Multidimensional Tug of War*

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Abstract

We analyze a contest in which players exert effort to pull a collective outcome in a multidimensional space in their preferred direction. We prove existence and uniqueness of equilibrium and perform comparative statics on the cost of effort and risk aversion of the players. As risk aversion increases, the equilibrium outcome converges to the Rawlsian outcome that maximizes the payoff of the worst off player; as risk aversion decreases and players become risk loving, the equilibrium outcome converges to the mean of the players' ideal points.

1 Introduction

In this paper, we envision a d-dimensional world in which a finite set of individuals compete to influence the location of a collective outcome. For a motivating application, it may be that a number of lobbyists or interest groups exert pressure on a legislature to influence public policy in a particular issue area. We view such competition as a contest among lobbyists, but in much of the literature on contest theory, the final outcome is imposed by a single winner who is determined stochastically by effort exerted by contest participants: although the winner of the contest is affected by effort choices, the final outcome reflects only the preferences of the winner, independent of effort expended by losers (cf. Corchón 2007). In the lobbying example, however, we view public policy as a continuous variable, and it is natural to expect that the final location of policy will be the result of a compromise that reflects the efforts of all participants, so that the effort exerted by any one lobbyist generates externalities on all others. Thus, we model lobbying as a "tug of war," where the collective outcome is a weighted combination of individual ideal points, weights are determined by effort choices via a standard contest success function, and effort is subject to a quadratic cost.

Formally, each player $i = 1, \ldots, n$ has Euclidean preferences over outcomes $x \in \Re^d$ with ideal point $\hat{x}^i$, which are not all the same for all players, and each i chooses effort

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$\alpha_i \geq 0$. The collective outcome is

$$x^* = \frac{\sum_j \alpha_j \hat{x}^j}{\sum_j \alpha_j},$$

and player i's payoff from effort profile $(\alpha_1, \ldots, \alpha_n)$ is $u(\|\hat{x}^i - x^*\|) - c\alpha_i^2$, where $c > 0$ is a cost parameter and u is a strictly decreasing function. When the sum of efforts is equal to zero, we specify that the outcome is an arbitrary status quo; since we assume it is not the case that all players share the same ideal point, total effort is necessarily positive in equilibrium, and the location of the status quo is immaterial. This paper characterizes the unique Nash equilibrium of this d-dimensional game of tug of war as the unique solution to a particular system of equations. In general: the equilibrium outcome is unaffected by the cost parameter, individual effort goes to zero as the cost parameter c becomes large; and total effort increases without bound as c goes to zero. We consider the parametric class of power utility functions $-\|x - \hat{x}^i\|^r$, and we show that the conditions for existence and uniqueness hold for all $r > 0$. As an application of the general characterization result, we show that the unique equilibrium outcome converges to the mean of the individual ideal points as $r \to 0$, i.e., as the coefficient of relative risk aversion goes to -1; and it converges to the Rawlsian optimum as relative risk aversion goes to infinity. In the concluding section, we discuss a number of extensions of the model, including heterogeneous cost of effort and a more general cost structure.

A distinguishing feature of our model is the nature of the externality generated by a player's effort. Each player pulls the collective outcome in a different direction, and thus the externality generated by the efforts of players j and k may affect player i differentially, e.g., j's effort may produce a positive externality to i, whereas k's effort moves the outcome further from player i's ideal point. For a fourth player, the externalities may be reversed, with j's effort producing a negative externality and k's a positive one. In short, externalities are "identity-dependent," a feature of Linster's (1993) generalized contest model and Esteban and Ray's (1999) analysis of conflict and redistribution. In both of the latter papers, the outcome of the contest is a lottery over winners, and each player i derives some utility u_{ij} in case j wins, with payoffs given by expected utility, i.e.,

$$\sum_j \left(\frac{\alpha_j}{\sum_k \alpha_k} \right) u_{ij} - c(\alpha_i),$$

where Linster (1993) assumes linear cost, and Esteban and Ray (1999) assume a more general cost function. In particular, Esteban and Ray (1999) establish existence and, under weak conditions, uniqueness of equilibrium; their main focus is the effect of group sizes on the level and pattern of conflict, while we exploit the structure of our model to consider comparative statics on preference and cost parameters.

Although the contest success function used in this note has a standard functional form (cf. Skaperdas 1996), our use of it differs from the contest literature in that it determines a specific outcome in Euclidean space—rather than a lottery over contestants—as a continuous function of individual effort. Some papers in the literature do consider models with a spatial structure; see Konrad (2000), Epstein and Nitzan (2004), and Münster (2006). In these papers, however, the positions of the players are determined endogenously in the first stage of the game, and the winner of the contest is determined stochastically in the second stage. In contrast, we fix the ideal points of the players, and we model the final outcome as a deterministic function of individual efforts, giving the model a structure something like a tug of war. Konrad and Kovenock (2005) analyze what they refer to as a tug of war, but they model a sequence of all-pay auctions in which two players attempt to pull a state variable to their preferred terminal points over time. In contrast, drawing an analogy to the well-known game, we model multiple players moving simultaneously, each exerting force on a segment of rope, and all segments meeting at a common nexus. The goal of each player is to pull the nexus point as close to her ideal point as possible, and a Nash equilibrium of the game is then a location of this nexus that is stable under the forces exerted by the n players. The literature appears to have omitted spatial contests with this strategic structure.

