Preference Exclusions for Social Rationality*

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Abstract

I develop sufficient conditions for transitivity and acyclicity of social preferences, continuing the investigation of restricted domains begun by Black (1948,1958), Arrow (1951), and Sen (1966,1969). The approach, which excludes certain triples of rankings over triples of alternatives, contributes to the literature in three ways. First, I generalize majority rule to classes of social preference relations defined by their decisiveness properties. Second, I consider not only transitivity of weak and strict social preference, but I provide conditions for acyclic strict preference as well. Third, the well-known conditions of value restriction, single peakedness, and order restriction are shown to satisfy corresponding exclusion conditions, so transitivity results on these domains follow from the more general analysis; in particular, the results are applied to weakly single-peaked preference profiles, and a result on acyclicity due to Austen-Smith and Banks (1999) is obtained as a special case. In contrast to the latter authors, the approach fixes a single preference profile and does not rely on the properties of social preferences as individual preferences are varied.

1 Introduction

The possibility of majority cycles, demonstrated in Condorcet's paradox, is well-known, as are the positive results on transitivity of majority preferences when restrictions are imposed on individual preferences. Perhaps the most familiar restriction under which transitivity is achieved is single-peakedness, investigated by Black (1948,1958) and Arrow (1951), which demands that among any three alternatives, there is one (or more) that is not bottom-ranked (among the three) for any individual. Sen (1966,1969) showed that the exclusion from being bottom-ranked is arbitrary: the desirable transitivity properties of majority voting persist if there is some alternative among

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the three and some rank (first, second, or third) such that the alternative does not attain that rank (by itself, or tied with another alternative). In fact, because of the consistency properties of majority rule, Sen allows individuals to be indifferent among all three alternatives and imposes the above condition only on the other, "concerned" individuals. A profile satisfying the above condition is value-restricted. In later work, Rothstein (1990,1991) investigated the condition of order restriction, which also delivers desirable transitivity properties of majority rule.[1] This restriction imposes an ordering of individuals such that given any pair of alternatives, the set of individuals with a given preference between the two alternatives is "connected" with respect to the ordering. Order restriction is logically unrelated to single-peakedness but implies value-restriction, delivering transitivity of majority preferences.

To clarify the content of the above transitivity results, it is important to distinguish between two transitivity properties and two versions of majority rule. Sen (1969) showed that value restriction implies transitivity of *strict* majority preferences when the number of concerned voters is unrestricted, and Sen (1966) deduced the stronger conclusion that *weak* majority preferences are transitive when the number of concerned voters is odd. Regarding the definition of majority rule, Arrow, Sen, and Rothstein define one alternative to be strictly (resp. weakly) majority preferred to another if strictly (resp. weakly) more individuals strictly (resp. weakly) prefer the first alternative to the second. A different definition, that used by Black, requires a majority of all individuals to strictly prefer the first alternative to the second in the above definition. I refer to the latter as *simple majority rule* and to the former as *relative majority rule.* The above work did not consider the implications of these preference restrictions for more general aggregation rules, nor did it consider preference restrictions that are more permissive yet still sufficient to ensure transitivity or the weaker rationality condition of acyclic strict social preference.

I extend the analysis to social preferences generated by three classes of aggregation rules: simple rules (which are generated by the rule's decisive groups), the larger class of relatively simple rules (which require the assent of a decisive group for a strict social preference), and resolute rules (which require the assent of a decisive group for a weak social preference). The relatively simple rules generalize relative majority rule and the "voting rules" of Austen-Smith and Banks (1999), but they need not be neutral with respect

[1] See also Gans and Smart (1996) for analysis of a single-crossing condition that is equivalent to order restriction.

to permutations of alternatives in the preferences of individuals; and the simple rules generalize simple majority rule and specialize the overlap condition of Schwartz (1986). Prototypical examples of resolute rules are relative or simple majority rule with an odd number of individuals, but resoluteness does not imply neutrality. I provide preference exclusion conditions under which these classes of rules deliver desirable transitivity properties, including transitivity of weak and strict social preference. All of the exclusion conditions are requirements that there do not exist triples of individuals and alternatives such that the individuals' preferences over the alternatives take a form similar to the classical Condorcet paradox. Regarding transitivity of strict social preferences, a comparable result is Schwartz's (1986) Theorem 4.1.2, which uses a stronger "Condorcet freedom" exclusion condition but applies to social preferences that generalize the simple rules. Of note, I also provide conditions sufficient for acyclicity of simple social preferences, a contribution of special interest because acyclicity is a key condition for ensuring the existence of socially maximal elements: it is weaker than transitivity of strict (and therefore weak) social preference but sufficient for existence of maximal elements in any given finite set of alternatives; and it is necessary for existence of maximal elements in all finite sets.

I then apply the above social rationality results to more familiar preference domains, in some cases showing that known results can be derived from the more general analysis, and in other cases extending known results to wider classes of social preferences under weaker preference restrictions. For example, value-restricted preferences satisfy the strongest preference exclusion considered in this paper, and as a corollary we conclude that if all individuals are concerned over triples of alternatives, then weak social preferences generated by resolute rules are transitive. A weakening of value restriction is sufficient for transitivity of strict social preferences generated by a simple rule. Results for single peakedness and order restriction follow immediately, including conditions for transitivity of weak social preferences generated by resolute rules. Interestingly, we obtain a result due to Austen-Smith and Banks (1999), showing that social preferences generated by a simple rule are acyclic when individual preferences are weakly single-peaked, as a corollary of our general analysis of acyclicity.

Work by Sen and Pattanaik (1969) is related but different in focus.[2] Those authors extend the earlier analysis of value restriction from majority rule to the more general class of neutral and monotonic rules; they consider

[2]See also Inada (1964,1969) for different sets of conditions that are necessary and/or sufficient for transitivity of majority preferences.

the preference restrictions of "limited agreement" and "extremal restriction," which are logically unrelated to value restriction; and they provide necessary conditions for acyclicity of majority preferences when the number of voters is variable.

Although the above discussion is couched in terms of aggregation rules, an advantage of the analysis of this paper is that it is based exclusively on single-profile properties of social preferences and is therefore domain-free; at the end of the paper, I show that the results extend in the expected way to aggregation rules defined on a domain of possible individual preferences. Section 2 sets up the single-profile framework and defines classes of social preference of interest. Section 3 establishes the main results of the paper on value restriction conditions for rationality. Section 4 applies these results to value-restricted, single-peaked, weakly single-peaked, and order-restricted environments. Finally, Section 5 extends the analysis to aggregation rules when individual preferences are allowed to vary, and Section 6 concludes.

2 Formal Preliminaries

Let N be a finite set of individual decision makers, and let be X a set of alternatives from which a collective choice must be made. Let P_i be the strict preference relation on X of individual i, and let R_i be i's weak preference relation. Assume that P_i is asymmetric and negatively transitive, that R_i is complete and transitive, and that the relations are *dual:* for all $x, y \in X$, xP_iy if and only if not yR_ix, or equivalently, xR_iy if and only if not yP_ix. As usual, I_i denotes the indifference relation of individual i, so that xI_iy if and only if neither xP_iy nor yP_ix, or equivalently xR_iy and yR_ix. To denote the groups strictly and weakly preferring one alternative to another and indifferent between two alternatives, respectively, write

$$\begin{aligned} P(x,y) &= \{i \in N \mid xP_iy\} \\ R(x,y) &= \{i \in N \mid xR_iy\} \\ I(x,y) &= \{i \in N \mid xI_iy\}, \end{aligned}$$

and note that $P(x,y) \subseteq R(x,y)$. In Sections 2–4, a single profile of individual preferences is fixed and denoted $\pi = (P_1, \ldots, P_n)$, with weak preferences determined via duality.

In addition, let P and R be, respectively, strict and weak social preference relations on X such that P is asymmetric, R is complete, and the two relations are dual. We can interpret P as a "dominance relation," so that xPy indicates that society would choose x over y, perhaps as the result of a binary vote between the two alternatives. Then we can write I for the corresponding social indifference relation, so that xIy if and only if neither xPy nor yPx, or equivalently xRy and yRx. Note that this concept of social indifference does not distinguish between indifference and non-comparability. Moreover, P is negatively transitive if and only if R is transitive, and negative transitivity of P implies transitivity. It is well known that R is transitive if and only if P and I are both transitive. Thus, because social indifference incorporates non-comparability, transitivity of weak social preference is a demanding condition.[3]

Some familiar examples of strict social preferences are as follows, with weak social preferences are determined from the above definitions via duality.

simple majority:

$$xP_{SM}y \ \text{ if and only if } \ |P(x,y)| > n/2.$$

relative majority:

$$xP_{RM}y \ \text{ if and only if } \ |P(x,y)| > |P(y,x)|.$$

simple Pareto:

$$xP_{SP}y \ \text{ if and only if } \ P(x,y) = N.$$

relative Pareto:

$$xP_{RP}y \ \text{ if and only if } \ R(x,y) = N \text{ and } P(x,y) \neq \emptyset.$$

Given preference profile π and social preference P, we say $G \subseteq N$ is *decisive* for P if for all $x, y \in X$, $G \subseteq P(x,y)$ implies xPy, and G is *blocking* if for all $x, y \in X$, $G \subseteq R(x,y)$ implies xRy. Let $\mathcal{D}(P)$ denote the collection of decisive groups, and let $\mathcal{B}(P)$ denote the collection of blocking groups. Note that these collections are *monotonic,* in the sense that if a group is decisive (resp. blocking), then so is any superset of that group. We say that relative to π, the social preference P is...

