In fact, for all $s > t$, conditions (C4$_{t+1}$)–(C4$_s$) imply that f^s is payoff-equivalent to f^{t-1} at $\hat{x}_{t-1}$, and thus $f^{t-1}_t(\hat{x}_{t-1})$ is a Nash equilibrium of the auxiliary game $\hat{G}_t(\hat{x}_{t-1}, \hat{U}_{t+1}((\hat{x}_{t-1}, \cdot), f^s))$. Note that by construction of f^s, play in periods $t, t+1, \ldots, s$ is determined by $f^{t-1}_t, f^t_{t+1}, \ldots, f^{s-1}_s$, respectively. By continuity at infinity, it follows that f^{∞} is payoff equivalent to f^{t-1} at $\hat{x}_{t-1}$, and thus $f^{t-1}_t(\hat{x}_{t-1})$ is a Nash equilibrium of the auxiliary game $\hat{G}_t(\hat{x}_{t-1}, \hat{U}_{t+1}((\hat{x}_{t-1}, \cdot), f^{\infty}))$. By the one-shot deviation principle, it follows that f^{∞} is a SPE of $\hat{G}$. Since the strategy profile f^{∞} is independent of the payoff-irrelevant public signals, we conclude that f is a SPE of G. Finally, setting $t = 1$ in the above discussion, it follows that f^{∞} is payoff-equivalent to $\hat{f}$ at $\hat{x}_0$, and, therefore, f is payoff equivalent to $\hat{f}$.

A Proof of Lemma 1

Consider any $\delta_i \in [0,1)$ and continuous, summable $u_{t,i}$. Define $u_i: X_\infty \to \mathbb{R}$ by $u_i(x) = \sum_{t \in T} \delta_i^{t-1} u_{t,i}(x_t)$, where $x_t \in X_t$ is the t-initial segment of $x \in X_\infty$. This establishes (1). Moreover, u_i is bounded, by summability of $u_{t,i}$. To prove continuity of u_i, let $\{x^\alpha\}$ be any net of infinite histories, directed by $\succ$, converging to x. Define $\overline{u}_i = \sup_{x \in X_\infty} \sum_{t \in T} \delta_i^{t-1} |u_{t,i}(x_t)| < \infty$. For each $\varepsilon > 0$, we can choose t sufficiently large so that $\frac{\delta_i^t}{1-\delta_i} \overline{u}_i < \frac{\varepsilon}{3}$ and $|u_i(x) - \sum_{s=1}^t \delta_i^{s-1} u_{s,i}(x_s)| < \frac{\varepsilon}{3}$. For each α, we have

$$\Big(\sum_{s=1}^t \delta_i^{s-1} u_{s,i}(x_s^\alpha) \Big) - \frac{\delta^t \overline{u}_i}{1-\delta_i} \quad \leq \quad u_i(x^\alpha) \quad \leq \quad \Big(\sum_{s=1}^t \delta_i^{s-1} u_{s,i}(x_s^\alpha) \Big) + \frac{\delta^t \overline{u}_i}{1-\delta_i}.$$

By continuity of each $u_{t,i}$, we can choose $\overline{\alpha}$ such that for all $\alpha \succ \overline{\alpha}$, we have $|(\sum_{s=1}^t \delta_i^{s-1} u_{s,i}(x_s^\alpha)) - (\sum_{s=1}^t \delta_i^{s-1} u_{s,i}(x_s))| < \frac{\varepsilon}{3}$. Then for $\alpha \succ \overline{\alpha}$, we have

$$u_i(x) - \varepsilon \quad \leq \quad u_i(x^\alpha) \quad \leq \quad u_i(x) + \varepsilon.$$

Since ε was arbitrary, we conclude that $u_i(x^\alpha) \to u_i(x)$, as required.

Conversely, consider any continuous function $u_i : X_\infty \to \mathbb{R}$ and discount factor $\delta_i \in (0,1)$. For each $x_t \in X_t$, let

$$H_t(x_t) = \{x' \in X_\infty \mid x'_t = x_t \text{ and for all } s > t, x'_s \in A_s(x'_{s-1})\}$$

be the set of infinite histories that extend x_t. This set is closed, and it is a subset of $\prod_{t \in T_0}(Y_t \times Z_t)$, which is compact by Tychonoff's theorem. Thus, $H_t(x_t)$ is nonempty and compact. Moreover, by continuity of $A_{s,0}$ for all $s > t$, the correspondence $H_t : X_t \rightrightarrows Z_t$ is continuous. Now, define the correspondence $U_t : X_t \rightrightarrows \mathbb{R}$ by

$$U_t(x_t) \quad = \quad \{u_i(x') \mid x' \in H_t(x_t)\},$$

which gives the set of possible payoffs from every infinite history that embeds x_t. Note that U_t is a continuous correspondence.

Next, we define a bounded, continuous function $\underline{u}_{t,i} : X_t \to \mathbb{R}$ such that for all $x \in X_\infty$ and all $t \in T$, we have $\lim_{t\to\infty} \underline{u}_{t,i}(x_t) = u_i(x)$. This is straightforward if the values of $A_{0,t}$ are compact, for then we can apply the theorem of the maximum to deduce continuity of $\min\{u_i(x') \mid x' \in H_t(x_t)\}$. More generally, however, we can apply Michael's selection theorem to take $\underline{u}_{t,i}$ as a continuous selection from coU_t, so that for all $x_t \in X_t$, we have $\underline{u}_{t,i}(x_t) \in \text{co}U_{t,i}(x_t)$. To verify the desired limiting property, consider any $x \in X_\infty$. For each t, there exist $x^{1,t}, x^{2,t} \in H_t(x_t)$ such that $\underline{u}_{t,i}(x)$ is a convex combination of $u_i(x^{1,t})$ and $u_i(x^{2,t})$. Then for each $j \in \{1,2\}$, we have $x^{j,t} \to x$ as $t \to \infty$, and by continuity of u_i, we have $u_i(x^{j,t}) \to u_i(x)$ as $t \to \infty$. We conclude that $\lim_{t\to\infty} \underline{u}_{t,i}(x_t) = u_i(x)$, as desired.

Finally, define $u_{t,i} : X_t \to \mathbb{R}$ by $u_{t,i}(x_t) = \delta_i^{1-t}(\underline{u}_{t,i}(x_t) - \underline{u}_{t-1,i}(x_{t-1}))$, so that for all $\xi \in \Delta(X_\infty)$, we have

$$\underline{u}_{0,i}(x_0) + \int_x \sum_{t\in T} \delta_i^{t-1} u_{t,i}(x_t)\xi(dx) \quad = \quad \int_x \lim_{t\to\infty} \underline{u}_{t,i}(x_t)\xi(dx) \quad = \quad \int_x u_i(x)\xi(dx),$$

where the second equality follows from the above claim and Lebesgue's dominated convergence theorem. Thus, the mappings $(u_{t,i})_{t\in T}$ are continuous, summable, and fulfill (1).

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