2 The Tug of War Game

Assume n players, denoted $i = 1, \ldots, n$, simultaneously choose effort levels $\alpha_i \geq 0$ to determine an outcome x^* in d-dimensional Euclidean space, $\Re^d$. Each player i has an ideal point $\hat{x}^i \in \Re^d$, and effort levels determine the collective outcome via a standard contest success function,

$$x^* = \frac{\sum_j \alpha_j \hat{x}^j}{\sum_j \alpha_j}.$$

Intuitively, each player pulls the collective outcome toward her ideal point, the weight on that outcome being determined by the ratio of individual to total effort. When all players exert zero effort, the above expression is not well-defined, and in this case, we set x^* equal to an arbitrary status quo outcome in the convex hull of ideal points. To avoid trivial cases, we will assume that at least two players have distinct ideal points; this renders the choice of status quo immaterial.

Payoffs are determined by a common utility applied to the distance of the collective outcome from a player's ideal point, minus a quadratic cost of effort; that is, for some twice differentiable function $u\colon \Re_+ \to \Re$, each player i's payoff is $u(\|\hat{x}^i - x^*\|) - c\alpha_i^2$. We impose the following assumptions on payoffs; in Assumption 2 and elsewhere, it is understood that the outcome x^* is a function of effort choices.

Assumption 1. *For all* $z > 0$, $u'(z) < 0$.

Assumption 2. *For all* α_j, $j \neq i$, $u(\|x^* - \hat{x}^i\|) - c\alpha_i^2$ *is quasi-concave in* α_i.

Assumption 3. *For all* $z > 0$, $u''(z)z^2 + 2u'(z)z < 0$.

Assumption 1, in the current context, is a differentiable formulation of strictly quasi-concavity. Assumption 2 is a standard condition used to guarantee existence of Nash equilibria in pure strategies, and Assumption 3, which is used in the uniqueness proof, limits the extent of non-concavity of the players' utilities. An obvious sufficient condition for the latter assumptions is concavity—specifically, $u'' \leq 0$—but the conditions hold even more generally. In Section 3, we show that they hold for power utility $u(z) = -z^r$ with arbitrary positive parameter $r > 0$. Note that the power function is actually convex for $r < 1$, showing that the analysis can accommodate substantially non-concave utility over distance.

Given the effort levels of the other players, player i's best response problem is therefore:

$$\max_{\alpha_i} u\left(\left\|\hat{x}^i - \frac{\sum_j \alpha_j \hat{x}^j}{\sum_j \alpha_j}\right\|\right) - c\alpha_i^2.$$

We focus on pure strategy Nash equilibria, and by the assumption that at least two players have distinct ideal points, it is immediate that all equilibria feature positive total effort, i.e., $A \equiv \sum_i \alpha_i > 0$ in equilibrium. Furthermore, it is clear that if $x^* = \hat{x}^i$ in equilibrium, then it must be that player i exerts zero effort, i.e., $\alpha_i = 0$.

For every other player i such that $x^* \neq \hat{x}^i$, optimal effort must satisfy the following first order condition:

$$-u'(\|\hat{x}^i - x^*\|)\|\hat{x}^i - x^*\| \leq 2c\alpha_i A,$$

with equality if $\alpha_i > 0$. By Assumption 1, the left-hand side of the above inequality is positive, and we conclude that $\alpha_i > 0$ for all i with $x^* \neq \hat{x}^i$. Thus, we have

$$\alpha_i = -\frac{1}{2cA} u'(\|\hat{x}^i - x^*\|)\|\hat{x}^i - x^*\| \tag{1}$$

for all such players. Substituting (1) into the definition of x^*, it follows that in equilibrium, we must have

$$\begin{aligned} 0 &= x^* - \frac{\sum_i \alpha_i \hat{x}^i}{A} \\ &= \frac{\sum_i \alpha_i (x^* - \hat{x}^i)}{A} \\ &= \frac{\sum_i -\frac{1}{2cA} u'(\|\hat{x}^i - x^*\|)\|\hat{x}^i - x^*\|(x^* - \hat{x}^i)}{A}. \end{aligned}$$

Simplifying further, we then have

$$\sum_i u'(\|\hat{x}^i - x^*\|)\|\hat{x}^i - x^*\|(x^* - \hat{x}^i) = 0. \tag{2}$$

In words, the equilibrium outcome must balance the forces exerted on it, where each players exerts force in the direction of her ideal point $\hat{x}^i$, with magnitude equal to the squared distance to $\hat{x}^i$ weighted by the marginal utility of distance. For the case of concave u, this means that in equilibrium, players further from the collective outcome x^* exert considerably greater effort, implying that x^* will resist being too distant from the ideal point of the furthest player; we return to this topic in Theorem 4 of the next section.

Our main result is that the tug of war game admits a unique equilibrium, which is characterized by the above analysis.