- *simple* if for all $x, y \in X$, xPy implies $P(x,y) \in \mathcal{D}(P) \cap \mathcal{B}(P)$,
- *resolute* if for all distinct $x, y \in X$, xRy implies $R(x,y) \in \mathcal{D}(P)$,

[3]For example, it precludes Pareto social preferences, defined subsequently. It is customary to define Pareto indifference by xIy if and only if $I(x,y) = N$, and this relation is actually transitive. But when non-comparable pairs are added to this relation, it may become intransitive.

- *relatively simple* if for all $x, y \in X$, xPy implies $R(x, y) \in \mathcal{D}(P)$.

Clearly, if P is simple or resolute, then it is relatively simple. For examples, the simple majority preference P_{SM} is always simple, as is the simple Pareto preference P_{SP}. The relative majority preference P_{RM} is relatively simple but need not be simple: suppose there are three alternatives, a, b, c, and two individuals with preferences aP_1bP_1c and cP_2aI_2b, and note that $aP_{RM}b$ and $P(a, b) \subseteq P(a, c)$, but $cR_{RM}a$. The relative Pareto preference is relatively simple, but the same example shows that the relative Pareto preference need not be simple. Both simple and relative majority preferences are resolute when n is odd, but they are not generally when n is even, as demonstrated by the latter example: there, $cR_{SM}b$ and $cR_{RM}b$, and $R(c, b) \subseteq P(c, a)$, but $aR_{SM}c$ and $aR_{RM}c$. More generally, the simple rules of Austen-Smith and Banks (1999), which include quota rules and but also non-anonymous rules (such as weighted majority rule), always generate simple social preferences. And the voting rules of those authors, which can be sensitive to individual indifferences, always generate relatively simple social preferences.[4]

The definition of simple social preference demands that the group $P(x, y)$ of individuals with a strict preference for x over y be not only decisive but blocking as well. This requirement is used in Lemma 3, below, and is not implied by the lone requirement that $P(x, y) \in \mathcal{D}(P)$. This is demonstrated by the next example.

Example 1. Assume $n = 3$ and $X = \{a, b, c\}$, let individual preferences be as follows,

1	2	3
a	bc	b
bc	a	c
		a

and let strict social preferences be $P = P_3$. Here, the minimal decisive groups are $\{1, 2\}$, vacuously so, and $\{3\}$. Note that cR_1b and cR_2b, yet bPc, so $\{1, 2\}$ is not blocking.

The above example also shows that decisive groups need not be blocking, i.e., we may have $\mathcal{D} \not\subseteq \mathcal{B}$. Nevertheless, the next lemma shows that if P is simple and a weak social preference holds, then the group of individuals with that weak preference is blocking.

[4]See the α-rules of Ferejohn and Grether (1974) or counting rules of Austen-Smith and Banks (1999). Note that simple rules and voting rules satisfy Arrow's independence of irrelevant alternatives, and aggregation rules that violate IIA (such as Borda) will not in general generate simple or relatively simple social preferences.

Lemma 1. *If P is simple and xRy for some $x, y \in X$, then $R(x, y) \in \mathcal{B}(P)$.*

Proof. Assume P is simple and xRy, and suppose toward a contradiction that $R(x, y) \notin \mathcal{B}(P)$. Then there exist $a, b \in X$ such that $R(x, y) \subseteq R(a, b)$ but bPa. Since P is simple, it follows that $P(b, a) \in \mathcal{D}(P)$. But $P(b, a) \subseteq P(y, x)$, and thus yPx, a contradiction. □

It is sometimes useful to consider social preferences satisfying neutrality and monotonicity conditions. In the single-profile framework, I combine these conditions and say P is *neutral-monotonic* relative to π if for all $x, y, a, b \in X$ such that xPy, $P(x, y) \subseteq P(a, b)$, and $R(x, y) \subseteq R(a, b)$, we have aPb. Social preferences generated by neutral and monotonic aggregation procedures (the voting rules of Austen-Smith and Banks (1999)) are neutral-monotonic. The next lemma implies that they are, therefore, relatively simple as well.

Lemma 2.

1. *If P is simple, then it is neutral-monotonic.*

2. *If P is neutral-monotonic, then it is relatively simple.*

Proof. 1. Assume P is simple, and consider $x, y, a, b \in X$ such that xPy, $P(x, y) \subseteq P(a, b)$, and $R(x, y) \subseteq R(a, b)$. Since P is simple, xPy implies that $P(x, y) \in \mathcal{D}(P)$, which implies that $P(a, b) \in \mathcal{D}(P)$, and therefore aPb.
2. Assume P is neutral-monotonic, and consider distinct $x, y \in X$ such that xPy and $a, b \in X$ such that $R(x, y) \subseteq P(a, b)$; then since $P(x, y) \subseteq R(x, y) \subseteq P(a, b) \subseteq R(a, b)$, neutrality-monotonicity implies aPb, and therefore $R(x, y) \in \mathcal{D}(P)$, as required. □

The converse direction does not generally hold in either part of Lemma 2: for part 1, relative majority preferences are neutral-monotonic when n is even, but we have seen that P_{RM} need not be simple in this case; for part 2, note that in Example 10, below, the social preference is relatively simple, and we have aPb, $P(a, b) \subseteq P(c, b)$, and $R(a, b) \subseteq R(c, b)$, but not cPb, so P is not neutral-monotonic. In fact, in that example, social preferences are resolute, so the converse of part 2 of the lemma does not generally hold even for resolute social preferences.

The next lemma establishes connections between decisive groups in the presence of certain social preferences among alternatives. It plays a key role in the analysis of social rationality in the next section.

Lemma 3. *For all $x, y, z, w \in X$,*

1. *if P is simple, xPy, and zRw, then $P(x,y) \cap R(z,w) \neq \emptyset$,*

2. *if P is simple, xPy and zPw, then $P(x,y) \cap P(z,w) \neq \emptyset$,*

3. *if P is resolute, xRy, and zRw, then $R(x,y) \cap R(z,w) \neq \emptyset$,*

4. *if P is relatively simple, xPy, and zRw, then $R(x,y) \cap R(z,w) \neq \emptyset$.*

Proof. Consider any $x, y, z, w \in X$. 1. Assume P is simple, xPy, and zRw, and suppose $P(x,y) \cap R(z,w) = \emptyset$. Then $P(x,y) \in \mathcal{D}(P)$ and $P(x,y) \subseteq P(w,z)$, which implies wPz, a contradiction. 2. Assume P is simple, xPy, zPw, and suppose $P(x,y) \cap P(z,w) = \emptyset$. Then $P(x,y) \in \mathcal{B}(P)$ and $P(x,y) \subseteq R(w,z)$, which implies wRz, a contradiction. 3. Assume P is resolute, xRy, and zRw, and suppose $R(x,y) \cap R(z,w) = \emptyset$. Thus, $R(x,y) \neq N$, so $x \neq y$. Then $R(x,y) \in \mathcal{D}(P)$ and $R(x,y) \subseteq P(w,z)$, which implies wPz, a contradiction. 4. Assume P is relatively simple, xPy, and zRw, and suppose $R(x,y) \cap R(z,w) \neq \emptyset$. Then $R(x,y) \in \mathcal{D}(P)$ and $R(x,y) \subseteq P(w,z)$, which implies wPz, a contradiction. □

3 Social Rationality

We seek general restrictions on individual preferences that are sufficient for a number of transitivity properties of social preferences. We first explore conditions under which the strongest rationality condition—transitivity of weak social preference—can be achieved. We say $\pi = (P_1, \ldots, P_n)$ satisfies *exclusion condition NT* if for all distinct $x_1, x_2, x_3 \in X$, there do not exist $i, j, k \in N$ such that

(1) $x_1 R_i x_2 R_i x_3$,

(2) $x_2 R_j x_3 P_j x_1$, and

(3) $x_3 P_k x_1 R_k x_2$.

Note that the individuals i, j, k described by (1)–(3) must be distinct. Resoluteness is needed for the desired result, as the majority weak preference may be intransitive if the number of individuals is even, even when exclusion condition NT is satisfied. In fact, individual preferences in the following simple example satisfy all of the preference restrictions that I introduce in the sequel; they do not, however, satisfy the extremal restriction condition of Sen and Pattanaik (1969).

Example 2. Assume there are just two individuals, three alternatives, and preferences are aP_1bP_1c and cP_2aP_2b, then we have $aP_{SM}b$ yet $bR_{SM}cR_{SM}a$.

The next proposition establishes that given simple and resolute social preferences, exclusion condition NT, is sufficient for transitive weak social preference.

