Directional Equilibria*

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March 28, 2017

Abstract

We propose the solution concept of directional equilibrium for the multidimensional model of voting with general spatial preferences. This concept isolates alternatives that are stable with respect to forces applied by all voters in the directions of their gradients, and it extends a widely (but not well-) known concept from statistics for Euclidean preferences. We establish connections to the majority core, Pareto optimality, existence and closed graph, and generic local uniqueness and stability of the solution, and we provide non-cooperative foundations in terms of a local contest game played by voters.

*We are grateful for very helpful feedback from Chris Chambers, Jean-Francois Laslier, and Bill Zwicker, along with the anonymous referees of the paper.

Contents

1 Introduction

The multidimensional spatial model of politics provides an abstract framework for collective choice, where points in a Euclidean space can represent vectors of positions on different policy issues. Given an odd number of voters with single-peaked preferences over a single policy issue, the median voter theorem dictates that the median voter's ideal point is the unique element of the majority core. An impediment to the analysis of the general model is the instability of majority rule in multiple dimensions, formalized in the symmetry conditions of Plott (1967) and the genericity result of Schofield (1983). In reaction to the indeterminacy of majority rule, several solution concepts have provided alternatives (or sets of alternatives) with special properties as having positive or normative significance. We propose a concept of directional equilibrium that isolates alternatives based on their stability with respect to "forces" applied by voters in the directions of their gradients. This solution generalizes a widely (but not well-) known concept from statistics for Euclidean preferences, and it possesses desirable core consistency, existence, efficiency, and stability properties.

To convey the idea of directional equilibrium, we first consider the problem of maximizing the sum of strictly concave voter utilities over alternatives $x \in \mathbb{R}^m$, which has first order condition

$$\sum_{i=1}^{n} \nabla u_i(x) \quad = \quad 0. \tag{1}$$

One interpretation of the utilitarian alternative obtained as a solution to the first order condition is that it is stable, in the sense that each voter subjects it to a force equal to her gradient, and that the forces applied by voters cancel each other out. But this definition of stability is sensitive to scalings of voter utilities and implicitly assumes the interpersonal comparison of utilities. We seek a criterion that is free of interpersonal utility comparisons—in terms of the tug of war analogy, we want to assume that all voters pull with equal force—and we therefore consider the *normalized* gradients of the voters. Thus, for an alternative x that is not an ideal point of any voter, we say x is a directional equilibrium if it solves the system

$$\sum_{i=1}^{n} \frac{1}{\|\nabla u_i(x)\|} \nabla u_i(x) \quad = \quad 0 \tag{2}$$

of equations of m equations in m unknowns. If the alternative is the ideal point of a voter, then we assume that voter can resist a net pull in any direction, up to a force of one unit.

This formulation has several desirable properties. We first establish that the concept of directional equilibrium extends the majority core, in the sense that if the majority core is nonempty, then the core alternative is a directional equilibrium. The proof of this result is an immediate corollary of Plott's theorem, for a core alternative must satisfy radial symmetry, which means that each voter is balanced by a voter whose gradient pulls in the opposite direction. In general, the core extension result allows for the possibility of directional equilibria in addition to the core, but we prove that if the set of alternatives is one-dimensional, then the equivalence is exact: if the majority core is nonempty, then the core alternative is the unique directional equilibrium. We then verify that directional equilibria are Pareto optimal and exist in great generality, and that when voter preferences are continuously paramerized, the directional equilibrium correspondence has closed graph. These attributes are appealing for any solution to the spatial model, and existence in particular opens potential for wide applicability of the concept.

There is the danger, however, that multiplicity and instability of solutions preclude the possibility of comparative statics. If a solution concept isolates an open set of alternatives, then even if that set varies in a nice way with respect to parameters, we cannot meaningfully perform the experiment of tracking a single alternative as we vary parameters of the model. And even if the alternatives specified by a solution concept are locally unique, we cannot perform comparative statics if those alternatives do not vary in a continuous way with parameters. To address these issues, we establish that for generic preferences, the directional equilibria are locally unique and can be expressed as a continuous function of additional parameters of the model; technically, for a quite general ambient space of utilities, the subset at which these properties fail is finitely shy (cf. Anderson and Zame (2001)). Finally, we explore the tug-of-war analogy in greater detail, expressing it in terms of a "contest game" at status quo alternative x, where voters' strategies consist of the application of an amount of force to move the outcome from x. We prove that under weak background conditions, an alternative x is a directional equilibrium if and only if it is the equilibrium outcome of a contest game at x, i.e., the status quo of x maintained in equilibrium, providing non-cooperative foundations for the solution concept.

The directional equilibria generalize a notion of multidimensional median from the statistics literature that has been applied to the spatial model when voter preferences are Euclidean and utilities are linear in distance from a voter's ideal point, i.e., $u_i(x) = -\|\hat{x}^i - x\|$. In this setting, because the norm of a voter's gradient is independent of the alternative $x \neq \hat{x}^i$ at which it is evaluated, the distinction between (1) and (2) disappears: an alternative maximizes the sum of voter utilities if and only if the sum of normalized gradients is equal to zero.[1] Put differently, in this special case, the utilitarian alternatives that maximize the sum of utilities are just the directional equilibria. Moreover, if the number of voters is odd or the voters' ideal points are not collinear, then it is known that there is a unique utilitarian alternative (cf. Baranchuk and Dybvig (2009)), and thus a unique directional equilibrium. Thus, when utilities are linear functions of distance to the voters' ideal points, the directional equilibrium is well-understood and consistent with the utilitarian welfare criterion.

In the statistical context, the minimizer of total distance to a given number of points has a long history as a notion of centrality. Weber (1909) introduced the idea in the context of locating a warehouse to minimize transportation costs, and it was imported to the statistics literature by Gini and Galvani (1929) and rediscovered by Haldane (1948). The concept has received various names: mediancentre (Gower (1974)), geometric median (Haldane (1948)), L_1-median (Small (1990)), spatial median (Brown (1983)).[2] Recently, Baranchuk and Dybvig (2009) use the term "consensus" derive some earlier results independently and use the concept to analyze decision making by a board of directors. Cervone et al. (2012) use the terminology of mediancentre and "Fermat-Weber point," and they discuss computational issues and cite earlier work on the topic. Brady and Chambers (2014) use the term geometric median and show that when there are three individuals and all have Euclidean preferences, the geometric median is the unique rule satisfying Maskin monotonicity, anonymity, and neutrality.

When voters have general spatial preferences, however, the systems of equations in (1) and (2) become distinct, and thus the utilitarian alternative is typically not contained among the directional equilibria. As long as voter utilities are strictly concave, the utilitarian alternative, which solves (1), will generally be unique, but as mentioned above, it is sensitive to scalings of individual utilities and requires interpersonal comparisons of utilities. We follow a different route to a solution for the general spatial model: we intuitively assume that each voter pulls with equal force in the direction of her gradient, and so our concept of directional equilibrium is defined by the system of equations in (2). Existence becomes a more subtle issue, as directional equilibria cannot generally be obtained as solutions to a well-behaved maximization problem; rather, we use a fixed point argument to establish existence. Another difference, as one would then expect, is that uniqueness of directional equilibrium is lost, as the system (2) can admit multiple solutions, even when utilities are strictly concave. A benefit of our route

[1]This observation must be phrased in terms of supergradients if the alternative is the ideal point of some voter, due to non-differentiability of voter utilities.

[2]See Small (1990) for a survey of multidimensional medians.

is that the resulting solution concept is invariant with respect to smooth transformations of voter utilities with positive derivative, so that it does not rely on interpersonal comparisons.

Several competing solution concepts have been proposed and explored in the context of the spatial model. Shepsle (1979) defines the notion of structure-induced equilibrium, which isolates alternatives that are stable with respect to majority voting on each dimension separately. This solution concept extends the core in the same way that directional equilibria do; structure-induced equilibria exist generally; and they can be shown to possess the same generic local uniqueness and stability properties. A drawback of this concept is that, unlike directional equilibrium, structure-induced equilibria may (in three or more dimensions) be Pareto inefficient.[3] McKelvey (1986) extends the concept of uncovered set to the spatial model and defines the yolk as a means of bounding the uncovered set. Whereas the yolk is defined only for Euclidean voter preferences, the uncovered set exists generally and is contained among the Pareto optimal alternatives. A drawback of this concept is that it can contain an open set of alternatives, leading to a limited form of indeterminacy and creating difficulties for comparative statics.[4] Grofman et al. (1987) extend the idea of Copeland winner to the spatial model and define the strong point as the alternative whose win set has minimal Lebesgue measure, and Owen and Shapley (1989) prove uniqueness of the strong point assuming Euclidean voter preferences and a two-dimensional set of alternatives. The existence of at least one strong point follows from elementary arguments, and Pareto optimality is straightforward to show. However, uniqueness (local or otherwise) of the strong point is not known in higher dimensions or when voter preferences are non-Euclidean. Wuffle et al. (1989) define the finagle point, and more recently, Van Wesep (2010) defines the defensive optimum in a model with a continuum of voters, both papers using geometric arguments that rely on the assumption of Euclidean voter preferences.

2 Model and Definitions

Let $N = \{1, \ldots, n\}$ be a set of voters, and let $X \subseteq \mathbb{R}^m$ be a set of alternatives, identified with a subset of m-dimensional Euclidean space. Assume that the preferences of voter i are represented by the utility function $u_i \colon X \to \mathbb{R}$, which is

[3]See Duggan and Fey (2013) for a proof of local uniqueness and an example of Pareto inefficiency of structure-induced equilibria.

[4]See Banks, Duggan, and Le Breton (2006) and Duggan (2013) for further results on the uncovered set in general environments.

assumed to be continuously differentiable.[5] For simplicity, we also assume that for all $i \in N$, there is a unique *ideal point* $\hat{x}^i \in X$ such that for all $x \in X \setminus \{\hat{x}^i\}$, we have $u_i(\hat{x}^i) > u_i(x)$; and we assume that the ideal point of voter i is the unique critical point of her utility function, i.e., for all $x \in X$, we have $\nabla u_i(x) = 0$ if and only if $x = \hat{x}^i$.

We seek to isolate alternatives that are stable when subject to the exertion of force by voters, without restricting the directions of these forces but assuming that each voter pulls with up to one unit of force. We hypothesize that a voter whose gradient at x is non-zero pulls x in the direction in which utility increases at the highest rate, while a voter whose ideal point is equal to x resists the net pull of other voters. That is, the pull exerted by voter i with $x \neq \hat{x}^i$ is

$$p^i(x) \quad = \quad \frac{1}{\|\nabla u_i(x)\|} \nabla u_i(x),$$

and if $x = \hat{x}^i$ for voter i, then we set $p^i(x) = 0$. When x is not the ideal point of any voter, it remains in place as long as the net force exerted on it is equal to zero, i.e., $\sum_{i=1}^n p^i(x) = 0$. And if $x = \hat{x}^i$ for a single voter, then i can resist a move in any direction, and so x remains in place as long as i can overcome the net force on x. The magnitude of this net force is just $\| \sum_{j \neq i} p^j(x)\|$, so our hypothesis is that x is in equilibrium when $\| \sum_{j \neq i} p^j(x)\| \leq 1$. Extending this informal story, if x is the ideal point of k voters, then each of those voters can overcome a net force of up to one unit, and we require that $\| \sum_j p^j(x)\| \leq k$.

Formally, an alternative $x \in X$ is a *directional equilibrium* if

$$\left\| \sum_{i=1}^n p^i(x) \right\| \quad \leq \quad \#\{i \in N : x = \hat{x}^i\},$$

and we let C^*_{DE} denote the set of directional equilibria. Note that, consistent with the above discussion, if x is not the ideal point of any voter, then it is a directional equilibrium if and only if $\| \sum_{i=1}^n p^i(x)\| = 0$, i.e., the sum of normalized gradients is equal to zero. See Figure 1 for two examples of such directional equilibria with five voters. In the example on the left, the normalized gradients are placed symmetrically to form angles of 120 degrees and clearly sum to zero; while the example on the right is asymmetric, but the normalized gradients at x nevertheless

[5] In general, given a set $X \subseteq \mathbb{R}^m$ and a function $f\colon X \to \mathbb{R}^n$, we say f is (continuously) differentiable if it extends to a (continuously) differentiable function defined on an open superset $Z \supseteq X$. We use the convention that $Df(x)$ denotes the $n \times m$ Jacobian matrix of f at x; and when $n = 1$, $D_j f(x)$ denotes the jth partial derivative of f at x, and $\nabla f(x)$ is the gradient vector of f at x.

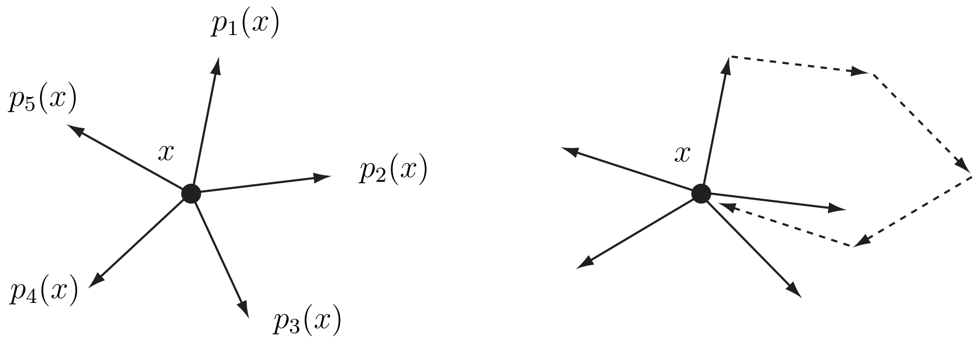

Figure 1: Directional equilibria

sum to zero. Note that because we normalize the gradient of each voter, all interested parties exert equal force. An advantage of this is that $p^i(x)$ is invariant with respect to smooth, monotonic transformations of any voter's utility function, and thus the concept of directional equilibrium is essentially ordinal.

It is interesting to compare directional equilibria to the structure-induced equilibria of Shepsle (1979), which are stable with respect to deviations in each coordinate separately. Given any structure-induced equilibrium, say x, and any coordinate, say the jth, there is some voter k such that $D_j u_k(x) = 0$, and assuming for simplicity that this is true of just one voter and n is odd, we furthermore have

$$\#\Big\{i \in N : D_j u_i(x) > 0\Big\} \quad = \quad \#\Big\{i \in N : D_j u_i(x) < 0\Big\}.$$

That is, the voters with partial derivatives at x that "pull" the alternative in the positive direction on coordinate j are exactly offset by the voters with partial derivatives that pull the alternative in the negative direction. More concisely,

$$\sum_{i \neq k} \frac{D_j u_i(x)}{|D_j u_i(x)|} \quad = \quad 0,$$

where dividing by the magnitude of the partial derivative essentially means that each voter pulls with the same force, so that the summation on the left-hand side simply counts the net number of voters pulling in the positive direction. Accordingly, we restrict the directions in which voters exert force to the coordinate axes, normalizing partial derivatives in each such direction. A disadvantage of this, in light of the "pulling" analogy, is that even if an voter's marginal rate of substitution between, say, coordinates 1 and 2, is very far from one, the stability property implicit in the equilibrium concept essentially views the force exerted by the voter on the two coordinates as the same.