Theorem 1. *Under (A1)–(A3), the tug of war game admits a unique Nash equilibrium, and equilibrium effort levels are characterized by equation (2).*

Proof. The existence argument must account for the facts that the outcome x^* is discontinuous at $(\alpha_1, \ldots, \alpha_n) = 0$ and that the players' strategy sets are unbounded above. To address the latter difficulty, note that the outcome x^* always belongs to the convex hull of $\{\hat{x}^1, \ldots, \hat{x}^n\}$, and since u is quasi-concave in x, it follows that the minimum of $u(\|\hat{x}^i - x^*\|)$ over profiles $(\alpha_1, \ldots, \alpha_n)$ is achieved when some player, say j_i, exerts positive effort and all others exert zero effort. That is, the worst feasible outcome for player i is $\hat{x}^{j_i}$; of course, the best feasible outcome for i is $\hat{x}^i$. Let $\overline{a}$ be such that for all players i, we have

$$u(0) - c\overline{a}^2 \quad < \quad u(\|\hat{x}^i - \hat{x}^{j_i}\|),$$

so that each player would prefer to choose zero effort and obtain the worst possible outcome, rather than receive the best possible outcome while exceeding the level $\overline{a}$. Clearly, an equilibrium of the tug of war game with strategy sets restricted to the compact interval $[0, \overline{a}]$ for all players will persist as an equilibrium of the unrestricted game.

To address the discontinuity of payoffs, we invoke Reny's (1999) Theorem 3.1 to obtain an equilibrium of the restricted game, and to this end, we must verify his condition of better-reply security. Write $x^*(\alpha)$ to bring out dependence of the outcome on the profile $\alpha = (\alpha_1, \ldots, \alpha_n)$ of effort levels. Let Γ be the closure of the graph of the vector payoff function of the game, i.e.,

$$\Gamma \quad = \quad \text{clos}\left(\{(\alpha, y_1, \ldots, y_n) \in [0, \overline{a}]^n \times \Re^n \mid \text{for all } i,\ y_i = u(\|\hat{x}^i - x^*(\alpha)\|)\} \right).$$

Because payoffs are continuous at every profile of effort levels not equal to the zero vector, it suffices to show that for every $(y_1, \ldots, y_n)$ with $(0, y_1, \ldots, y_n) \in \Gamma$, some player i has a profitable deviation to a positive effort level, $\alpha_i > 0$. Indeed, for such a vector $(y_1, \ldots, y_n)$, there is at least one player i whose payoff is less than the payoff from the ideal outcome, i.e., $y_i < u(0)$. For every deviation $\alpha_i > 0$, the resulting outcome is the player's ideal point, $x^*(\alpha_i, 0_{-i}) = \hat{x}^i$. Then for $\alpha_i > 0$ sufficiently small, we have

$$u(x^*(\alpha_i, 0_{-i})) - c\alpha_i^2 \quad = \quad u(0) - c\alpha_i^2 \quad > \quad y_i,$$

as required. Because the outcome x^* is continuous at $(\alpha_i, 0_{-i})$, there exists $\epsilon > 0$ such that for all $\alpha_{-i} = (\alpha_j)_{j \neq i} \in [0, \epsilon)^{n-1}$, we have $u(x^*(\alpha_i, \alpha_{-i})) - c\alpha_i^2 > y_i$. Thus, Reny's theorem delivers existence of an equilibrium. Necessity of (2) follows from first order arguments above. Sufficiency follows from the fact that equation (2) admits at most one solution, as argued below.

To establish that equation (2) has a unique solution, note first that if there are two solutions, say x^* and x^{**}, then the equality

$$t \cdot \sum_i u'(\|\hat{x}^i - x^* - \beta t\|)\|\hat{x}^i - x^* - \beta t\|(x^* - \hat{x}^i + \beta t) = 0.$$

must hold for the direction $t = \frac{1}{\|x^{**}-x^*\|}(x^{**} - x^*)$ and both $\beta = 0$ and $\beta = \|x^{**} - x^*\|$. Thus, to preclude the possibility of multiple solutions, it suffices to show that

$$t \cdot \sum_i u'(\|\hat{x}^i - x^* - \beta t\|)\|\hat{x}^i - x^* - \beta t\|(x^* - \hat{x}^i + \beta t) \tag{3}$$

is strictly monotone in $\beta > 0$ for every direction t. Let $\xi^i(\beta) = \hat{x}^i - x^* - \beta t$, and note that

$$\frac{d}{d\beta}\|\xi^i(\beta)\| \;=\; -\frac{t \cdot \xi^i(\beta)}{\|\xi^i(\beta)\|}.$$

Then we write the derivative of (3) with respect to β as the sum $\sum_i (A_i + B_i + C_i)$, where for each player i, we define

$$\begin{aligned} A_i &= u''(\|\xi^i(\beta)\|)[t \cdot \xi^i(\beta)]^2 \\ B_i &= \frac{u'(\|\xi^i(\beta)\|)[t \cdot \xi^i(\beta)]^2}{\|\xi^i(\beta)\|} \\ C_i &= u'(\|\xi^i(\beta)\|)\|\xi^i(\beta)\|. \end{aligned}$$

By Assumption 1, we have $A_i + B_i + C_i < 0$ when $t \cdot \xi^i(\beta) = 0$, so consider the remaining case. Using the identity $z + \frac{1}{z} \geq 2$ for all $z > 0$, we have

$$\frac{[t \cdot \xi^i(\beta)]^2}{\|\xi^i(\beta)\|} + \|\xi^i(\beta)\| \;\geq\; 2|t \cdot \xi^i(\beta)|.$$

Therefore, with Assumption 1, we have

$$A_i + B_i + C_i \;\leq\; u''(\|\xi^i(\beta)\|)[t \cdot \xi^i(\beta)]^2 + 2u'(\|\xi^i(\beta)\|)|t \cdot \xi^i(\beta)| \;<\; 0,$$

where the second inequality follows from Assumption 3. We conclude that (3) is strictly monotone in $\beta > 0$ for each player i, and this implies that there is a unique solution to (2) and thus a unique Nash equilibrium, as required. □

