May's Theorem in One Dimension*

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Abstract

This paper provides three versions of May's theorem on majority rule, adapted to the one-dimensional model common in formal political modeling applications. The key contribution is that single-peakedness of voter preferences allows us to drop May's restrictive positive responsiveness axiom. The simplest statement of the result holds when voter preferences are single-peaked and linear (no indifferences), in which case a voting rule satisfies anonymity, neutrality, Pareto, and transitivity of weak social preference if and only if the number of individuals is odd and the rule is majority rule.

Majority rule occupies a central role in democratic decision making, and it has accordingly received close attention in formal political theory. A well-known axiomatization due to May (1952) provides some theoretical justification for the status of majority rule: when individual preferences are unrestricted, it is the only voting rule that is unbiased toward individuals, unbiased toward alternatives, and positively responsive to changes individuals preferences. In the terminology of this paper, majority rule is uniquely characterized by the axioms of anonymity, neutrality, and strong tie break. The latter axiom is indeed strong: it requires that if social indifference holds between two alternatives and if even just one individual increases her support for one of these alternatives (e.g., indifference is replaced by strict preference for one alternative), then the social "tie" between the alternatives is broken in favor of the one with increased support. Although desirable in a world where each vote counts and social ties represent an exact balancing of individual preferences, the strong tie break axiom is violated by most all

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voting rules—for example, it is violated by all quota rules other than majority rule—and its normative importance is less compelling than the other axioms.

In this paper, I provide an alternative to May's axiomatization that is closer to the subject matter of political science and eschews his strong tie break axiom. The key structure added to the problem is the assumption of single-peakedness: whereas May assumes individual preferences are unrestricted, consistent with the social choice literature, formal modeling in political science often assumes alternatives are one-dimensional and individual preferences are single-peaked. This preference restriction is prevalent in work on electoral modeling in the tradition of Calvert (1983); it was used in the analysis of agenda control by Romer and Rosenthal (1978) and introduced to legislative bargaining by Banks and Duggan (2000); and Penn, Patty, and Gailmard (2011) recently explore the implications of single-peakedness in an axiomatic analysis of strategy-proof voting mechanisms. A compelling property of single-peaked preferences, established by Black (1948,1958) and Arrow (1951), is that it precludes the Condorcet paradox and confers desirable transitivity properties on majority rule: majority voting generates transitive strict social preferences, and when the number of individuals is odd, the majority indifference relation is also transitive. This full transitivity of majority rule with an odd number of individuals is a highly specialized property—for example, it is violated by all quota rules other than majority rule—and in this sense, transitivity with single-peaked individual preferences seems to impose restrictions similar to strong tie break with unrestricted preferences.

I maintain May's other axioms and investigate the question, "Assuming single-peaked preferences, does May's characterization carry over if we replace the tie break axiom with transitivity?" A positive answer to this question would be of interest because it would provide further justification for majority rule in many applications considered in political science, and because, compared to the tie break axiom, transitivity arguably has greater normative relevance: when a voting rule generates an ordering of alternatives, we can view social choices as deriving from a representative agent; this simplifies the process of choosing from a finite set of feasible options, as we simply choose the top-ranked of the available options; and it precludes inconsistencies when options are added or deleted. The answer to the above question depends on the details—on the possibility of individual indifferences between alternatives, on the form of majority rule considered, and on the nature of the transitivity condition imposed. Regarding the definition of majority rule, a majority preference for x over y may hold if more than half of all individuals strictly prefer x to y (simple majority rule), or it may hold if more individuals strictly prefer x than strictly prefer y (relative majority rule). One transitivity condition is that strict majority preferences are transitive, and a stronger one is that weak majority preferences are transitive (i.e., both strict majority preference and indifference are transitive). The sharpest result holds under the stronger transitivity condition in the absence of individual indifference, where an exact characterization of majority rule is obtained, and May's theorem carries over in the conjectured manner.

The conclusion distills the findings of this paper into three corollaries, each of which provides a version of May's theorem in one dimension. First, assuming there are at least three alternatives and all single-peaked preferences are possible, if we consider a voting rule that satisfies anonymity, neutrality, and the Pareto axiom, then the weak social preference relation is transitive if and only if the number of individuals is odd and the voting rule is nested between strict and weak simple majority rule. Second, assuming there are at least three alternatives and only the single-peaked preferences without indifferences are possible (so simple and relative majority rule coincide), if we consider a voting rule that satisfies anonymity, neutrality, and the Pareto axiom, then strict social preferences are transitive if and only if strict social preference implies majority preference and an additional non-negative responsiveness axiom (used implicitly by May) is satisfied. The third corollary continues to preclude individual indifference and provides an exact characterization of majority rule: a voting rule satisfies anonymity, neutrality, Pareto and always generates transitive weak social preference if and only if the number of individuals is odd and it is exactly equal to majority rule. This result provides a close analogue of May's theorem that exploits the unidimensionality prevalent in formal models of politics and fits naturally within the analytical framework of political science.

Section 1 sets the formal framework of the analysis, and Section 2 states May's axioms and provides two statements of May's theorem from the social choice literature, depending on the domain of individual preferences. Section 3 deduces implications of the two transitivity conditions mentioned above assuming single-peaked preferences, depending on whether individual preferences may admit indifferences. Section 4 presents sufficient conditions for the transitivity conditions, and Section 5 contains a number of technical examples to confirm that the results of the paper are tight. The conclusion states three versions of May's theorem for the one-dimensional model in the form of three corollaries, and it compares the results to those of Dasgupta and Maskin (2008) on the robustness of majority rule.

1 Formal Preliminaries

A collective choice *environment* is defined by four things: a set Θ of social states, which represent information that we want to treat as variable and which capture the uncertainty of the analyst; a set N of individual decision makers; a set X of alternatives from which they must make a collective choice; and finally individual preferences $(P_i(\theta), R_i(\theta))$ over alternatives in each state θ, where $P_i(\theta)$ represents strict preference and $R_i(\theta)$ represents weak preference. We assume there is a finite number n of individuals, who are indexed $1, \ldots, n$, and we assume that strict preference is asymmetric and weak preference is complete,[1] and we assume these relations are *dual,* in the sense that for all $x, y \in X$, $xP_i(\theta)y$ if and only if not $yR_i(\theta)x$. We define the indifference relation $I_i(\theta)$ by the standard convention, i.e., for all $x, y \in X$, $xI_i(\theta)y$ if and only if neither $xP_i(\theta)y$ nor $yP_i(\theta)x$. We always assume that individual preferences form an ordering of alternatives; formally, we say a dual pairing (P, R) of an asymmetric relation P with a complete relation R is an *order* if P is negatively transitive and R is transitive.[2] A stronger restriction is that (P, R) is a *linear order,* which adds the requirement that indifference cannot hold between any two distinct alternatives.

To denote the groups for which strict preference, weak preference, and indifference holds between any two alternatives, we use

$$\begin{aligned} P(x,y|\theta) &= \{i \in N \mid xP_i(\theta)y\} \\ R(x,y|\theta) &= \{i \in N \mid xR_i(\theta)y\} \\ I(x,y|\theta) &= \{i \in N \mid xI_i(\theta)y\}, \end{aligned}$$

and we write $p(x,y|\theta) = |P(x,y|\theta)|$ and $r(x,y|\theta) = |R(x,y|\theta)|$ for the numbers of individuals strictly and weakly preferring one alternative to another.