To further elucidate the concept of directional equilibrium, and to contrast it with structure-induced equilibrium, we note that the system of equations in (2) depends on the units in which the axes are measured. Given one unit of measure and a solution to (2), we can define an equivalent model by "stretching" the first coordinate axis by the diffeomorphism $f_1 : \mathbb{R} \to \mathbb{R}$, where the quantity x_1 is translated as $\tilde{x}_1 = f(x_1)$ in the new model. Then voter i's utility is given by $\tilde{u}_i(\tilde{x}_1, x_{-1}) = u_i(f^{-1}(\tilde{x}_1), x_{-1})$. Directional equilibria in the new model are given by the new system of equations

$$\sum_{i=1}^{n} \frac{1}{\|\nabla \tilde{u}_i(\tilde{x}_1, x_{-1})\|} \nabla \tilde{u}_i(\tilde{x}_1, x_{-1}) \quad = \quad 0,$$

which will generally have solutions that differ from the original.[6] Application of the directional equilibrium concept therefore presumes commitment to units of measurement along the coordinate axes,[7] though a common linear transformation of the units of measure leaves the directional equilibria unchanged. Of course, if all axes are used to measure similar quantities, such as money or distance, then it is natural to use the same unit of measure for all axes, and then the directional equilibria will be invariant to linear transformations of the unit of measure. Note, however, that direction equilibria *are* invariant with respect to rigid Euclidean transformations, such as translation or rotation, of the coordinate axes. In contrast, the structure-induced equilibria are invariant with respect to stretching of the axes, as this does not affect the median on any given coordinate. But structure induced equilibria are not generally invariant with respect to rotations of the coordinate axes. These solution concepts both select a (typically) small number of alternatives, but each incorporates a different aspect of the Euclidean structure of the spatial model.

3 Core Extension and Pareto Optimality

In this section, we show that the directional equilibria extend the core, in the sense that quite generally, if there is a majority core alternative, then it is necessarily a

[6] We thank an anonymous referee for this observation.

[7] This commitment to units of measure is implicitly made in the large literature on spatial modeling with Euclidean voter preferences. Indeed, many solutions are not even defined for non-Euclidean preferences, so the stretching of an axis will render them inapplicable. This is true more generally in modeling where a specific functional form, e.g., quadratic utility, is assumed.

directional equilibrium. The proof of this follows easily from Plott's (1967) radial symmetry characterization of the majority core. Indeed, let n be odd, consider any majority core alternative $x \in \text{int}X$, and assume there is at most one voter k such that $\nabla u_k(x) = 0$. By Plott's theorem, radial symmetry must be satisfied at x, so there is a permutation $\pi\colon (N \setminus \{k\}) \to (N \setminus \{k\})$ such that for each $i \neq k$, the gradients of i and $\pi(i)$ point in opposite directions. Then

$$\sum_{i \neq k} p^i(x) \;=\; \frac{1}{2}\left(\sum_{i \neq k}(p^i(x) + p^{\pi(i)}(x))\right) \;=\; 0,$$

which proves the result.

Theorem 1 *Assume that n is odd. Let $x \in \text{int}X$, and assume there is at most one voter k such that $x = \hat{x}^k$. If x is a majority core alternative, then it is a directional equilibrium.*

The preceding argument allows for the possibility that even when the majority core is non-empty, there may be other directional equilibria, but it is straightforward to verify that when the set of alternatives is one-dimensional and voter preferences are single-peaked, the median ideal point is the unique directional equilibrium. Recall that an alternative x is a *median* if $\#\{i \in N : x < \hat{x}^i\} \leq \frac{n}{2}$ and $\#\{i \in N : \hat{x}^i < x\} \leq \frac{n}{2}$. Note that when n is odd, there is a unique median alternative.

Theorem 2 *Assume that n is odd, that $X \subseteq \mathbb{R}$ is convex, and that for all $i \in N$, the utility function u_i is strictly quasi-concave. Then the median alternative is the unique directional equilibrium.*

Proof: We first verify that the median, say x, is a directional equilibrium. Let $G^> = \{i \in N : x < \hat{x}^i\}$ be the voters whose ideal point is above the median, and let $G^< = \{i \in N : \hat{x}^i < x\}$ be the voters whose ideal point is below x. Assume without loss of generality that $\#G^> \geq \#G^<$. Note that for all $i \in G^>$ and all $j \in G^<$, we have $Du_i(x) > 0$ and $Du_j(x) < 0$, which implies

$$\left\|\sum_{i=1}^{n} p^i(x)\right\| \;=\; \left\|\sum_{i \in G^> \cup G^<} p^i(x)\right\| \;=\; \#G^> - \#G^<.$$

Using the identity $\#G^> + \#G^< + \#\{i \in N : x = \hat{x}^i\} = n$, this yields

$$\left\|\sum_{i=1}^{n} p^i(x)\right\| \;=\; 2\#G^> - n + \#\{i \in N : x = \hat{x}^i\}.$$

Since x is the median, we have $\#G^{>} \leq \frac{n}{2}$, i.e., $2\#G^{>} - n \leq 0$, and we conclude from the above that x is a directional equilibrium. Next, assume x is a directional equilibrium. If x is not the median, then using the notational convention above, we can assume without loss of generality that $\#G^{>} > \frac{n}{2}$, but then

$$\left\| \sum_{i=1}^{n} p^i(x) \right\| = \#G^{>} - \#G^{<} = 2\#G^{>} - n + \#\{i \in N : x = \hat{x}^i\}$$
$$> \#\{i \in N : x = \hat{x}^i\},$$

a contradiction. ∎

Whereas the preceding strong core equivalence result assumes one dimension and allows quite general preferences, we next specialize to Euclidean voter preferences and allow any number of dimensions. It is known that, quite generally, there is a unique alternative that minimizes total distance to voter ideal points, and that this corresponds with the unique directional equilibrium. We provide a self-contained proof of uniqueness of directional equilibrium that relies on the following lemma, which is also used in a key step in the analysis of non-cooperative foundations in Section 6.

Lemma 1 *Assume X is convex and voter preferences are Euclidean. Given any distinct $x, y \in X$ and any $\alpha \in (0,1)$, let $t = \frac{1}{\|y-x\|}(y-x)$ be the direction pointing from x to y. Then for all voters i, the dot product $p^i((1-\alpha)x + \alpha y) \cdot t$ is non-increasing in α; and if $\{\hat{x}^i, x, y\}$ are not collinear, then it is strictly decreasing in α.*

Proof: Consider any voter i and $\alpha \in (0,1)$. If $\hat{x}^i = (1-\alpha)x + \alpha y$, then the dot product $p^i((1-\alpha)x + \alpha y) \cdot t$ is discontinuous and jumps down at α. Otherwise, if $\hat{x}^i \neq (1-\alpha)x + \alpha y$, then it is differentiable. The first part of the lemma follows by confirming that this derivative is non-positive. Write $z(\alpha) = (1-\alpha)x + \alpha y$, so that $\nabla z(\alpha) = y - x$. Note that

$$\frac{d}{d\alpha} \frac{(\hat{x}^i - z(\alpha)) \cdot t}{\|\hat{x}^i - z(\alpha)\|} \;\propto\; -\|\hat{x}^i - z(\alpha)\|\|y - x\| - [(\hat{x}^i - z(\alpha)) \cdot t] \frac{d}{d\alpha}\|\hat{x}^i - z(\alpha)\|,$$

where $\propto$ indicates that the left-hand side has the same sign as the right-hand side. Here,

$$\frac{d}{d\alpha}\|\hat{x}^i - z(\alpha)\| = \frac{-(\hat{x}^i - z(\alpha)) \cdot (y - x)}{\|\hat{x}^i - z(\alpha)\|} = -\frac{\|y - x\|\,(\hat{x}^i - z(\alpha)) \cdot t}{\|\hat{x}^i - z(\alpha)\|}.$$

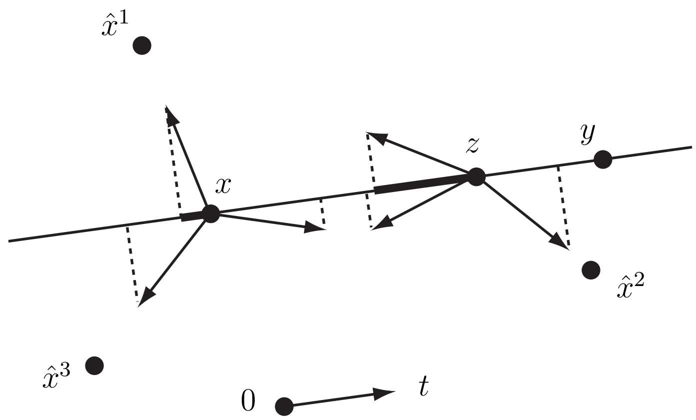

Figure 2: Decreasing dot products

Thus, the derivative is non-positive if

$$\|\hat{x}^i - z(\alpha)\|^2 \;\; \geq \;\; [(\hat{x}^i - z(\alpha)) \cdot t]^2.$$

By the Cauchy-Schwartz inequality, we have $|(\hat{x}^i - z(\alpha)) \cdot t| \leq \|\hat{x}^i - z(\alpha)\| \, \|t\| = \|\hat{x}^i - z(\alpha)\|$, and the desired inequality follows. Moreover, if $\{\hat{x}^i, x, y\}$ are not collinear, then $\{(\hat{x}^i - z(\alpha), t, 0\}$ are not collinear, and the inequality holds strictly, delivering the second part of the lemma. ∎

The idea of the lemma is illustrated in Figure 2. Starting with alternative x, the figure indicates the normalized gradients of three voters, which are projected onto the line through x and y. The projection of voter 1's normalized gradient, e.g., is the thick line segment emanating from x. Letting t be the direction pointing from x to y, the fact that the projection of $p^1(x)$ onto the line is on the side of x opposite y indicates that the dot product $t \cdot p^i(x)$ is negative. As we move the alternative toward y, voter gradients continue to point toward their ideal point, and these will tip away from y; this is evident in the case of voter 1, because the line segment emanating from z is longer, indicating that the dot product $t \cdot p^i(z)$ is negative and of greater magnitude.

The next theorem uses this lemma to establish uniqueness of directional equilibrium when voter preferences are quadratic, under otherwise quite general conditions. The statement here is from Baranchuk and Dybvig (2009).

Theorem 3 (Baranchuk and Dybvig) *Assume X is convex, that voter preferences are Euclidean, and that either n is odd or the voter ideal points $\{\hat{x}^i : i \in N\}$ are not collinear. Then there is at most one directional equilibrium.*

Proof: In case ideal points are collinear, then n is odd, and since voter preferences are Euclidean, uniqueness follows from Theorem 2, applied to the one-dimensional model induced by voter preferences. In the remaining case that voter ideal points are not collinear, suppose there are distinct directional equilibria, say x and y, and let $t = \frac{1}{\|x-y\|}(y-x)$ be the direction pointing from x to y. Let $G = \{i \in N : \hat{x}^i = x\}$ and $H = \{i \in N : \hat{x}^i = y\}$ be the coalitions of voters with ideal points at x and y, respectively. Note that by the Cauchy-Schwartz inequality, we have

$$t \cdot \sum_{i \in N \setminus G} p^i(x) \leq \left\| \sum_{i \in N \setminus G} p^i(x) \right\|.$$

Moreover, for all $\alpha \in (0,1)$ and all $i \in G$, we have $p^i((1-\alpha)x + \alpha y) = -t$, where we use the fact that $(1-\alpha)x + \alpha y = x + \alpha\|y - x\|t$. Thus, we have

$$\lim_{\alpha \downarrow 0} t \cdot \sum_{i=1}^{n} p^i((1-\alpha)x + \alpha y) \leq \left\| \sum_{i \in N \setminus G} p^i(x) \right\| - \#G \leq 0,$$

where the second inequality follows from the definition of directional equilibrium. A similar argument, starting from y, yields

$$\lim_{\alpha \uparrow 1} t \cdot \sum_{i=1}^{n} p^i((1-\alpha)x + \alpha y) \geq \#H - \left\| \sum_{i \in N \setminus H} p^i(x) \right\| \geq 0. \quad (3)$$

But starting from x, as we increase α from zero to one, Lemma 1 implies that for each voter i, the quantity $t \cdot p^i((1-\alpha)x + \alpha y)$ weakly decreases; this change is continuous unless $\hat{x}^i = (1-\alpha)x + \alpha y$, in which case the quantity jumps down discontinuously at $\hat{x}^i$. Moreover, by the assumption that ideal points are not collinear, there is at least one voter j whose ideal point does not lie on the line through x and y, and for this voter, Lemma 1 implies that $t \cdot p^j((1-\alpha)x + \alpha y)$ is strictly decreasing in α. We conclude that

$$\lim_{\alpha \uparrow 1} t \cdot \sum_{i=1}^{n} p^i((1-\alpha)x + \alpha y) < 0,$$

contradicting (3). ∎

The preceding theorem uses the assumption of quadratic utility to deduce that a majority core alternative is the unique directional equilibrium, raising the

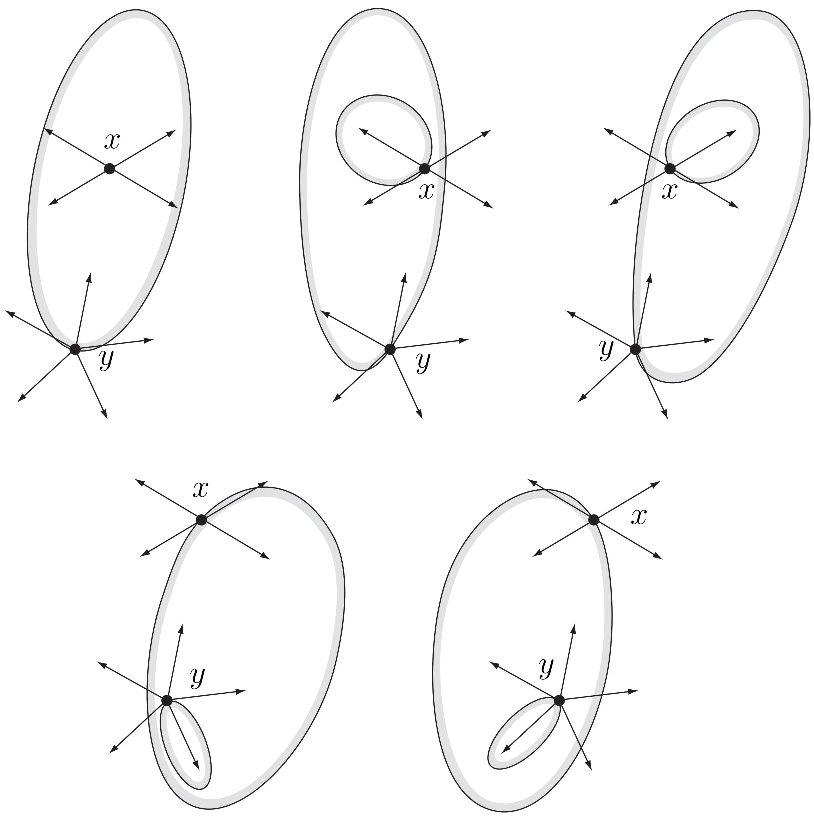

Figure 3: Directional equilibrium outside core

question of whether exact equivalence between the majority core and directional equilibria can be obtained for multiple dimensions under weaker assumptions. In Figure 3, we give an example of five voters with continuous and convex (but non-Euclidean) preferences in which x is a majority core alternative, yet there is another directional equilibrium, y. To reduce clutter, we depict indifference curves of the voters in separate panels. Although multiplicity of directional equilibria is unavoidable, we will see in Section 5 that for almost all specifications of utilities, each directional equilibrium is isolated from the others.

An advantage of the directional equilibria is that they are quite generally Pareto optimal; for purposes of comparison, Pareto optimality of structure-induced equilibria is guaranteed only when the set of alternatives is one- or two-dimensional. In addition to differentiability, we assume *strict pseudo-concavity* of voter utilities, i.e., that for all $x \in X$ and all $y \in X \setminus \{x\}$ such that $u_i(y) \geq u_i(x)$, we have $\nabla u_i(x) \cdot (y - x) > 0$. Note that this condition is satisfied if, for example, each u_i

is strictly concave.