An implication of Theorem 1 is that the equilibrium outcome is independent of the cost parameter. Summing over i in equation (1) and solving for the equilibrium total effort level, we directly obtain:

$$A = \sqrt{-\frac{1}{2c}\sum_i u'(\|\hat{x}^i - x^*\|)\|\hat{x}^i - x^*\|}. \tag{4}$$

Substituting this expression back into (1), we see that the equilibrium effort level of each player i with $\hat{x}^i \neq x^*$ varies inversely with the cost parameter:

$$\alpha_i = \frac{-u'(\|\hat{x}^i - x^*\|)\|\hat{x}^i - x^*\|}{\sqrt{-2c\sum_j u'(\|\hat{x}^j - x^*\|)\|\hat{x}^j - x^*\|}}. \tag{5}$$

In particular, as the cost parameter c varies, the outcome x^* in the right-hand side of the above expression is constant, and it follows that the effort level of all such players increases without bound as c becomes small, and effort goes to zero as c becomes large. The total cost of effort, in contrast, is monotonic in the cost parameter.

Theorem 2. *In the tug of war game, as c becomes large, equilibrium effort goes to zero, and for all players i with $\hat{x}^i \neq x^*$, the cost of equilibrium effort increases without bound, i.e., $\lim_{c\to\infty} \alpha_i = 0$ and $\lim_{c\to\infty} c\alpha_i = \infty$. As c becomes small, the cost of equilibrium effort goes to zero, and for all players i with $\hat{x}^i \neq x^*$, equilibrium effort increases without bound, i.e., $\lim_{c\to 0} \alpha_i = \infty$ and $\lim_{c\to 0} c\alpha_i = 0$.*

The equilibrium analysis can be illustrated in two special case of interest. First, assume utility over distance is linear, i.e., $u(z) = -z$. Then equation (2) reduces to

$$\sum_i \|x^* - \hat{x}^i\|(x^* - \hat{x}^i) \;=\; 0,$$

or equivalently,

$$x^* \;=\; \sum_i \left(\frac{\|x^* - \hat{x}^i\|}{\sum_j \|x^* - \hat{x}^i\|}\right)\hat{x}^i,$$

so that x^* is a weighted combination of ideal points, with weights equal to the ratio of a player's distance to x^* over the total distance. In one dimension, the equilibrium outcome therefore equalizes the total of squared distance from x^* to the left and to the right of this outcome. Second, assume quadratic utility, i.e., $u(z) = -z^2$. Now, equation (2) reduces to

$$\sum_i \|x^* - \hat{x}^i\|^2(x^* - \hat{x}^i) = 0,$$

or equivalently,

$$x^* \;=\; \sum_i \left(\frac{\|x^* - \hat{x}^i\|^2}{\sum_j \|x^* - \hat{x}^i\|^2}\right)\hat{x}^i,$$

and total effort is given by

$$A = \sqrt{\frac{1}{c}\sum_i \|x^* - \hat{x}^i\|^2}.$$

Thus, in one dimension, the equilibrium outcome equalizes the total of cubed distance to x^* to the left and to the right of this outcome; and total effort is the standard deviation of individual distance to x^*, normalized by $\sqrt{n/c}$.

3 Power utility

In this section, we analyze the effects of curvature of utilities on the equilibrium outcome of the tug of war game. To this end, we assume power utility of the form $u(z) = -z^r$ for some $r > 0$, so that the coefficient of relative risk aversion is $r - 1$. Write player i's utility for distance-effort pairs as $-z^r - c\alpha_i^2$, so that the marginal rate of substitution of effort for distance is

$$MRS_i(r) \quad = \quad \frac{rz^{r-1}}{2\alpha_i}.$$

In ordinal terms, the parameter r affects the marginal rate of substitution as follows: if $z \geq 1$, then $MRS_i(r)$ increases without bound as r becomes large; and if $z < 1$, then when r is sufficiently high, $MRS_i(r)$ decreases to zero with r. In the power utility case, the first order condition in (1) can be written as

$$MRS_i(r) \quad = \quad \frac{cA}{z_i},$$

where $z_i = \|\hat{x}^i - x^*\|$ is the distance between the player's ideal point and the equilibrium outcome. Here, the appearance of z_i on the right-hand side of the first order condition indicates that the trade off between effort and distance is given by the contest success function, which is non-linear. A direct implication is a restriction on ratios of effort and distance across players: in equilibrium, we must have

$$\frac{\alpha_i}{\alpha_j} \quad = \quad \left(\frac{z_i}{z_j}\right)^r, \tag{6}$$

so that players further from x^* must exert greater effort, with the ratio of efforts increasing in r.

Power utility clearly satisfies (A1), and the next result confirms (A2) and (A3).