Proposition 1. *Assume that π satisfies exclusion condition NT. If P is simple and resolute, then it is negatively transitive.*

Proof. Consider any $x, y, z \in X$ such that $xRyRz$, and suppose toward a contradiction that zPx; in particular, the alternatives are distinct. Let $G_1 = R(x, y)$, $G_2 = R(y, z)$, and $G_3 = P(z, x)$. By Lemma 3, since P is simple and resolute, there exist $i \in G_1 \cap G_2$, $j \in G_2 \cap G_3$, and $k \in G_1 \cap G_3$. In particular, we have xR_iyR_iz, yR_jzP_jx, and zP_kxR_ky. But letting $x_1 = x$, $x_2 = y$, and $x_3 = z$, this violates exclusion condition NT. We conclude that R is transitive. □

Example 2 shows that resoluteness cannot be dropped from the preceding result, even when social preferences are simple. To show that the assumption of simple social preferences cannot be dropped, the next example offers a version of majority rule with neutral tie breaking and demonstrates a preference profile that satisfies exclusion condition NT yet for which social weak preferences are intransitive.

Example 3. Assume $n = 3$ and $X = \{a, b, c\}$, and define P as a variation of simple majority rule with neutral tie breaking. Specifically, define

$$xPy \quad \Leftrightarrow \quad xP_{SM}y \text{ or both } xR_2y \text{ and } xP_3y \text{ hold}$$

for all $x, y \in X$. This definition entails resolute and neutral-monotonic, but not necessarily simple, social preferences. Now consider the following profile of preferences.

1	2	3
ab	ac	bc
c	b	a

Because no individual has a unique top-ranked alternative, (3) is automatically falsified, so this profile satisfies exclusion condition NT, but $aRbRcPa$, violating transitivity of weak preference.

Note that exclusion condition NT precludes more than just the Condorcet rankings, in which preferences of individuals i, j, k over x_1, x_2, x_3 are strict. The next example demonstrates that some individual preferences exhibiting indifference must be excluded in order to obtain the transitivity result, even when social preferences are simple.

Example 4. Assume $n = 3$, $X = \{a, b, c\}$, with preferences below.

1	2	3
abc	bc	c
	a	ab

Here, exclusion condition NT is violated with $x_1 = a$, $x_2 = b$, $x_3 = c$ and $i = 1$, $j = 2$, and $k = 3$, and we have $aR_{SM}bR_{SM}c$ but $cP_{SM}a$, violating transitivity of simple majority weak preference.

To obtain the weaker condition of transitive strict social preference, we say $\pi = (P_1, \ldots, P_n)$ satisfies *exclusion condition T* if for all distinct $x_1, x_2, x_3 \in X$, there do not exist $i, j, k \in N$ such that

(4) $x_1 P_i x_2 P_i x_3$,

(5) $x_2 P_j x_3 R_j x_1$, and

(6) $x_3 R_k x_1 P_k x_2$.

Note that the individuals i, j, k described by (4)–(6) must be distinct. Furthermore, (4)–(6) implies (1)–(3), so exclusion condition NT implies exclusion condition T. Specifically, suppose that for distinct alternatives x_1, x_2, x_3, there exist individuals i, j, k satisfying (4)–(6). Set

$$\begin{array}{ll} x_1' = x_2 & i' = j \\ x_2' = x_3 & j' = k \\ x_3' = x_1 & k' = i. \end{array}$$

Rewriting and rearranging (4)–(6), we then have

(5) $x_1' P_{i'} x_2' R_{i'} x_3'$,

(6) $x_2' R_{j'} x_3' P_{j'} x_1'$, and

(4) $x_3' P_{k'} x_1' P_{k'} x_2'$,

implying (1)–(3) and verifying the claim. Moreover, Sen's (1966) value restriction and the limited agreement condition of Sen and Pattanaik (1969) both imply exclusion condition T. Finally, Schwartz's (1986) Condorcet freedom essentially implies exclusion condition T, the logical gap being that his exclusion condition adds a fourth condition that involves three arbitrary individuals who may be distinct from i, j, k.

Transitivity of strict social preferences does not always hold under exclusion condition T; in fact, the following example shows that when social preferences are resolute (but not simple) and exclusion condition T is satisfied, strict social preferences may actually be cyclic.

Example 5. Assume $n = 7$ and $X = \{a, b, c\}$, with preferences below.

1	2	3	4	5	6	7
a	a	a	c	c	bc	bc
b	b	b	ab	ab	a	a
c	c	c				

To see that this profile satisfies exclusion condition T, note that the only possible way of fulfilling (4)–(6) is setting $x_1 = a$, $x_2 = b$, and $x_3 = c$. Then because b is not uniquely bottom ranked by any individual, (6) does not hold. Furthermore, because $n = 7$ is odd, P_{RM} is resolute. But we have $aP_{RM}bP_{RM}cP_{RM}a$, violating acyclicity of strict relative majority preference.

The next proposition replaces the assumption of resoluteness with the assumption that P is simple, and it establishes that under this modification, exclusion condition T indeed implies transitivity of strict preference. The result differs from Theorem 4.1.2 of Schwartz (1986) as follows: his Condorcet freedom condition adds a fourth property to the profiles excluded, and his transitivity result accordingly adds a strong Pareto condition; in other ways his condition is more restrictive than exclusion condition T, allowing him to weaken the assumption of simple social preferences somewhat to an overlap condition that relaxes part 1 of Lemma 3.

Proposition 2. *Assume π satisfies exclusion condition T. If P is simple, then it is transitive.*

Proof. Consider any $x, y, z \in X$ such that $xPyPz$, and suppose toward a contradiction that zRx. Let $G_1 = P(x, y)$, $G_2 = P(y, z)$, and $G_3 = R(z, x)$. By Lemma 3, since P is simple, there exist $i \in G_1 \cap G_2$, $j \in G_2 \cap G_3$ and $k \in G_1 \cap G_3$. In particular, we have xP_iyP_iz, yP_jzR_jx, and zR_kxP_ky. But letting $x_1 = x$, $x_2 = y$, and $x_3 = z$, this violates exclusion condition T. We conclude that P is transitive. □

Note that Example 4 shows that exclusion condition T is not sufficient for transitivity of weak social preference, even when social preferences are simple and resolute. The condition precludes some preference profiles in which individuals are indifferent between alternatives. The next example demonstrates that some such restrictions, beyond elimination of the Condorcet rankings, is needed to obtain the transitivity result.

Example 6. Assume $n = 3$, $X = \{a, b, c\}$, with preferences below.

1	2	3
a	b	ac
b	ac	b
c		

Here, exclusion condition T is violated in light of restrictions on individual preferences exhibiting indifferences, and we have $aP_{SM}bP_{SM}c$ but $cR_{SM}a$, violating transitivity of strict social preference.

A strengthening of the above two preference exclusions can be used to obtain the above transitivity results under weaker background conditions. We say $\pi = (P_1, \ldots, P_n)$ satisfies *exclusion condition T^** if for all distinct $x_1, x_2, x_3 \in X$, there do not exist $i, j, k \in N$ such that

(4*) $x_1 R_i x_2 R_i x_3$,

(5*) $x_2 R_j x_3 R_j x_1$, and

(6*) $x_3 R_k x_1 R_k x_2$.

Note that the individuals i, j, k described by (4*)–(6*) must be distinct. Furthermore, the above condition implies that there is no individual who is indifferent over any triple of alternatives, for in that case we could set $i = j = k$ equal to that individual and $\{x_1, x_2, x_3\}$ equal to that triple to fulfill (4*)–(6*).

The next result drops the assumption that P is simple from Proposition 2 and shows that under resoluteness alone, exclusion condition T* is sufficient for transitivity of weak social preference.

Proposition 3. *Assume π satisfies exclusion condition T^*. If P is resolute, then it is negatively transitive.*

Proof. Consider any $x, y, z \in X$ such that $xRyRz$, and suppose toward a contradiction that zPx. Let $G_1 = R(x, y)$, $G_2 = R(y, z)$, and $G_3 = R(z, x)$.

By Lemma 3, since P is resolute and thus relatively simple, there exist $i \in G_1 \cap G_2$, $j \in G_2 \cap G_3$ and $k \in G_1 \cap G_3$. In particular, we have xR_iyR_iz, yR_jzR_jx, and zR_kxR_ky. But letting $x_1 = x$, $x_2 = y$, and $x_3 = z$, this violates exclusion condition T*. We conclude that R is transitive. □

We have noted that exclusion condition T* implies that no individual is indifferent over any triple of alternatives, i.e., all individuals are "concerned over triples." A weaker preference exclusion would rule out (4*)–(6*) only if none of the individuals i, j, k are indifferent over all three alternatives. This weaker condition is not sufficient for transitivity of weak social preference, however, as the next example shows.

Example 7. Assume $n = 3$ and $X = \{a, b, c\}$, with preferences below.