Theorem 4 *Assume that for all $i \in N$, the utility function u_i is strictly pseudo-concave. If an alternative x is a directional equilibrium, then it is Pareto optimal.*

Proof: Consider a directional equilibrium x, and let $y \in X \setminus \{x\}$ be any other alternative. We consider two cases. First, if $\nabla u_k(x) = 0$ for some voter k, then it follows that $x = \hat{x}^k$ and $u_k(x) > u_k(y)$, so y does not Pareto dominate x, and we conclude that x is Pareto optimal. Second, suppose $\nabla u_i(x) \neq 0$ for all $i \in N$. For each i, define the quantity

$$\alpha_i \;=\; \frac{1}{\|\nabla u_i(x)\| \sum_{i=1}^n \frac{1}{\|\nabla u_i(x)\|}},$$

and note that $\alpha_i > 0$ for each i and $\sum_{i=1}^n \alpha_i = 1$. Since x is a directional equilibrium, we have

$$\sum_{i=1}^n \alpha_i \nabla u_i(x) \;=\; \frac{1}{\sum_{i=i}^n \frac{1}{\|\nabla u_i(x)\|}} \sum_{i=i}^n \frac{1}{\|\nabla u_i(x)\|} \cdot \nabla u_i(x) \;=\; \beta \sum_{i=1}^n p^i(x) \;=\; 0,$$

where β is a constant. Then we have

$$0 \;=\; (y-x)\cdot 0 \;=\; (y-x)\cdot \sum_{i=1}^n \alpha_i \nabla u_i(x) \;=\; \sum_{i=1}^n \alpha_i (y-x)\cdot \nabla u_i(x),$$

and this in turn implies that there is an voter i such that $\alpha_i(y-x)\cdot \nabla u_i(x) \leq 0$. And since $\alpha_i > 0$, it follows that $(y-x)\cdot \nabla u_i(x) \leq 0$. Since u_i is strictly pseudo-concave, we conclude that $u_i(x) > u_i(y)$, and once again, y does not Pareto dominate x, so x is Pareto optimal. ∎

4 Existence and Upper Hemicontinuity

We now address the issue of existence of directional equilibrium, and we verify that the equilibrium correspondence has closed graph and, under compactness of X, thereby upper hemicontinuous. The latter property is desirable because it confers a type of robustness of equilibria: if model parameters are varied slightly, it cannot be that directional equilibria are introduced "far" from the equilibrium set for the initial parameterization. The dual robustness notion of lower hemicontinuity is addressed in the stability results of the following section.

General existence is established in the next theorem. The proof, which is located after the closed graph result, is based on an application of Kakutani's fixed point theorem and roughly proceeds by updating any alternative x by adding the sum of normalized gradients, i.e., we map x to $x + \sum_i \epsilon p^i(x)$, suitably scaled by a factor of $\epsilon > 0$. A fixed point of this mapping would of course be a directional equilibrium. But the proof must address two technical difficulties. First, by the assumption that u_i is continuously differentiable, the normalized gradient $p^i(x)$ is continuous at all $x \neq \hat{x}^i$, but it is discontinuous (jumping from norm one to zero) at the ideal point of voter i. To solve this problem, we replace $p^i(\hat{x}^i)$ by a closed disc of radius ϵ, giving us a correspondence, Φ, as depicted in Figure 4. The second difficulty is that the updated alternative may take us outside the set of alternatives, as in Figure 5, where we depict seven voters with $x + \sum_i \epsilon p^i(x) \notin X$. In this case, we scale the shift by a factor of $g^\epsilon(x)$ that brings us to the boundary of the set of alternatives.

To facilitate the above proof approach, and with no essential loss of generality, the existence theorem adds two restrictions on the properties of utility functions over the boundary of the set of alternatives. First, we assume that ideal points do not belong to the boundary of X. Second, we assume that the sum of gradients points to the interior of X from any point on the boundary of that set. Under these assumptions, directional equilibria exist generally.

Theorem 5 *Assume that X is compact and convex, that for all $i \in N$, $\hat{x}^i \notin bdX$, and that for all $x \in bdX$, there exists $\alpha > 0$ such that*

$$x + \alpha \sum_{i=1}^{n} p^i(x) \in intX.$$

Then there is a directional equilibrium, i.e., $C^*_{DE} \neq \emptyset$.

Before proceeding to the proof of Theorem 5, we note the correspondence of directional equilibria possesses the closed graph property. To formalize this result, let Π be a metric space of parameters, and let utility functions u_i depend on elements $\pi \in \Pi$ of this space parametrically; that is, we now have $u_i\colon X \times \Pi \to \mathbb{R}$. Assume that $u_i(x, \pi)$ is jointly continuous, that for all $\pi \in \Pi$, $u_i(\cdot, \pi)$ is differentiable in x, and that the gradient $\nabla_x u_i(x, \pi)$ is jointly continuous in (x, π). Finally, assume that for each $\pi \in \Pi$, $u_i(\cdot, \pi)$ admits a unique ideal point $\hat{x}^i(\pi)$

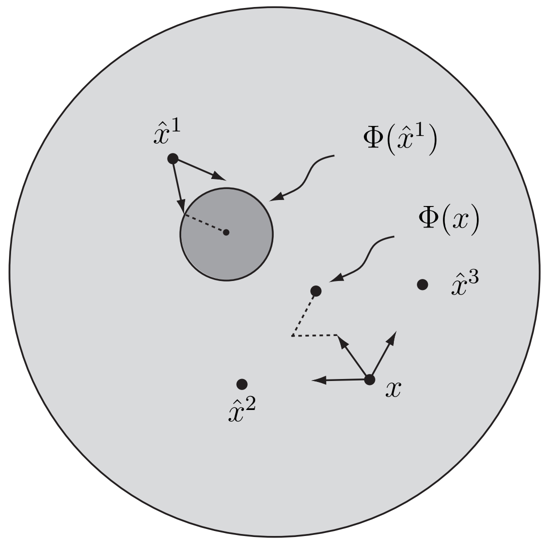

Figure 4: Existence proof

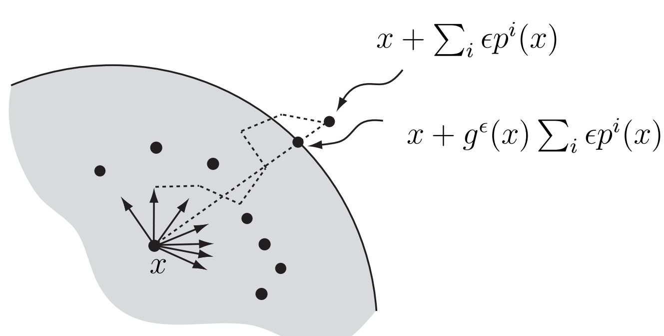

Figure 5: Existence proof again

and that this is the unique critical point of $u_i(\cdot,\pi)$. We extend our notation in the obvious way: for a voter with non-zero gradient at x, let

$$p^i(x,\pi) \quad = \quad \frac{1}{\|\nabla_x u_i(x,\pi)\|}\nabla_x u_i(x,\pi),$$

and if $\nabla_x u_i(x,\pi) = 0$, then we set $p^i(x,\pi) = 0$, and say an alternative $x \in X$ is a *directional equilibrium at* π if

$$\left\| \sum_{i=1}^n p^i(x,\pi) \right\| \quad \leq \quad \#\{i \in N : x = \hat{x}^i(\pi)\},$$

and we let $C^*_{DE}(\pi)$ denote the set of directional equilibria at π. Closed graph of the directional equilibrium correspondence $C^*_{DE}\colon \Pi \rightrightarrows X$ follows immediately from the above definition, and when X is compact, it implies that the correspondence is upper hemicontinuous.

Theorem 6 *The directional equilibrium correspondence* $C^*_{DE}\colon \Pi \rightrightarrows X$ *has closed graph.*

Proof: Let $\{x^m\}$ and $\{\pi^m\}$ be sequences in X and in Π, respectively, such that for all m, we have $x^m \in C^*_{DE}(\pi^m)$. If $\nabla_x u_i(x^m,\pi^m) = 0$ for all m, then by continuity we have $\nabla_x u_i(x,\pi) = 0$. Contrapositively, if $\|p^i(x,\pi)\| = 1$, then $\|p^i(x,\pi)\| = 1$ for large enough m. Combining these observations, we have

$$\begin{aligned} \left\| \sum_{i=1}^n p^i(x,\pi) \right\| \quad &\leq \quad \lim\inf_{m\to\infty} \left\| \sum_{i=1}^n p^i(x^m,\pi^m) \right\| \\ &\leq \quad \lim\sup_{m\to\infty} \#\{i \in N : x = \hat{x}^i(\pi^m)\} \\ &\leq \quad \#\{i \in N : x = \hat{x}^i(\pi)\}. \end{aligned}$$

We conclude that $x \in C^*_{DE}(\pi)$, as required. ∎

The remainder of this section consists of the proof of Theorem 5. Since we assume $\hat{x}^i \in \text{int}X$ for all $i \in N$, there exists an $\epsilon > 0$ such that for all $i \in N$, the closed disc of radius $n\epsilon$ around $\hat{x}^i$ is contained in the interior of the set of alternatives, i.e., $D_{n\epsilon}(\hat{x}^i) \subseteq \text{int}X$. Given any $x \in X$, consider the following constrained maximization problem,

$$\begin{gathered} \max_{\alpha\in\mathbb{R}} \alpha \\ \text{s.t. } x + \alpha \textstyle\sum_{i=1}^n \epsilon p^i(x) \in X \\ \alpha \in [0,1]. \end{gathered}$$

Since the constraint set is nonempty (it contains $\alpha = 0$) and compact, this problem has a solution, which must be unique. We denote the solution by $g^\epsilon(x)$. Note that by construction of ϵ, we have $g^\epsilon(x) = 1$ when $x = \hat{x}^i$ for some voter i, and by the assumption that the sum of normalized gradients points to the interior of X, we have $g^\epsilon(x) > 0$.

Claim 1: For all $x \in X$, g^ϵ is continuous at x.

We first show that the constraint correspondence $\Gamma\colon X \rightrightarrows [0,1]$ defined by

$$\Gamma(x) \;=\; \left\{\alpha \in [0,1] : x + \alpha \sum_i \epsilon p^i(x) \in X\right\}$$

is continuous at all $x \in X' = X \setminus \{\hat{x}^1, \ldots, \hat{x}^n\}$. The correspondence obviously has closed graph on X', and since the unit interval is compact, it is upper hemicontinuous. To prove lower hemicontinuity, note that the correspondence $\Gamma^\circ\colon X' \rightrightarrows [0,1]$ defined by

$$\Gamma^\circ(x) \;=\; \left\{\alpha \in [0,1] : x + \alpha \sum_i \epsilon p^i(x) \in \mathrm{int} X\right\}$$

has open graph and is therefore lower hemicontinuous. We claim that for all $x \in \mathrm{int}X'$, we have $\Gamma(x) = \mathrm{clos}\Gamma^\circ(x)$. Indeed, given $\alpha \in \Gamma(x)$, we have $x + \alpha \sum_i \epsilon p^i(x) \in X$. Since X is convex and $x \in \mathrm{int}X$, it follows that for each k,

$$x + \frac{(k-1)\alpha}{k} \sum_i \epsilon p^i(x) \in \mathrm{int} X, \tag{4}$$

so $\frac{(k-1)\alpha}{k} \in \Gamma^\circ(x)$ and $\lim_{k\to\infty} \frac{(k-1)\alpha}{k} = \alpha$, as claimed. Furthermore, we claim that for all $x \in X' \setminus \{\hat{x}^1, \ldots, \hat{x}^n\}$, we also have $\Gamma(x) = \mathrm{clos}\Gamma^\circ(x)$. Indeed, now (4) holds by convexity of X and our assumption that the sum of normalized gradients points to the interior of X, and the argument proceeds as above, as claimed. We conclude that for all $x \in X'$, $\Gamma(x)$ is the closure of the value of a lower hemi-continuous correspondence; thus, Γ is lower hemi-continuous on X'. By the theorem of the maximum, it follows that $g^\epsilon\colon X' \to [0,1]$ is continuous. We have left to verify that g^ϵ is continuous at the ideal point $\hat{x}^i$ of a voter. We have noted that $g^\epsilon(\hat{x}^i) = 1$. Moreover, by choice of ϵ, we have $\hat{x}^i + D_{n\epsilon}(\hat{x}^i) \subseteq \mathrm{int}X$. Given any sequence $\{x^k\}$ in X such that $x^k \to \hat{x}^i$, we have

$$\left\| x^k + \sum_j \epsilon p^j(x^k) - \hat{x}^i \right\| \;\le\; \|x^k - \hat{x}^i\| + \sum_j \epsilon \|p^j(x^k)\| \;\le\; \|x^k - \hat{x}^i\| + n\epsilon \;\to\; n\epsilon.$$

Therefore, for high enough k, we have $x^k + \sum_j \epsilon p^j(x^k) \in \text{int}X$, so $g^\epsilon(x^k) = 1$, and we conclude that g^ϵ is continuous. □

Given any $x \in X$, let $G(x) = \{i \in N : \hat{x}^i \neq x\}$ be the set of voters whose ideal point is distinct from x, and let $H(x) = \{i \in N : \hat{x}^i = x\}$ be the set of voters whose ideal point is equal to x. Next, define the correspondence $\Phi : X \rightrightarrows X$ as follows: for all $x \in X$,

$$\Phi(x) \;=\; \left[x + g^\epsilon(x) \sum_{j \in G(x)} \epsilon p^j(x) + \sum_{j \in H(x)} D_\epsilon(0)\right] \cap X,$$

where $D_\epsilon(0)$ is the closed disc of radius ϵ around zero, and we let the summation over the empty set equal zero or the singleton consisting of the zero vector, as appropriate.

Claim 2: For all $x \in X$, $\Phi(x)$ is non-empty, convex, and closed.

By construction, $x + g^\epsilon(x) \sum_{j \in G(x)} \epsilon p^j(x) \in X$. Since $0 \in D_\epsilon(0)$, we have $\Phi(x) \neq \emptyset$. Since $D_\epsilon(0)$ is convex and the sum of convex sets is convex, we immediately obtain convexity of $\Phi(x)$. Next, consider any sequence $\{y^k\}$ in $\Phi(x)$ converging to y. In case $H(x) = \emptyset$, we have $y^k = x + g^\epsilon(x) \sum_{i=1}^n \epsilon p^i(x)$ for all k, and thus $y = y^k \in \Phi(x)$. Otherwise, in case $H(x) \neq \emptyset$, we use the fact that $D_\epsilon(0)$ is compact, and therefore the sum of closed discs is closed. □

Claim 3: The correspondence Φ is upper hemi-continuous.

Consider any sequence $\{x^k\}$ in X converging to $x \in X$, and let $\{y^k\}$ be a sequence such that $y^k \in \Phi(x^k)$ for all k and $y^k \to y$. We must show that $y \in \Phi(x)$. In case $H(x) = \emptyset$, we have $H(x^k) = \emptyset$ for high enough k, which implies

$$y^k \;=\; x^k + g^\epsilon(x^k) \sum_{i=1}^n \epsilon p^i(x^k) \;\to\; x + g^\epsilon(x) \sum_{i=1}^n \epsilon p^i(x) \;=\; y \in \Phi(x),$$

where the limit uses continuous differentiability of u_i and continuity of g^ϵ from Claim 1. In case $H(x) \neq \emptyset$, if $x^k = x$ for sufficiently high k, then the claim follows by closedness of $\Phi(x)$ from Claim 2. Thus, suppose that $x^k \neq x$ for arbitrarily high k, so (going to a subsequence if necessary) we can assume that $H(x^k) = \emptyset$ for all k. Thus, we have

$$y^k \;=\; x^k + g^\epsilon(x^k) \sum_{j=1}^n \epsilon p^j(x^k) \;=\; x^k + g^\epsilon(x^k) \sum_{j \in G(x)} \epsilon p^j(x^k) + g^\epsilon(x^k) \sum_{j \in H(x)} \epsilon p^j(x^k)$$

for all k. We can assume (going to a subsequence if necessary) that for all $j \in H(x)$, the sequence $\{p^j(x^k)\}$ converges to a limit $w^j \in D_1(0)$. Thus, taking limits, we obtain

$$y = \lim_{k\to\infty} y^k = x + g^\epsilon(x) \sum_{j\in G(x)} \epsilon p^j(x) + g^\epsilon(x) \sum_{j\in H(x)} \epsilon w^j.$$

Noting that $g^\epsilon(x) = 1$ and $\epsilon w^j \in D_\epsilon(0)$, we then have $y \in \Phi(x)$, as required. □

Thus, Kakutani's fixed point theorem guarantees the existence of a fixed point, i.e., an alternative $x \in X$ such that $x \in \Phi(x)$. The final step of our proof is to show that the fixed point is a directional equilibrium. First, suppose $x \in \Phi(x)$ is not the ideal point of any voter, so $H(x) = \emptyset$. Then

$$x = x + g^\epsilon(x) \sum_{i=1}^{n} \epsilon p^i(x),$$

which implies

$$g^\epsilon(x) \sum_{i=1}^{n} \epsilon p^i(x) = 0.$$

Dividing by $g^\epsilon(x)\epsilon > 0$, we then have

$$\sum_{i=1} p^i(x) = 0,$$

and we conclude that x is a directional equilibrium. Next, suppose $x \in \Phi(x)$ is an ideal point, so that $H(x) \neq \emptyset$. Then for each $j \in H(x)$, there exists a $v^j \in D_\epsilon(0)$ such that

$$x = x + \sum_{j\in G(x)} \epsilon p^j(x) + \sum_{j\in H(x)} v^j,$$

which implies

$$\left\| \sum_{j\in G(x)} \epsilon p^j(x) \right\| = \left\| \sum_{j\in H(x)} v^j \right\| \leq \sum_{j\in H(x)} \|v^j\| \leq \epsilon \, \#H(x).$$

Dividing by $\epsilon > 0$, this implies

$$\left\| \sum_{j=1}^{n} p^j(x) \right\| = \left\| \sum_{j\in G(x)} p^j(x) \right\| \leq \#\{i \in N : x = \hat{x}^i\},$$

and we again conclude that x is a directional equilibrium.