Proposition 1. *In the tug of war game, power utility with* $r > 0$ *satisfies (A1)–(A3).*

Proof. We verify (A2) and (A3). For the former, it suffices to show that the second derivative of $u(\|x^* - \hat{x}^i\|) - c\alpha_i^2$ with respect to α_i is negative. For any player i, the first derivative with respect to α_i is the following:

$$\frac{r}{\sum_j \alpha_j}\left[\sum_{m=1}^{d}(x_m^* - \hat{x}_m^i)^2\right]^{\frac{r}{2}} - 2c\alpha_i$$

The second derivative simplifies to:

$$\begin{aligned}
&-\frac{1}{(\sum_j \alpha_j)^2}\left(r^2\left[\sum_{m=1}^{d}(x_m^* - \hat{x}_m^i)^2\right]^{\frac{r}{2}} + r\left[\sum_{m=1}^{d}(x_m^* - \hat{x}_m^i)^2\right]^{\frac{r}{2}}\right) - 2c \\
&= \quad -\frac{1}{(\sum_j \alpha_j)^2}(r^2 + r)\left[\sum_{m=1}^{d}(x_m^* - \hat{x}_m^i)^2\right]^{\frac{r}{2}} - 2c \\
&< \quad 0,
\end{aligned}$$

as required. To verify (A3), write

$$\begin{aligned} 2u'(z)z + u''(z)z^2 &= -2rz^{r-1}z - r(r-1)z^{r-2}z^2 \\ &= -2rz^r - r(r-1)z^r \\ &= -(r^2+r)z^r \\ &< 0, \end{aligned}$$

as required. □

By Theorem 1, the Nash equilibrium of the tug of war game is given by the unique solution to

$$0 = \sum_i \|x^* - \hat{x}^i\|^r (x^* - \hat{x}^i), \tag{7}$$

where we specialize equation (2) to the power utility case. In turn, the equilibrium effort of player i is determined by equation (5), which becomes:

$$\alpha_i = \frac{\|\hat{x}^i - x^*\|^r}{\sqrt{\frac{2c}{r}\sum_j \|\hat{x}^j - x^*\|^r}}. \tag{8}$$

It is then straightforward to confirm that as the coefficient of relative risk aversion goes to negative one (i.e., $r \to 0$), the equilibrium outcome converges to the mean of the players' ideal points, and total effort goes to zero.

Theorem 3. *In the tug of war game with power utility and $r > 0$, the limit of equilibrium outcomes as $r \to 0$ is the mean of the players' ideal points, and total effort goes to zero as $r \to 0$, i.e., $\lim_{r\to 0} x^* = \frac{1}{n}\sum \hat{x}^i$ and $\lim_{r\to 0} A = 0$.*

Proof. Because equilibrium outcomes belong to the convex hull of $\{\hat{x}^1, \ldots, \hat{x}^n\}$, and because this convex hull is compact, we can assume without loss of generality that equilibrium outcomes converge as $r \to 0$, so that $x^* \to \tilde{x}$ for some outcome $\tilde{x}$. Let $\tilde{N}$ be the set of players i such that $\hat{x}^i = \tilde{x}$, and note that $N \setminus \tilde{N}$ is nonempty by assumption. Rewriting (7), we have

$$\begin{aligned} 0 &= \sum_{i\in\tilde{N}} \|x^* - \hat{x}^i\|^r (x^* - \hat{x}^i) + \sum_{i\in N\setminus\tilde{N}} \|x^* - \hat{x}^i\|^r (x^* - \hat{x}^i) \\ &\to \sum_{i\in N\setminus\tilde{N}} (\tilde{x} - \hat{x}^i) \\ &= \sum_i (\tilde{x} - \hat{x}^i), \end{aligned}$$

and it follows that $\tilde{x} = \frac{1}{n}\sum_i \hat{x}^i$, as required. For any player i, since there is some j such that $\hat{x}^j \neq \tilde{x}$, the summation $\sum_j \|\hat{x}^j - x^*\|^r$ in the denominator of (8) has positive liminf. Thus, equation (8) implies that equilibrium effort goes to zero, i.e., $\alpha_i \to 0$ as $r \to 0$, as required. □

Notice that as r goes to 0, the power utility function becomes increasingly steep near the player's ideal point and flat at outcomes distinct from the ideal point, so that the player is increasingly indifferent toward outcomes that are not ideal. In other words, the situation becomes "all-or-nothing," with players caring only about getting their own preferred policy outcomes. Theorem 3 establishes that in this case, equilibrium outcomes balance the players' preference by locating close to the mean of their ideal points. Players whose ideal points are distinct from the mean fail to achieve their ideal outcomes and thus expend vanishingly small effort in equilibrium, and players with ideal points at the mean (if any) achieve their ideal outcome with effort that also goes to zero.

Next, we show that as the coefficient of relative risk aversion increases, the equilibrium outcome converges to the outcome that maximizes the minimum utility of the players, i.e, to the maximal element of the Rawlsian social welfare ordering, $x^R = \text{argmax}_x \min_i u(\|x - \hat{x}^i\|)$. Moreover, we show that total effort becomes arbitrarily large as $r \to \infty$. Before proceeding to the analysis, we establish that the Rawlsian outcome is uniquely defined, so that the limit of equilibria is pinned down uniquely as risk aversion increases. Uniqueness of the Rawlsian outcome is formulated in general terms in the following proposition to emphasize that it does not rely on the structure of power utility, but only on strict quasi-concavity of the players' utility over distance.

Proposition 2. *Let $f_i \colon \Re^d \to \Re$, $i = 1, \ldots, n$, be strictly quasi-concave functions. Then*

$$\max_x \min_i f_i(x)$$

has at most one solution.