[1] A relation P on X is *asymmetric* if for all $x, y \in X$, we do not have both xPy and yPx; and R is *complete* if for all $x, y \in X$, either xRy or yRx or both.

[2] A relation P is *negatively transitive* if for all $x, y, z \in X$, xPy implies either xPz or zPy; and R is *transitive* if for all $x, y, z \in X$, xRy and yRz together imply xRz. We use the standard shorthand $xRyRz$ for the conjunction of xRy and yRz.

The profile of individual preferences at θ is denoted by

$$PR(\theta) \quad = \quad ((P_1(\theta), R_1(\theta)), \ldots, (P_n(\theta), R_n(\theta))),$$

and the set $PR(\Theta) = \{PR(\theta) \mid \theta \in \Theta\}$ is the domain of possible preference profiles. I impose the following minimal richness condition on the domain of preferences: there exists a *free pair* $\{s,t\} \subseteq X$ such that for all $x, y \in X$ and all $\theta \in \Theta$, there exists $\theta' \in \Theta$ such that $P(s,t|\theta') = P(x,y|\theta)$ and $P(t,s|\theta') = P(y,x|\theta)$. This assumption is satisfied in most all environments of interest in formal political theory, including the domains defined below.[3]

A classical example of a collective choice environment is that of a group N of voters who must choose from a finite set X of candidates to fill a political office. Each voter has preferences over the candidates, and states simply index the set of possible preference profiles. Alternatively, a voting body (e.g., an electorate, a city council, a board of directors) must choose from a finite set of projects to undertake. Or a department faculty must choose from a finite set of job candidates to fill a position. In these kinds of environments, it is often difficult to impose *a priori* restrictions on individual preferences, and it is natural to allow for every profile of orderings on X. Let

$$\mathbf{U} \quad = \quad \{((P_1, R_1), \ldots, (P_n, R_n)) \mid \forall i \in N : (P_i, R_i) \text{ is an order of } X\}$$

denote the set of all profiles of orders of X. We refer to the assumption that $PR(\Theta) = \mathbf{U}$ as *unrestricted domain.* Sometimes, especially when the set of alternatives is small, we may wish to consider the possibility that individuals can always discern a strict preference between distinct candidates. Let

$$\mathbf{L} \quad = \quad \{((P_1, R_1), \ldots, (P_n, R_n)) \mid \forall i \in N : (P_i, R_i) \text{ is a linear order of } X\}.$$

We refer to the assumption that $PR(\Theta) = \mathbf{L}$ as *linear domain.*

Another classical example of a collective choice environment is that of a society that must choose a single proportional income tax rate between 0 and 1. Or that of a society that must choose the level of provision of a single public good. Or, suppose that public policies can be ordered according to a single summary statistic that contains all information relevant for voter preferences, with the standard interpretation that policies corresponding to smaller values of the statistic are more "liberal," while those corresponding to greater values of the statistic are more "conservative." Given this structure, it is natural to assume individual preferences are single-peaked, in the

[3]The existence of a free pair is used only in the proof of Proposition 8.

sense defined formally as follows. Letting $(\prec, \preceq)$ be a linear order of X, a dual pair (P, R) is *single-peaked with respect to* $(\prec, \preceq)$ if there exists $\hat{x} \in X$ satisfying the following conditions:

- for all $y \in X$ with $y \neq \hat{x}$, $\hat{x} P y$;
- for all $y, z \in X$, if $z \prec y \prec \hat{x}$, then yPz;
- for all $y, z \in X$, if $\hat{x} \prec y \prec z$, then yPz.

We say a profile $((P_1, R_1), \ldots, (P_n, R_n))$ of individual preferences is *single-peaked with respect to* $(\prec, \preceq)$ if for each $i \in N$, (P_i, R_i) is single-peaked with respect to $(\prec, \preceq)$. In this case, we denote the unique maximal element of (P_i, R_i) by $\hat{x}^i$, and we refer to $\hat{x}^i \in X$ as the individual's *ideal point.* We say $((P_1, R_1), \ldots, (P_n, R_n))$ is *single-peaked* if it is single-peaked with respect to some $(\prec, \preceq)$. Let

$$\mathbf{S} \quad = \quad \{((P_1, R_1), \ldots, (P_n, R_n)) \mid ((P_1, R_1), \ldots, (P_n, R_n)) \text{ is single-peaked}\}.$$

We refer to the assumption that $PR(\Theta) = \mathbf{S}$ as *single-peaked domain.* An implication of single-peakedness is that given any three distinct alternatives $\{x, y, z\}$, there is one alternative among the three, namely the alternative that is ordered in the middle, that is not bottom ranked among the three for any individual, e.g., if $x \prec y \prec z$, then for every individual i, we have either $yP_i(\theta)x$ or $yP_i(\theta)z$.

To analyze collective decisions, we model social preferences in each state as determined by a mapping F, where we denote the values of F by $F(\theta) = (P_F(\theta), R_F(\theta))$. Thus, we have

$$\theta \overset{F}{\mapsto} (P_F(\theta), R_F(\theta)),$$

and we call F a *social preference rule,* or simply an *SPR.* Here, $P_F(\theta)$ is the strict social preference relation at θ, and we interpret a preference $xP_F(\theta)y$ to indicate that the alternative x is normatively superior to y. As well, $R_F(\theta)$ is the weak social preference relation at θ, and $xR_F(\theta)y$ indicates that x is normatively at least as desirable as y. We assume that the latter relation is complete, the former is asymmetric, and the two relations are dual. We denote by $I_F(\theta)$ the social indifference relation defined from strict or weak social preference in the usual way.

The following examples give two versions of well-known SPRs based on majority voting and Pareto dominance.

simple majority, F_{SM}

$$xP_{SM}(\theta)y \text{ if and only if } p(x,y|\theta) > n/2.$$

relative majority, F_{RM}

$$xP_{RM}(\theta)y \text{ if and only if } p(x,y|\theta) > p(y,x|\theta).$$

simple Pareto, F_{SP}

$$xP_{SP}(\theta)y \text{ if and only if } p(x,y|\theta) = n.$$

relative Pareto, F_{RP}

$$xP_{RP}(\theta)y \text{ if and only if } r(x,y|\theta) = n \text{ and } p(x,y|\theta) > 0.$$

Weak social preferences are obtained via duality, e.g., $xR_{SM}(\theta)y$ if and only if not $yP_{SM}(\theta)x$, which holds if and only if not $p(y,x|\theta) > n/2$, which is equivalent to $r(x,y) \geq n/2$. Of course, under linear domain, the distinction between simple and relative majority rule disappears, so that $F_{SM} = F_{RM}$, and similarly for simple and relative Pareto.

2 Axioms for Majority Rule

This section states axioms that are standard for the characterization of majority rule and in the broader analysis of social choice, and it presents two axiomatizations for majority rule that are essentially due to May (1954). We begin with an axiom that formalizes the idea that social preferences are not biased for or against any particular alternative. Note that it strengthens Arrow's independence of irrelevant alternatives (IIA).[4]

neutrality For all $x, y, w, z \in X$ and all $\theta, \theta' \in \Theta$,

$$\left.\begin{array}{c} P(x,y|\theta) = P(w,z|\theta'), \\ R(x,y|\theta) = R(w,z|\theta'), \\ \text{and } xP_F(\theta)y \end{array}\right\} \Rightarrow \quad wP_F(\theta')z.$$

[4]Arrow's IIA is the condition that for all $x, y \in X$ and all $\theta, \theta' \in \Theta$, if individual preferences between x and y are unchanged across the two states, i.e., $P(x,y|\theta) = P(x,y|\theta')$ and $P(y,x|\theta) = P(y,x|\theta')$, then the social preference between x and y is the same in both states.

