A Note on Pareto Manifolds in the Spatial Model of Politics

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September 6, 2016

This note works out some properties of Pareto optimal alternatives—or what I call the infinitesimal Pareto optimal alternatives—in a model that subsumes the spatial model of political science. These results could be known, but their derivation appears to be difficult to find in the literature. By an infinitesimal Pareto optimum, I mean an alternative x at which zero can be written as a positive linear combination of the gradients of the agents, i.e.,

$$\sum_{i=1}^{n} \alpha_i \nabla u_i(x) \quad = \quad 0$$

for some vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{++}$ of strictly positive multipliers. This set, here denoted PO^*, typically forms a subset of the strong Pareto optima, and it is relatively amenable to techniques from differential topology.

Smale (1976) has shown that under standard conditions in an exchange economy, the Pareto optimal set is a manifold with dimension equal to the number of consumers minus one, and he has investigated the structure of this set. His result assumes, however, that consumption is private and preferences are monotonic. It is also known that for generic utility functions in exchange economies, the infinitesimal Pareto optima have a nice manifold structure; see the "main proposition" of Smale (1974, p.113) for the generic manifold structure of his infinitesimal Pareto optima in an exchange economy.[1]

The goal here is to assume a general framework that includes the spatial model of politics as a special case (i.e., there may be no transferable private good), to take utility functions as given, and to find general restrictions on those utilities that are sufficient for the manifold structure of the Pareto optima. Or, rather, given utility functions and a particular infinitesimally Pareto optimal alternative x, I'm concerned that there is an open set around x in which PO^* has a manifold structure. Beyond that, I would like to understand how we can parameterize PO^* around x, and I'm interested in the properties of parameterized utility functions. Such are the goals of this note.

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[1] See also the discussion of Smale (1975) on pp.349–350.

1 Preliminaries

Consider $X \subseteq \mathbb{R}^m$ and a differentiable mapping $f\colon X \to \mathbb{R}^n$, I use the notation $Df(x)$ for the $n \times m$ Jacobian matrix of f evaluated at x. Thus, if f is real-valued, then $Df(x)$ is a $1 \times m$ row matrix. If the derivative is calculated only letting a subset of coordinates of x vary, I indicate this by subscripting the derivative operator. I view the gradient $\nabla f(x)$ as a column vector and do not distinguish between it and a $m \times 1$ matrix with the same entries, i.e., $\nabla f(x)$ is just the transpose of $Df(x)$. When applicable, $D^2 f(x)$ denotes the Hessian matrix of f at x, and given a function whose domain has a product structure, $D_1 f(x, y)$ and $D_2 f(x, y)$ indicate the derivatives with respect to x and y, respectively.

The analysis takes as given n functions $u_i\colon X \to \mathbb{R}$, $i = 1, \ldots, n$, which are assumed to be differentiable. We interpret $N = \{1, \ldots, n\}$ as a set of decision making agents, X as a set of alternatives, and u_i as the utility function of an agent i. A special case is that $X = \mathbb{R}^d$ and each u_i is quadratic, i.e., there is an ideal point $\hat{x}^i \in X$ such that $u_i(x) = -\|x - \hat{x}^i\|^2$ for all x; more generally, X may be a general issue space and u_i may be strictly quasi-concave with a unique ideal point, as in the spatial model of politics. Assuming utility functions are C^2, say utilities are *negative definite in aggregate* at x if for all $\alpha \in \mathbb{R}^n_{++}$ such that $\sum_{i=1}^n \alpha_i \nabla u_i(x) = 0$, the weighted sum of Hessian matrices, $\sum_{i=1}^n \alpha_i D^2 u_i(x)$, is negative definite at x. A sufficient condition for this is that each $D^2 u_i(x)$ is negatively semi-definite, and that some $D^2 u_i(x)$ is negative definite.[2]

An alternative x is *Pareto optimal* if there is no $y \in X$ such that for all i, $u_i(y) > u_i(x)$. It is *strongly Pareto optimal* if there is no $y \in X$ such that $u_i(y) \geq u_i(x)$, with strict inequality for at least one i. It is *infinitesimally Pareto optimal* if $x \in \text{int}X$ and there exist coefficients $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{++}$ such that

$$\sum_{i=1}^n \alpha_i \nabla u_i(x) \quad = \quad 0.$$

Letting $PO^\bullet$, PO°, and PO^* denote these sets, respectively, we clearly have $PO^\circ \subseteq PO^\bullet$.

An easy implication of the separating hyperplane theorem is that at each Pareto optimal alternative, zero is contained in the convex cone generated by the gradients of the agents. In particular, the gradients of the agents are linearly dependent.