Proof. Suppose toward a contradiction that there exist distinct outcomes x' and x'' such that $\min_i f_i(x') = \min_i f_i(x'') = \max_x \min_i f_i(x)$, and let $\underline{u}$ denote this value. Set $x''' = \frac{1}{2}x' + \frac{1}{2}x''$, and let k solve $\min_i f(x''')$, so that f_k is among the functions taking the lowest value at x'''. By construction, we have

$$\min\{f_k(x'), f_k(x'')\} \quad \geq \quad \underline{u}.$$

Then strict quasi-concavity implies $f_k(x''') > \underline{u}$, but this implies that x' and x'' do not solve $\max_x \min_i f_i(x)$, a contradiction. We conclude that the latter problem has a unique solution, as required. □

In one dimension, the Rawlsian outcome is easily characterized as the average of the most extreme ideal points, i.e.,

$$x^R \quad = \quad \frac{1}{2}\min_j \hat{x}^j + \frac{1}{2}\max_j \hat{x}^j.$$

In particular, it belongs to the convex hull of the players who are worst off at x^R. The next proposition establishes that this property holds in general and in fact characterizes the Rawlsian outcome. Again, the characterization result does not depend on the power utility functional form, and thus it is stated for the general game.

Proposition 3. *In the tug of war game, an outcome x is the Rawlsian outcome if and only if it is a convex combination of the ideal points of the players who are worst off at x, i.e., $x = x^R$ if and only if*

$$x \in conv\{\hat{x}^i \mid i \in N(x)\},$$

where

$$N(x) \quad = \quad \left\{ i \in N \mid u(\|x - \hat{x}^i\|) = \min_j u(\|x - \hat{x}^j\|) \right\}$$

is the set of players with lowest utility at x.

Proof. First, assume $x = x^R$, and suppose toward a contradiction that x is not a convex combination of ideal points of players belonging to $N(x)$. Then x is not Pareto optimal for the players in this set, and there exists x' such that for all $i \in N(x)$, we have $u(\|x' - \hat{x}^i\|) > u(\|x - \hat{x}^i\|)$. For each $\epsilon \in (0, 1)$, define $x^\epsilon = (1 - \epsilon)x + \epsilon x'$, and note that strict quasi-concavity implies that for all such ϵ and all $i \in N(x)$, we have $u(\|x^\epsilon - \hat{x}^i\|) > u(\|x - \hat{x}^i\|)$. Furthermore, by continuity of u, we can choose $\epsilon > 0$ sufficiently small that for all $i \in N \setminus N(x)$, we have $u(\|x^\epsilon - \hat{x}^i) > u(\|x - \hat{x}^i\|)$. But then we have

$$\min_j u(\|x^\epsilon - \hat{x}^j\|) \quad > \quad \min_j u(\|x^R - \hat{x}^j\|),$$

a contradiction. We conclude that x is a convex combination of ideal points of players in $N(x)$.

For the converse, assume that x is a convex combination of ideal points of players in $N(x)$, and suppose toward a contradiction that $x \neq x^R$. Then we have

$$\min_j u(\|x^R - \hat{x}^j\|) \quad > \quad \min_j u(\|x - \hat{x}^j\|).$$

Since $x^R \neq x$ and $x \in \text{conv}\{\hat{x}^i \mid i \in N(x)\}$, it follows that if $\|\hat{x}^i - x^R\| < \|\hat{x}^i - x\|$ for some $i \in N(x)$, then there is some $k \in N(x)$ such that $\|\hat{x}^k - x\| < \|\hat{x}^k - x^R\|$. But this would imply

$$\min_j u(\|x - \hat{x}^j\|) \quad = \quad u(\|x - \hat{x}^k\|) \quad > \quad u(\|x^R - \hat{x}^k\|) \quad \geq \quad \min_j u(\|x^R - \hat{x}^j\|),$$

a contradiction. Thus, we must have $u(\|x - \hat{x}^i\|) = u(\|x^R - \hat{x}^i)$ for all $i \in N(x)$. For each $\epsilon \in (0, 1)$, define $x^\epsilon = (1 - \epsilon)x + \epsilon x^R$, and note that strict quasi-concavity implies that for all such ϵ and all $i \in N(x)$, we have $u(\|x^\epsilon - \hat{x}^i\|) > u(\|x^R - \hat{x}^i\|)$. Furthermore, by continuity of u, we can choose $\epsilon > 0$ sufficiently small that for all $i \in N \setminus N(x)$, we have $u(\|x^\epsilon - \hat{x}^i\|) > u(\|x^R - \hat{x}^i\|)$, which leads to a contradiction, as above. We conclude that $x = x^R$, as required. □

We now proceed to the analysis of increasing risk aversion. The limit of total equilibrium effort turns out to depend on the distance of the worst off players from the Rawlsian outcome: if this distance is greater than or equal to one, then total effort goes to infinity; and otherwise, it goes to zero. Using equation (6), however, we can also say something about the relative rates of change of individual effort levels: given two players i and j with $\|\hat{x}^i - x^R\| > \|\hat{x}^j - x^R\|$, equation (6) implies that the ratio α_i/α_j of equilibrium efforts goes to infinity as $r \to \infty$.