In particular, the above axiom captures the idea that if individual preferences between x and y in one state are the same as individual preferences between w and z in another, then social preferences should be the same too (with w playing the same role as x, z playing the same role as y).

The next axiom formalizes the notion that social preferences are not biased for or against any individual: the social preference between x and y depends only on the numbers of individuals who prefer (strictly and weakly) x over y, and not on their identities. Note that it also strengthens IIA.

anonymity For all $x, y \in X$ and all $\theta, \theta' \in \Theta$

$$\left.\begin{array}{c} p(x,y|\theta) = p(x,y|\theta'), \\ r(x,y|\theta) = r(x,y|\theta'), \\ \text{and } xP_F(\theta)y \end{array}\right\} \quad \Rightarrow \quad xP_F(\theta')y.$$

The next axiom says, essentially, that social indifferences are sensitive to changes in individual preferences: if x and y are socially indifferent and if some individual increases their support for x, then, after the change, x is strictly socially preferred to y. Note that the axiom only applies when two alternatives are initially socially indifferent, so it does not generally imply that a strict social preference is preserved by an increase in support for the preferred alternative. Nevertheless, this initial formulation of the axiom does impose restrictive conditions on the way individual indifferences are handled.

strong tie break For all $x, y \in X$ and all $\theta, \theta' \in \Theta$,

$$\left.\begin{array}{c} P(x,y|\theta) \subseteq P(x,y|\theta'), \\ R(x,y|\theta) \subseteq R(x,y|\theta'), \\ \text{at least one inclusion strict,} \\ \text{and } xI_F(\theta)y \end{array}\right\} \quad \Rightarrow \quad xP_F(\theta')y.$$

In particular, if x and y are socially indifferent, if one individual initially prefers y to x strictly, and if we consider another state where the individual is just indifferent between the two alternatives (all other individuals' preferences between x and y held constant), then this breaks the social indifference in favor of x. The same is true if the individual is initially indifferent and then moves x above y. Of the above SPRs, only relative majority rule generally satisfies this condition.

The first proposition provides a formal statement of the restrictiveness of strong tie break among the anonymous and neutral SPRs. It is the closer to May's original theorem of two results presented in this section. Note, however, that the axiom of strong tie break used here is weaker than May's Condition IV (positive responsiveness), which incorporates the monotonicity condition of non-negative responsiveness, defined below, as well.

Proposition 1 (May) *Assume* $PR(\Theta) = \mathbf{U}$. *A SPR* F *satisfies anonymity, neutrality, and strong tie break if and only if* $F = F_{RM}$.

Proof: It is clear that F_{RM} satisfies these conditions. Now assume F does, and consider any $a, b \in X$ and any $\theta \in \Theta$. I claim that $p(a, b|\theta) = p(b, y|\theta)$ implies $aI_F(\theta)b$, for suppose $aP_F(\theta)b$. Consider a state $\theta' \in \Theta$ such that $P(a, b|\theta') = P(b, a|\theta)$ and $P(b, a|\theta') = P(a, b|\theta)$. By anonymity, $aP_F(\theta')b$. But by neutrality, $bP_F(\theta')a$, a contradiction. It now suffices to show that $p(a, b|\theta) > p(b, a|\theta)$ implies $aP_F(\theta)b$. Assuming the inequality holds, let θ' be a state such that $P(b, a|\theta') = P(b, a|\theta)$ and $P(a, b|\theta') \subsetneq P(a, b|\theta)$ with $p(a, b|\theta') = p(b, a|\theta')$. From the above claim, we have $aI_F(\theta')b$, and strong tie break implies $aP_F(\theta)b$, as required. ∎

As we have mentioned, Proposition 1 uses a strong tie break condition, one that exploits the possibility of individual indifferences in a perhaps undesirable way. When individual indifference is precluded, we can prove a similar result using a weaker version of the tie break condition that requires a strict preference reversal in order to break a social indifference.

tie break For all $x, y \in X$ and all $\theta, \theta' \in \Theta$,

$$\left.\begin{array}{c} P(x, y|\theta) \subseteq P(x, y|\theta'), \\ R(x, y|\theta) \subseteq R(x, y|\theta'), \\ P(y, x|\theta) \cap P(x, y|\theta') \neq \emptyset, \\ \text{and } xI_F(\theta)y \end{array}\right\} \quad \Rightarrow \quad xP_F(\theta')y.$$

The next axiom, which strengthens IIA, imposes a monotonicity condition to the effect that increased support for any x over any y should preserve a social preference for x over y. Note that in comparing individual preferences across states θ and θ', the antecedent assumes that no individuals are indifferent between x and y in θ', so in particular every individual who is indifferent in θ has a strict preference for x in θ'. This leads to a weaker NNR condition than normally used.

non-negative responsiveness (NNR) For all distinct $x, y \in X$ and all $\theta, \theta' \in \Theta$,

$$\left.\begin{array}{c} R(x,y|\theta) \subseteq P(x,y|\theta') \\ \text{and } xP_F(\theta)y \end{array}\right\} \quad \Rightarrow \quad xP_F(\theta')y.$$

The following proposition assumes that only profiles of linear orders are possible and uses the weaker tie break axiom, above. Now, however, we must add the NNR axiom separately; see Example 1 in Section 5 for a counterexample to the statement of the proposition without NNR.

Proposition 2 *Assume $PR(\Theta) = \mathbf{L}$. A SPR F satisfies anonymity, neutrality, NNR, and tie break if and only if $F = F_{SM} = F_{RM}$.*

Proof: It is clear that F_{SM} satisfies these conditions. Now assume F does, and consider any $a, b \in X$ and any $\theta \in \Theta$. We need to show $aP_F(\theta)b$ if and only if $p(a,b|\theta) > p(b,a|\theta)$. First, assume $aP_F(\theta)b$, and suppose $p(a,b|\theta) \leq p(b,a|\theta)$. Letting $\theta' \in \Theta$ be such that $P(b,a|\theta') = P(a,b|\theta)$ and $P(a,b|\theta') = P(b,a|\theta)$, neutrality implies $bP_F(\theta')a$. Since $p(a,b|\theta') \geq p(b,a|\theta')$, there is a state θ'' in which $p(a,b|\theta') - p(b,a|\theta') \geq 0$ members of $P(a,b|\theta')$ have their ab-preferences reversed (so now they prefer b to a), while all others' ab-preferences are unchanged. By NNR, $bP_F(\theta'')a$. But

$$\begin{aligned} p(a,b|\theta'') &= p(a,b|\theta') - (p(a,b|\theta') - p(b,a|\theta')) \\ &= p(b,a|\theta') \\ &= p(a,b|\theta) \end{aligned}$$

and

$$\begin{aligned} p(b,a|\theta'') &= p(b,a|\theta') + (p(a,b|\theta') - p(b,a|\theta')) \\ &= p(a,b|\theta') \\ &= p(b,a|\theta), \end{aligned}$$