Lemma 1 *For $x \in X$, if $x \in PO^\bullet \cap \text{int}X$, then there exist non-negative coefficients $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\sum_{i=1}^n \alpha_i \nabla u_i(x) = 0$.*

Proof: Assume $x \in$ X is Pareto optimal, and suppose there do not exist such weights. Then $0 \notin \text{co}\{\nabla u_i(x) \mid i \in N\}$. By the separating hyperplane theorem,

[2] In exchange economies, due to the fact that consumption is private, negative definiteness of $D^2 u_i(x)$ will not hold for the agents, but aggregate negative definiteness remains a weak condition.

there exists $p \in \mathbb{R}^n$ such that $p\nabla u_i(x) > 0$ for all i. Then for $\epsilon > 0$ sufficiently small, we have $x + \epsilon p \in X$ and for all i, $u_i(x + \epsilon p) > u_i(x)$, contradicting the Pareto optimality of x. ∎

Say u_i is *pseudo-concave at* x if for all $y \in X$, $u_i(y) > u_i(x)$ implies $\nabla u_i(x) \cdot (y - x) > 0$. It is *pseuodo-concave around* x if there is an open neighborhood $G \subseteq \mathbb{R}^d$ containing x such that u_i is pseudo-concave at every $z \in G \cap X$. It is *pseudo-concave* if it is pseudo-concave at every $x \in X$. Say u_i is *strictly pseudo-concave at* x if for all $y \in X \setminus \{x\}$, $u_i(y) \geq u_i(x)$ implies $\nabla u_i(x) \cdot (y - x) > 0$. As a slight digression, the next lemma presents a converse to Lemma 1.

Lemma 2 *For $x \in X$, assume each u_i is pseudo-concave at x. If there exist non-negative coefficients $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\sum_{i=1}^n \alpha_i \nabla u_i(x) = 0$, then $x \in PO^\bullet$.*

Proof: Assume there exist such coefficients, and suppose x is not Pareto optimal, i.e., there exists $y \in X$ such that for all i, $u_i(y) > u_i(x)$. By pseudo-concavity, we have $\nabla u_i(x) \cdot (y - x) > 0$ for all i, but then

$$\left(\sum_{i=1}^n \alpha_i \nabla u_i(x) \right) \cdot (y - x) \;=\; \sum_{i=1}^n \alpha_i (\nabla u_i(x) \cdot (y - x)) \;>\; 0,$$

a contradiction. ∎

Our interest is in PO^*, a subset of the strong Pareto optima, but we note in passing that under strict pseudo-concavity, the Pareto optima coincide with the strong Pareto optima.

Lemma 3 *For $x \in X$, assume each u_i is strictly pseudo-concave at x. Then $x \in PO^\bullet$ if and only if $x \in PO^\circ$.*

Proof: One direction is immediate. For the other, suppose x is Pareto optimal, but not strongly so, i.e., there exists $y \in X$ such that $u_i(y) \geq u_i(x)$ for all i, with at least one inequality strict. By Lemma 1, there are coefficients $\alpha \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\sum_{i=1}^n \alpha_i \nabla u_i(x) = 0$. But strict pseudo-concavity implies that $\nabla u_i(x) \cdot (y - x) > 0$ for all i, and then again we have

$$\left(\sum_{i=1}^n \alpha_i \nabla u_i(x) \right) \cdot (y - x) \;=\; \sum_{i=1}^n \alpha_i (\nabla u_i(x) \cdot (y - x)) \;>\; 0,$$

a contradiction. ∎

Under pseudo-concavity, the infinitesimally Pareto optimal alternatives, which are defined locally, are actually strongly Pareto optimal.

Lemma 4 *For $x \in X$, assume that each u_i is pseudo-concave around x. If $x \in PO^*$, then $x \in PO^\circ$.*

Proof: Suppose x is infinitesimally Pareto optimal but not strongly so, and let $y \in X$ be such that $u_i(y) \geq u_i(x)$, with at least one strict inequality. By pseudo-concavity at x, if $u_i(y) > u_i(x)$, then of course $\nabla u_i(x) \cdot (y - x) > 0$. I claim that if $u_i(y) = u_i(x)$, then $\nabla u_i(x) \cdot (y - x) \geq 0$, for suppose not. Then $\nabla u_i(x) \cdot (y - x) < 0$, and thus i's utility is decreasing in direction $y - x$. Thus, we can choose $\epsilon > 0$ small enough that $z = (1 - \epsilon)x + \epsilon y \in X$ and $u_i(x) > u_i(z)$. Moreover, because u_i is pseudo-concave around x, we can choose ϵ small enough that u_i is pseudo-concave at z. Then $u_i(x) > u_i(z)$ implies $\nabla u_i(z) \cdot (x - z) > 0$, and $u_i(y) > u_i(z)$ implies $\nabla u_i(z) \cdot (y - z) > 0$. But by construction, the vectors $x - z = \epsilon(x - y)$ and $y - z = (1 - \epsilon)(y - x)$ point in opposite directions, a contradiction. This establishes the claim. Since $x \in PO^*$, there are strictly positive coefficient α_i such that $\sum_{i=1}^n \alpha_i \nabla u_i(x) = 0$, but we have argued that $\alpha_i(\nabla u_i(x) \cdot (y - x)) \geq 0$ for all i, with at least one inequality strict. Thus, we once again have

$$\left(\sum_{i=1}^{n} \alpha_i \nabla u_i(x) \right) \cdot (y - x) \quad = \quad \sum_{i=1}^{n} \alpha_i (\nabla u_i(x) \cdot (y - x)) \quad > \quad 0,$$

a contradiction. ∎

There can certainly be strongly Pareto optimal alternatives that are not infinitesimally Pareto optimal. Indeed, in the spatial model, let $X = \mathbb{R}$, and assume there are two agents, each with quadratic utility: $u_1(x) = -x^2$ and $u_2(x) = -(1 - x)^2$. Then $x = 1$ is strongly Pareto optimal, yet there do not exist strictly positive weights $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 u_1'(1) + \alpha_2 u_2'(1) = 0$. At issue here is the fact that $x = 1$ is the critical point of one agent's utility function. If this possibility is precluded—or in more general environments, if every collection of $n - 1$ gradients is linearly independent—then we obtain full equivalence.[3]