5 Local Uniqueness and Stability

In this section, we examine generic properties of directional equilibria. We first establish that for almost all specifications of utilities, these alternatives are locally unique, i.e., given any directional equilibrium, there is an open set around it that contains no other directional equilibrium; subsequently, we establish that for generic utilities, each directional equilibrium is stable, in the sense that it is locally expressible as a continuous (or smooth) function of parameters. In particular, stability implies lower hemicontinuity of the directional equilibrium correspondence: if model parameters are varied slightly, then it must be that directional equilibria exist "close to" any given equilibrium at the initial parameterization. Thus, these results complement Theorem 6 by establishing, generically, full continuity of the directional equilibrium correspondence.

The framing of these issues immediately runs into the technical difficulties that voter utility functions are naturally imbedded in an infinite-dimensional space, and that there are different possible ambient spaces that can be used to host voter utility functions. The analyst may prefer to leave utility functions unrestricted, subject to some smoothness requirement; or perhaps only voter utilities that respect some form of concavity are of interest; or in more structured applications, it may be that the quadratic functional form is imposed. The problem posed by infinite-dimensional genericity statements is that "smallness" can be formalized in different ways, and genericity results may depend on the ambient space chosen for voter utility functions.

It is common to take a topological approach to smallness in infinite-dimensional spaces (e.g., Schofield (1983)). Here, we would let the space $\mathcal{U}$ of possible utility vector functions consist of all smooth mappings $u\colon X \to \mathbb{R}^n$, topologized appropriately. Then we would show that the subset of $\mathcal{U}$ on which a desired property fails is closed and nowhere dense; or in lieu of that, that it is *meagre,* i.e., the countable union of closed sets with empty interior.[8] This approach identifies a property as generic if it holds in a *residual* set, i.e., a set that is the complement of a meagre set. In finite-dimensional spaces, of course, there is also a measure-theoretic concept of genericity, namely, that a desired property holds almost everywhere with respect to Lebesgue measure. The latter approach does not directly extend to the infinite-dimensional setting, because the construction of Lebesgue measure assumes a finite-dimensional Euclidean structure. However, Anderson and Zame (2001) provide a notion of smallness, called "shyness," that generalizes the

[8] In a complete metric space, the Baire Category theorem ensures that such sets do have empty interior, as we would want.

finite-dimensional notion of Lebesgue measure zero and possesses many familiar properties, and importantly they allow for a general, convex ambient subset $\mathcal{U}$ of a topological vector space.

In fact, we use an even stronger notion that they refer to as "finite shyness." To define this, let $\mathcal{U}$ be a subset of a topological vector space V. A subset $S \subseteq \mathcal{U}$ is *finitely shy* relative to $\mathcal{U}$ if there is a finite-dimensional subspace $L \subseteq V$ such that (i) for every $u \in V$, $L \cap (S - u)$ has Lebesgue measure zero in L, and (ii) for some $u \in V$, the set $L \cap (\mathcal{U} - u)$ has positive Lebesgue measure in L. Thus, we pass a finite-dimensional subspace L through S, evaluating the Lebesgue measure of S for different slices through S. By condition (ii), the ambient space $\mathcal{U}$ is non-trivial as evaluated by L, but by (i), the set S of interest has zero measure, regardless of how it is sliced by L.[9]

The way finite shyness is established in the proofs of Theorems 7–9 is as follows. We set V to be the normed linear space of C^2 functions $u\colon X \to \mathbb{R}^n$ with the C^2-sup norm. Given vectors $\theta^i \in \mathbb{R}^m$, $i = 1, \ldots, n$, write $\theta = (\theta^1, \ldots, \theta^n)$. We then define the linear function $f_\theta\colon X \to \mathbb{R}^n$ by

$$f_\theta(x) \quad = \quad (\theta^1 \cdot x, \ldots, \theta^n \cdot x),$$

and we let $L = \{f_\theta : \theta \in \mathbb{R}^{mn}\}$ be a finite-dimensional subspace of utility vector functions generated by varying the linear parameters θ. In the proof of Theorem 7, to show that directional equilibria are locally unique for generic preferences, the critical step is to demonstrate condition (i) in the definition of finite shyness. Technically, we would define S as the set of utility vector functions for which the directional equilibria are *not* locally finite, and then we would show that for all $u \in V$, the set $L \cap (S - u)$ has Lebesgue measure zero in L. This is operationalized by showing simply that for Lebesgue almost all $\theta \in \mathbb{R}^{mn}$, the directional equilibria for utilities $u + f_\theta$ are locally finite. [10]

This measure-theoretic approach offers a number of advantages. First, the concept of finite shyness directly generalizes the notion of Lebesgue measure zero, which in turn has an interpretation in terms of probability theory: if a selection

[9] See Patty (2007) for another application of finite shyness in formal political science.

[10] The mapping $u + f_\theta$ determines the utility function $u_i(x) + \theta^i \cdot x$ for each voter i. Thus, it may appear that our genericity analysis informs us of the properties of a specific (linear) parameterization of voter utilities, but this functional form is simply a manifestation of the linear subspace we have used to slice the set S, and this subspace is largely arbitrary: any finite-dimensional subspace may be used to verify conditions (i) and (ii) in the definition of finite shyness, and the linear parameterization is merely an efficacious choice of subspace L. The key is that *regardless* of the initial utilities u_i, the set of perturbations θ for which the directional equilibria violate local uniqueness has Lebesgue measure zero.

is made at random according to a density, then a measure zero set has probability zero.[11] Second, we can posit a quite general ambient space of utility vector functions, requiring in addition to convexity only that for all $u \in \mathcal{U}$, there is a positive measure set of $\theta \in \mathbb{R}^{mn}$, such that $u + f_\theta \in \mathcal{U}$. This of course holds if we impose minimal structure and let $\mathcal{U}$ be the set of all C^2 utility vector functions. It also holds if $\mathcal{U}$ consists of all profiles of C^2, concave utility functions. And if the focus is on the special case of Euclidean preferences, we can let $\mathcal{U}$ consist of all profiles of utility functions with a quadratic functional form. Third, the mathematical structure of the genericity proofs is essentially finite-dimensional, leveraging the finite-dimensional version of the transversality theorem and avoiding explicit use of infinite-dimensional spaces.

For simplicity, we assume X is open in the following analysis. Let $\mathcal{U}$ be a convex subset of the set of C^2 utility vector functions $u\colon X \to \mathbb{R}^n$ such that for all $u \in \mathcal{U}$, there is a positive measure set of $\theta \in \Theta$ such that $u + f_\theta \in \mathcal{U}$. For general $u \in \mathcal{U}$, define the normalized gradient

$$p^i(x,u) \quad = \quad \begin{cases} \frac{1}{\|\nabla u_i(x)\|}\nabla u_i(x) & \text{if } \nabla u_i(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then an alternative x is a *directional equilibrium at u* if

$$\left|\left| \sum_{i=1}^{n} p^i(x,u) \right|\right| \quad \leq \quad \#\{i \in N : \nabla u_i(x) = 0\}, \tag{5}$$

and we let $C^*_{DE}(u)$ denote the set of directional equilibria at u. A directional equilibrium x at u is *locally unique* if there is some $\epsilon > 0$ such that $B_\epsilon(x) \cap C^*_{DE}(u) = \{x\}$. Let $\mathcal{U}_{LU}$ denote the subset consisting of $u \in \mathcal{U}$ such that the directional equilibria are locally unique. The next theorem establishes that local uniqueness is a generic property.

Theorem 7 *Assume that X is open and that n is odd. Then $\mathcal{U} \setminus \mathcal{U}_{LU}$ is finitely shy relative to $\mathcal{U}$.*

To state the generic stability result, again let $\mathcal{U}$ be a convex subset of the C^2 utility vector functions such that for all $u \in \mathcal{U}$, there is a positive measure set of θ such that $u + f_\theta \in \mathcal{U}$. We assume a metric space Π of parameters, and a *parameterization* is any continuous mapping $U\colon \Pi \to \mathcal{U}$, where we abuse

[11] Note that there are residual sets that have Lebesgue measure zero, and there are sets of full measure that are meagre.

notation slightly and write $U_i(x,\pi)$ for $U_i(\pi)(x)$, the value of the ith component of $U(\pi)$ evaluated at x, and $\nabla_x U_i(x,\pi)$ for the gradient of $U_i(\pi)$. Note that with the C^2-sup norm, $U_i(x,\pi)$ is jointly continuous in (x,π), and the first- and second-order derivatives of $U_i(x,\pi)$ with respect to x are also jointly continuous in (x,π). A directional equilibrium $x \in C^*_{DE}(u)$ is *stable* if for every parameterization $U\colon \Pi \to \mathcal{U}$ and every $\pi \in \Pi$ with $U(\pi) = u$, there exist an open set $\Gamma \subseteq \Pi$ with $\pi \in \Gamma$, an open set $G \subseteq \mathbb{R}^m$ with $x \in G$, and a continuous mapping $F\colon \Gamma \to G$ such that for all $\pi' \in \Gamma$ and all $y \in G$, y is a directional equilibrium at $U(\pi')$ if and only if $y = F(\pi')$. The next result states that for utility vector functions u outside a finitely shy subset $\mathcal{U}'$, the directional equilibria at u are stable. An implication is that for every π and every parameterization U such that utilities $U(\pi)$ belong to a generic set, the correspondence $C^*_{DE}\colon \Pi \rightrightarrows X$ is lower hemicontinuous at π. Note that, technically, stability implies local uniqueness, so Theorem 8 actually implies Theorem 7.

Theorem 8 *Assume that X is open and that n is odd. Then there is a subset $\mathcal{U}' \subseteq \mathcal{U}$ that is finitely shy relative to $\mathcal{U}$ such that for all $u \in \mathcal{U} \setminus \mathcal{U}'$, every $x \in C^*_{DE}(u)$ is stable.*

The preceding theorem is stated with the assumption that the parameter π enters voter utility functions in a continuous way, giving us a continuous form of stability: generically, each directional equilibrium is a continuous function of parameters. In some contexts, we expect or desire a smooth stability result, and we obtain this result after strengthening our assumptions so that Π is a complete normed linear space, in which case we say that a parameterization $U\colon \Pi \to \mathcal{U}$ is *smooth* if each $u_i(x,\pi)$ is twice continuously differentiable jointly as a function of (x,π). Formally, we say a directional equilibrium $x \in C^*_{DE}(u)$ is *smoothly stable* if for every smooth parameterization $U\colon \Pi \to \mathcal{U}$ and every $\pi \in \Pi$ with $U(\pi) = u$, there exist an open set $\Gamma \subseteq \Pi$ with $\pi \in \Gamma$, an open set $G \subseteq \mathbb{R}^m$ with $x \in G$, and a continuously differentiable mapping $F\colon \Gamma \to G$ such that for all $\pi' \in \Gamma$ and all $y \in G$, y is a directional equilibrium at π' if and only if $y = F(\pi')$.

Theorem 9 *Assume that X is open, that n is odd, and that Π is a Banach space. Then there is a subset $\mathcal{U}' \subseteq \mathcal{U}$ that is finitely shy relative to $\mathcal{U}$ such that for all $u \in \mathcal{U} \setminus \mathcal{U}'$, every $x \in C^*_{DE}(u)$ is smoothly stable.*

The proofs of Theorems 7–9 are located in the appendix. Although the last two results generalize the first, they follow directly from an application of the implicit function theorem applied to a system of equations that characterizes the directional equilibria for given parameter values. Most of the work for this is required for Theorem 7, so the proof focuses on this result. The proof distinguishes between several types of directional equilibrium: given a directional equilibrium at u, denoted x, we say x is *critical* if $\nabla u_i(x) = 0$ for some voter i, and otherwise it is *non-critical.* Furthermore, a critical directional equilibrium x is *slack* if the inequality in (5) holds strictly, and otherwise it is *tight.* Note that by continuous differentiability of u_i, the set of critical directional equilibria at u is closed.

The proof begins by taking an arbitrary utility vector function $u \in \mathcal{U}$ as given. We let $\Theta^i = \mathbb{R}^m$ be the set of possible linear parameters for voter i, so the set of possible vectors $\theta = (\theta^1, \ldots, \theta^n)$ is $\Theta = \Theta^1 \times \cdots \times \Theta^n$. A key to the argument is the mapping $f \colon X \times \mathbb{R}^n_{++} \times \Theta^n \to \mathbb{R}^{m+n}$ defined by:

$$f(x, \alpha, \theta) \;=\; \begin{bmatrix} \sum_{i=1}^{n-1} \alpha_i (D_1 u_i(x) + \theta^i_1) \\ \vdots \\ \sum_{i=1}^{n-1} \alpha_i (D_m u_i(x) + \theta^i_m) \\ \|\nabla u_1(x) + \theta^1\|^2 - \frac{1}{\alpha_1^2} \\ \vdots \\ \|\nabla u_n(x) + \theta^n\|^2 - \frac{1}{\alpha_n^2} \end{bmatrix}. \tag{6}$$

Clearly, given θ, a pair $(x, \alpha) \in X \times \mathbb{R}^n_{++}$ satisfies $f(x, \alpha, \theta) = 0$ if and only if x is a non-critical directional equilibrium at $u + f_\theta$. The proof of Theorem 7 then proceeds with an application of the transversality theorem to this mapping, showing that for Lebesgue almost all θ, the set of solutions to $f(\cdot, \theta) = 0$ is a zero-dimensional manifold, which implies that non-critical directional equilibria are locally unique. Additional arguments are then used to show that for Lebesgue almost all θ, the critical equilibria are locally unique, and that these sets cannot have accumulation points. Thus, the subset of utility vector functions violating local uniqueness of directional equilibria is finitely shy, as required.

As an aid to the reader, we provide a guide to the structure of the proof:

- **Step 1:** We show that 0 is a regular value of f, i.e., the Jacobian $Df(x, \alpha, \theta)$ at every solution to $f = 0$ has full row rank. Then the transversality theorem establishes that for almost all θ, 0 is a regular value of $f(\cdot, \theta)$. Thus, for almost all θ, the set of non-critical directional equilibria at $u + f_\theta$ is a manifold of dimension $m + n - (m + n) = 0$, so the non-critical equilibria are locally unique (among non-critical equilibria).

- **Step 2:** We show that for almost all θ, every critical directional equilibrium at $u + f_\theta$ is a critical point of exactly one voter, and critical points of voter utility functions are non-degenerate, in the sense that the second derivative $D^2u_i(x)$ is non-singular at every critical point of u_i (i.e., voter utilities are Morse functions). This implies, after a brief argument, that the critical directional equilibria are locally unique (among critical equilibria).