so $aP_F(\theta)b$ and anonymity imply $aP_F(\theta'')b$, a contradiction. Second, assume $p(a,b|\theta) > p(b,a|\theta)$, and suppose $bR_F(\theta)a$. Let $\theta' \in \Theta$ be such that $p(a,b|\theta) - p(b,a|\theta) > 0$ members of $P(a,b|\theta)$ have their ab-preferences reversed (so now they prefer b to a), while all others' ab-preferences are unchanged. In case $bP_F(\theta)a$, NNR implies $bP_F(\theta')a$, and in case $bI_F(\theta)a$, tie break implies $bP_F(\theta')a$ again. Letting $\theta'' \in \Theta$ be such that $P(a,b|\theta'') = P(b,a|\theta')$ and $P(b,a|\theta'') = P(a,b|\theta')$, neutrality implies $aP_F(\theta'')b$. But

$$\begin{aligned} p(a,b|\theta'') &= p(b,a|\theta') \\ &= p(b,a|\theta) + (p(a,b|\theta) - p(b,a|\theta)) \\ &= p(a,b|\theta) \end{aligned}$$

and

$$\begin{aligned} p(b,a|\theta'') &= p(a,b|\theta') \\ &= p(a,b|\theta) - (p(a,b|\theta) - p(b,a|\theta)) \\ &= p(b,a|\theta), \end{aligned}$$

so $aP_F(\theta'')b$ and anonymity implies $aP_F(\theta)b$, a contradiction. ∎

Thus, the ostensibly intuitive conditions of neutrality and anonymity, together with the monotonicity conditions of NNR and tie break, lead us to majority rule. It is arguably for this reason, at least in part, that majority rule has occupied a central role in democratic collective choice. Before proceeding, we define a well-known axiom that plays a central role in normative social choice analysis.

Pareto For all $x, y \in X$ and all $\theta \in \Theta$, if $p(x,y|\theta) = n$, then $xP_F(\theta)y$.

That is, if every individual strictly prefers one alternative to another, then social preferences must agree with the unanimous assessment of the members of society. In what follows, we examine the implications of dropping the tie break axioms, while maintaining the Pareto axiom, when individual preferences are restricted to being single-peaked.

3 Necessary Conditions for Transitivity

We begin with the observation that if all profiles of single-peaked linear orders are possible, then neutrality and transitivity of strict social preferences imply NNR. This elementary result is well known under unrestricted domain or linear domain; thus, Proposition 3 confirms that it carries over to the one-dimensional model, a fact that will be useful in drawing together results from this section and the next.

Proposition 3 *Assume* $|X| \geq 3$ *and* $PR(\Theta) = \mathbf{S} \cap \mathbf{L}$*. Let* F *be an SPR satisfying neutrality and Pareto. If* $P_F(\theta)$ *is transitive for all* $\theta \in \Theta$*, then* F *satisfies NNR.*

Proof: Consider $a, b \in X$ and $\theta, \theta' \in \Theta$ such that $R(a,b|\theta) \subseteq P(a,b|\theta')$ and $aP_F(\theta)b$. Let $\{G_1, G_2, G_3\}$ be the partition of N such that $G_1 = P(a,b|\theta)$, $G_2 = P(b,a|\theta) \cap P(a,b|\theta')$, and $G_3 = P(b,a|\theta) \cap P(b,a|\theta')$. Now let $c \in X \setminus \{a,b\}$ be a distinct alternative, and consider a state $\theta'' \in \Theta$ such that preferences restricted to $\{a,b,c\}$ are as follows; in particular, individual preferences between a and b are unchanged compared to θ.

G_1	G_2	G_3
c	c	b
a	b	c
b	a	a

Note that these preferences are single-peaked with the ordering $a \prec c \prec b$, so this specification is permissible. Since $P(a,b|\theta'') = G_1 = P(a,b|\theta)$ and $P(b,a|\theta'') = G_2 \cup G_3 = P(b,a|\theta)$, neutrality and $aP_F(\theta)b$ imply $aP_F(\theta'')b$. By Pareto, $cP_F(\theta'')a$. Then transitivity and $cP_F(\theta'')aP_F(\theta'')b$ imply $cP_F(\theta'')b$. Since $P(a,b|\theta') = G_1 \cup G_2 = P(c,b|\theta'')$ and $P(b,a|\theta') = G_3 = P(b,c|\theta'')$, neutrality and $cP_F(\theta'')b$ imply $aP_F(\theta')b$, as required. ∎

For completeness, the next result shows that the implication of Proposition 3 carries over to single-peaked domain with indifferences, as long as there are at least four alternatives. The assumption that there are at least four alternatives is needed for the result, as Section 5 contains the three-alternative Example 2 of an SPR satisfying anonymity, neutrality, Pareto, and transitivity of strict social preferences, yet violating NNR.[5]

Proposition 4 *Assume* $|X| \geq 4$ *and* $PR(\Theta) = \mathbf{S}$*. Let* F *be an SPR satisfying neutrality and Pareto. If* $P_F(\theta)$ *is transitive for all* $\theta \in \Theta$*, then* F *satisfies NNR.*

Proof: Consider $a, b \in X$ and $\theta, \theta' \in \Theta$ such that $R(a,b|\theta) \subseteq P(a,b|\theta')$ and $aP_F(\theta)b$. Let $\{G_1, G_2, G_3, G_4, G_5\}$ be the partition of N such that $G_1 = P(a,b|\theta)$, $G_2 = I(a,b|\theta)$, and $G_3 = P(b,a|\theta) \cap P(a,b|\theta')$, $G_4 = P(b,a|\theta) \cap$

[5] See comments following Proposition 6, however, for more on the three-alternative case.

$I(a,b|\theta')$, and $G_5 = P(b,a|\theta) \cap P(b,a|\theta')$. Now let $c, d \in X \setminus \{a,b\}$ be distinct alternatives, and consider a state $\theta'' \in \Theta$ such that preferences restricted to $\{a,b,c,d\}$ are as follows; in particular, individual preferences between a and b are unchanged compared to θ.

G_1	G_2	G_3	G_4	G_5
d	d	d	d	d
c	c	c	bc	b
a	ab	b	a	c
b		a		a

Note that these preferences are single-peaked with the ordering $a \prec c \prec d \prec b$, so this specification is permissible. Since $P(a,b|\theta'') = G_1 = P(a,b|\theta)$ and $P(b,a|\theta'') = G_3 \cup G_4 \cup G_5 = P(b,a|\theta)$, neutrality and $aP_F(\theta)b$ imply $aP_F(\theta'')b$. By Pareto, $cP_F(\theta'')a$. Then transitivity and $cP_F(\theta'')aP_F(\theta'')b$ imply $cP_F(\theta'')b$. Since $P(a,b|\theta') = G_1 \cup G_2 \cup G_3 = P(c,b|\theta'')$ and $P(b,a|\theta') = G_5 = P(b,c|\theta'')$, neutrality and $cP_F(\theta'')b$ imply $aP_F(\theta')b$, as required. ∎

Next, we investigate the majoritarian structure implied by transitive strict social preference under linear, single-peaked domain: a strict social preference for one alternative over another can hold only if the first is majority preferred to the second. In this result and others, we view a relation P on X as a subset of ordered pairs, i.e., $P \subseteq X \times X$. In line with this, the set inclusion $P \subseteq P'$ indicates that if xPy holds for a pair of alternatives, then $xP'y$ also holds, so that preferences under P carry over to P'. Thus, Proposition 5 instructs us that under anonymity, neutrality, and Pareto, if an SPR always generates transitive strict social preferences, then $xP_F(\theta)y$ holds only if a strict majority preference also holds in the same direction.