Lemma 5 *For $x \in intX$, assume that for all $i \in N$, the gradients $\{\nabla u_j(x) \mid j \in N \setminus \{i\}\}$ are linearly independent. If $x \in PO^\circ$, then $x \in PO^*$.*

Proof: Consider any $x \in PO^\circ$. Since x is Pareto optimal, Lemma 1 yields coefficients $\alpha \in \mathbb{R}^n_+$, not all zero, such that $\sum_{i=1}^n \alpha_i \nabla u_i(x) = 0$. If $\alpha_i = 0$ for any i, then $\sum_{j \neq i} \alpha_j \nabla u_j(x) = 0$, contradicting linear independence of $\{\nabla u_j(x) \mid j \in N \setminus \{i\}\}$. We conclude that $\alpha_i > 0$ for all i, and thus $x \in PO^*$. ∎

The focus of this note is on the infinitesimal Pareto optima, which have tight connections to the strong Pareto optima, given Lemmas 4 and 5, and are typically well-behaved; in particular, they are hospitable to calculus-based techniques.

Note that if x is infinitesimally Pareto optimal, then the gradients $\{\nabla u_i(x) \mid i \in N\}$ are linearly dependent. If the remaining gradients are linearly independent after deleting any one $\nabla u_i(x)$, as in Lemma 5, then we say it is "minimally

[3]Note that this gradient restriction is satisfied in exchange economies, or more generally, in any environments where alternatives have a private good component.

dependent." Formally, a set $\{v^i \mid i \in N\}$ of vectors in $\mathbb{R}^d$ is *minimally dependent* if it is linearly dependent, and for all $j \in N$, the set $\{v^i \mid i \in N \setminus \{j\}\}$ is linearly independent. The next lemma presents some algebraic preliminaries on minimally dependent sets.

Lemma 6 *If a set $\{v^i \mid i \in N\} \subseteq \mathbb{R}^d$ is minimally dependent, then there are unique constants $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\sum_{j=1}^n \alpha_j v^j = 0$, $\sum_{j=1}^n \alpha_j^2 = 1$, and $\alpha_n \geq 0$. Furthermore, we must have $\alpha_j \neq 0$ for all $j \in N$. Conversely, if there are unique constants $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\sum_{j=1}^n \alpha_j v^j = 0$, $\|\alpha\| = 1$, $\alpha_j \neq 0$ for all $j = 1, \ldots, n$, and $\alpha_n > 0$, then $\{v^i \mid i \in G\}$ is minimally dependent.*

Proof: First, assume $\{v^1, \ldots, v^n\}$ is minimally dependent. Then since the set has rank $n-1$, the set of solutions $\alpha = (\alpha_1, \ldots, \alpha_n)$ to $\sum_{j=1}^n \alpha_j v^j = 0$ is a one-dimensional linear subspace of $\mathbb{R}^n$. If there is a solution with $||\alpha|| = 1$ and $\alpha_i = 0$ for some i, then $\alpha_h \neq 0$ for some $h \neq i$, and $\sum_{j \neq i} \alpha_j v^j = 0$ implies $\{v^1, \ldots, v^n\} \setminus \{v^i\}$ is linearly dependent, a contradiction. Therefore, $\alpha_i \neq 0$ for all i, and it follows that there are two solutions satisfying $||\alpha|| = 1$, and there is a unique solution α satisfying $||\alpha|| = 1$ and $\alpha_n \geq 0$. For the converse, assume there is a unique solution satisfying $\sum_{j=1}^n \alpha_j v^j = 0$, $||\alpha|| = 1$, $\alpha_j \neq 0$ for all j, and $\alpha_n > 0$. Then $\{v^1, \ldots, v^n\}$ has rank $n-1$. Next, suppose that $\{v^1, \ldots, v^{n-1}\}$ is linearly dependent, so there exist constants $\beta_1, \ldots, \beta_{n-1}$, not all zero, such that $\sum_{j=1}^{n-1} \beta_j v^j = 0$. Define $\gamma = (\alpha_1 + \beta_1, \ldots, \alpha_{n-1} + \beta_{n-1}, \alpha_n)$, and scale $\beta_1, \ldots, \beta_{n-1}$ so that $||\gamma|| \neq 1$ and so that for all j, we have $\beta_j + \alpha_j \neq 0$. Normalize this vector as $\gamma' = \frac{1}{||\gamma||}\gamma$. Then $\sum_{j=1}^n \gamma'_j v^j = 0$, $||\gamma'|| = 1$, $\gamma'_j \neq 0$ for all j, and $\gamma'_n > 0$. Furthermore, $\gamma'_n \neq \alpha_n$, so $\gamma' \neq \alpha$, contradicting uniqueness of α. Since v^n was an arbitrary element of $\{v^1, \ldots, v^n\}$, it follows that every proper subset is linearly independent, and we conclude that the set is minimally dependent. ∎

This immediately implies that if x is infinitesimally Pareto optimal, and if every collection of $n-1$ gradients is linearly independent, then the set $\{\nabla u_i(x) \mid i \in N\}$ of all gradients is minimally dependent; thus, the coefficients associated with x are uniquely determined.