1	2	3
ac	abc	b
b		c
		a

To see that exclusion condition NT is satisfied, it suffices to check the case in which $i = 3$, so that $x_1 = b$, $x_2 = c$, and $x_3 = a$. But then there does not exist an individual k whose preferences satisfy (3), fulfilling the condition. Nevertheless, we have $aR_{RM}bR_{RM}cP_{RM}a$, violating transitivity of weak relative majority preference.

The next proposition establishes that, given relatively simple social preferences, exclusion condition T* implies transitivity of strict preference.

Proposition 4. *Assume π satisfies exclusion condition T*. If P is relatively simple, then it is transitive.*

Proof. Consider any $x, y, z \in X$ such that $xPyPz$, and suppose toward a contradiction that zRx. Let $G_1 = R(x, y)$, $G_2 = R(y, z)$, and $G_3 = R(z, x)$. By Lemma 3, since P is relatively simple, there exist $i \in G_1 \cap G_2$, $j \in G_2 \cap G_3$ and $k \in G_1 \cap G_3$. In particular, we have xR_iyR_iz, yR_jzR_jx, and zR_kxR_ky. But letting $x_1 = x$, $x_2 = y$, and $x_3 = z$, this violates exclusion condition T*. We conclude that P is transitive. □

To extend the analysis to provide sufficient conditions for the fundamental condition of acyclicity of strict social preference, we say $\pi = (P_1, \ldots, P_n)$ satisfies *exclusion condition A* if for all natural numbers $m \geq 3$ and all distinct $x_1, \ldots, x_m \in X$, we have the following:

- if $m = 3$, then there do not exist $i, j, k \in N$ such that
  - (i) $x_1 P_i x_2 P_i x_3$,
  - (ii) $x_2 P_j x_3 P_j x_1$, and
  - (iii) $x_3 P_k x_1 P_k x_2$,
- if $m \geq 4$, then there exists $\ell \in \{1, \ldots, m\}$ such that there do not exist $i, j, k \in N$ such that
  - (iv) $x_\ell P_i x_{\ell+1} P_i x_{\ell+2}$,
  - (v) $x_{\ell+1} P_j x_{\ell+2} R_j x_\ell$, and
  - (vi) $x_{\ell+2} R_k x_\ell P_k x_{\ell+1}$,

  where addition to subscripts is understood to be modulo m.

Note that (i)–(iii) rule out the possibility that the preferences of three individuals over three alternatives form a Condorcet profile, as in the well-known Condorcet paradox. Conditions (iv)–(vi) are equivalent to conditions (4)–(6) in exclusion condition T. Essentially, when $m \geq 4$, we require that at least one consecutive triple $\{x_\ell, x_{\ell+1}, x_{\ell+2}\}$ satisfy the sufficient condition for transitivity of strict social preference.

The next result establishes that when social preferences are simple, exclusion condition A is sufficient for acyclicity of strict social preference.

Proposition 5. *Assume π satisfies exclusion condition A. If P is simple, then it is acyclic.*

Proof. Assuming P is simple and $\pi = (P_1, \ldots, P_n)$ satisfies exclusion condition A, suppose toward a contradiction that P is not acyclic. Then there exist $m \geq 3$ and $x_1, \ldots, x_m$ such that $x_1 P x_2 P \cdots x_m P x_1$. Furthermore, by selecting a smallest such cycle, we can assume without loss of generality that the m alternatives are distinct. In case $m = 3$, by Lemma 3, since P is simple, there exist $i, j, k \in N$ such that (i)–(iii) hold, contradicting exclusion condition A.

In case $m \geq 4$, note that by choice of a smallest cycle, it follows that for all $\ell \in \{1, \ldots, m\}$, we have $x_\ell P x_{\ell+1} P x_{\ell+2} R x_\ell$. Indeed, if this were not the case, then we would have $x_\ell P x_{\ell+2}$, but then we could remove $x_{\ell+1}$ to create a smaller cycle. Now consider any $\ell \in \{1, \ldots, m\}$. By Lemma 3, since P is simple, there exist i, j, k such that $i \in P(x_\ell, x_{\ell+1}) \cap P(x_{\ell+1}, x_{\ell+2})$, $j \in P(x_{\ell+1}, x_{\ell+2}) \cap R(x_{\ell+2}, x_1)$, and $k \in R(x_{\ell+2}, x_\ell) \cap P(x_\ell, x_{\ell+1})$. Then (iv)–(vi) hold, and since ℓ was arbitrary, this contradicts exclusion condition A. □

To see that exclusion condition A is not sufficient for transitivity of strict social preference, even when social preferences are simple and resolute, consider Example 6. There, preferences satisfy exclusion condition A, but the strict simple majority preference is intransitive. Example 5 shows that the acyclicity result does not hold when the assumption of simple social preferences is replaced by resoluteness.

4 Special Cases

This section draws out implications of the analysis of preference exclusions for the familiar preference restrictions of value restriction, single peakedness, and order restriction. In addition, we specialize to the weak single-peaked preferences of Austen-Smith and Banks (1999) and show that their result for acyclic social preferences follows from the more general results above; in particular, if a preference profile is weakly single-peaked, then it satisfies exclusion condition A. As corollaries, we obtain sufficient conditions for transitive social preferences that are in some cases known for majority rule or the simple rules and voting rules of Austen-Smith and Banks (1999). The contribution of this section is to extend known results to the single-profile framework and to more general classes of simple and relatively simple social preferences.

4.1 Value Restriction

Whereas the preference restrictions introduced in the previous section exclude triples of orderings over triples of alternatives, we now consider restrictions that limit the ranks of alternatives in each triple. The *value* in the finite set $Y \subseteq X$ of alternative $x \in Y$ for individual i is

$$V_i^Y(x) \quad = \quad \{|P_i^{-1}(x) \cap Y| + 1, \ldots, |R_i^{-1}(x) \cap Y|\},$$

where $P_i^{-1}(x) = \{y \in X \mid xP_iy\}$ and $R_i^{-1}(x) = \{y \in X \mid xR_iy\}$ are the lower sections of P_i and R_i, respectively. In words, it is the "rank" of x in i's ordering restricted to the set Y.[5] Technically, to account for individual indifferences, the value is a set of ranks: we give x a least rank equal to the number of alternatives in Y strictly worse than it (plus one for x); and we give it a greatest rank equal to the number of alternatives in Y it is weakly preferred to; and the set consists of all ranks in this range. For concreteness, let $Y =$

[5] Depending on terminology, one might think of this as the inverse of rank; it is roughly the number of alternatives below x in i's ordering, plus one.

$\{a, b, c, d\}$, and let individual i's preferences be given by the ranking below.

$$\begin{array}{c} a \\ bc \\ d \end{array}$$

Then $V_i^Y(a) = \{4\}$ and $V_i^Y(d) = \{1\}$ are singleton, whereas $V_i^Y(b) = V_i^Y(c) = \{2, 3\}$, reflecting the fact that indifference between b and c could be resolved in either direction. Note that if xR_iyR_iz, then we immediately have $3 \in V_i^Y(x)$, $2 \in V_i(y)$, and $1 \in V_i(z)$.

An individual i is *concerned* over a subset $Y \subseteq X$ if there exist $x, y \in Y$ such that xP_iy, i.e., the individual is not entirely indifferent across the set Y. Let N^Y denote the set of individuals concerned over Y. The profile π satisfies *concern over triples* if for every $Y \subseteq X$ with $|Y| = 3$, we have $N^Y = N$.

The preference profile π is *value-restricted* if for every subset $Y \subseteq X$ with $|Y| = 3$, there exists $x \in Y$ and $r \in \{1, 2, 3\}$ such that for all $i \in N^Y$, we have $r \notin V_i^Y(x)$. Note that the latter exclusion only applies to individuals who are concerned over a given triple, so the condition of value restriction does not preclude the possibility that some individual is indifferent over three alternatives. The profile π is *weakly value-restricted* if for all $Y \subseteq X$ with $|Y| = 3$, either

(i) there exist $x \in Y$ and $r \in \{1, 2, 3\}$ such that for all $i \in N^Y$, we have $r \notin V_i^Y(x)$, or

(ii) for every $x \in Y$, there exists $r \in \{1, 2, 3\}$ such that for all $i \in N^Y$, we have $\{r\} \neq V_i^Y(x)$.

Thus, a profile can be weakly value-restricted but not value-restricted, as long as it is the case that when the latter restriction is not satisfied for some Y, it is the case that for every alternative $x \in Y$, there is some rank $r \in \{1, 2, 3\}$ such that x does not uniquely attain value r for any concerned individuals.

To see that the value restriction conditions are not equivalent, consider the following example.

Example 8. Assume $n = 3$ and $X = \{a, b, c\}$, with preferences below.

1	2	3
c	bc	ab
ab	a	c

Here, all individuals are concerned over X, and $V_1^X(a) = V_1^X(b) = \{1, 2\}$, $V_1^X(c) = \{3\}$, $V_2^X(a) = \{1\}$, $V_2^X(b) = V_2^X(c) = \{2, 3\}$, $V_3^X(a) = V_3^X(b) = \{2, 3\}$, and $V_3^X(c) = \{1\}$. In particular, each alternative attains each value, so this profile is not value-restricted. Nevertheless, we have $V_i^X(a) \neq \{3\}$ for all individuals i, $V_i^X(b) \neq \{3\}$ for all i, and $V_i^X(c) \neq \{2\}$ for all i, and therefore the profile is weakly value-restricted. Note also that $aR_{SM}bR_{SM}c$ yet $cP_{SM}a$, violating transitivity of weak simple majority preference.