Proposition 5 *Assume $|X| \geq 3$ and $PR(\Theta) = \mathbf{S} \cap \mathbf{L}$. Let F be an SPR satisfying anonymity, neutrality, and Pareto. If $P_F(\theta)$ is transitive for all $\theta \in \Theta$, then $P_F(\theta) \subseteq P_{SM}(\theta) = P_{RM}(\theta)$ for all $\theta \in \Theta$.*

Proof: Consider $a, b \in X$ and $\theta \in \Theta$ such that $aP_F(\theta)b$, and suppose in order to deduce a contradiction that $bR_{SM}(\theta)a$, so that $p(a,b|\theta) \leq p(b,a|\theta)$. Let $\{G_1, G_2, G_3\}$ be the partition of N such that $G_1 = P(a,b|\theta)$ and $G_2 \cup G_3 = P(b,a|\theta)$, with $|G_2| = p(a,b|\theta)$, and $|G_3| = p(b,a|\theta) - p(a,b|\theta)$. Consider a state θ' such that preferences restricted to $\{a,b\}$ are as follows.

G_1	G_2	G_3
a	b	a
b	a	b

By Proposition 3, the SPR F satisfies NNR. Since $R(a,b|\theta) \subseteq P(a,b|\theta')$, NNR and $aP_F(\theta)b$ imply $aP_F(\theta')b$. Now consider state θ'' such that preferences over a and b are as follows.

G_1	G_2	G_3
b	a	a
a	b	b

Since $p(a,b|\theta'') = p(a,b|\theta')$ and $p(b,a|\theta'') = p(b,a|\theta')$, anonymity and $aP_F(\theta')b$ imply $aP_F(\theta'')b$. But since $P(b,a|\theta'') = G_1 = P(a,b|\theta)$ and $P(a,b|\theta'') = G_2 \cup G_3 = P(b,a|\theta)$, $aP_F(\theta)b$ and neutrality imply $bP_F(\theta'')a$, contradicting asymmetry of strict social preference. We conclude that $aP_{SM}(\theta)b$, as required. ∎

Section 5 provides two examples to illustrate the boundaries of Proposition 5. First, Example 3 shows that the result does not hold for single-peaked domain when individual indifferences are allowed, even if the transitivity condition is strengthened to transitivity of weak social preference: there are SPRs satisfying anonymity, neutrality, and Pareto and that always generate transitive weak social preferences, yet admit the possibility that $P_F(\theta) \not\subseteq P_{RM}(\theta)$. Thus, the implications of transitive strict social preference are dulled by the presence of individual indifferences. Second, Example 4 shows that the result cannot be strengthened to the equality $P_{SM}(\theta) = P_{RM}(\theta) = P_F(\theta)$. Thus, transitivity of strict social preference is consistent with SPFs based on criteria that are more demanding than majority rule, and our axioms must be strengthened to obtain an exact characterization of majority rule.

In accordance with these observations, to deduce necessary conditions for transitivity under single-peaked domain while permitting individual indifferences, we consider transitivity of weak social preferences. With this stronger transitivity condition, we show that the number of individuals is odd, and that strict social preferences are nested between strict and weak simple majority preferences. Note that Example 3, in Section 5, shows that the conclusion of Proposition 6 cannot be strengthened to the equality $P_F(\theta) = P_{SM}(\theta)$, or even to the nesting $P_{SM}(\theta) \subseteq P_F(\theta) \subseteq P_{RM}(\theta)$.[6]

[6]That said, if the inclusion $P_{SM}(\theta) \subseteq P_F(\theta)$ holds for a state θ such that individual preferences are linear orders, then asymmetry of $P_F(\theta)$ immediately implies the stronger statement that $P_F(\theta) = P_{SM}(\theta) = P_{RM}(\theta)$ for that state.

Proposition 6 *Assume* $|X| \geq 3$ *and* $PR(\Theta) = \mathbf{S}$. *Let* F *be an SPR satisfying anonymity, neutrality, and Pareto. If* $R_F(\theta)$ *is transitive for all* $\theta \in \Theta$, *then* n *is odd and* $P_{SM}(\theta) \subseteq P_F(\theta) \subseteq R_{SM}(\theta)$ *for all* $\theta \in \Theta$.

Proof: We first show that under the conditions of the proposition, for all $\theta \in \Theta$ and all $x, y \in X$, $xR_F(\theta)y$ implies $r(x, y|\theta) > \frac{n}{2}$. To this end, consider $a, b \in X$ and $\theta \in \Theta$ such that $aR_F(\theta)b$, and suppose in order to deduce a contradiction that $r(a, b|\theta) \leq \frac{n}{2}$. Let $\{G_1, G_2, G_3\}$ be the partition of N such that $G_1 = P(a, b|\theta)$, $G_2 = I(b, a|\theta)$, and $G_3 = p(b, a|\theta)$, and note that $|G_3| \geq \frac{n}{2}$. Let $c \in X \setminus \{a, b\}$ be a distinct alternative, and consider a state θ' such that preferences restricted to $\{a, b, c\}$ are as follows; in particular, individual preferences between a and b are unchanged compared to θ.

G_1	G_2	G_3
c	c	b
a	ab	c
b		a

Note that these preferences are single-peaked with respect to the ordering $a \prec c \prec b$, so this specification is permissible. Since $p(a, b|\theta') = p(a, b|\theta)$ and $aR_F(\theta)b$, anonymity implies $aR_F(\theta')b$. By Pareto, $cP_F(\theta')a$. Then transitivity and $cP_F(\theta')aR_F(\theta')b$ imply $cP_F(\theta')b$. Now let $\{H_1, H_2, H_3\}$ be a partition of N such that $|H_1| = |G_1| + |G_2|$, $|H_2| = |G_1| + |G_2|$, and $|H_3| = |G_3| - |G_1| - |G_2|$, and consider a state θ'' with preferences restricted $\{b, c\}$ as follows.

H_1	H_2	H_3
c	b	b
b	c	c

Since $p(c, b|\theta'') = p(c, b|\theta')$ and $p(b, c|\theta'') = p(b, c|\theta')$, anonymity and $cP_F(\theta')b$ imply $cP_F(\theta'')b$. Now consider a state θ''' with preferences restricted to $\{a, b, c\}$ as follows.

H_1	H_2	H_3
a	b	b
b	c	a
c	a	c

These preferences are single-peaked with respect to the ordering $a \prec b \prec c$, so this specification is admissible. Since $P(a, b|\theta''') = P(c, b|\theta'')$ and $P(b, a|\theta''') = P(b, c|\theta'')$, neutrality and $cP_F(\theta'')b$ imply $aP_F(\theta''')b$. By Pareto,

$bP_F(\theta''')c$. Then transitivity and $aP_F(\theta''')bP_F(\theta''')c$ imply $aP_F(\theta''')c$. Finally, consider a state θ'''' with preferences restricted to $\{a, c\}$ as follows.