Lemma 7 *If $x \in PO^*$ and for all $i \in N$, the gradients $\{\nabla u_j(x) \mid j \in N \setminus \{i\}\}$ are linearly independent, then $\{\nabla u_i(x) \mid i \in N\}$ is minimally dependent.*

2 Properties of Pareto Optima and Associated Multipliers

The first theorem on the Pareto manifold—or specifically, the infinitesimal Pareto optima—connects these optima and their associated multipliers in a $(n-1)$-dimensional submanifold of $\mathbb{R}^{d+n}$. At this point we do not employ any assumption of minimal dependence, and it is possible that gradients at some infinitesimal Pareto optima near x may sum to zero with multiple vectors of coefficients.

Theorem 1 *Assume that each u_i is C^2. Let $x \in PO^*$ be such that utilities are negative definite in aggregate at x. Then there are open sets $G_1 \subseteq X$ with $x \in G_1$ and $G_2 \subseteq \mathbb{R}^n_{++}$ such that*

$$W(x) \quad = \quad \left\{(z,\alpha) \in G_1 \times G_2 \mid \sum_{i=1}^{n} \alpha_i \nabla u_i(z) = 0 \text{ and } \|\alpha\| = 1\right\}$$

is a manifold of dimension $n-1$.

Proof: Since x is infinitesimally Pareto optimal, there are coefficients $\alpha^* \in \mathbb{R}^n_{++}$ such that $\sum_{i=1}^n \alpha_i^* \nabla u_i(x) = 0$. Since utilities are negative definite in aggregate, $\sum_{i=1}^n \alpha_i^* D^2 u_i(x)$ is negative definite. Let G_1 be an open set around x and $G_2 \subseteq \mathbb{R}^n_{++}$ be an open set around α^* such that for all $(z,\alpha) \in G_1 \times G_2$, the matrix $\sum_{i=1}^n \alpha_i D^2 u_i(z)$ is negative definite. Define the function $f\colon G_1 \times G_2 \to \mathbb{R}^{d+1}$ by

$$f(z,\alpha) \quad = \quad \left[\begin{array}{c} \sum_{i=1}^n \alpha_i \nabla u_i(z) \\ ||\alpha||^2 - 1 \end{array}\right],$$

for all $z \in G_1$ and all $\alpha \in G_2$. Then the derivative at (z,α) is the $(d+1) \times (d+n)$ matrix

$$Df(z,\alpha) \quad = \quad \left[\begin{array}{cccc} \sum_{i=1}^n \alpha_i D^2 u_i(z) & \nabla u_1(z) & \cdots & \nabla u_n(z) \\ 0 & 2\alpha_1 & \cdots & 2\alpha_n \end{array}\right].$$

For all $(z,\alpha) \in f^{-1}(0)$, it follows that $\sum_{i=1}^n \alpha_i D^2 u_i(z)$ is negative definite, so the first d rows of the derivative are linearly independent. Given that the first d coordinates of the last row are zero, and that $\alpha_1, \ldots, \alpha_n$ are non-zero, we conclude that $Df(z,\alpha)$ has full row rank at all $(z,\alpha) \in f^{-1}(0)$, so f is transversal to zero. Thus, $f^{-1}(0) = W(x)$ is a manifold of dimension $d+n-(d+1) = n-1$. ∎

The subset of $W(x)$ at which the gradients are minimally dependent is also a $(n-1)$-dimensional manifold. See Figure 1 for a depiction of of this set in the space of (z,α) pairs.

Theorem 2 *Assume that each u_i is C^2. Let $x \in PO^*$ be such that utilities are negative definite in aggregate at x. Define*

$$W^*(x) \quad = \quad \left\{(z,\alpha) \in W(x) \mid \begin{array}{c} \text{for all } i \in N,\ \{\nabla u_j(x) \mid j \in N \setminus \{i\}\} \\ \text{is linearly independent} \end{array}\right\}.$$

Then $W^*(x)$ is a manifold of dimension $n-1$.

Proof: Theorem 1 establishes that $W(x)$ is a manifold of dimension $n-1$. Since

$$W^<(x) \quad = \quad \{(z,\alpha) \in W(x) \mid \text{ for some } C \subsetneq N, \operatorname{rank}\{\nabla u_i(z) \mid i \in C\} < |C|\}$$

is closed, it follows that $W^*(x) = W(x) \setminus W^<(x)$ is a $(n-1)$-dimensional manifold. ∎

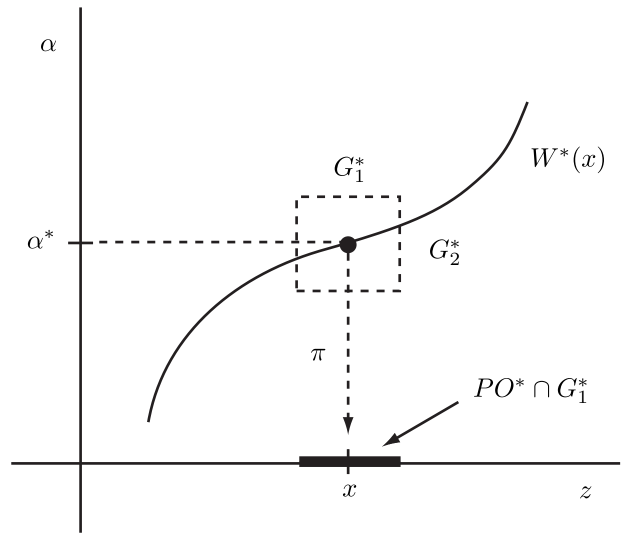

Figure 1: Projecting onto the Pareto Manifold

3 Projecting onto the Pareto Manifold

Next, with the structure of minimal dependence, I show that the mapping that projects each $(z, \alpha) \in W(x)$ to z is in fact an embedding, i.e., the projection is an immersion that is injective and proper, the latter meaning that the preimage of every compact subset of G_1 is compact.