- **Step 3:** We define a mapping h such that the solutions to $h(x, \alpha, \beta, \theta) = 0$ characterize the tight directional equilibria such that exactly one voter's gradient is zero. We show that 0 is a regular value of h, so by the transversality theorem, for almost all θ, 0 is a regular value of $h(\cdot, \theta)$. Because the system $h(\cdot, \theta) = 0$ is over-specified, this implies that for almost all θ, there do not exist tight, critical directional equilibria at $u + f_\theta$.

- **Step 4:** We argue that for almost all θ, the non-critical directional equilibria cannot accumulate at a critical equilibrium; indeed, if this were the case, the critical equilibrium would be tight, and by Step 3, such directional equilibria generically fail to exist. Furthermore, the critical directional equilibria cannot accumulate at a non-critical equilibrium because voter gradients are continuous (so the set of critical equilibria are closed). This implies that for almost all θ, the directional equilibria at $u + f_\theta$ are locally unique, and we conclude that the set of utility vector functions violating local uniqueness is finitely shy, establishing Theorem 7.

- **Steps 5 and 6:** For almost all θ, we use Steps 1–4 to apply the implicit function theorem separately to non-critical directional equilibria and to slack, critical directional equilibria. Stability and smooth stability are obtained by the continuity argument from Step 4, which proves Theorems 8 and 9.

6 Non-cooperative Stability

The concept of directional equilibrium has rather direct game-theoretic foundations. Given any alternative $x \in X$ and any $\eta > 0$ such that the closed disc of radius $n\eta$ lies in the set of alternatives, i.e., $D_{n\eta}(x) \subseteq X$, we define the *η-contest game at x* as a strategic form game among the voters such that each voter i's strategy set is the closed disc $D_\eta(0)$ of radius η around zero, and given a profile $s = (s_1, \ldots, s_n)$ of strategies, voter i's payoff is

$$U_i(s) \quad = \quad u_i\left(x + \sum_j s_j\right).$$

That is, each voter has a "budget" of norm η and can use this budget to pull the outcome in any desired direction. If we interpret x as a status quo outcome, which obtains if the forces applied by the voters cancel each other out, then it is of interest to understand when the status quo is stable, in this sense. Formally, given $\eta > 0$, we say x is *non-cooperatively η-stable* if there is a (pure strategy) Nash equilibrium s in the η-contest game at x such that $x = x + \sum_j s_j$, i.e., $\sum_j s_j = 0$. It is *non-cooperatively strictly-η stable* if there is a strict Nash equilibrium s such that $\sum_j s_j = 0$. Finally, it is *non-cooperatively strongly η-stable* if x is non-cooperatively strictly η-stable, and for every Nash equilibrium s of the η-contest game at x, we have $\sum_j s_j = 0$.

Assuming voter utilities are quasi-concave, the conditions of the Debreu-Fan-Glicksberg theorem are satisfied in the η-contest game at x, and thus there exists a pure strategy equilibrium, say s. The first part of the next theorem establishes that if the voters' strategies cancel each other out, so $\sum_j s_j = 0$, then x must be a directional equilibrium. While quasi-concavity is not actually needed for this direction of implication, it is used to obtain the opposite direction: if utilities are quasi-concave, then every directional equilibrium can be supported as a strict Nash equilibrium outcome of every η-contest game.

Theorem 10 *For all $x \in X$, if x is non-cooperatively η-stable for some $\eta > 0$ with $D_{n\eta}(x) \subseteq X$, then it is a directional equilibrium. Conversely, assume voter utilities are quasi-concave. If x is a directional equilibrium, then for all $\eta > 0$ with $D_{n\eta}(x) \subseteq X$, x is non-cooperatively strictly η-stable.*

Proof: First, let s be a Nash equilibrium of the η-contest game at x with $\sum_j s_j = 0$, so for all voters i, the strategy s_i solves

$$\begin{array}{c} \max_{y \in \mathbb{R}^m} u_i\left(x + (\sum_{j:j\neq i} s_j) + y\right) \\ \text{s.t. } \|y\|^2 \leq \eta^2. \end{array}$$

Let $G = \{i \in N : s_i = 0\}$ be the coalition of voters who do not exert effort in equilibrium, which implies $x = \hat{x}^i$ and $\nabla u_i(x) = 0$. For all other voters, the constraint qualification holds at $y = s_i$, so the necessary first order condition is that there exists $\lambda_i \geq 0$ such that

$$\begin{aligned} \nabla u_i(x) &= 2\lambda_i s_i \\ \lambda_i(\eta^2 - \|s_i\|^2) &= 0. \end{aligned}$$

Thus, we have $\|\nabla u_i(x)\| = 2\lambda_i \|s_i\|$, and $\lambda_i = 0$ holds if and only if $x = \hat{x}^i$. Letting

$H = \{i \in N \setminus G : \lambda_i > 0\}$, we then have

$$0 \;=\; \sum_{i=1}^{n} s_i \;=\; \sum_{i \in H} \frac{1}{2\lambda_i} \nabla u_i(x) + \sum_{i \notin H} s_i. \tag{7}$$

By complementary slackness, it follows that for all $i \in H$, we have $\|s_i\| = \eta$. Thus, for all $i \in H$, we have $\|\nabla u_i(x)\| = 2\eta\lambda_i$, and this implies

$$\sum_{i=1}^{n} p^i(x) \;=\; \sum_{i \in H} p^i(x) \;=\; \sum_{i \in H} \frac{1}{2\eta\lambda_i} \nabla u_i(x) \;=\; -\frac{1}{\eta} \sum_{i \notin H} s_i,$$

where the last equality follows from (7). Therefore, using the fact that $\|s_i\| \leq \eta$ for all $i \notin H$, we have

$$\left\| \sum_{i=1}^{n} p^i(x) \right\| \;=\; \left\| \frac{1}{\eta} \sum_{i \notin H} s_i \right\| \;\leq\; \frac{1}{\eta} \sum_{i \notin H} \|s_i\| \;\leq\; \#\{i \in N : x = \hat{x}^i\},$$

and we conclude that x is a directional equilibrium.

For the converse, assume voter utility functions are quasi-concave, let x be a directional equilibrium, and consider any $\eta > 0$ with $D_{n\eta}(x) \subseteq X$. Define the strategy profile s by $s_i = \eta p^i(x)$ for all i, and note that $\sum_{i=1}^{n} s_i = 0$. To see that s is a Nash equilibrium of the η-contest game, consider any voter i. If $x = \hat{x}^i$, then clearly $s_i = 0$ is a best response. Otherwise, if $x \neq \hat{x}^i$, then we return to the above constrained maximization problem.

$$\begin{array}{c} \max_{y \in \mathbb{R}^m} u_i \left(x + (\sum_{j: j \neq i} s_j) + y \right) \\ \text{s.t. } \|y\|^2 \leq \eta^2. \end{array}$$

Note that the constraint function $g(y) = \|y\|^2$ is quasi-convex, with $\nabla g(y) = 2y$. Setting $y = s_i$ and $\lambda_i = \frac{1}{2}\|\nabla u_i(x)\| \geq 0$, we have

$$\begin{aligned} \nabla u_i(x) &= 2 s_i \lambda_i \\ \lambda(\eta^2 - \|s_i\|^2) &= 0. \end{aligned}$$

Since u_i is quasi-concave and has non-zero gradient at x, the first order condition is sufficient for a global maximum. Thus, s_i is a best response to the strategies of other voters, and x is non-cooperatively η-stable.

It is straightforward to deduce that s_i specified above is in fact the unique best response for voter i, for suppose there is another best response s_i'. Then in the

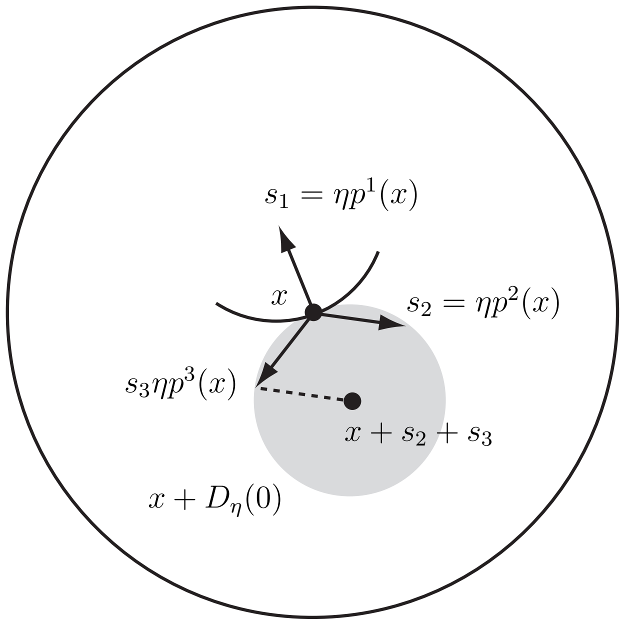

Figure 6: Non-cooperative stability of directional equilibrium

context of the above constrained maximization problem, there are two optima, $y = s_i$ and $y' = s'_i$. Note that s_i is the unique maximizer of the linear function $f(y) \cdot p^i(x)$ subject to $y \in D_\eta(0)$. Since u_i is quasi-concave and $u_i(x + s'_i + \sum_{j \neq i} s_j) \geq u_i(x)$, it follows that $p^i(x) \cdot (s'_i - s_i) \geq 0$, but then s'_i is an additional maximizer of $f(y)$ subject to $y \in D_\eta(0)$, a contradiction. We conclude that x is non-cooperatively strictly η-stable. ∎

The second part of Theorem 10 is depicted in Figure 6. Here, x is a directional equilibrium, and voter 1's indifference curve through x is shown. Given that the other voters exert effort in the direction of their gradients, i.e., voters 2 and 3 use s_2 and s_3, the set of outcomes that voter 1 can obtain is the shaded disk around $x+s_2+s_3$. By construction, voter 1's indifference curve through x is tangent to the disk; more precisely, the first order condition for voter 1 is satisfied at $s_1 = \eta p^1(x)$. And since the voter's utility function is quasi-concave (with non-zero gradient at x), it follows that x is the unique best outcome achievable for voter 1. The same arguments apply to other voters, and we conclude that $s = (s_1, s_2, s_3)$ is a strict Nash equilibrium of the η-contest game, and that x is indeed non-cooperatively strictly η-stable.

Theorem 10 leaves open the possibility that x is non-cooperatively η-stable,

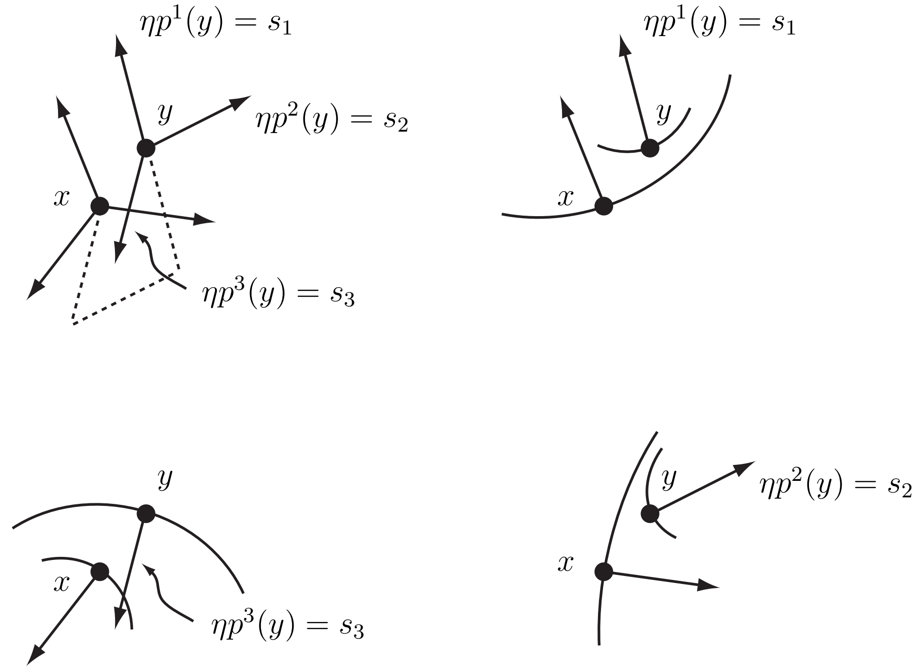

Figure 7: Contest game with multiple equilibria

and thereby a directional equilibrium, but that the η-contest game at x admits other Nash equilibria; that is, x may not be non-cooperatively strongly η-stable. This possibility is illustrated in Figure 7, where x is a directional equilibrium, y is not, but the strategy profile $s = (\eta p^1(y), \eta p^2(y), \eta p^3(y))$ is a Nash equilibrium of the η-contest game at x. To see this in the figure, note that the outcome of this strategy profile is indeed y, and thus (by an argument similar to that for the proof of the first part of Theorem 10) the strategies of the voters are mutual best responses. The equilibrium is depicted in the top-left panel, while the remaining panels depict indifference curves for each voter, showing that the example is consistent with concave utilities. Thus, strong stability is strictly more restrictive than stability.

Nevertheless, the two stability concepts coincide when voter preferences are Euclidean: the properties of Euclidean preferences underpinning the uniqueness result of Theorem 3 can be used here to show that if x is a directional equilibrium, then it is the unique Nash equilibrium of the η-contest game at x. The result in fact uses Lemma 1, but it holds even without the additional background conditions of Theorem 3.

Theorem 11 *Assume voter utilities are Euclidean. For all $x \in X$, if x is a directional equilibrium, then for all $\eta > 0$ with $D_{n\eta}(x) \subseteq X$, x is non-cooperatively strongly η-stable.*

Proof: Let x be a directional equilibrium, let $\eta > 0$ be such that $D_{n\eta}(x) \subseteq X$, and suppose there is a Nash equilibrium s of the η-contest game at x such that $\sum_j s_j \neq 0$. Let $y = x + \sum_j s_j$, and let $t = \frac{1}{\|\sum_j s_j\|}\sum_j s_j = \frac{1}{\|y-x\|}(y-x)$ be the direction pointing from x to y. Since x is a directional equilibrium, we have $t \cdot \sum_{i=1}^n p^i(x) = 0$. For each voter i, Lemma 1 implies that $t \cdot p^i(y) \leq t \cdot p^i(x)$, and thus we have

$$t \cdot \sum_{i=1}^n p^i(y) \quad \leq \quad 0.$$

In fact, we can strengthen this inequality to account for voters with ideal point at x. Note that for a voter i with $\hat{x}^i = y$, we have,

$$t \cdot p^i(x) \quad = \quad \frac{1}{\|y-x\|} t \cdot (y-x) \quad = \quad 1,$$

where we use $t = \frac{1}{\|y-x\|}(y-x)$. Similarly, for a voter i with $\hat{x}^i = x$, we have $t \cdot p^i(y) = -1$. Letting $G = \{i \in N : y = \hat{x}^i\}$ and $H = \{i \in N : x = \hat{x}^i\}$, we can then use Lemma 1 to deduce

$$t \cdot \sum_{i=1}^n p^i(y) \;\leq\; t \cdot \sum_{i \in G} p^i(x) - \#G + t \cdot \sum_{i \in H} p^i(x) - \#H + t \cdot \sum_{i \notin G \cup H} p^i(x) \;=\; -\#G,$$

where the last equality uses the assumption that x is a directional equilibrium. But as in the proof of the first part of Theorem 10, for each voter i, s_i solves

$$\begin{array}{c} \max_{y \in \mathbb{R}^m} u_i(x + \sum_{j:j \neq i} s_j + y) \\ \text{s.t. } \|y\|^2 \leq \eta^2, \end{array}$$

and for all $i \notin G$, we thus have $s_i = \eta p^i(y)$. Since $t \cdot p^i(y) = 0 \geq t \cdot s_i - \eta$ for all $i \in G$, we then have

$$t \cdot \sum_{i=1}^n p^i(y) \quad \geq \quad \frac{1}{\eta} t \cdot \sum_{i=1}^n s_i - \#G \quad = \quad \frac{\|\sum_j s_j\|}{\eta} - \#G \quad > \quad -\#G,$$

a contradiction. ∎

It is clear from Figure 6 that quasi-concavity of voter utilities plays an important role in the second part of Theorem 10. Even without this convex structure, however, we can typically provide local non-cooperative support for directional equilibria. We next consider directional equilibria x that are *non-degenerate,* in the sense that for all voters i, the Hessian $D^2u_i(x)$ of the voter's utility function is negative definite at x. Such non-degenerate alternatives should be expected to be common; indeed, if voter utilities are quadratic, then every alternative is non-degenerate. Substituting non-degeneracy for quasi-concavity, the next result establishes that directional equilibrium implies non-cooperative η-stability for η sufficiently small.