G_1	G_2	G_3
c	c	a
a	a	c

Since $p(a, c|\theta'''') = |G_3| = |H_1| + |H_3| = p(a, c|\theta''')$ and $p(c, a|\theta'''') = |G_1| + |G_2| = |H_2| = p(c, a|\theta''')$, anonymity and $aP_F(\theta''')c$ imply $aP_F(\theta'''')c$. But since $P(c, a|\theta'''') = P(c, b|\theta')$ and $P(a, c|\theta'''') = P(b, c|\theta')$, neutrality and $cP_F(\theta')b$ imply $cP_F(\theta'''')a$, contradicting asymmetry of $P_F(\theta'''')$. We conclude that for all $x, y \in X$ and all θ, $xR_F(\theta)y$ implies $r(x, y|\theta) > \frac{n}{2}$, or equivalently, $p(y, x|\theta) \geq \frac{n}{2}$ implies $yP_F(\theta)x$. An implication is that $P_{SM}(\theta) \subseteq P_F(\theta) \subseteq R_{SM}(\theta)$. Another implication is that n is odd: otherwise, we can choose state θ and alternatives x and y such that $p(x, y|\theta) = p(y, x|\theta) = \frac{n}{2}$, and then $xP_F(\theta)y$ and $yP_F(\theta)x$, contradicting asymmetry. ∎

An implication of Proposition 6 is an extension of Proposition 4 to the case of three alternatives. While Example 2 shows that the result does not carry over directly to the three-alternative case while maintaining only transitivity of strict social preference, it actually does extend if we strengthen the transitivity assumption to transitivity of weak social preference. In this case, retaining the background assumptions of anonymity, neutrality, and Pareto, Proposition 6 implies that n is odd and for all states θ, we have $P_{SM}(\theta) \subseteq P_F(\theta) \subseteq R_{SM}(\theta)$. Now, suppose $R(x, y|\theta) \subseteq P(x, y|\theta')$ and $xP_F(\theta)y$, as in the antecedent of NNR. We then have $\frac{n}{2} < r(x, y|\theta) \leq p(x, y|\theta')$, which implies $xP_{SM}(\theta)y$, which implies $xP_F(\theta')y$, fulfilling NNR. Because the case of exactly three alternatives is quite special, the formal statement of this corollary is omitted.

Finally, we return to single-peakedness in the absence of individual indifference, and we continue to maintain transitivity of weak social preference. In case only the single-peaked linear orders are possible, indifference cannot arise in the proof of Proposition 6, and the proof goes through (with $G_2 = \emptyset$). With Proposition 5, moreover, we can state the next proposition as an equivalence result: the only SPR that satisfies anonymity, neutrality, and Pareto and generates transitive weak social preferences in every state is majority rule.

Proposition 7 *Assume* $|X| \geq 3$ *and* $PR(\Theta) = \mathbf{S} \cap \mathbf{L}$*. Let* F *be an SPR satisfying anonymity, neutrality, and Pareto. If* $R_F(\theta)$ *is transitive for all* $\theta \in \Theta$*, then* n *is odd and* $F = F_{SM} = F_{RM}$*.*

4 Sufficient Conditions for Transitivity

We now focus on the opposite logical direction and provide sufficient conditions under which single-peakedness implies desirable transitivity properties for social preferences. First, we consider transitivity of strict preference and establish that for every SPR satisfying neutrality and NNR, if strict social preference implies strict relative majority preference, then the SPR always generates transitive strict social preferences. Note that anonymity and Pareto are not used. Example 5, in Section 5, defines a class of SPRs that satisfy neutrality, NNR, anonymity, and Pareto and are intermediate between simple and relative majority rule; an implication of Proposition 8 is that the strict social preferences generated by this class are transitive.

Proposition 8 *Assume* $PR(\Theta) \subseteq \mathbf{S}$*. Let* F *be an SPR satisfying neutrality and NNR. For all* $\theta \in \Theta$*, if* $P_F(\theta) \subseteq P_{RM}(\theta)$*, then* $P_F(\theta)$ *is transitive.*

Proof: Consider any $\theta \in \Theta$ such that $PR(\theta)$ is single-peaked with respect to $\prec$, and consider any distinct $a, b, c \in X$. Assume $aP_F(\theta)bP_F(\theta)c$. Then one of the following cases must obtain.

1. $a \prec b \prec c$	3. $b \prec c \prec a$	5. $c \prec a \prec b$
2. $b \prec a \prec c$	4. $c \prec b \prec a$	6. $a \prec c \prec b$

Case 1: Consider any $i \in R(a, b|\theta)$. By single-peakedness, we have $\hat{x}^i \prec b$, and then single-peakedness implies $bP_i(\theta)c$. Since $aR_i(\theta)bP_i(\theta)c$, transitivity implies $aP_i(\theta)c$. Therefore, $R(a, b|\theta) \subseteq P(a, c|\theta)$. Let $\{G_1, G_2, G_3, G_4, G_5\}$ be the partition of N such that $G_1 = P(a, b|\theta)$, $G_2 = I(a, b|\theta)$, $G_3 = P(b, a|\theta) \cap P(a, c|\theta)$, $G_4 = P(b, a|\theta) \cap I(a, c|\theta)$, and $G_5 = P(b, a|\theta) \cap P(c, a|\theta)$. Let $\{s, t\}$ be a free pair, and consider states $\theta', \theta'' \in \Theta$ such that individual preferences restricted to $\{s, t\}$ are

G_1	G_2	G_3	G_4	G_5
s	st	t	t	t
t		s	s	s

(1)

in state θ', and they are

$$\begin{array}{ccccc} G_1 & G_2 & G_3 & G_4 & G_5 \\ \hline s & s & s & st & t \\ t & t & t & & s \end{array} \tag{2}$$

in state θ''. Since $P(s,t|\theta') = G_1 = P(a,b|\theta)$ and $P(t,s|\theta) = G_3 \cup G_4 \cup G_5 = P(b,a|\theta)$, neutrality and $aP_F(\theta)b$ imply $sP_F(\theta')t$. Since $R(s,t|\theta') \subseteq P(s,t|\theta'')$, NNR then implies $sP_F(\theta'')t$. Since $P(a,c|\theta) = G_1 \cup G_2 \cup G_3 = P(s,t|\theta'')$ and $P(c,a|\theta) = G_5 = P(t,s|\theta'')$, neutrality then implies $aP_F(\theta)c$, as required. Case 2: Consider any $i \in R(b,c|\theta)$. By single-peakedness, $\hat{x}^i \prec c$. If $\hat{x}^i \preceq a$, then single-peakedness implies $aP_i(\theta)c$. If $a \prec \hat{x}^i \prec c$, then single-peakedness implies $aP_i(\theta)b$. Since $aP_i(\theta)bR_i(\theta)c$, transitivity implies $aP_i(\theta)c$. Therefore, $R(b,c|\theta) \subseteq P(a,c|\theta)$. Let $\{G_1, G_2, G_3, G_4, G_5\}$ be the partition of N such that $G_1 = P(b,c|\theta)$, $G_2 = I(b,c|\theta)$, $G_3 = P(c,b|\theta) \cap P(a,c|\theta)$, $G_4 = P(c,b|\theta) \cap I(a,c|\theta)$, and $G_5 = P(c,b|\theta) \cap P(c,a|\theta)$. Let $\{s,t\}$ be a free pair, and consider states $\theta', \theta'' \in \Theta$ such that individual preferences restricted to $\{s,t\}$ are as in (1) in state θ' and as in (2) in state θ''. Since $P(s,t|\theta') = G_1 = P(b,c|\theta)$ and $P(t,s|\theta') = G_3 \cup G_4 \cup G_5 = P(c,b|\theta)$, neutrality and $bP_F(\theta)c$ imply $sP_F(\theta')t$. Since $R(s,t|\theta') \subseteq P(s,t|\theta'')$, NNR then implies $sP_F(\theta'')t$. Since $P(a,c|\theta) = G_1 \cup G_2 \cup G_3 = P(s,t|\theta'')$, neutrality then implies $aP_F(\theta)c$, as required. Case 3: Since $aP_F(\theta)b$, we have $aP_{RM}(\theta)b$, implying $p(a,b|\theta) > p(b,a|\theta)$ and therefore $r(a,b|\theta) > \frac{n}{2}$. Since $bP_F(\theta)c$, we similarly have $r(b,c|\theta) > \frac{n}{2}$. Then there is some individual $i \in R(a,b|\theta) \cap R(b,c|\theta)$, but then c is bottom ranked among $\{a,b,c\}$ for individual i, contradicting single-peakedness with respect to $(\prec, \preceq)$. Thus, this case cannot occur. Case 4 is symmetric to 1, Case 5 is symmetric to 2, and Case 6 is symmetric to 3. ∎