Theorem 3 *Assume that each u_i is C^2. Let $x \in PO^*$ be such that utilities are negative definite in aggregate at x. Let $G_1^* \subseteq G_1$ be an open set with $x \in G_1^*$ such that for all $z \in G_1^*$ and all $i \in N$, the gradients $\{\nabla u_j(z) \mid j \in N \setminus \{i\}\}$ are linearly independent. Then the projection mapping $\pi\colon W^*(x) \cap (G_1^* \times \mathbb{R}^n) \to X$ defined by $\pi(z, \alpha) = z$ is an embedding.*

Proof: First, I claim that the projection mapping $\pi\colon W^*(x) \cap (G_1^* \times \mathbb{R}^n) \to X$ defined by $\pi(z, \alpha) = z$ is one-to-one. Indeed, for all $(z, \alpha) \in W^*(x) \cap (G_1^* \times \mathbb{R}^n)$, Lemma 7 implies that $\{\nabla u_i(z) \mid i \in N\}$ is minimally dependent. For all $\tilde{\alpha} \in G_2$, if $\pi(z, \tilde{\alpha}) = z$, then $\sum_{i=1}^n \tilde{\alpha}_i \nabla u_i(z) = 0$, and Lemma 6 implies that $\tilde{\alpha} = \alpha$, as required.

Second, I claim that π is an immersion, i.e., $D\pi(z, \alpha)$ has rank $n - 1$ for all $(z, \alpha) \in W^*(x) \cap (G_1^* \times \mathbb{R}^n)$. Given any $(z, \alpha) \in W^*(x)$ with $z \in G_1^*$, let $T_{(z,\alpha)}(W^*(x))$ denote the tangent space of $W^*(x)$ at (z, α), and note that this is $(n - 1)$-dimensional, and that it is the space orthogonal to the $d + 1$ linearly independent rows of $Df(z, \alpha)$. I must show that $D\pi(z, \alpha)T_{(z,\alpha)}(W^*(x)) = \pi(T_{(z,\alpha)}(W^*(x)))$ has dimension $n - 1$. Let $\{v^1, \ldots, v^{n-1}\}$ be a basis for the space $T_{(z,\alpha)}(W^*(x))$, and decompose each vector v^ℓ as $v^\ell = (p^\ell, q^\ell)$, where

p^ℓ is the first d coordinates and q^ℓ is the last n coordinates of v^ℓ, so that $\pi(v^\ell) = p^\ell$. I claim that $\{p^1, \ldots, p^{n-1}\}$ is linearly independent. Indeed, consider any $\beta_1, \ldots, \beta_{n-1}$ such that $\sum_{\ell=1}^{n-1} \beta_\ell p^\ell = 0$. Since $Df(z,\alpha)v^\ell = 0$, we have

$$\left[\begin{array}{cccc} \sum_{i=1}^n \alpha_i D^2 u_i(z) & \nabla u_1(z) & \cdots & \nabla u_n(x) \\ 0 & 2\alpha_1 & \cdots & 2\alpha_n \end{array} \right] \left(\begin{array}{c} p^\ell \\ q^\ell \end{array} \right) \quad = \quad 0$$

for each ℓ, which implies

$$\left[\begin{array}{ccc} \nabla u_1(z) & \cdots & \nabla u_n(z) \\ \alpha_1 & \cdots & \alpha_n \end{array} \right] \cdot \sum_{\ell=1}^{n-1} \beta_\ell q^\ell \quad = \quad 0.$$

The first d equations in the preceding system imply that $\sum_{\ell=1}^{n-1} \beta_\ell q^\ell$ belongs to the subspace

$$L(z) \quad = \quad \left\{ \tilde{\alpha} \in \mathbb{R}^n \mid \sum_{i=1}^n \tilde{\alpha}_i \nabla u_i(z) = 0 \right\},$$

which is one-dimensional, since $z \in G_1^*$ and $\{\nabla u_i(z) \mid i \in N\}$ has rank $n-1$. Moreover, $(z,\alpha) \in W^*(x)$ implies $\alpha \in L(z) \setminus \{0\}$. Note that equation $d+1$ implies that $\alpha \cdot \sum_{\ell=1}^{n-1} \beta_\ell q^\ell = 0$, which then implies that $(\sum_{\ell=1}^{n-1} \beta_\ell q^\ell) \perp L(z)$. Thus, we have $\sum_{\ell=1}^{n-1} \beta_\ell q^\ell = 0$, but then it must be that

$$\sum_{\ell=1}^{n-1} \beta_\ell v^\ell \quad = \quad \sum_{\ell=1}^{n-1} (\beta_\ell p^\ell, \beta_\ell q^\ell) \quad = \quad 0,$$

and linear independence of $\{v^\ell\}$ then implies $\beta_\ell = 0$ for all ℓ. This yields the claimed linear independence of $\{p^1, \ldots, p^{n-1}\}$, and we conclude that the vectors $\{\pi(v^1), \ldots, \pi(v^{n-1})\} = \{p^1, \ldots, p^{n-1}\}$ form a basis for $D\pi(z,\alpha) T_{(z,\alpha)}(W^*(x))$, as required.