Theorem 12 *For all $x \in intX$, if x is a non-degenerate directional equilibrium, then there exists $\eta > 0$ such that for all $\epsilon \in (0, \eta)$, x is non-cooperatively strictly ϵ-stable.*

Proof: Assume x is a non-degenerate directional equilibrium, and to simplify accounting, assume $D_n(x) \subseteq X$. We initially construct a strategy profile s for the 1-contest game at x as follows. For all $i \in N$ such that $\nabla u_i(x) \neq 0$, we set $s_i^* = p^i(x)$. Letting $G = \{i \in N : \nabla u_i(x) \neq 0\}$, the definition of directional equilibrium implies $\| \sum_{i \in G} p^i(x) \| \leq \#\{i \in N : x = \hat{x}^i\}$, so we can specify s_i^* for all $i \notin G$ such that $\|s_i^*\| \leq 1$ and $\sum_{i \notin G} s_i^* = -\sum_{i \in G} p^i(x)$, which implies $\sum_i s_i^* = 0$. Consider the following maximization problem.

$$\begin{array}{c} \max_{y \in \mathbb{R}^m} u_i \left(x + (\sum_{j:j \neq i} s_j^*) + y \right) \\ \text{s.t. } \|y\|^2 \leq 1. \end{array}$$

For all $i \in G$, set $\lambda_i = \frac{1}{2}\|\nabla u_i(x)\|$, and note that the first order condition at s_i^*,

$$\begin{aligned} \nabla u_i(x) &= 2\lambda_i s_i^* \\ \lambda_i(1 - \|s_i^*\|^2) &= 0, \end{aligned}$$

is satisfied. Furthermore, because the constraint function is quadratic and x is non-degenerate, the second order sufficient condition holds, so s_i^* is a strict local solution to the above problem. Thus, there exists $\epsilon_i' > 0$ such that for all s_i' with $\|s_i'\| \leq 1$ and $\|s_i' - s_i^*\| < \epsilon_i'$, we have

$$u_i(x) \;>\; u_i \left(x + (\sum_{j:j \neq i} s_j^*) + s_i' \right).$$

Set $\epsilon_i = \epsilon_i'/3$. For all $i \notin G$, we have $x = \hat{x}^i$. Of course, if $G = \emptyset$, then x is clearly non-cooperatively strictly ϵ-stable for all $\epsilon > 0$.

Assuming, then, that $G \neq \emptyset$, we will show that x is non-cooperatively strictly ϵ-stable for $\epsilon < \eta = \min\{\epsilon_i : i \in G\}$. We specify the strategy profile s in the ϵ-contest game such that $s_i = \epsilon s_i^*$ for all voters i. To see that s_i is the unique best response for each $i \in G$, consider any $\tilde{s}_i$ such that $\|\tilde{s}_i\| \leq \epsilon$. Then define

$$s_i' \;=\; \tilde{s}_i - (1-\epsilon)\sum_{j:j\neq i} s_j^*,$$

and note that since $s_i^* = -\sum_{j:j\neq i} s_j^*$, we have

$$\|s_i'\| \;=\; \left\|\tilde{s}_i - (1-\epsilon)\sum_{j:j\neq i} s_j^*\right\| \;\leq\; \|\tilde{s}_i\| + (1-\epsilon)\left\|\sum_{j:j\neq i} s_j^*\right\| \;\leq\; 1,$$

and furthermore,

$$\|s_i' - s_i^*\| \;=\; \left\|\tilde{s}_i - (1-\epsilon)\sum_{j:j\neq i} s_j^* - s_i^*\right\| \;=\; \|\tilde{s}_i + \epsilon s_i^*\| \;\leq\; 2\epsilon \;<\; \epsilon_i'.$$

By construction of ϵ_i', we therefore have

$$\begin{aligned} u_i(x) \;&>\; u_i\left(x + (\sum_{j:j\neq i} s_j^*) + s_i'\right) \;=\; u_i\left(x + \epsilon(\sum_{j:j\neq i} s_j^*) + \tilde{s}_i\right) \\ &=\; u_i\left(x + (\sum_{j:j\neq i} s_j) + \tilde{s}_i\right), \end{aligned}$$

and we conclude that s is a strict Nash equilibrium of the ϵ-contest game at x with $\sum_i s_i = 0$, as required. ∎

As a matter of interest, we note that quasi-concavity in the second part of Theorem 10 and non-degeneracy in Theorem 12 are needed for the results: it may be that an alternative is a (degenerate) directional equilibrium, yet not non-cooperatively ϵ-stable for any $\epsilon > 0$. To see this, let $m = 2$ and $n = 3$, and consider the following example:

$$\begin{aligned} u_1(x,y) \;&=\; y + x^3 + x^3\sin(1/x) \\ u_2(x,y) \;&=\; -x^2 - y^2 \\ u_3(x,y) \;&=\; -x^2 - (1-y)^2, \end{aligned}$$

where $\sin(1/x)$ and $\cos(1/x)$ are evaluated as zero when $x = 0$. Of course, voters 2 and 3 have Euclidean preferences with ideal points at the origin $(0,0)$ and at

$(0, -1)$, respectively, while voter 1's utility function violates quasi-concavity. Noting that $\frac{d}{dx}x^3\sin(1/x) \to 0$ as $x \to 0$, voter 1's utility is continuously differentiable, and the gradient of u_1 is

$$\nabla u_1(x, y) = (3x^2 + 3x^2\sin(1/x) + x\cos(1/x), 1),$$

so evaluated at zero, it is $\nabla u_1(0, 0) = (0, 1)$. Thus, the origin is a directional equilibrium. Note, however, that the Hessian of u_1 is singular, so Theorem 12 cannot be applied.

Now suppose there exists $\epsilon > 0$ such that there is a Nash equilibrium of the ϵ-local game at $(0, 0)$ such that $\sum_{i=1}^{3} s_i = 0$. In equilibrium, each voter i solves the following maximization problem.

$$\begin{array}{c} \max_{(x,y)\in\mathbb{R}^2} u_i\left((\sum_{j:j\neq i} s_j^*) + (x, y)\right) \\ \text{s.t. } x^2 + y^2 \leq \epsilon^2. \end{array}$$

Clearly, the solution for voters 1 and 3 is non-zero, so the constraint qualification holds for both. For voter 1, for example, the necessary first order condition is that there exists $\lambda_1 \geq 0$ satisfying

$$\begin{aligned} \nabla u_1(0, 0) &= 2\lambda_1 s_1^* \\ \lambda_1(\epsilon - \|s_1^*\|^2) &= 0. \end{aligned}$$

Thus, we obtain that s_1^* points in the direction of $(0, 1)$ and has norm ϵ, i.e., $s_1^* = (0, \epsilon)$. Similarly, $s_3^* = (0, -\epsilon)$, and thus $s_2^* = (0, 0)$.

Note, however, that while s_1^* solves the necessary first order condition for voter 1, it is not a best response to $s_2^* + s_3^* = (0, -\epsilon)$. Using the functional form of the voter's utility and changing variables, voter 1 solves

$$\begin{array}{c} \max_{(x,y)\in\mathbb{R}^2} y + x^3 + x^3\sin(1/x) \\ \text{s.t. } \|(x, y) - (0, -\epsilon)\| \leq \epsilon, \end{array}$$

and s_1^* corresponds to a potential solution $(0, \epsilon)$. Thus, the outcome from following s_1^* is the origin, and voter 1's utility is zero. But consider the voter's utility from (x', y') with $x' = \alpha > 0$ and $y' = -\epsilon + \sqrt{\epsilon^2 - \alpha^2}$. This is the quantity

$$-\epsilon + \sqrt{\epsilon^2 - \alpha^2} + \alpha^3 + \alpha^3\sin(1/\alpha),$$

which is positive if

$$1 > \frac{\epsilon - \sqrt{\epsilon^2 - \alpha^2}}{\alpha^3 + \alpha^3\sin(1/\alpha)}.$$

Applying L'Hôpital's rule twice, the limit of the right-hand side of the above inequality for small $\alpha > 0$ is the limit of

$$\frac{(\epsilon^2-\alpha^2)^{-\frac{1}{2}}+\alpha^2(\epsilon^2-\alpha^2)^{-\frac{3}{2}}}{6\alpha+6\alpha\sin(1/\alpha)-2\cos(1/\alpha)+\frac{1}{\alpha}\sin(1/\alpha)}$$

as $\alpha \to 0$. Choosing the sequence $\{\alpha_k\}$ with $\alpha_k = 2/\pi k$, the limit of the right-hand side is therefore zero. We conclude that for any given $\epsilon > 0$, voter 1's utility from choosing (x', y') with $\alpha > 0$ appropriately small is positive, contradicting the assumption that s_1^* is a best response.

A Proof of Genericity Results

The proof follows the outline given in Section 5, taking an arbitrary utility vector function $u \in \mathcal{U}$ as given. Recall that $\Theta^i = \mathbb{R}^m$ for each voter i, and that $\Theta = \Theta^1 \times \cdots \times \Theta^n$ is the set of all possible vectors $\theta = (\theta^1, \ldots, \theta^n)$. For linear parameters $\theta \in \Theta$, we extend earlier notation in the obvious way: define the normalized gradient

$$p^i(x,\theta^i) \quad = \quad \begin{cases} \frac{1}{\|\nabla u_i(x)+\theta^i\|}(\nabla u_i(x)+\theta^i) & \text{if } \nabla u_i(x)+\theta \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

so that an alternative x is a directional equilibrium at $u + f_\theta$ if

$$\left\|\sum_{i=1}^{n} p^i(x,\theta^i)\right\| \quad \leq \quad \#\{i \in N : \nabla u_i(x)+\theta^i = 0\}.$$

Henceforth, we refer to such an alternative as a *directional equilibrium at θ,* and we denote the set of such equilibria by $C^*_{DE}(\theta)$. Steps 1–4 show that the set

$$\{\theta \in \Theta : \text{ every } x \in C^*_{DE}(\theta) \text{ is locally unique}\}$$

has full Lebesgue measure, from which we conclude that $\mathcal{U} \setminus \mathcal{U}_{LU}$ is finitely shy relative to $\mathcal{U}$, and Theorem 7 is proved. Theorems 8 and 9 are proved in Steps 5 and 6, respectively.

Step 1: We show that 0 is a regular value of f. Let $(x, \alpha, \theta) \in X \times \mathbb{R}^n_{++} \times \Theta^n$ be a solution to $f = 0$. Taking the derivative of f at (x, α, θ), we have the following block matrix:

$$Df(x,\alpha,\theta) \quad = \quad \left[\begin{array}{c|c|c} \mathcal{A} & \mathcal{B} & \mathcal{C} \\ \hline \mathcal{D} & \mathcal{E} & \mathcal{F} \end{array}\right],$$

where in particular,

$$\underset{m\times n}{\mathcal{B}} = \begin{bmatrix} D_1u_1(x)+\theta_1^1 & D_1u_2(x)+\theta_1^2 & \cdots & D_1u_n(x)+\theta_1^n \\ D_2u_1(x)+\theta_2^1 & D_2u_2(x)+\theta_2^2 & \cdots & D_2u_n(x)+\theta_2^n \\ \vdots & \vdots & \ddots & \vdots \\ D_mu_1(x)+\theta_m^1 & D_mu_2(x)+\theta_m^2 & \cdots & D_mu_n(x)+\theta_m^n \end{bmatrix}$$

$$\underset{m\times mn}{\mathcal{C}} = \begin{bmatrix} \alpha_1 I_m & \cdots & \alpha_n I_m \end{bmatrix} \text{ (where } I_m \text{ is the } m\times m \text{ identity matrix)}$$

$$\underset{n\times n}{\mathcal{E}} = \begin{bmatrix} \frac{2}{\alpha_1^3} & 0 & \cdots & 0 \\ 0 & \frac{2}{\alpha_2^3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2}{\alpha_n^3} \end{bmatrix}$$

$$\underset{n\times mn}{\mathcal{F}} = \begin{bmatrix} \underbrace{2(\nabla u_1(x)+\theta^1)^T}_{(1\times m)} & 0\cdots\cdots 0 & \cdots & 0\cdots\cdots 0 \\ 0\cdots\cdots 0 & \underbrace{2(\nabla u_2(x)+\theta^2)^T}_{(1\times m)} & \cdots & 0\cdots\cdots 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0\cdots\cdots 0 & 0\cdots\cdots 0 & \cdots & \underbrace{2(\nabla u_n(x)+\theta^n)^T}_{(1\times m)} \end{bmatrix}.$$

We show that the Jacobian $Df(x,\alpha,\theta)$ has full row rank, i.e., letting $Df_i(x,\alpha,\theta)$ denote the i^{th} row of the Jacobian, we consider arbitrary scalars $\gamma_1,\ldots,\gamma_{m+n}\in\mathbb{R}$ such that

$$\gamma_1 Df_1(x,\alpha,\theta)+\cdots+\gamma_{m+n}Df_{m+n}(x,\alpha,\theta) = 0,$$

and we deduce

$$\gamma_1 = \cdots = \gamma_{m+n} = 0.$$

To derive a contradiction, suppose that the scalars $\gamma_1,\ldots,\gamma_{m+n}$ are not all zero.

Claim 1: At least one of $\gamma_1,\gamma_2,\ldots,\gamma_m$ is non-zero.

Suppose $\gamma_1=\gamma_2=\cdots=\gamma_m=0$, implying $\gamma_{m+k}\neq 0$ for some $k=1,\ldots,n$. Consider the set of columns of $Df(x,\alpha,\theta)$ from the $(m+n+m(k-1)+1)^{th}$ column to the $(m+n+mk)^{th}$ column, corresponding to the derivative with respect to the coordinates of θ^k. Since $\gamma_1 = \gamma_2 = \cdots = \gamma_m = 0$, the weighted sum of entries of these columns is

$$2\gamma_{m+k}(\nabla u_k(x) + \theta^k) \;=\; 0,$$

contradicting $\|\nabla u_k(x) + \theta^k\| = \frac{1}{\alpha_k} > 0$ and establishing the claim. □

Claim 2: For all $i = 1, \ldots, n$, $\gamma_{m+i} \neq 0$.