The preceding example shows that weak value restriction is not sufficient for transitivity of weak social preference, even when concern over triples is satisfied and social preferences are simple and resolute. Moreover, the following example shows that simple majority rule with n odd can generate intransitive weak social preferences if indifference over triples is not ruled out, even if individual preferences satisfy value restriction.

Example 9. Assume $n = 3$ and $X = \{a, b, c\}$, with preferences below.

1	2	3
b		c
c	abc	a
a		b

This preference profile is value-restricted, because $1 \notin V_i^X(c)$ for concerned individuals $i = 1, 3$, and we have $aR_{SM}bR_{SM}c$ yet $cP_{SM}a$, violating transitivity of simple majority weak social preference.

The next proposition establishes that value restriction and concern over triples imply the strongest preference exclusion introduced above, i.e., exclusion condition T*, precluding the latter two examples.

Proposition 6. *If π is value-restricted and satisfies concern over triples, then it satisfies exclusion condition T^*.*

Proof. Consider any distinct $x_1, x_2, x_3 \in X$, and suppose toward a contradiction that (4*)–(6*) hold for some i, j, k. Setting $Y = \{x_1, x_2, x_3\}$, concern over triples implies that $\{i, j, k\} \subseteq N^Y$, and we furthermore have $V_i^Y(x_1) \cup V_j^Y(x_1) \cup V_k^Y(x_1) = \{1, 2, 3\}$, $V_i^Y(x_2) \cup V_j^Y(x_2) \cup V_k^Y(x_2) = \{1, 2, 3\}$, and $V_i^Y(x_3) \cup V_j^Y(x_3) \cup V_k^Y(x_3) = \{1, 2, 3\}$, contradicting value restriction. □

The immediate corollary of Propositions 3 and 6 is an analogue of Sen's (1966) Theorem 1; whereas he assumes that the number of "concerned" individuals is odd for each triple, we impose concern over triples to capture non-majoritarian social preferences.

Corollary 1. *Assume π is value-restricted and satisfies concern over triples. If P is resolute, then it is negatively transitive.*

Example 8 shows that weak value restriction is not sufficient for exclusion condition NT or, for that matter, transitivity of weak social preference, even when social preferences are simple and resolute and concern over triples is satisfied. Dropping concern over triples, Example 9 shows that value restriction is not by itself sufficient for exclusion condition NT or, for that matter, transitivity of weak social preferences, even when social preferences are simple and resolute.

Next, we investigate the possibility of transitive strict social preference. In short, weak value restriction implies exclusion condition T.

Proposition 7. *If π is weakly value-restricted, then it satisfies exclusion condition T.*

Proof. Consider any distinct $x_1, x_2, x_3 \in X$, and suppose toward a contradiction that (4)–(6) hold for some i, j, k. Setting $Y = \{x_1, x_2, x_3\}$, note that $\{i, j, k\} \subseteq N^Y$, and we have $V_i^Y(x_1) \cup V_j^Y(x_1) \cup V_k^Y(x_1) = \{1, 2, 3\}$, $V_i^Y(x_2) \cup V_j^Y(x_2) \cup V_k^Y(x_2) = \{1, 2, 3\}$, and $V_i^Y(x_3) \cup V_j^Y(x_3) \cup V_k^Y(x_3) = \{1, 2, 3\}$. Moreover, we have $V_i^Y(x_2) = \{2\}$, $V_j^Y(x_2) = \{3\}$, and $V_k^Y(x_2) = \{1\}$, contradicting weak value restriction. □

As a corollary of Propositions 2 and 7, it follows that when social preferences are simple, weak value restriction immediately delivers transitivity of strict social preference; concern over triples is unneeded.

Corollary 2. *Assume π is weakly value-restricted. If P is simple, then it is transitive.*

The assumption that social preferences are simple in Corollary 2 cannot be weakened to the assumption that they are merely relatively simple, even when social preferences are resolute and concern over triples is satisfied, as demonstrated in Example 5. There, individual preferences are weakly value-restricted and satisfy concern over triples, the number of individuals is odd, yet the relative majority strict preference P_{RM} is intransitive, in fact cyclic. Indeed, to verify that weak value restriction is satisfied, note that $V_i^X(a) \neq \{2\}$ for all i, $V_i^X(b) \neq \{1\}$ for all i, and $V_i^X(c) \neq \{2\}$ for all i.

Of course, Proposition 6 establishes that value restriction and concern over triples together imply exclusion condition T*, so by strengthening the assumption on individual preferences and invoking Proposition 4, we obtain a corollary on transitivity of P when social preferences are relatively simple.

Corollary 3. *Assume π is value-restricted and satisfies concern over triples. If P is relatively simple, then it is transitive.*

Nevertheless, Corollary 3 invites speculation that under value restriction and the stronger assumption of resoluteness, transitivity of P may hold even without the assumption of concern over triples. In fact, Sen's (1969) Theorem 8 (and proof thereof) establishes that Corollary 3 extends to relative majority rule when individual preferences are value-restricted, even when concern over triples is violated and the number of individuals is even. Sen's result exploits special consistency properties of relative majority rule, however, and it does not hold for all relatively simple social preferences. Indeed, the next example modifies Example 3 by using non-neutral tie breaking to show that the transitivity properties of resolute social preferences can be quite poor, even when individual preferences are value-restricted, if concern over triples is not satisfied.

Example 10. Assume $n = 3$ and $X = \{a, b, c\}$, and define P as a variation of simple majority rule with non-neutral tie-breaking. Specifically, define $i(a, b) = 1$, $i(a, c) = 2$, and $i(b, c) = 3$, and specify

$$xPy \quad \Leftrightarrow \quad \begin{array}{c} xP_{SM}y \text{ or in case } (x, y) \in \{(a, b), (a, c), (b, c)\}, \\ \text{both } xR_{SM}y \text{ and } xP_{i(x,y)}y \text{ hold} \end{array}$$

for all $x, y \in X$. This entails resolute, but not necessarily simple, social preferences. Now consider the following profile of preferences.

1	2	3
c		b
a	abc	c
b		a

This profile is value-restricted, as in Example 9, but $aPbPcPa$, violating acyclicity.

Because Example 5 uses a profile that violates value restriction, and because Example 10 uses a social preference relation that is non-neutral, there is the possibility that Corollary 3 may yet carry over to neutral-monotonic social preferences when individual preferences are value-restricted, even when concern over triples fails. The next result establishes that this is so, generalizing Theorem 8 of Sen (1969) and providing a single-profile version of Sen and Pattanaik's (1969) Theorem 1. I use the fact that if P is neutral-monotonic, then it is relatively simple, and the result is then deduced from

Corollary 3 by considering each triple of alternatives separately, deleting any individuals who are indifferent over the triple, and applying the earlier proposition.

Proposition 8. *Assume π is value-restricted. If P is neutral-monotonic, then it is transitive.*

Proof. Consider any three alternatives $a, b, c \in X$ such that $aPbPc$, let $Z = \{a, b, c\}$, and define the restricted profile $\pi|_Z = (P_1|_Z, \ldots, P_n|_Z)$, which is value-restricted. Then the restricted social preference $P|_Z$ is neutral-monotonic relative to $\pi|_Z$, and by Lemma 2, it is therefore relatively simple with respect to $\pi|_Z$. With $aP|_Z bP|_Z c$, Corollary 3 implies $aP|_Z c$ if $\pi|_Z$ satisfies concern over triples, i.e., no individual is indifferent over Z, and thus aPc. Suppose that $J = \{i \in N \mid aI_i bI_i c\}$ is nonempty, and let $\pi' = (P_i)_{i \in N \setminus J}$ be the profile of preferences of individuals who are not indifferent over Z.

I claim that $P|_Z$ is relatively simple with respect to π'. To see this, consider any $x, y \in Z$ such that $xP|_Z y$. Letting $R'(x, y) = \{i \in N \setminus J \mid xR_i y\}$, it suffices to show that $R'(x, y)$ belongs to the collection $\mathcal{D}'(P)$ of decisive groups given π' and $P|_Z$. To this end, consider any $w, z \in Z$, let $P'(w, z) = \{i \in N \setminus J \mid wP_i z\}$, and assume $R'(x, y) \subseteq P'(w, z)$. Note that

$$P(x, y) \subseteq R'(x, y) \subseteq P'(w, z) = P(w, z)$$

and

$$R(x, y) = R'(x, y) \cup J \subseteq P'(w, z) \cup J \subseteq R(w, z),$$

and since P is neutral-monotonic, xPy implies wPz. Thus, $R'(x, y) \subseteq \mathcal{D}'(P)$, as claimed.