Last, I claim that for every compact $K \subseteq G_1^*$, the preimage $\pi^{-1}(K)$ is compact. Note that $\pi^{-1}(K) \subseteq K \times \{\alpha \in G_2 \mid \|\alpha\| = 1\}$, so the preimage is bounded. Consider any sequence $\{(z^m, \alpha^m)\}$ in $\pi^{-1}(K)$. Then some subsequence (still indexed by m) is convergent with limit, say, $(z,\alpha) \in K \times \{\alpha \in G_2 \mid \|\alpha\| = 1\}$. It remains to be shown that $(z,\alpha) \in \pi^{-1}(K)$, and since $z \in K$, this follows if $(z,\alpha) \in W^*(x)$. For all m, we have $\sum_{i=1}^n \alpha_i^m \nabla u_i(z^m) = 0$, and thus $\sum_{i=1}^n \alpha_i \nabla u_i(z) = 0$. Since $\alpha^m \in G_2$ for all m, we have $\alpha_i \geq 0$ for all i. And since $z \in G_1^*$, it follows that for all $i \in N$, the gradients $\{\nabla u_j(z) \mid j \in N \setminus \{i\}\}$ are linearly independent. Thus, by Lemmas 6 and 7, we have $\alpha_i > 0$ for all i, and we conclude that $(z,\alpha) \in W^*(x)$, as required. ∎

As a matter of interest, the projection maps relatively open sets in $W^*(x)$ to relatively open sets $\pi(W^*(x))$.

Theorem 4 *Assume that each u_i is C^2. Let $x \in PO^*$ be as in Theorem 3. Then for every open set $G \subseteq \mathbb{R}^{d+n}$, the image $\Pi(G \cap W^*(x))$ is open relative to $\pi(W^*(x))$.*

Proof: Now let $G \subseteq \mathbb{R}^{d+n}$ be open. I claim that $\pi(G \cap W^*(x))$ is open in $\pi(W^*(x))$, or equivalently, its complement is closed. Consider any sequence $\{x^m\}$ in $\pi(W^*(x)) \setminus \pi(G \cap W^*(x))$ converging to x, and suppose toward a contradiction that $x \in \pi(G \cap W^*(x))$. Then there exists $\alpha \in G_2$ such that $(x, \alpha) \in G \cap W^*(x)$. For each m, there is a unique $\alpha^m \in G_2$ with $(x^m, \alpha^m) \in W^*(x)$ such that $f(x^m, \alpha^m) = 0$, and in particular, $||\alpha^m|| = 1$. Since the unit sphere is compact, we may go to a subsequence (still indexed by m) such that $\alpha^m \to \tilde{\alpha}$ and $||\tilde{\alpha}|| = 1$. Since $(x, \alpha) \in W^*(x)$, it follows that $x \notin M_G^{<}(s)$. Thus, $\tilde{\alpha} \in (\mathbb{R} \setminus \{0\})^{|G|-1} \times \mathbb{R}_{++}$, and we then have $f(x, \tilde{\alpha}) = 0$. Therefore, $(x, \tilde{\alpha}) \in W^*(x)$, and injectivity of π yields $\alpha = \tilde{\alpha}$. In particular, $(x, \tilde{\alpha}) \in G \cap W^*(x)$, but then we have $(x^m, \alpha^m) \in G \cap W^*(x)$, i.e., $x^m \in \pi(G \cap W^*(x))$, for sufficiently high m, a contradiction. We conclude that π maps relatively open sets to open sets. ∎

Finally, under the conditions established above, a theorem of Guillemin and Pollack (1974, p.17) implies that the image of $W^*(x) \cap (G_1^* \times \mathbb{R}^n)$ under the projection, namely, the infinitesimal Pareto optima in G_1^*, is a $(n-1)$-dimensional manifold. See Figure 1.

Theorem 5 *Assume that each u_i is C^2. Let $x \in PO^*$ be as in Theorem 3. Then $PO^* \cap G_1^*$ is a manifold of dimension $n-1$.*

Proof: The theorem of Guillman and Pollack immediately implies that the image $\pi(W^*(x) \cap (G_1^* \times \mathbb{R}^n))$ is a $(n-1)$-dimensional manifold. I claim that $PO^* \cap G_1^* = \pi(W^*(x) \cap (G_1^* \times \mathbb{R}^n))$. Indeed, if $z \in PO^* \cap G_1^*$, then there exist coefficients $\alpha \in \mathbb{R}^n_{++}$ such that $\sum_{i=1}^n \alpha_i \nabla u_i(z) = 0$. Since $z \in G_1^*$, it follows that for all $i \in N$, the gradients $\{\nabla u_j(z) \mid j \in N \setminus \{i\}\}$ have full rank. Then $(z, \alpha) \in W^*(x) \cap (G_1^* \times \mathbb{R}^n)$, and we have $\pi(z, \alpha) = z$, i.e., z belongs to the image. For the remaining inclusion, if z belongs to the projection, then there exist $\alpha \in \mathbb{R}^n_{++}$ such that $f(z, \alpha) = 0$. This implies $\sum_{i=1}^n \alpha_i \nabla u_i(z) = 0$, i.e., z is infinitesimally Pareto optimal, as required. ∎