From Claim 1, we have $\gamma_j \neq 0$ for some $j = 1, \ldots, m$. Consider the columns $m + n + (k-1)m + j$, for $k = 1, \ldots, n$, corresponding to the partial derivative with respect to θ^k_j for each individual. Taking the weighted sum of entries of these columns, we have

$$\begin{aligned} \gamma_j\alpha_1 + 2\gamma_{m+1}(D_j u_1(x) + \theta^1_j) &= 0 \\ &\vdots \\ \gamma_j\alpha_n + 2\gamma_{m+n}(D_j u_n(x) + \theta^n_j) &= 0. \end{aligned}$$

Since $\gamma_j > 0$, and using $\alpha_i > 0$ for all $i = 1, \ldots, n$, this implies that each γ_{m+i} is non-zero, as claimed. □

Claim 3: Letting $\gamma = (\gamma_1, \ldots, \gamma_m)$, we have for all $i = 1, \ldots, n$,

$$\nabla u_i(x) + \theta^i \;=\; -\frac{\alpha_i}{2\gamma_{m+i}}\gamma.$$

Given any $i = 1, \ldots, n$, consider the $(m + n + m(i-1) + 1)^{th}$ column to the $(m+n+mi)^{th}$ column, corresponding to the derivative with respect to θ^i. Taking the weighted sum of entries of these columns, we have

$$\begin{aligned} \gamma_1\alpha_i + 2\gamma_{m+i}(D_1 u_i(x) + \theta^i_1) &= 0 \\ &\vdots \\ \gamma_m\alpha_i + 2\gamma_{m+i}(D_m u_i(x) + \theta^i_m) &= 0. \end{aligned}$$

Rearranging, and using Claim 2, we have the claim. □

Claim 4: For all $i = 1, \ldots, n$,

$$\frac{\alpha_i^2}{|\gamma_{m+i}|} \;=\; \frac{2}{\|\gamma\|}.$$

Given any $i = 1, \ldots, n$, the weighted sum of entries in column $m + i$ is

$$\gamma \cdot (\nabla u_i(x) + \theta^i) + \gamma_{m+i} \frac{2}{\alpha_i^3} \quad = \quad 0.$$

By Claim 3, this is equivalent to

$$-\frac{\alpha_i}{2\gamma_{m+i}} \gamma \cdot \gamma + \gamma_{m+i} \frac{2}{\alpha_i^3} \quad = \quad 0.$$

By Claim 1, $\gamma \neq 0$. Thus, rearranging and taking square roots, we obtain the desired expression. □

To deduce a final contradiction, note that since $f(x, \alpha, \theta) = 0$, we have

$$\sum_{i=1}^{n} \alpha_i (\nabla u_i(x) + \theta^i) \quad = \quad 0. \tag{8}$$

By Claim 3, this is

$$\sum_{i=1}^{n} \frac{\alpha_i^2}{2\gamma_{m+i}} \gamma \quad = \quad 0,$$

and using Claim 4, we have

$$0 \quad = \quad \sum_{i=1}^{n} \frac{\text{sgn}(\gamma_{m+i})}{\|\gamma\|} \gamma \quad = \quad \frac{1}{\|\gamma\|} \left(\sum_{i=1}^{n} \text{sgn}(\gamma_{m+i}) \right) \gamma.$$

By Claim 2 and the assumption that n is odd, this implies $\gamma = 0$, contradicting Claim 1, and we conclude that 0 is a regular value of f. By the transversality theorem, there is a measure zero set $\tilde{\Theta}^1 \subseteq \mathbb{R}^{nm}$ such that for all $\theta \in \Theta \setminus \tilde{\Theta}^1$, 0 is a regular value of $f(\cdot, \theta)$, i.e., for all (x, α) such that $f(x, \alpha, \theta) = 0$, the derivative $D_{(x,\alpha)} f(x, \alpha, \theta)$ with respect to (x, α) has full row rank.

Step 2: Given any $i \in N$, it is well-known that there is a measure zero set $\tilde{\Theta}^{2,i} \subseteq \mathbb{R}^m$ such that for all $\theta^i \in \Theta^i \setminus \tilde{\Theta}^{2,i}$, the function $x \mapsto u_i(x) + \theta^i \cdot x$ is a Morse function (see Guilllemin and Pollack, p.53), i.e., for all $x \in X$ such that $\nabla u_i(x) + \theta^i = 0$, the Hessian matrix $D^2 u_i(x)$ is non-singular. Next, we show, generically, no two voters share a critical point. For all $i, j \in N$, define $g^{i,j} \colon X \times \Theta^i \times \Theta^j \to \mathbb{R}^{2m}$ by

$$g^{i,j}(x, \theta^i, \theta^j) \quad = \quad \begin{bmatrix} D_1 u_i(x) + \theta_1^i \\ \vdots \\ D_m u_i(x) + \theta_m^i \\ D_1 u_j(x) + \theta_1^j \\ \vdots \\ D_m u_j(x) + \theta_m^j \end{bmatrix}.$$

Note that $x \in X$ is a critical point for i and j if and only if $g^{i,j}(x, \theta^i, \theta^j) = 0$. To show that 0 is a regular value of $g^{i,j}$, we assume $g^{i,j}(x, \theta^i, \theta^j) = 0$ and take the derivative of $g^{i,j}$ at (x, θ^i, θ^j) to obtain the following block matrix

$$Dg^{i,j}(x, \theta^i, \theta^j) \quad = \quad \left[\begin{array}{c|c|c} \mathcal{A} & I_m & 0 \\ \hline \mathcal{B} & 0 & I_m \end{array}\right],$$

which clearly has full row rank. The transversality theorem then yields a measure zero set $\tilde{\Theta}^{2,i,j} \subseteq \mathbb{R}^{2m}$ that for all $(\theta^i, \theta^j) \in (\Theta^i \times \Theta^j) \setminus \tilde{\Theta}^{2,i,j}$, the set of solutions to $g^{i,j}(\cdot, \theta^i, \theta^j) = 0$ is a manifold of dimension $m - 2m = -m < 0$, i.e., it is empty. Letting Θ^{-i} be the projection of Θ onto the parameters of voters other than i, and letting $\Theta^{-i,j}$ be the projection onto parameters of voters other than i and j, the set

$$\tilde{\Theta}^2 \quad = \quad \left(\bigcup_i \tilde{\Theta}^{2,i} \times \Theta^{-i}\right) \cup \left(\bigcup_{i,j:i \neq j} \tilde{\Theta}^{2,i,j} \times \Theta^{-i,j}\right)$$

is a measure zero subset of Θ that fulfills the requirements of Step 2.

Step 3: We show that for almost all θ, there does not exist a critical directional equilibrium at which the gradient of exactly one voter is zero and the sum of normalized gradients itself has norm one. Define $h^n \colon X \times \mathbb{R}^{n-1}_{++} \times \mathbb{R}^m \times \Theta^n \to \mathbb{R}^{2m+n}$ as:

$$h^n(x, \alpha, \beta, \theta) \quad = \quad \left[\begin{array}{c} \sum_{i=1}^{n-1} \alpha_i (D_1 u_i(x) + \theta^i_1) - \beta_1 \\ \vdots \\ \sum_{i=1}^{n-1} \alpha_i (D_m u_i(x) + \theta^i_m) - \beta_m \\ \|\nabla u_1(x) + \theta^1\|^2 - \frac{1}{\alpha_1^2} \\ \vdots \\ \|\nabla u_{n-1}(x) + \theta^{n-1}\|^2 - \frac{1}{\alpha_{n-1}^2} \\ D_1 u_n(x) + \theta^n_1 \\ \vdots \\ D_m u_n(x) + \theta^n_m \\ \|\beta\|^2 - 1 \end{array}\right]. \qquad (9)$$

Clearly, given θ, a triple $(x, \alpha, \beta) \in X \times \mathbb{R}^{n-1}_{++} \times \mathbb{R}^m$ satisfies $h^n(x, \alpha, \beta, \theta) = 0$ if and only if x is a critical directional equilibrium at θ such that the gradient of voter n equals zero, and the sum of normalized gradients of the other voters has norm one. We show that 0 is a regular value of h^n, i.e., the Jacobian $Dh^n(x, \alpha, \beta, \theta)$ at every solution to $h^n = 0$ has full row rank.

Let $(x, \alpha, \beta, \theta) \in X \times \mathbb{R}^{n-1}_{++} \times \mathbb{R}^m \times \Theta^n$ be a solution to $h^n = 0$. Taking the derivative of h^n with at $(x, \alpha, \beta, \theta)$, we have the following block matrix:

$$Dh^n(x, \alpha, \beta, \theta) = \left[\begin{array}{c|c|c|c|c} \mathcal{A} & \mathcal{B} & -I_m & \mathcal{C} & 0 \\ \hline \mathcal{D} & \mathcal{E} & 0 & \mathcal{F} & 0 \\ \hline \mathcal{G} & 0 & 0 & 0 & I_m \\ \hline 0 & 0 & \mathcal{H} & 0 & 0 \end{array}\right],$$

where in particular,

$$\underset{m\times(n-1)}{\mathcal{B}} = \begin{bmatrix} D_1u_1(x)+\theta_1^1 & D_1u_2(x)+\theta_1^2 & \cdots & D_1u_{n-1}(x)+\theta_1^{n-1} \\ D_2u_1(x)+\theta_2^1 & D_2u_2(x)+\theta_2^2 & \cdots & D_2u_{n-1}(x)+\theta_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ D_mu_1(x)+\theta_m^1 & D_mu_2(x)+\theta_m^2 & \cdots & D_mu_{n-1}(x)+\theta_m^{n-1} \end{bmatrix}$$

$$\underset{m\times m(n-1)}{\mathcal{C}} = \begin{bmatrix} \alpha_1 I_m & \cdots & \alpha_{n-1} I_m \end{bmatrix}$$

$$\underset{(n-1)\times(n-1)}{\mathcal{E}} = \begin{bmatrix} \frac{2}{\alpha_1^3} & 0 & \cdots & 0 \\ 0 & \frac{2}{\alpha_2^3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2}{\alpha_{n-1}^3} \end{bmatrix}$$

$$\underset{(n-1)\times m(n-1)}{\mathcal{F}} = \begin{bmatrix} \underbrace{2(\nabla u_1(x)+\theta^1)^T}_{1\times m} & 0\cdots\cdots 0 & \cdots & 0\cdots\cdots 0 \\ 0\cdots\cdots 0 & \underbrace{2(\nabla u_2(x)+\theta^2)^T}_{1\times m} & \cdots & 0\cdots\cdots 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0\cdots\cdots 0 & 0\cdots\cdots 0 & \cdots & \underbrace{2(\nabla u_{n-1}(x)+\theta^{n-1})^T}_{1\times m} \end{bmatrix}$$

$$\underset{1\times m}{\mathcal{H}} = \begin{bmatrix} 2\beta_1 \cdots 2\beta_m \end{bmatrix}.$$

We show that $Dh^n(x, \alpha, \beta, \theta)$ has full row rank. Letting $Dh^n_i(x, \alpha, \theta)$ denote the ith row of the Jacobian, consider arbitrary scalars $\gamma_1, \ldots, \gamma_{2m+n} \in \mathbb{R}$ such that

$$\gamma_1 Dh_1^n(x,\alpha,\beta,\theta) + \cdots + \gamma_{2m+n} Dh_{2m+n}^n(x,\alpha,\beta,\theta) \;=\; 0.$$

To deduce a contradiction, suppose that it is not the case that $\gamma_1 = \cdots = \gamma_{2m+n} = 0$. Note that submatrices $\mathcal{B}$, $\mathcal{C}$, $\mathcal{E}$, and $\mathcal{F}$ correspond exactly to the matrices in Step 1 for the set $N' = \{1, \ldots, n-1\}$ of voters.

Claim 1: At least one of $\gamma_1, \gamma_2, \ldots, \gamma_m$ is non-zero.

Suppose $\gamma_1 = \gamma_2 = \cdots = \gamma_m = 0$. Since $\|\beta\| = 1$, there is some $k = 1, \ldots, m$ such that $\beta_k \neq 0$. The sum of entries in column $m+(n-1)+k$ is then $\gamma_{2m+n} 2\beta_k = 0$, which implies $\gamma_{2m+n} = 0$. Furthermore, for all $k = 1, \ldots, m$, the weighted sum of entries in column $m + (n-1) + m + m(n-1) + k$ is then $\gamma_{m+n-1+k} = 0$. We conclude that $\gamma_{m+i} \neq 0$ for some $i = 1, \ldots, n-1$. Consider the set of columns of $Dh^n(x,\alpha,\theta)$ from the $(m + (n-1) + m + m(i-1) + 1)^{th}$ column to the $(m + (n-1) + m + mi)^{th}$ column, corresponding to the derivative with respect to the coordinates of θ^i. Since $\gamma_1 = \gamma_2 = \cdots = \gamma_m = 0$, the weighted sum of entries of these columns is

$$2\gamma_{m+i}(\nabla u_i(x) + \theta^i) \;=\; 0,$$

contradicting $\|\nabla u_i(x) + \theta^i\| = \frac{1}{\alpha_i} > 0$ and establishing the claim. $\square$

Claim 2: For all $i = 1, \ldots, n-1$, $\gamma_{m+i} \neq 0$.

Note that all entries below $\mathcal{B}$ and $\mathcal{E}$ are zero, and all entries below $\mathcal{C}$ and $\mathcal{F}$ are zero. Since Claim 2 in Step 1 does not rely on the assumption that n is odd, the claim holds here by the same argument. $\square$

Claim 3: For all $k = 1, \ldots, m$, $\gamma_k = 2\gamma_{2m+n}\beta_k$.

For all $k = 1, \ldots, m$, the weighted sum of entries in column $m + (n-1) + k$ is $-\gamma_k + 2\gamma_{2m+n}\beta_k = 0$, which implies $\gamma_k = 2\gamma_{2m+n}\beta_k$. $\square$

Claim 4: For all $i = 1, \ldots, n-1$,

$$\nabla u_i(x) + \theta^i \;=\; \left(-\frac{\gamma_{2m+n}\alpha_i\beta_1}{\gamma_{m+i}}, \ldots, -\frac{\gamma_{2m+n}\alpha_i\beta_m}{\gamma_{m+i}}\right).$$

Given coordinate $k = 1, \ldots, m$, the weighted sum of entries in column $m + (n-1) + m + m(i-1) + k$ is

$$\gamma_k\alpha_i + 2\gamma_{m+i}(D_k u_i(x) + \theta^i_k) \;=\; 0.$$

Substituting $\gamma_k = 2\gamma_{2m+n}\beta_k$ from Claim 3, and using $\gamma_{m+i} \neq 0$ from Claim 2, the claim follows. □

Claim 5: For all $i = 1, \ldots, n-1$,

$$\alpha_i^2 \;=\; \frac{|\gamma_{m+i}|}{|\gamma_{2m+n}|}.$$

Given any $i = 1, \ldots, n-1$, we combine $\frac{1}{\alpha_i} = \|\nabla u_i(x) + \theta^i\|$ with Claim 4 to obtain

$$\alpha_i \;=\; \frac{1}{\alpha_i |\gamma_{2m+n}/\gamma_{m+i}| \; \|\beta\|}.$$

Substituting $\|\beta\| = 1$ and solving for α_i, we have the claim. □

To deduce a final contradiction, note that since $h^n(x, \alpha, \beta, \theta) = 0$, we have

$$\left\| \sum_{i=1}^{n-1} \alpha_i(\nabla u_i(x) + \theta^i) \right\| \;=\; 1. \tag{10}$$

By Claim 4, this is

$$|\gamma_{2m+n}| \left| \sum_{i=1}^{n-1} \frac{\alpha_i^2}{\gamma_{m+i}} \right| \|\beta\| \;=\; 1. \tag{11}$$

By Claim 5, and the fact that $\|\beta\| = 1$, this is equivalent to

$$\left| \sum_{i=1}^{n-1} \operatorname{sgn}(\gamma_{m+i}) \right| \;=\; 1,$$

which is impossible by Claim 2 and the assumption that $n-1$ is even, and we conclude that 0 is a regular value of h^n. By the transversality theorem, we conclude that for all θ outside a measure zero set $\tilde{\Theta}^{3,n}$, 0 is a regular value of $h^n(\cdot, \theta)$, i.e., for all (x, α, β) such that $h^n(x, \alpha, \beta, \theta) = 0$, the derivative $D_{(x,\alpha,\beta)} h^n(x, \alpha, \beta, \theta)$ with respect to (x, α, β) has full row rank.

The solutions to $h^n(\cdot, \theta) = 0$ characterize the tight directional equilibria at θ such that the gradient of just voter n equals zero, so we index the above exceptional set by n. The same argument can be applied for any pre-specified voter i using a mapping h^i, giving us a measure zero set $\tilde{\Theta}^{3,i}$ for each voter. Then $\tilde{\Theta}^3 = \bigcup_i \tilde{\Theta}^{3,i}$ is a measure zero subset such that for all $\theta \in \Theta \setminus \tilde{\Theta}^3$ and for all $i \in N$, 0 is a regular value of $h^i(\cdot, \theta)$, i.e., for all (x, α, β) such that $h^i(x, \alpha, \beta, \theta) = 0$, the derivative $D_{(x,\alpha,\beta)}h^i(x, \alpha, \beta, \theta)$ with respect to (x, α, β) has full row rank. This implies that the set of solutions to $h^i(\cdot, \theta) = 0$ is a manifold of dimension $m + n - 1 + m - (2m + n) = -1$, i.e., the set of solutions is empty. Therefore, we conclude that for all $\theta \in \Theta \setminus \tilde{\Theta}^3$, there do not exist tight, critical directional equilibria such that the gradient of exactly one voter equals zero.