Theorem 4. *In the tug of war game with power utility and $r > 0$, the limit of equilibrium outcomes as $r \to \infty$ is the Rawlsian outcome, i.e., $\lim_{r\to\infty} x^* = x^R$. If $\max_i \|\hat{x}^i - x^R\| \geq 1$, then total effort increases without bound as $r \to \infty$, i.e., $\lim_{r\to\infty} A = \infty$; and if $\max_i \|\hat{x}^i - x^R\| < 1$, then total effort goes to zero as $r \to \infty$, i.e., $\lim_{r\to\infty} A = 0$.*

Proof. Because equilibrium outcomes belong to the convex hull of the players' ideal points, we can assume without loss of generality that equilibrium outcomes converge as $r \to \infty$, so that $x^* \to \tilde{x}$ for some outcome $\tilde{x}$. We argue that $\tilde{x} = x^R$. Let N_r be the set of worst-off players given parameter r, i.e.,

$$N_r \;=\; \left\{ i \in N \mid u(\|x^* - \hat{x}^i\|) = \min_j u(\|x^* - \hat{x}^j\|) \right\},$$

where x^* implicitly depends on r through equation (7). Given that the number of players is finite, we may go to a subsequence (still indexed by r) such that there exists some player k such that $k \in N_r$ for all r in the subsequence. Since the ratios $\|x^* - \hat{x}^i\|/\|x^* - \hat{x}^k\|$ lie in the unit interval $[0,1]$, we can go to a further subsequence (still indexed by r) such that for all players $i \neq k$, the ratios converge, so that the limit

$$\rho_i \;=\; \lim_{r\to\infty} \frac{\|x^* - \hat{x}^i\|}{\|x^* - \hat{x}^k\|} \;\in\; [0,1]$$

is well-defined. Now, rewrite equation (7) as:

$$0 = (x^* - \hat{x}^k) + \sum_{i\neq k} \left(\frac{\|x^* - \hat{x}^i\|}{\|x^* - \hat{x}^k\|} \right)^r (x^* - \hat{x}^i). \tag{9}$$

Taking limits in (9), we obtain the expression

$$0 = (\tilde{x} - \hat{x}^k) + \sum_{i\neq k} \rho_i (\tilde{x} - \hat{x}^i),$$

which after manipulation becomes

$$\frac{\hat{x}^k + \sum_{i\neq k} \rho_i \hat{x}^i}{1 + \sum_{i\neq k} \rho_i} = \tilde{x}.$$

Let $\gamma_k = \frac{1}{1+\sum_{i\neq k}\rho_i}$, and for $i \neq k$, let $\gamma_i = \frac{\rho_i}{1+\sum_{j\neq k}\rho_i}$. Then we have

$$\sum_i \gamma_i \hat{x}^i = \tilde{x}.$$

Note that for every player j, if $\|\tilde{x} - \hat{x}^j\| < \|\tilde{x} - \hat{x}^k\|$, then $\rho_j = 0$; equivalently, if $\rho_j > 0$, then $\|\tilde{x} - \hat{x}^j\| = \|\tilde{x} - \hat{x}^k\|$. We conclude that the limit $\tilde{x}$ of equilibrium outcomes is a convex combination of the ideal points of the players who are worst off at $\tilde{x}$. Therefore, by Proposition 3, $x^* \to \tilde{x} = x^R$, proving the first part of the theorem.

For the second part of the theorem, note that for every player i, we have

$$\alpha_i = \frac{\|\hat{x}^i - x^*\|^{r/2}}{\sqrt{\frac{2c}{r}\sum_j \left(\frac{\|\hat{x}^j - x^*\|}{\|\hat{x}^i - x^*\|}\right)^r}}.$$

First assume that $\max_i \|\hat{x}^i - x^R\| \geq 1$. Going to a subsequence (still indexed by r) if necessary, we may choose a player j such that $\rho_j = \max_i \rho_i$ for every r in the subsequence. Then in the above expression for equilibrium effort, the denominator goes to zero, while the numerator is bounded below by one, and we have $\lim_{r\to\infty} \alpha_j = 0$, and thus total effort increases without bound. Next, assume that $\max_i \|\hat{x}^i - x^R\| < 1$. For each player i, the sum $\sum_j (\frac{\|\hat{x}^j - x^*\|}{\|\hat{x}^i - x^*\|})^r$ includes one term equal to one, and thus the summation is bounded below by one. Thus, the limit of equilibrium effort is determined by $r\|\hat{x}^i - x^*\|^r$, which goes to zero, and total effort goes to zero, as required. □

4 Discussion

A special case of the model is that with just two players, who could be construed as partisan lobbyists engaged in a contest to pull the final outcome toward their ideal points. In contrast to an all-pay auction approach, the model of this paper has the nature of a tug of war between the lobbyists, with both sides exerting effort and determining the outcome in a continuous way. It is clear that in any symmetric Nash equilibrium, the equilibrium outcome must be the midpoint between the ideal points, $(\hat{x}^1 + \hat{x}^2)/2$, but this simple observation leaves open the question of existence and the possibility of asymmetric equilibria. The results of this paper resolve these issues by establishing existence and uniqueness of equilibrium in the lobbying model, and moreover they provide a characterization of the equilibrium effort levels of the lobbyists. This differs from the models of Epstein and Nitzan (2004) and Münster (2006), in which lobbyists choose which policies they stand for, and following that, the lobbyists play a contest in which the outcome is the position of the winner; in the latter paper, the probability of winning is a discontinuous function of efforts, while in the former paper it is continuous.