4 Parameterizing the Pareto Manifold

Theorem 5 means that around x, we can parameterize the infinitesimal Pareto optima by a coordinate system in $\mathbb{R}^{n-1}$. The precise nature of the parameterization is open, but an obvious possibility is to use the first $n-1$ coordinates of α. Define $g\colon G_1^* \times \mathbb{R}^{n-1}_{++} \to \mathbb{R}^d$ by

$$g(z, \tilde{\alpha}) \quad = \quad \sum_{i=1}^{n-1} \tilde{\alpha}_i \nabla u_i(z) + \nabla u_n(z),$$

where $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1})$. Given $(z, \alpha) \in W^*(x)$, let $\tilde{\alpha}^* = (\frac{\alpha_1}{\alpha_n}, \ldots, \frac{\alpha_{n-1}}{\alpha_n})$, and note that $g(z, \tilde{\alpha}^*) = 0$. The derivative of g at $(z, \tilde{\alpha})$ with respect to z is

$$D_1 g(z, \tilde{\alpha}) \quad = \quad \sum_{i=1}^{n-1} \tilde{\alpha}_i D^2 u_i(z) + D^2 u_n(z),$$

which is negative definite and, thus, has full row rank. By the implicit function theorem, there is an open set $\tilde{G} \subseteq \mathbb{R}^{n-1}$ with $\tilde{\alpha} \in \tilde{G}$ and a differentiable mapping $\xi \colon \tilde{G} \to X$ such that for all $\tilde{\beta} \in \tilde{G}$,

$$g(\xi(\tilde{\beta}), \tilde{\beta}) \quad = \quad 0.$$

This mapping may serve as a parameterization around x, but we must verify that it is a diffeomorphism between $\tilde{G}$ and a relatively open neighborhood of $\tilde{\alpha}^*$ in $W^*(x) \cap (G_1^* \times \mathbb{R}^{n-1}_{++})$.

Theorem 6 *Assume that each u_i is C^2. Let $x \in PO^*$ be as in Theorem 3. Then the mapping $\xi \colon \tilde{G} \to X$ is a local diffeomorphism at x.*

Proof: At every $\tilde{\beta} \in \tilde{G}$, we have

$$D_1 g(z, \tilde{\beta}) D\xi(\tilde{\beta}) + D_2 g(z, \tilde{\beta}) \quad = \quad 0,$$

where $z = \xi(\tilde{\beta})$ and

$$D_2 g(z, \tilde{\beta}) \quad = \quad [\ \nabla u_1(z) \quad \cdots \quad \nabla u_{n-1}(z)\ ].$$

Of course, this implies

$$D\xi(\tilde{\beta}) \quad = \quad -[D_1 g(z, \tilde{\beta})]^{-1} D_2 g(z, \tilde{\beta}).$$

By the inverse function theorem (Guillemin and Pollack, 1974, p.13), it suffices to show that $D\xi(\tilde{\alpha}^*)$, viewed as a linear mapping from $\mathbb{R}^{n-1}$ to the $(n-1)$-dimensional tangent space $T_{(x,\alpha^*)}(W^*(x))$, is an isomorphism.

In the proof of Theorem 3, given $(z, \alpha) \in W^*(x) \cap (G_1^* \times \mathbb{R}^n)$, we obtained a basis $\{v^1, \ldots, v^{n-1}\}$ of the tangent space $T_{(z,\alpha)}(W^*(x))$, where each v^ℓ is decomposed as $v^\ell = (p^\ell, q^\ell)$, with $p^\ell \in \mathbb{R}^d$ and $q^\ell \in \mathbb{R}^n$. We argued that

$$\left[\begin{array}{cccc} \sum_{i=1}^n \alpha_i D^2 u_i(z) & \nabla u_1(z) & \cdots & \nabla u_n(x) \\ 0 & 2\alpha_1 & \cdots & 2\alpha_n \end{array} \right] \left(\begin{array}{c} p^\ell \\ q^\ell \end{array} \right) \quad = \quad 0,$$

and that the vectors $\{p^1, \ldots, p^{n-1}\}$ are linearly independent. The vectors q^ℓ lie in n-dimensional Euclidean space, but the above restrictions allow us to delete a coordinate. Since $\sum_{i=1}^n \alpha_i q_i^\ell = 0$, we can write $q_n^\ell = \sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} q_i^\ell$. We can then rewrite $\sum_{i=1}^n q_i^\ell \nabla u_i(z) = 0$ as $\sum_{i=1}^{n-1} q_i^\ell (\nabla u_i(z) + \frac{\alpha_i}{\alpha_n} \nabla u_n(z)) = 0$. Making this substitution in the above system of equations, we obtain