Step 4: The set $\tilde{\Theta} = \tilde{\Theta}^1 \cup \tilde{\Theta}^2 \cup \tilde{\Theta}^3$ is a measure zero subset of Θ. For all $\theta \notin \tilde{\Theta}$, Step 1 shows that 0 is a regular value of $f(\cdot, \theta)$, so the set of solutions to $f(\cdot, \theta) = 0$ is a manifold of dimension $m + n - m - n = 0$. Thus, every element of $\{(x, \alpha) \in X \times \mathbb{R}^n_{++} \mid f(x, \alpha, \theta) = 0\}$ is locally isolated. In turn, we claim that the non-critical directional equilibria at θ are locally unique among non-critical equilibria, i.e., for every such equilibrium x, there exists $\epsilon > 0$ such that no other non-critical directional equilibria at θ belong to the ball of radius $\epsilon > 0$ around x. Indeed, consider any such equilibrium x and suppose toward a contradiction that there is a sequence $\{x^k\}$ of non-critical equilibria converging to x. For each k, set $\alpha^k = (\alpha^k_1, \dots, \alpha^k_n)$, where $\alpha^k_i = \|\nabla u_i(x) + \theta^i\|^{-1}$. Then we have $f(x^k, \alpha^k, \theta) = 0$ for all k, and we have have $(x^k, \alpha^k) \to (x, \alpha)$, a contradiction. We conclude that the non-critical directional equilibria at θ are locally unique among non-critical equilibria, as claimed.

Next, by Step 2, because $u_i(x) + \theta^i \cdot x$ is Morse, the critical directional equilibria at which a given voter i has zero gradient are locally isolated. Furthermore, it cannot be that a sequence of such equilibria converge to a critical directional equilibrium x at which voter $j \neq i$ has zero gradient, for then continuous differentiability of u_i implies that $\nabla u_i(x) + \theta^i = \nabla u_j(x) + \theta^j = 0$, which is precluded by $\theta \notin \tilde{\Theta}^2$. Thus, the critical directional equilibria at θ are locally unique among critical equilibria, i.e., for every such equilibrium x, there exists $\epsilon > 0$ such that no other critical directional equilibria at θ belong to the ball of radius $\epsilon > 0$ around x.

We have left to argue that directional equilibria of one type cannot accumulate around directional equilibria of the other type. First, note that by continuous differentiability, there do not exist a sequence $\{x^m\}$ of critical directional equilibria at θ converging to a non-critical directional equilibrium: if there were such a sequence, we could move to a subsequence such that for some voter i, $\nabla u_i(x^m) + \theta^i = 0$ for all m, which implies $\nabla u_i(x) + \theta^i = 0$, a contradiction. Next, suppose there exist a critical directional equilibrium at θ, say x, and a sequence $\{x^m\}$ of non-critical directional equilibria at θ such that $x^m \to x$. Since x is critical, there is a voter k such that $\nabla u_k(x) + \theta^k = 0$, and since $\theta \notin \tilde{\Theta}^2$, we have $\nabla u_i(x) + \theta^i \neq 0$ for all $i \neq k$. To ease notation, we henceforth assume without loss of generality that $k = n$.

Note that since $\theta \notin \tilde{\Theta}^3$, Step 3 implies that the set of solutions to $h^n(\cdot,\theta) = 0$ is empty. Since each x^m is non-critical, we have

$$\sum_{i=1}^{n} p^i(x^m, \theta^i) \;\; = \;\; 0,$$

or equivalently,

$$\sum_{i:i<n} p^i(x^m, \theta^i) \;\; = \;\; -p^n(x^m, \theta^n)$$

for all m. This implies that for all m,

$$\left\| \sum_{i:i<n} p^i(x^m, \theta^i) \right\| \;\; = \;\; 1,$$

and since each u_i is continuously differentiable, we then have

$$\left\| \sum_{i:i<n} p^i(x, \theta^i) \right\| \;\; = \;\; 1.$$

But this implies that x is a tight directional equilibria at θ such that the gradient of just voter n equals zero, or equivalently,

$$h^n\left(x, \frac{1}{\|\nabla u_1(x) + \theta^1\|}, \ldots, \frac{1}{\|\nabla u_{n-1}(x) + \theta^{n-1}\|}, \sum_{i:i<n} p^i(x, \theta^i), \theta\right) \;\; = \;\; 0,$$

a contradiction.

Therefore, for all $\theta \in \Theta \setminus \tilde{\Theta}$, every directional equilibrium at θ is locally unique, and the proof of Theorem 7 is complete.

Step 5: We argue that for a given utility vector function $u \in \mathcal{U}$, the set

$$\{\theta \in \Theta : \text{ every } x \in C^*_{DE}(\theta) \text{ is stable}\}$$

has full Lebesgue measure. We again set $\tilde{\Theta} = \tilde{\Theta}^1 \cup \tilde{\Theta}^2 \cup \tilde{\Theta}^3$, a measure zero subset of Θ generated by Steps 1–4. Consider any $\theta \in \Theta \setminus \tilde{\Theta}$, any parameterization $U\colon \Pi \to \mathcal{U}$, and any $\pi^* \in \Pi$ such that $U(\pi^*) = u + f_\theta$. The key mapping is now $f\colon X \times \mathbb{R}^n_{++} \times \Pi \to \mathbb{R}^{m+n}$, defined by:

$$f(x, \alpha, \pi) \;\; = \;\; \begin{bmatrix} \sum_{i=1}^{n-1} \alpha_i D_1 U_i(x,\pi) \\ \vdots \\ \sum_{i=1}^{n-1} \alpha_i D_m U_i(x,\pi) \\ \|\nabla_x U_1(x,\pi)\|^2 - \frac{1}{\alpha_1^2} \\ \vdots \\ \|\nabla_x U_n(x,\pi)\|^2 - \frac{1}{\alpha_n^2} \end{bmatrix}. \qquad (12)$$

In particular, an alternative x is a non-critical directional equilibrium at $U(\pi)$ if and only if $f(x,\alpha,\pi)=0$, where α is the vector of inverses of norms of gradients. Furthermore, when $\pi=\pi^*$, the values of f have the form

$$f(x,\alpha,\pi^*) \quad = \quad \left[\begin{array}{c} \sum_{i=1}^{n-1}\alpha_i(D_1u_i(x)+\theta_1^i) \\ \vdots \\ \sum_{i=1}^{n-1}\alpha_i(D_mu_i(x)+\theta_m^i) \\ \|\nabla_x u_1(x)+\theta^1\|^2-\frac{1}{\alpha_1^2} \\ \vdots \\ \|\nabla_x u_n(x)+\theta^n\|^2-\frac{1}{\alpha_n^2} \end{array}\right], \tag{13}$$

and since $\theta\notin\tilde{\Theta}^1$, Step 1 implies that 0 is a regular value of $f(\cdot,\pi^*)$.

Now, consider any non-critical directional equilibrium x^* at $U(\pi^*)=u+f_\theta$, so that $f(x^*,\alpha^*,\pi^*)=0$, where α^* is the vector of inverses of norms of gradients. By a version of the implicit function theorem (see Theorem 9.3, p.230, of Loomis and Sternberg (1968)), it follows that there exist an open set $\tilde{\Gamma}\subseteq\Pi$ with $\pi^*\in\tilde{\Gamma}$, an open set $\tilde{G}\subseteq\mathbb{R}^m$ with $x^*\in\tilde{G}$, an open set $V\subseteq\mathbb{R}^n$ with $\alpha^*\in V$, and continuous mappings $F\colon\tilde{\Gamma}\to\tilde{G}$ and $H\colon\tilde{\Gamma}\to V$ such that for all $\pi\in\tilde{\Gamma}$ and all $(x,\alpha)\in\tilde{G}\times V$, we have $f(x,\alpha,\pi)=0$ if and only if $x=F(\pi)$ and $\alpha=H(\pi)$. Since utility functions are continuously differentiable, we can furthermore take subsets $\hat{\Gamma}\subseteq\tilde{\Gamma}$ with $\pi^*\in\hat{\Gamma}$ and $G\subseteq\tilde{G}$ with $x^*\in G$ such that for all $(\pi,x)\in\hat{\Gamma}\times G$, we have

$$\left(\frac{1}{\|\nabla_x U_1(x,\pi)\|},\ldots,\frac{1}{\|\nabla_x U_n(x,\pi)\|}\right)\in V.$$

We set $\Gamma=\hat{\Gamma}\cap F^{-1}(G)$, and we conclude that for all $\pi\in\Gamma$ and all $x\in G$, x is a non-critical directional equilibrium at $U(\pi)$ if and only if $x=F(\pi)$. This construction allows for the possibility that there exist $\pi\in\Gamma$ and a critical directional equilibrium x at $U(\pi)$ with $x\in G$. But we claim there does not exist a sequence $\{\pi^k\}$ in Π converging to π^* and a sequence $\{x^k\}$ such that x^k is a critical directional equilibrium at $U(\pi^k)$ for all k and such that $x^k\to x^*$. Indeed, because voter utilities are continuously differentiable, this would imply that x^* is critical, a contradiction. Thus, we can specify open subsets $\Gamma'\subseteq\Gamma$ with $\pi^*\in\Gamma'$ and $G'\subseteq G$ with $x^*\in G'$ such that for all $\pi\in\Gamma'$, there are no critical directional equilibria at $U(\pi)$ in G'. Letting $\Gamma''=\Gamma'\cap F^{-1}(G')$, the restriction $F\colon\Gamma''\to G'$ fulfills the definition of stability. We conclude that all non-critical directional equilibria at $U(\pi)=u+f_\theta$ are stable.

Next, consider a critical directional equilibrium, say x^*, at $U(\pi^*)=u+f_\theta$. By Step 2, there is exactly one voter i such that $\nabla_x U_i(x^*,\pi^*)=\nabla_x u_i(x^*)+\theta^i=0$, and without loss of generality, we assume $i = n$. Let

$$p^i(x,\pi) \;=\; \begin{cases} \frac{1}{\|\nabla_x U_i(x,\pi)\|}\nabla_x U_i(x,\pi) & \text{if } \nabla_x U_i(x,\pi) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Step 3, x^* is slack, so we have

$$\left\|\sum_{i=1}^{n-1} p^i(x^*,\pi^*)\right\| \;=\; \left\|\sum_{i=1}^{n-1} p^i(x^*,\theta^i)\right\| \;<\; 1.$$

By continuity of voter gradients, there is an open set $\tilde{\Gamma} \subseteq \Pi$ with $\pi^* \in \tilde{\Gamma}$ and an open set $\tilde{G} \subseteq \mathbb{R}^m$ with $x^* \in \tilde{G}$ such that and for all $\pi \in \tilde{\Gamma}$ and all $x \in \tilde{G}$, the strict inequality continues to hold, i.e.,

$$\left\|\sum_{i=1}^{n-1} p^i(x,\pi)\right\| \;<\; 1,$$

and gradients of all voters other than n are non-zero, i.e., for all $i < n$, we have $\nabla_x U_i(x,\pi) \neq 0$. Also by Step 2, the Hessian $D_x^2 U_n(x^*,\pi^*)$ is negative definite, so the implicit function theorem yields an open set $\hat{\Gamma} \subseteq \Pi$ with $\pi^* \in \hat{\Gamma}$, an open set $\hat{G} \subseteq \mathbb{R}^m$ with $x^* \in \hat{G}$, and a continuous mapping $F\colon \hat{\Gamma} \to \hat{G}$ such that for all $\pi \in \hat{\Gamma}$ and all $x \in \hat{G}$, we have $\nabla_x U_n(x,\pi) = 0$ if and only if $x = F(\pi)$. Setting $\Gamma = \tilde{\Gamma} \cap \hat{\Gamma}$ and $G = \tilde{G} \cap \hat{G}$, it follows that for all $\pi \in \Gamma$, $F(\pi)$ is the unique critical directional equilibrium at $U(\pi)$ in G. This allows for the possibility that there exist $\pi \in \Gamma$ and a non-critical directional equilibrium x at $U(\pi)$ with $x \in G$. But we claim that there cannot exist a sequence $\{\pi^k\}$ in Π converging to π^* and a sequence $\{x^k\}$ such that x^k is a non-critical directional equilibrium at $U(\pi^k)$ for all k and such that $x^k \to x^*$. As in Step 4, it follows that

$$\left\|\sum_{i=1}^{n-1} p^i(x^*,\pi^*)\right\| \;=\; 1,$$

contradicting the assumption that x^* is slack. Thus, we can specify open subsets $\Gamma' \subseteq \Gamma$ with $\pi^* \in \Gamma'$ and $G' \subseteq G$ with $x^* \in G'$ such that for all $\pi \in \Gamma'$, there are no non-critical directional equilibria at $U(\pi)$ in G'. Letting $\Gamma'' = \Gamma' \cap F^{-1}(G')$, the restriction $F\colon \Gamma'' \to G'$ fulfills the definition of stability. We conclude that all critical directional equilibria at $U(\pi^*) = u + f_\theta$ are stable. This completes the proof of Theorem 8.

Step 6: Adapting Step 5, we argue that for a given utility vector function $u \in \mathcal{U}$, the set

$$\{\theta \in \Theta : \text{ every } x \in C^*_{DE}(\theta) \text{ is smoothly stable}\}$$

has full Lebesgue measure. We again set $\tilde{\Theta} = \tilde{\Theta}^1 \cup \tilde{\Theta}^2 \cup \tilde{\Theta}^3$, a measure zero subset of Θ generated by Steps 1–4. To prove Theorem 9, we consider any $\theta \in \Theta \setminus \tilde{\Theta}$, any smooth parameterization $U: \Pi \to \mathcal{U}$, and any $\pi^* \in \Pi$ such that $U(\pi^*) = u + f_\theta$. By Step 5, every directional equilibrium x at $U(\pi^*) = u + f_\theta$ is stable, so there exist open $\Gamma \subseteq \Pi$ with $\pi^* \in \Gamma$, open $G \subseteq \mathbb{R}^m$ with $x^* \in G$, and a continuous mapping $F: \Gamma \to G$ such that for all $\pi \in \Gamma$ and all $x \in G$, x is a directional equilibrium at $U(\pi)$ if and only if $x = F(\pi)$. We first consider the case that x^* is non-critical, so that $f(x^*, \alpha^*, \pi^*) = 0$, where α^* is the vector of inverses of norms of gradients. Since $f(x, \alpha, \pi)$ is continuously differentiable in (x, α, π) and 0 is a regular value at $f(\cdot, \pi^*)$, the implicit function theorem (see Theorem 11.2, p.167, of Loomis and Sternberg (1968)) implies that there are an open set $\Gamma' \subseteq \Gamma$ with $\pi^* \in \Gamma'$ and an open set $G' \subseteq G$ with $x^* \in G'$ such that the restriction $F: \Gamma' \to G'$ is continuously differentiable. Thus, x^* is smoothly stable. In case x^* is a critical equilibrium, it is slack, and there is exactly one voter, say n, whose gradient is zero at x^*. Note that for all $\pi \in \Gamma$, we have $\nabla_x U_n(F(\pi), \pi) = 0$. Since the Hessian $D^2 U_n(x^*, \pi^*)$ is non-singular, the implicit function theorem implies that there are an open set $\Gamma' \subseteq \Gamma$ with $\pi^* \in \Gamma'$ and an open set $G' \subseteq G$ with $x^* \in G$ such that the restriction $F: \Gamma' \to G'$ is continuously differentiable. Thus, x^* is smoothly stable. This completes the proof of Theorem 9.

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