Since π' is value-restricted and satisfies concern over triples, Corollary 3 implies that $P|_Z$ is transitive. With $aP|_Z bP|_Z c$, this implies $aP|_Z c$, or equivalently, aPc, as required. □

The above result is an analogue of Sen and Pattanaik's (1969) Theorem 1 on transitivity of strict social preferences generated by neutral, monotonic rules. I do not, however, attempt to generalize their Theorem 2 (limited agreement is sufficient for transitivity of strict social preference) or their Theorem 4 (extremal restriction is sufficient for transitivity of weak social preference). Those authors show that if the number of voters is allowed to vary, with the voters' preferences belonging to a common admissible set, then a necessary condition for strict relative majority preferences to always be acyclic is that for every profile of admissible voter preferences and every triple of alternatives, the profile satisfies either value restriction, limited agreement, or extremal restriction over that triple. The sufficient condition for acyclic social preference in Proposition 5 does not imply their necessary condition, an apparent inconsistency that is resolved upon noting that relative majority social preferences are not generally simple.

4.2 Single-peakedness

Next, we explore the familiar condition of single-peakedness, which imposes natural restrictions on individual preferences when triples of alternatives can be ordered along a single dimension. Using the definition of Inada (1964), the preference profile π is *single-peaked* if for all distinct $a, b, c \in X$, there is a linear order $\prec$ of $\{a, b, c\}$ such that for all $i \in N$ and all $x, y, z \in \{a, b, c\}$, if xR_iy and either $x \prec y \prec z$ or $z \prec y \prec x$, then yP_iz. This definition refers to a linear order that depends on the triple of alternatives, but sometimes a narrower definition that fixes a linear order of X independently of the triple is used in the literature. In the one-dimensional spatial model, $X \subseteq \mathbb{R}$ is convex and each P_i has utility representation $u_i\colon X \to \mathbb{R}$, and under these conditions, it is straightforward to show that the profile $(P_1, \ldots, P_n)$ is single-peaked with respect to the usual greater than relation if and only if for all $i \in N$, u_i is strictly quasi-concave.[6]

As a matter of interest, it is straightforward to show that the definition with a fixed linear order is strictly narrower than the classical definition. Assume $X = \{a, b, c, d\}$ and $n = 2$, and consider the following preference profile.

1	2
a	a
b	d
c	c
d	b

This profile is single-peaked with respect to the definition here: given a subset of three alternatives, if a belongs to the set, we use a linear order with a in the middle; and otherwise, use a linear order with c in the middle. This profile is not, however, single-peaked with respect to any linear ordering of X fixed independently of the triple.[7]

[6] The definition of single-peakedness in the spatial model sometimes also requires existence of an ideal point, which maximizes u_i. When preferences are single-peaked and X is convex, an ideal point must be unique, but in general the definition used here allows for two ideal points. For example, if $X = \{a, b, c\}$ and aI_ibP_ic for each individual, then π is single-peaked.

[7] For such a linear ordering $\prec$, because d and b are worst for individuals 1 and 2,

As is well-known, single-peakedness implies value-restriction and concern over triples, a fact that is recorded next. The proof follows immediately by noting that under single-peakedness, the middle alternative according to $\prec$ attains a value of one for no individuals.

Proposition 9. *If π is single-peaked, then it satisfies value restriction and satisfies concern over triples.*

A direct implication of Corollary 1 and Proposition 9 is that when individual preferences are single-peaked, resoluteness is sufficient for transitivity of weak social preference. The following corollary is a single-profile generalization of Arrow's (1951) Theorem 4 on the method of majority decision to the class of resolute social preferences.

Corollary 4. *Assume π is single-peaked. If P is resolute, then it is negatively transitive.*

That resoluteness is needed for Corollary 4, even if P is simple, can be seen from Example 2, in which individual preferences are single-peaked with respect to the ordering $b \prec a \prec c$, but the simple majority weak preference is intransitive.

Replacing the assumption of resoluteness in Corollary 4 with the weaker assumption that social preferences are relatively simple, we obtain the conclusion that strict social preferences are transitive. The next result follows immediately from Corollary 3 and Proposition 9. It is the single-profile generalization of Theorem 4.1 of Austen-Smith and Banks (1999), which considers social preferences generated by a voting rule.

Corollary 5. *Assume π is single-peaked. If P is relatively simple, then it is transitive.*

Again, Example 2 shows that the assumption that social preferences are relatively simple, or even simple, is not sufficient for the stronger conclusion of transitivity of weak social preference.

4.3 Weak Single-peakedness

The preceding analysis shows that single peakedness is sufficient for transitivity of social preferences for quite general social preferences, but a more

respectively, these alternatives must be at the extremes of $\prec$. Then up to reversals, the linear ordering must be $b \prec a \prec c \prec d$ or $d \prec a \prec c \prec b$. The first case is precluded by individual 2, and the second is precluded by individual 1.

fundamental rationality condition is acyclicity of strict social preference. Austen-Smith and Banks (1999) define a relaxation of single peakedness that has a natural interpretation in one-dimensional spatial models and yet is sufficient for acyclicity. Let $\prec$ be a weak linear order of the set X of alternatives. We say $\pi = (P_1, \ldots, P_n)$ is *weakly single-peaked* with respect to $\prec$ if for each $i \in N$, there exists $\tilde{x}_i \in X$ such that

(1) for all $y \in X \setminus \{\tilde{x}_i\}$, we have $\tilde{x}_i R_i y$;

(2) for all $y, z \in X$, $\tilde{x}_i \prec y \prec z$ implies $y R_i z$;

(3) for all $y, z \in X$, $y \prec z \prec \tilde{x}_i$ implies $z R_i y$.

Clearly, assuming $X \subseteq \mathbb{R}$ is convex and each P_i has utility representation $u_i \colon X \to \mathbb{R}$, the profile $(P_1, \ldots, P_n)$ is weakly single-peaked with respect to the usual less than relation if and only if for all $i \in N$, u_i is quasi-concave.

The next proposition establishes that weak single-peakedness implies exclusion condition A, which is a key sufficient condition for acyclicity of social preferences.

Proposition 10. *If π is weakly single-peaked, then it satisfies exclusion condition A.*

Proof. Let $\pi = (P_1, \ldots, P_n)$ be weakly single-peaked with respect to $\prec$. Consider any $m \geq 3$ and any distinct $x_1, \ldots, x_m \in X$. In case $m = 3$, suppose toward a contradiction that there exist $i, j, k \in N$ such that $x_1 P_i x_2 P_i x_3$, $x_2 P_j x_3 P_j x_1$, and $x_3 P_k x_1 P_k x_2$. Assume without loss of generality that $x_1 \prec x_2 \prec x_3$. By weak single-peakedness, $x_1 P_k x_2$ implies that $\tilde{x}_k \prec x_2$, but similarly $x_3 P_k x_2$ implies that $x_2 \prec \tilde{x}_k$, a contradiction.

In case $m \geq 4$, I claim that there exists $\ell \in \{1, \ldots, m\}$ such that one of the following holds (addition is modulo m):

(a) $x_\ell \prec x_{\ell+1} \prec x_{\ell+2}$,

(b) $x_{\ell+2} \prec x_{\ell+1} \prec x_\ell$,

(c) $x_\ell \prec x_{\ell+2} \prec x_{\ell+1}$,

(d) $x_{\ell+1} \prec x_{\ell+2} \prec x_\ell$.

Note that (a) and (b) are symmetric, as are (c) and (d). To prove the claim, suppose otherwise, and without loss of generality assume $x_1 \prec x_2$. Then it must be that $x_3 \prec x_1 \prec x_2$, and similarly $x_3 \prec x_1 \prec x_2 \prec x_4$, and so on. Finally, we have either

$$x_{m-1} \prec \cdots \prec x_3 \prec x_1 \prec x_2 \prec x_4 \prec \cdots \prec x_m.$$

or

$$x_m \prec \cdots \prec x_3 \prec x_1 \prec x_2 \prec x_4 \prec \cdots \prec x_{m-1}$$

Setting $\ell = m - 1$, we then obtain either (c) or (d), respectively, because $\ell + 2 = m + 1 = 1 \,(\text{mod}\ m)$. This contradiction establishes the claim. Thus, we may select ℓ such that one of (a)–(d) hold.

To verify exclusion condition A, suppose toward a contradiction that there exist $i, j, k \in N$ such that $x_\ell P_i x_{\ell+1} P_i x_{\ell+2}$, $x_{\ell+1} P_j x_{\ell+2} R_j x_\ell$, and $x_{\ell+2} R_k x_\ell P_k x_{\ell+1}$. In case (a) holds, by weak single-peakedness, $x_{\ell+2} P_k x_{\ell+1}$ implies $x_{\ell+1} \prec \tilde{x}_k$, but similarly $x_\ell P_k x_{\ell+1}$ implies $\tilde{x}_k \prec x_{\ell+1}$, a contradiction. A contradiction is derived analogously in case (b) holds. In case (c) holds, by weak single-peakedness, $x_\ell P_i x_{\ell+2}$ implies $\tilde{x}_i \prec x_{\ell+2}$, but similarly $x_{\ell+1} P_i x_{\ell+2}$ implies $x_{\ell+2} \prec \tilde{x}_i$, a contradiction. A contradiction is derived analogously in case (d) holds. We conclude that for the selected ℓ, in each possible case (a)–(d), there do not exist i, j, k whose preferences over $\{x_\ell, x_{\ell+1}, x_{\ell+2}\}$ are given by (iv)–(vi) in the definition of exclusion condition A, as required. □

It follows from Propositions 5 and 10 that for a simple social preferences, weak single peakedness is sufficient for acyclicity. This observation is sated in the next corollary, which extends Theorem 4.2 of Austen-Smith and Banks (1999) to the single-profile framework.