The next result shows that the converse direction of Proposition 6 holds, even without the ancillary assumptions of anonymity, neutrality, and Pareto, and without the NNR axiom used in the preceding result. That is, if the number of individual is odd, if individual preferences are single-peaked, and if strict social preferences respect simple majority rule, then the weak social preference is transitive. Compared to Proposition 8, we add the assumption that the number of individuals is odd and deduce the stronger conclusion that weak social preferences are transitive. Obviously, the result implies the well-known result that weak social preference for simple and relative majority rule are transitive.

Proposition 9 *Assume $PR(\Theta) \subseteq \mathbf{S}$ and n is odd. Let F be an SPR. For all $\theta \in \Theta$, if $P_{SM}(\theta) \subseteq P_F(\theta)$, then $R_F(\theta)$ is transitive.*

Proof: Consider any $\theta \in \Theta$ such that $PR(\theta)$ be single-peaked with respect to $(\prec, \preceq)$, and consider any distinct $a, b, c \in X$. Assume $aR_F(\theta)bR_F(\theta)c$. Then one of the six cases in the proof of Proposition 8 must obtain. Case 1: Since $aR_F(\theta)b$, we have $aR_{SM}(\theta)b$, which means $r(a, b|\theta) \geq \frac{n}{2}$. Since n is odd, the latter inequality must in fact hold strictly. Consider any $i \in R(a, b|\theta)$. By single-peakedness, we have $\hat{x}^i \prec b$, and then single-peakedness implies $bP_i(\theta)c$. By transitivity, $aP_i(\theta)c$. Therefore, $R(a, b|\theta) \subseteq P(a, c|\theta)$. Thus, $p(a, c|\theta) \geq r(a, b|\theta) > \frac{n}{2}$, so $aP_{SM}(\theta)c$, which implies $aP_F(\theta)c$. Case 2: Since $bR_F(\theta)c$, we have $bR_{SM}(\theta)c$, so $r(b, c|\theta) > \frac{n}{2}$. Consider any $i \in R(b, c|\theta)$. By single-peakedness, $\hat{x}^i \prec c$. If $\hat{x}^i \preceq a$, then single-peakedness implies $aP_i(\theta)c$. If $a \prec \hat{x}^i \prec c$, then single-peakedness implies $aP_i(\theta)b$. By transitivity, $aP_i(\theta)c$. Therefore, $R(b, c|\theta) \subseteq P(a, c|\theta)$, and it follows that $p(a, c|\theta) > \frac{n}{2}$, so $aP_{SM}(\theta)c$, which implies $aP_F(\theta)c$. Case 3: Since $aR_F(\theta)b$, we have $r(a, b|\theta) > \frac{n}{2}$. Since $bR_F(\theta)c$, we similarly have $r(b, c|\theta) > \frac{n}{2}$. Then there is some individual $i \in R(a, b|\theta) \cap R(b, c|\theta)$, but then c is bottom ranked among $\{a, b, c\}$ for individual i, contradicting single-peakedness with respect to $(\prec, \preceq)$. Thus, this case cannot occur. Case 4 is symmetric to 1, Case 5 is symmetric to 2, and Case 6 is symmetric to 3. ∎

5 Examples

1. To see that Proposition 2 does not hold without NNR, even if we impose Pareto, consider the non-majoritarian SPR specifying that one alternative is strictly socially preferred to another if either every individual prefers it or, failing that, a minority of individuals prefer it. Formally, this SPR is defined as follows: for all $x, y \in X$ and all $\theta \in \Theta$,

$$xP_F(\theta)y \quad \Leftrightarrow \quad \begin{cases} p(x, y|\theta) = n \text{ or} \\ 0 < p(x, y|\theta) < \frac{n}{2}. \end{cases}$$

Assuming n is odd, tie break is vacuously satisfied by this SPR, since social indifference never obtains. Note also that replacing tie break with transitivity of strict social preference does not preserve Proposition 2, because simple Pareto F_{SP} satisfies anonymity, neutrality, NNR, and transitivity of strict social preferences.

2. Proposition 4 does not hold when there are just three alternatives. For example, assume $n \geq 5$ is odd, and define the SPR F so that for all $x, y \in X$ and all $\theta \in \Theta$, $xP_F(\theta)y$ if and only if both $xP_{RM}(\theta)y$ and $p(x, y|\theta) + p(y, x|\theta) < n$. This SPR satisfies anonymity, neutrality, and Pareto. I argue that strict social preferences are vacuously transitive. Consider any $a, b, c \in X$ and any state θ, and suppose in order to deduce a contradiction that $aP_F(\theta)bP_F(\theta)c$. Then there is at least one individual i such that $aI_i(\theta)b$, and for each such individual, it must be that $cP_i(\theta)aI_i(\theta)b$; otherwise, $aI_i(\theta)bP_i(\theta)c$, and the individual does not have an ideal point. Moreover, for every individual j with $aP_j(\theta)b$, we must have $cP_j(\theta)b$; otherwise, each alternative is bottom ranked by someone among $\{a, b, c\}$, contradicting single-peakedness. Thus, we have $R(a, b|\theta) \subseteq P(c, b|\theta)$. Note that n odd and $aP_F(\theta)b$ imply $r(a, b|\theta) > \frac{n}{2}$, which implies $p(c, b|\theta) > \frac{n}{2}$, and thus not $bP_F(\theta)c$, a contradiction. Nevertheless, F violates NNR. To see this for the $n = 5$ case, consider states $\theta, \theta' \in \Theta$ such that preferences restricted to $\{a, b\}$ are

1	2	3	4	5
a	a	a	ab	b
b	b	b		a

in θ and

1	2	3	4	5
a	a	a	a	b
b	b	b	b	a

in θ'. Then we have $aP_F(\theta)b$, and since $R(a, b|\theta) \subseteq P(a, b|\theta')$, NNR would imply $aP_F(\theta')b$, but this does not hold.