There are of course other possible extensions of the model that could be considered. A simple generalization of the above model is to allow the cost parameter $c_i > 0$ to vary across players, in which case equation (2) takes the form

$$0 = \sum_i \frac{1}{c_i} u'\left(\left\|\hat{x}^i - x^*\right\|\right) \|\hat{x}^i - x^*\|(\hat{x}^i - x^*),$$

so that players with lower cost have greater weight, and total effort is:

$$A = \sqrt{-\sum_i \frac{1}{2c_i} u'(\|\hat{x}^i - x^*\|)\|\hat{x}^i - x^*\|}.$$

Solving for the equilibrium effort of player i, we obtain:

$$\alpha_i = \frac{-u'(\|\hat{x}^i - x^*\|)\|\hat{x}^i - x^*\|}{\sqrt{\sum_j \frac{2c_i^2}{c_j} u'(\|\hat{x}^j - x^*\|)\|\hat{x}^j - x^*\|}}.$$

In contrast to the earlier analysis, the equilibrium outcome can now be affected by relative changes in cost parameters. Because the outcome x^* is restricted to the convex hull of the players' ideal points, a compact set, it follows that as player i's cost parameter c_i becomes large, her equilibrium effort goes to zero. Moreover, if we fix the parameters of all other players and let $c_i \to 0$, then player i's equilibrium effort goes to infinity, and the equilibrium outcome converges to i's ideal point.

A second extension generalizes the assumption of quadratic cost of effort, so that player i's payoff is $u(\|\hat{x}^i - x^*\|) - c\alpha_i^k$, where $k > 1$ is a cost parameter. Of course, the linear cost case is approximated by $k \downarrow 1$. Then the first order condition for player i with $\hat{x}^i \neq x^*$ is:

$$\alpha_i = \left(-\frac{1}{kcA} u'(\|\hat{x}^i - x^*\|)\|\hat{x}^i - x^*\| \right)^{\frac{1}{k-1}}.$$

Following the logic above, equation (2) becomes

$$\sum_i \left(u'(\|\hat{x}^i - x^*\|)\|\hat{x}^i - x^*\| \right)^{\frac{1}{k-1}} (x^* - \hat{x}^i) = 0, \tag{10}$$

and total effort is:

$$A = \left[\sum_i \left(-\frac{1}{kc} u'(\|\hat{x}^i - x^*\|)\|\hat{x}^i - x^*\| \right)^{\frac{1}{k-1}} \right]^{\frac{k-1}{k}}.$$

Once again, equilibrium effort goes to zero as c becomes large, and for all players i with $\hat{x}^i \neq x^*$, effort increases without bound as $c \to 0$. More interestingly, we can analyze the effect of varying the curvature of the cost function for the special case of power utility. Then (10) becomes

$$\sum_i \|\hat{x}^i - x^*\|^{\frac{r}{k-1}} (x^* - \hat{x}^i) \quad = \quad 0,$$

and it can be seen that varying k has the effect of varying r in the inverse direction: letting $k \downarrow 1$ is analogous to letting r go to infinity, and letting $k \to \infty$ is analogous to letting r go to zero. Thus, Theorems 4 and 3 show, respectively, that the equilibrium outcome goes to the Rawlsian outcome and the mean of ideal points in the two cases.

Third, we could consider a model in which each player chooses the effort for each dimension separately, i.e., $\alpha_m^i \geq 0$ for each $i = 1, \ldots, n$ and $m = 1, \ldots, d$, with the cost parameter for each dimension being $c_m > 0$ for $m = 1, \ldots, d$. In that case, equation (2) is replaced by

$$
\begin{aligned}
0 \quad = \quad & \sum_{i:\hat{x}^i_m < x^*_m}^{n} u'(\|\hat{x}^i - x^*\|) \frac{1}{\|\hat{x}^i - x^*\|} (x^*_m - \hat{x}^i_m)^2 \\
& - \sum_{i:\hat{x}^i_m > x^*_m}^{n} u'(\|\hat{x}^i - x^*\|) \frac{1}{\|\hat{x}^i - x^*\|} (x^*_m - \hat{x}^i_m)^2,
\end{aligned}
$$

for $m = 1, \ldots, d$, and the total effort on dimension m is characterized by

$$
A_m \quad = \quad \sqrt{-\frac{1}{2c_m} \sum_i u'(\|\hat{x}^i - x^*\|) \frac{1}{\|\hat{x}^i - x^*\|} (x^*_m - \hat{x}^i_m)^2}.
$$

Again, the equilibrium outcome on dimension m is unaffected by the cost parameter, and if the players' ideal points on dimension m are not identical, i.e., $\hat{x}^i_m \neq \hat{x}^j_m$ for distinct i and j, then total effort level in dimension m increases without bound as c_m becomes small.

A final variant considered here is the model with collective outcome as a linear function of effort, i.e., $x^* = q + \sum_j \alpha_j$, where q is an arbitrary status quo outcome. In this case, the best response problem of player i is

$$
\max_{\alpha_i} u_i(q + \sum_j \alpha_j) - c\alpha_i^2.
$$

Assuming u_i is strictly concave, this objective function is strictly concave in α_i, and the first order condition satisfied in equilibrium is then

$$
u'_i(q + \sum_j \alpha^*_j) \quad = \quad 2c\alpha^*_i.
$$

Letting $A^* = \sum_j \alpha^*_j$, this implies

$$
\sum_i u'_i(q + A^*) \quad = \quad 2cA^*,
$$

which has a unique solution. Moreover, to solve for status quos that are stable, in the sense that $A^* = 0$, we immediately obtain the expression

$$
\sum_i u'_i(q) \quad = \quad 0.
$$

That is, the unique stable status quo is the alternative that maximizes the sum of individual utilities.

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