Corollary 6. *Assume π is weakly single-peaked. If P is simple, then it is acyclic.*

Example 5 exhibits a preference profile that is weakly single-peaked with respect to $a \prec b \prec c$ and such that the strict relative majority preference is cyclic, despite the fact that n is odd. Thus, the assumption of simple social preferences in Corollary 6 cannot be replaced by the assumption of resoluteness. To see that weak single peakedness is not sufficient for transitive strict social preference, even for simple majority rule, note that individual preferences in Example 5 are weakly single-peaked with respect to $b \prec a \prec c$, yet the strict simple majority preference is intransitive.

4.4 Order Restriction

The preference restrictions in the previous subsections rely on the ordering of alternatives; in the spirit of Rothstein (1990,1991) and Gans and Smart (1996), we next explore the implications of a restriction that instead involves the ordering of individuals. We say the preference profile π is *order restricted* if for all distinct $a, b, c \in X$, there is a bijection $\phi\colon N \to N$ such that for all $x, y \in \{a, b, c\}$, either

$$\phi(P(x,y)) < \phi(I(x,y)) < \phi(P(y,x))$$

or

$$\phi(P(y,x)) < \phi(I(x,y)) < \phi(P(x,y)),$$

where $G < H$ means $i < j$ for all $i \in G$ and all $j \in H$. (Here, use the convention that the empty set is less than and greater than all groups.) In words, for each pair of alternatives, x and y, the group of individuals with any given preference between them is "connected" with respect to the ordering induced by ϕ.

Analogous to single peakedness, the definition of order restriction used in this this paper weakens the concept defined by the above authors by allowing the ordering of individuals to depend on the triple of alternatives considered. As a matter of interest, it is straightforward to show that the definition with a fixed ordering of individuals is strictly narrower than the one used here. Assume $X = \{a, b, c, d\}$ and $n = 3$, and consider the following preference profile.

1	2	3
a	d	c
b	b	b
d	c	d
c	a	a

This profile is order restricted with respect to the definition here: if the triple of alternatives is $\{a, b, c\}$ or $\{a, c, d\}$, then use $1 \prec 2 \prec 3$; if the triple is $\{a, b, d\}$, then use $1 \prec 3 \prec 2$; and if it is $\{b, c, d\}$, then use $2 \prec 1 \prec 3$. This profile is not, however, order restricted with respect to any linear ordering of individuals fixed independently of the triple.[8]

[8] Note that individual 1 prefers a to c, but the others have the opposite preference; individual 2 prefers d to b, but the others do not; and individual 3 prefers c to b, but the others do not. Thus, no individual can be middle ranked in a fixed ordering of individuals.

Order restriction engenders desirable transitivity properties in light of the fact that it implies value restriction. This result is stated in Theorem 2 of Rothstein (1990).[9]

Proposition 11. *If π is order-restricted, then it is value-restricted.*

Proof. Let π be order-restricted, consider any triple $Z = \{a, b, c\} \subseteq X$, and assume without loss of generality that ϕ is the identity function, i.e., preferences are order restricted with respect to the order in which individuals are indexed. Suppose toward a contradiction that for all $x \in Z$ and all $r \in \{1, 2, 3\}$, there exists $i \in N^Z$ such that $r \in V_i^Z(x)$. Since individuals who are indifferent among alternatives in Z play no role in the analysis, assume for simplicity that $N^Z = N$. Since individual 1 is concerned over Z, we can assume, by permuting a, b, c if necessary, that aP_1b, that $3 \in V_1^Z(a)$, and that $1 \in V_1^Z(b)$. Then aR_1cR_1b. Since a and b attain all values in Z, by supposition, it cannot be that all individuals have preferences over Z identical to 1's. Let j be the lowest indexed individual such that $P_j|_Z \neq P_1|_Z$. Then P_j must be characterized by a preference reversal among alternatives in Z that involve c.

We first consider the possible preference reversals between c and a, depending on the initial position of c relative to b. The first five cases involve c moving up relative to a. Case 1: $cP_{j-1}b$ and cP_ja. Then $V_i^Z(c) \subseteq \{2, 3\}$ for all $i = 1, \ldots, j-1$, and by order restriction, we have cP_ia for all $i = j, \ldots, n$. Therefore, $V_i^Z(c) \subseteq \{2, 3\}$ for all $i = j, \ldots, n$, which implies $1 \notin \bigcup_{i \in N^Z} V_i^Z(c)$, so c does not attain value 1. Case 2: $cI_{j-1}b$ and cP_jaR_jb. By order restriction, we have cP_ib for all $i = j, \ldots, n$, and therefore b does not attain value 3. Case 3: $cI_{j-1}b$, cP_ja, and bP_ja. By order restriction, a does not attain a value of 2. Case 4: $aP_{j-1}c$ and bP_jcI_ja. By order restriction, c does not attain value 3. Case 5: $aP_{j-1}c$ and cI_jaP_jb. Since c attains value 1, there exists $k > j$ such that bP_kcI_ka, and by order restriction and the assumption that all agents are concerned, we can assume cI_iaP_ib for $i = j, \ldots, k-1$. But then b does not attain value 2. Case 6: $cI_{j-1}a$ and aP_jc. By order restriction, a does not attain value 1.

The remaining possibilities concern preferences reversals between c and b. Note, however, that since π is order restricted, so is the profile in which each individual i's preference is the inverse of R_i. Then a preference reversal between c and b in the initial profile is analogous to a reversal between c and

[9] Rothstein's (1990) proof is incomplete, however. He states that if value-restriction is violated, then there exist three alternatives and three individuals in a specific relationship, but he does not prove the assertion.

a in the inverse profile, and the previous argument can be applied to deduce a contradiction in all possible cases. We conclude that π is value-restricted, as required. □

An immediate implication of Corollary 1 and Proposition 11 is that resoluteness implies transitivity of weak social preference when individual preferences are order-restricted and concern over triples is satisfied. The next proposition goes beyond this simple observation by showing that concern over triples can be dropped if social preferences are actually simple as well.

Proposition 12. *Let P be resolute. Assume that π is order-restricted and that either P is simple or π satisfies concern over triples. Then P is negatively transitive.*

Proof. If π satisfies concern over triples, then the result follows from Corollary 1 and Proposition 11. Assume that P is simple, as well as resolute, and consider any $x, y, z \in X$ such that $xRyRz$. By Lemma 1, we have $\{R(x, y), R(y, z)\} \subseteq \mathcal{B}(P)$, and by order restriction we can assume $R(x, y) = \{1, 2, \dots, k\}$ for some k. If $R(y, z) = \{1, \dots, \ell\}$ for some ℓ, then either $R(x, y) \cap R(y, z)$ equals $R(x, y)$ or it equals $R(y, z)$. In either case, $R(x, y) \cap R(y, z) \in \mathcal{B}(P)$. Then for all $i \in R(x, y) \cap R(y, z)$, we have $xR_i yR_i z$, which implies $xR_i z$, and therefore xRz.

Otherwise, $R(y, z) = \{\ell, \ell + 1, \dots, n\}$ for some ℓ. Now suppose toward a contradiction that zPx. Then because P is simple, we have $P(z, x) \in \mathcal{D}(P)$. Since P is resolute, we also have $\{R(x, y), R(y, z)\} \subseteq \mathcal{D}(P)$, and by the above argument (depending on whether $P(z, x)$ contains individuals 1 or n), we have either $R(x, y) \cap P(z, x) \in \mathcal{D}(P)$ or $R(y, z) \cap P(z, x) \in \mathcal{D}(P)$. In the latter case, we have $yR_i zP_i x$, and therefore $yP_i x$, for all $i \in R(y, z) \cap P(z, x)$, but this implies yPx, a contradiction; and in the former case, we have $zP_i xR_i y$, and therefore $zP_i y$, for all $i \in R(x, y) \cap P(z, x)$, but this implies zPy, a contradiction. We conclude that xRz, as required. □

That resoluteness is needed for Proposition 12, even if P is simple and concern over triples is satisfied, can be seen from Example 2, where individual preferences are trivially order restricted but the simple majority weak preference is intransitive. The following example demonstrates that concern over triples is needed for the result when P is not simple, even if it is resolute.

Example 11. Assume $n = 3$ and $X = \{a, b, c\}$, with preferences below.

1	2	3
a		c
b	abc	ab
c		

This profile is order-restricted, because $2 \notin V_i^X(c)$ for concerned individuals $i = 1, 3$, and relative majority rule is resolute since n is odd, but we have $aP_{RM}b$ and $bR_{RM}cR_{RM}a$, violating transitivity of weak preference.