3. Proposition 5 does not extend to single-peaked domain when individual indifferences are allowed, even if transitivity of $P_F(\theta)$ is increased to transitivity of $R_F(\theta)$. For example, assume $n \geq 7$ is odd, and define the SPR F so that for all $x, y \in X$ and all $\theta \in \Theta$, $xP_F(\theta)y$ if and only if either (i) $xP_{SM}(\theta)y$ or (ii) both $p(x, y|\theta) = 1$ and $p(y, x|\theta) = 2$. Note that in case (ii), we have $|I(x, y|\theta)| = n - 3 > \frac{n}{2}$, so $P_F(\theta)$ is asymmetric; in particular, we cannot have $xP_{SM}(\theta)y$ and $|I(x, y|\theta)| = n - 3$. This SPR is anonymous and neutral, and it satisfies Pareto. I claim that $R_F(\theta)$ is transitive, or equivalently that $P_F(\theta)$ is negatively transitive, for all θ. Indeed, consider distinct alternatives $a, b, c \in X$ and a state θ such that $aP_F(\theta)b$. There are two cases. First, if $aP_{SM}(\theta)b$, then by negative transitivity of $P_{SM}(\theta)$, from Proposition 9, it follows that either $aP_{SM}(\theta)c$ or $cP_{SM}(\theta)b$, and therefore either $aP_F(\theta)c$ or $cP_F(\theta)b$. Second, if $p(a, b|\theta) = 1$ and $p(b, a|\theta) = 2$, then $|I(a, b|\theta)| > \frac{n}{2}$. For every individual i such that $aI_i(\theta)b$, single-peakedness

implies that we cannot have $aI_i(\theta)c$. Thus, there are two possibilities: (i) $aP_i(\theta)c$ and (ii) $cP_i(\theta)a$. Furthermore, observe that by single-peakedness, it cannot be that (i) holds for one individual i with $aI_i(\theta)b$ but (ii) holds for another individual j with $aI_j(\theta)b$. Indeed, if (i) holds for i, then c is bottom ranked among $\{a, b, c\}$ for i, and if (ii) holds for j, then a and b are bottom ranked among $\{a, b, c\}$ for j, contradicting single-peakedness. This observation implies that if $aP_i(\theta)c$ holds for one individual with $aI_i(\theta)b$, then it holds for $|I(a, b|\theta)| > \frac{n}{2}$ individuals, and thus $aP_{SM}(\theta)c$, which implies $aP_F(\theta)c$. And similarly, if $cP_i(\theta)a$ holds for one individual with $aI_i(\theta)b$, then it hold for a majority of individuals, and we have $cP_F(\theta)a$. This establishes negative transitivity of $P_F(\theta)$, as required.

4. The conclusion of Proposition 5 cannot be strengthened by adding the inclusion $P_{SM}(\theta) \subseteq P_F(\theta)$, because strict simple Pareto preference, $P_{SP}(\theta)$, is transitive and we can have $P_{SM}(\theta) \nsubseteq P_{SP}(\theta)$.

5. To construct a class of SPRs with social preferences nested between simple and relative majority rule, and given any $k \in \{1, \ldots, \frac{n+1}{2}\}$, define F_k as follows: for all $x, y \in X$ and all $\theta \in \Theta$,

$$xP_{F_k}(\theta)y \quad \Leftrightarrow \quad p(x, y|\theta) \ \geq \ k \ > \ p(y, x|\theta).$$

This SPR clearly satisfies anonymity, neutrality, and Pareto, and if $1 < k < \frac{n+1}{2}$, then F_k is not equal to either version of majority rule. Since we have $P_{SM}(\theta) \subseteq P_{F_k}(\theta) \subseteq P_{RM}(\theta)$ for all $\theta \in \Theta$, Proposition 8 implies $P_{F_k}(\theta)$ is transitive in every state.

6 Conclusion

Combining the results of the previous sections, we obtain three versions of May's theorem that exploit the structure of single-peakedness, which plays a central role in formal political theory. Importantly, this structure allows us to replace the strong tie-break axiom of May with normatively compelling transitivity properties. When there are three or more alternatives and all profiles of single-peaked preferences are possible, Propositions 6 and 9 immediately yield the following corollary: among SPRs satisfying the background conditions of anonymity, neutrality, and Pareto, transitivity of weak social preference holds if and only if the number of individuals is odd and social preferences are nested between strict and weak simple majority rule. In particular, under anonymity, neutrality, and Pareto, Proposition 6 tells us that transitivity of weak majority preference implies that n is odd and that the inclusion $P_{SM}(\theta) \subseteq P_F(\theta) \subseteq R_{SM}(\theta)$ holds; and Proposition 9 provides the converse direction.

Corollary 10 *Assume $|X| \geq 3$ and $PR(\Theta) = \mathbf{S}$. Let F be an SPR satisfying anonymity, neutrality, and Pareto. Then $R_F(\theta)$ is transitive for all $\theta \in \Theta$ if and only if n is odd and $P_{SM}(\theta) \subseteq P_F(\theta) \subseteq R_{SM}(\theta)$ for all $\theta \in \Theta$.*

When only the single-peaked linear orders are possible, using Propositions 3, 5, and 8, we obtain the following corollary: among SPRs satisfying anonymity, neutrality, and Pareto, transitivity of strict social preferences holds if and only if an SPR satisfies NNR and strict social preference implies majority preference. For the result, we no longer need the full force of transitivity of weak social preference, and we obtain the result for an arbitrary (possibly even) number of individuals. Here, with anonymity, neutrality, and Pareto, Proposition 5 tells us that transitivity of strict social preference implies the inclusion $P_F(\theta) \subseteq P_{SM}(\theta) = P_{RM}(\theta)$, and Proposition 3 tells us that if strict social preferences are transitive, then F satisfies NNR; and Proposition 8 provides the converse direction.

Corollary 11 *Assume $|X| \geq 3$ and $PR(\Theta) = \mathbf{S} \cap \mathbf{L}$. Let F be an SPR satisfying anonymity, neutrality, and Pareto. Then $P_F(\theta)$ is transitive for all $\theta \in \Theta$ if and only if F satisfies NNR and $P_F(\theta) \subseteq P_{SM}(\theta) = P_{RM}(\theta)$.*

Finally, we return to transitivity of weak social preference and obtain an exact characterization of majority rule under single-peaked, linear domain that sharpens Corollary 11 and provides a close analogue of May's theorem that fits naturally within the rubric of formal political theory: an SPR satisfies anonymity, neutrality, Pareto and always generates transitive weak social preference if and only if the number of individuals is odd and it is exactly equal to majority rule. Indeed, Proposition 7 shows that transitivity of weak social preference implies n is odd and F is equal to majority rule; and Proposition 9 yields the converse direction.

Corollary 12 *Assume $|X| \geq 3$ and $PR(\Theta) = \mathbf{S} \cap \mathbf{L}$. Then the SPR F satisfies anonymity, neutrality, and Pareto and is such that $R_F(\theta)$ is transitive for all $\theta \in \Theta$ if and only if n is odd and $F = F_{SM} = F_{RM}$.*

The preceding corollary is related to Proposition 2 of Dasgupta and Maskin (2008), who show that majority rule satisfies anonymity, neutrality, Pareto, and "generic decisiveness" on more domains than any other SPR. (Note that our definitions of anonymity and neutrality imply IIA, whereas those of Dasgupta and Maskin do not.) The framework of the latter authors is, however, rather different from that of this paper: (i) they assume a finite set of alternatives and a continuum of voters, (ii) they restrict the possible preferences of each individual to be a linear ordering of alternatives, and (iii) their generic decisiveness axiom has the effect of excluding preference profiles for which an SPR generates social indifference. Outside this set of excluded profiles, the decisiveness axiom is equivalent to transitivity of weak (which is the same as strict) social preference. In light of (iii), it is most appropriate to compare their result with Corollary 12, where n is assumed odd. The second part of Dasgupta and Maskin's Theorem 2 states that for every SPR F distinct from majority rule, there is some domain on which majority rule satisfies their axioms, whereas F does not. Corollary 12 is more specific: it states that majority rules satisfies the axioms on the particular domain of profiles of single-peaked linear orders, whereas F does not.

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