$$\left[\sum_{i=1}^n \alpha_i D^2 u_i(z) \right] p^\ell + [\ \nabla u_1(z) + \tfrac{\alpha_1}{\alpha_n} \nabla u_n(z) \quad \cdots \quad \nabla u_{n-1}(z) + \tfrac{\alpha_{n-1}}{\alpha_n} \nabla u_n(z)\ ]\, \tilde{q}^\ell$$
$$= \quad 0$$

for each $\ell = 1, \ldots, n-1$, where $\tilde{q}^\ell = (q_1^\ell, \ldots, q_{n-1}^\ell)$ consists of the first $n-1$ coordinates of q^ℓ. I claim that $\{\tilde{q}^1, \ldots, \tilde{q}^{n-1}\}$ is linearly independent, for consider coefficients $\beta_1, \ldots, \beta_{n-1}$ such that $\sum_{\ell=1}^{n-1} \beta_\ell \tilde{q}^\ell = 0$. Then we have

$[\sum_{i=1}^n \alpha_i D^2 u_i(z)](\sum_{\ell=1}^{n-1} \beta_\ell p^\ell) = 0$. Since the weighted sum of Hessian matrices is negative definite, we have $\sum_{\ell=1}^{n-1} \beta_\ell p^\ell) = 0$, and since $\{p^1, \ldots, p^{n-1}\}$ is linearly independent, this implies $\beta_\ell = 0$ for all ℓ, as claimed.

Evaluating the system of equations above for (x, α^*), we have for all ℓ,

$$\Big[\sum_{i=1}^{n-1} \tilde{\alpha}_i^* D^2 u_i(x) + D^2 u_n(x)\Big] p^\ell + \big[\nabla u_1(x) + \tilde{\alpha}_1^* \nabla u_n(x) \ \cdots \ \nabla u_{n-1}(x) + \tilde{\alpha}_{n-1}^* \nabla u_n(x)\big] \tilde{q}^\ell = 0.$$

Therefore,

$$p^\ell = -[D_1 g(x, \tilde{\alpha}^*)]^{-1} \big[\nabla u_1(x) + \tilde{\alpha}_1^* \nabla u_n(x) \ \cdots \ \nabla u_{n-1}(x) + \tilde{\alpha}_{n-1}^* \nabla u_n(x)\big] \tilde{q}^\ell.$$

Since $\{\nabla u_i(x) \mid i \in N \setminus \{n\}\}$ is linearly independent, there exists $\tilde{\gamma} \in \mathbb{R}^{n-1}$ such that

$$\sum_{i=1}^{n-1} \tilde{\gamma}_i \nabla u_i(x) = \big[\nabla u_1(x) + \tilde{\alpha}_1^* \nabla u_n(x) \ \cdots \ \nabla u_{n-1}(x) + \tilde{\alpha}_{n-1}^* \nabla u_n(x)\big] \tilde{q}^\ell.$$

Thus, we have

$$p^\ell = -[D_1 g(x, \tilde{\alpha}^*)]^{-1} D_2(x, \tilde{\alpha}^*) \tilde{\gamma} = D\xi(\tilde{\alpha}^*) \tilde{\gamma},$$

so that each p^ℓ is in the image of $D\xi(\tilde{\alpha}^*)$. Since $\{p^1, \ldots, p^{n-1}\}$ is a basis for $T_{(x,\alpha^*)}(W^*(x))$, we conclude that $D\xi(\tilde{\alpha}^*)$ is an isomorphism, and that ξ is a local diffeomorphism. ∎

5 Parameterized Utilities

Last, I show that for all i, the gradients of the parameterized utility functions, $\tilde{u}_j \equiv u_j \circ \xi$, of the remaining agents $j \neq i$ are linearly independent.

Theorem 7 *Assume that each u_i is C^2. Let $x \in PO^*$ be as in Theorem 3. For all $i \in N$, the gradients $\{\nabla \tilde{u}_j(\tilde{\alpha}^*) \mid j \in N \setminus \{i\}\}$ evaluated at $\tilde{\alpha}^*$ are linearly independent.*

Proof: By the chain rule, $D\tilde{u}_j(\tilde{\alpha}^*) = Du_j(x) D\xi(\tilde{\alpha}^*)$. Thus,

$$D\tilde{u}_j(\tilde{\alpha}^*) = -Du_j(x)[D_1 g(x, \tilde{\alpha}^*)]^{-1} D_2 g(x, \tilde{\alpha}^*).$$

Letting $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_{n-1})$ be the vector-valued parameterized utility function, we must show that the $((n-1) \times (n-1))$-matrix $D\tilde{u}(\tilde{\alpha}^*)$ has full rank. Letting $u = (u_1, \ldots, u_{n-1})$, we have

$$\begin{aligned} D\tilde{u}(\tilde{\alpha}^*) &= -Du(x)[D_1 g(x, \tilde{\alpha}^*)]^{-1} D_2 g(x, \tilde{\alpha}^*) \\ &= -[D_2 g(x, \tilde{\alpha}^*)]^T [D_1 g(x, \tilde{\alpha}^*)]^{-1} D_2 g(x, \tilde{\alpha}^*) \end{aligned}$$

Because $D_1 g(x, \tilde{\alpha}^*)$ is negative definite, the inverse is as well. It follows that $D\tilde{u}(\tilde{\alpha}^*)$ is positive definite and, thus, has full rank, as required. ∎

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