As a corollary of Propositions 8 and 11, we obtain the result that if individual preferences are order-restricted and social preferences are neutral-monotonic, then the strict social preference relation is transitive. This is the single-profile version of Theorem 4.6 of Austen-Smith and Banks (1999), which considers social preferences generated by a neutral and monotonic preference aggregation rule. This result is established for relative majority rule in Theorems 1 and 2 of Rothstein (1990) and in Corollary 1 of Gans and Smart (1996). Of course, the result extends to simple social preferences, via Lemma 2.

Corollary 7. *Assume π is order-restricted. If P is neutral-monotonic, then it is transitive.*

By Corollary 3 and Proposition 11, the transitivity result also holds for relatively simple social preferences when no individuals are indifferent over any triples of alternatives.

Corollary 8. *Assume π is order-restricted and satisfies concern over triples. If P is relatively simple, then it is transitive.*

To show that concern over triples cannot be dropped from Corollary 8, we modify the preference profile in Example 10 for majority rule with non-neutral tie-breaking.

Example 12. Assume $n = 3$ and $X = \{a, b, c\}$, and define P as in Example 10, with preferences below.

1	2	3
a		b
c	abc	c
b		a

Here, individual preferences are order-restricted, but we have $aPbPc$ and cRa, violating transitivity of strict social preference.

5 Aggregation Procedures

Thus far, the analysis has fixed a profile π of individual preferences, and concepts of decisive and blocking groups and simple, relatively simple, and resolute social preference have been defined relative to π. The literature often, however, takes as given an aggregation rule defined on a domain of possible profiles and considers the social preferences generated by such a rule across possible preference profiles. Then decisive groups are typically defined across profiles: we would say a group is decisive if for every profile, if the members of the group strictly prefer one alternative to another, then the first alternative is strictly socially preferred to the second. In this section, I extend the earlier concepts to define simple, relatively simple, and resolute social preference rules in a manner that generalizes majority rule (and variants thereof), and I apply the single-profile results established above to the standard framework.

Let Θ denote a set of states, where each state θ determines a profile $\pi(\theta) = (P_1(\theta), \ldots, P_n(\theta))$ of individual preferences. Let $P_i(\theta)$ denote individual i's strict preference relation in state θ, assume $P_i(\theta)$ is an asymmetric and negatively transitive relation on X, and let $R_i(\theta)$ be the corresponding weak preference relation. Following earlier conventions, define

$$\begin{aligned} P(x, y|\theta) &= \{i \in N \mid xP_i(\theta)y\} \\ R(x, y|\theta) &= \{i \in N \mid yP_i(\theta)x\}. \end{aligned}$$

A *social preference rule* (SPR), denoted F, is a mapping $\theta \overset{F}{\mapsto} P_F(\theta)$, where $P_F(\theta)$ is the strict social preference relation determined by F in state θ. Assume $P_F(\theta)$ is asymmetric, and let $R_F(\theta)$ be the corresponding weak social preference. In addition, I adopt the following *maintained assumption:* there exist $x, y \in X$ such that for all $G \subseteq N$, there exists $\theta \in \Theta$ such that $P(x, y|\theta) = G$ and $P(y, x|\theta) = N \setminus G$; such a pair $\{x, y\}$ is a "free pair."

A group G is *decisive* for F if for all $\theta \in \Theta$ and all $x, y \in X$, $G \subseteq P(x, y|\theta)$ implies $xP_F(\theta)y$, and G is *blocking* for F if for all $\theta \in \Theta$ and all $x, y \in X$, $G \subseteq R(x, y|\theta)$ implies $xR_F(\theta)y$. Let $\mathcal{D}(F)$ denote the collection of groups decisive for F, and let $\mathcal{B}(F)$ be the blocking groups. We say F is...

- *simple* if for all $\theta \in \Theta$ and all $x, y \in X$, $xP_F(\theta)y$ implies $P(x, y|\theta) \in \mathcal{D}(F)$,
- *resolute* if for all $\theta \in \Theta$ and all $x, y \in X$, $xR_F(\theta)y$ implies $R(x, y|\theta) \in \mathcal{D}(F)$,

- *relatively simple* if for all $\theta \in \Theta$ and all $x, y \in X$, $xP_F(\theta)y$ implies $R(x, y|\theta) \in \mathcal{D}(F)$.

By definition, if G is decisive for F, then for every state θ, G is decisive for $P_F(\theta)$. It follows immediately that if F is resolute, then for every state θ, $P_F(\theta)$ is resolute; and likewise, if F is relatively simple, then $P_F(\theta)$ is relatively simple. The simple SPRs coincide with those of Austen-Smith and Banks (1999), while the relatively simple SPRs include the voting rules of Austen-Smith and Banks (1999) as special cases.

The next lemma establishes that simple SPRs generate simple social preferences, as the terminology suggests.

Lemma 4. *Let F be a simple SPR. Then for all θ, $P_F(\theta)$ is simple.*

Proof. Consider any $\theta \in \Theta$ and any $x, y \in X$ such that $xP_F(\theta)y$. It follows immediately from the definition of simple SPR that $P(x, y|\theta)$ is decisive for $P_F(\theta)$. To see that the group is also blocking for $P_F(\theta)$, consider any $w, z \in X$ with $G \subseteq R(w, z|\theta)$, and suppose toward a contradiction that $zP_F(\theta)w$. Since F is simple, it follows that $H = P(z, w|\theta) \in \mathcal{D}(F)$. Furthermore, $G \cap H = \emptyset$, and letting $\{a, b\}$ be any free pair, there is a state θ' such that $G = P(a, b|\theta')$ and $N \setminus G = P(b, a|\theta')$. But then $G \in \mathcal{D}(F)$ implies $aP_F(\theta')b$, and $H \in \mathcal{D}(F)$ implies $bP_F(\theta')a$, contradicting asymmetry of $P_F(\theta')$. We conclude that $wR_F(\theta)z$, as desired. □

The preceding observations permit the application of the fixed profile results of Section 3 to the current setting. Define the following domain restrictions.

- *NT-domain:* for all θ, $\pi(\theta)$ satisfies exclusion condition NT,
- *T-domain:* for all θ, $\pi(\theta)$ satisfies exclusion condition T,
- *T^*-domain:* for all θ, $\pi(\theta)$ satisfies exclusion condition T^*,
- *A-domain:* for all θ, $\pi(\theta)$ satisfies exclusion condition A.

We can now extend the results of Section 3 to social preference rules defined on restricted domains. The corollaries are immediate and stated without proof.

Corollary 9. *Assume NT-domain holds. If F is simple and resolute, then for all $\theta \in \Theta$, $P_F(\theta)$ is negatively transitive.*

Corollary 10. *Assume T-domain holds. If F is simple, then for all* $\theta \in \Theta$, $P_F(\theta)$ *is transitive.*

Corollary 11. *Assume* T^**-domain holds. If F is relatively simple, then for all* $\theta \in \Theta$, $P_F(\theta)$ *is transitive.*

Corollary 12. *Assume A-domain holds. If F is simple, then for all* $\theta \in \Theta$, $P_F(\theta)$ *is acyclic.*

6 Conclusion

This paper provides sufficient conditions for different forms of social rationality, including acyclicity of strict social preferences, and applies them to familiar domains including value-restricted, single-peaked, and order-restricted preference profiles. Advantages of the analysis are that results are stated for general classes of social preference, containing variants of majority rule and other examples as special cases, and that it takes as given a single preference profile. It does not rely on properties of social preferences as individual preferences are varied, and the results can be applied to social preferences generated by the simple rules and voting rules of Austen-Smith and Banks (1999). In addition, it is shown that the latter authors' acyclicity result for weakly single-peaked preferences stems from a deeper restriction on individual preferences, i.e., exclusion condition A.

The results on transitivity properties of social preferences and their application to single-peaked and order-restricted domains have not extracted some extremely useful implications of the latter two preference restrictions. The analysis has defined single peakedness in a general fashion, letting the ordering of alternatives depend on the triple (of alternatives) considered; and likewise, the definition of order restriction lets the ordering of individuals depend on the triple (of individuals) considered. In many applications, these preference restrictions are defined with respect to an ordering that is fixed independently of the triple considered, a strengthening that has useful implications. First, in the case of single-peaked preferences, such "globally" single-peaked preference profiles permit a characterization of the maximal elements of majority rule, i.e., the majority core, in terms of the ideal points of individuals; this is the well-known median voter theorem of Black (1948). Second, in the case of globally order-restricted preferences, Rothstein (1991) and Gans and Smart (1996) prove that the majority preference relation coincides with that of the median voter, providing a corresponding representative voter theorem. The median voter theorem is generalized by

Austen-Smith and Banks (1999) to simple aggregation rules, and they give a partial representative voter theorem for monotonic and neutral aggregation rules. Extensions of these results to general classes of social preference in the single-profile framework are contained in Duggan (2013), which is otherwise superseded by the current paper.

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