Lobbying and Policy Extremism in Repeated Elections

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Abstract

We study a citizen-candidate model of repeated elections that includes ideologically extreme lobby groups. Existence of a class of perfect Bayesian equilibria is established. In equilibrium, politicians are lobbied to more extreme policy choices than they would otherwise make. We show that when office incentives are high, all equilibria are polarized: liberal politicians all choose the same policy, as do all conservative politicians. When the effectiveness of money approaches zero, these equilibrium policies converge to the median, providing a dynamic version of the median voter theorem. When the effectiveness of money becomes large, however, we show that the most polarized equilibria become arbitrarily extremist. Thus, the influence of effective lobbying creates the possibility of arbitrarily extreme policy outcomes.

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1 Introduction

From 1999 to 2014, total lobbying spending in the United States increased from \$1.44 billion to \$3.24 billion.[1] Evidently, interest groups that hire lobbyists expect their investment to yield policy dividends, and lobbying is thus undoubtedly an important aspect of the political process. In the US, this importance has only increased in light of recent shifts in patterns of political contributions: for example, Citizens United v. FEC has led to weaker restrictions on contributions and to the rise of Super PACs and "dark money" organizations. A correspondingly large literature studies the impact of lobbying on policy outcomes (see, e.g., Grossman and Helpman (1994), De Figueiredo and Silverman (2006), Baumgartner et al. (2009), and Kang (2012)). Elections enable voters to remove incumbent politicians who enact unpopular policy, and politicians must balance their policy preferences against electoral considerations and pressure from private interests. These considerations indirectly influence interest groups, which account for electoral consequences when they choose which policies to advocate. These interactions create highly complex, dynamic incentives and, consequently, difficult analytical challenges.

This paper provides a formal analysis of elections over time that allows us to trace the causal mechanisms at work, which are, in turn, crucial for understanding the role and effects of money—and special interest lobbying, in particular—in electoral politics. We assume that elections are held over an infinite time horizon, and that in each period, a lobby group makes an offer to an incumbent politician; the incumbent either accepts this offer and chooses the proposed policy, or she rejects the offer and chooses policy independently; and an election between the incumbent and a challenger is held. Voters observe the policy choice of the incumbent but not the type (i.e., policy preferences) of either politician. We analyze stationary perfect Bayesian equilibria of the electoral game in which the incumbent always chooses the same policy if re-elected, and thus voters face the choice between a known incumbent and a relatively unknown challenger. The incumbent politician may face a trade off between the short term gains from choosing her ideal policy and the long term gains of compromising her choice, choosing a more moderate policy in order to gain re-election. A lobby group, anticipating voter choices and politician incentives, can make an offer to the incumbent politician that consists of a desired policy and a transfer to the politician; we model this transfer as a monetary payment from the lobby group to the politician, but it may more generally represent resources that are desirable to the politician, such as (unmodeled) campaign contributions or promises of future "revolving door" opportunities.[2]

In equilibrium, these strategic incentives determine a centrally located "win set," which consists of policy choices that are sufficient for re-election: if the incumbent chooses a policy in the win set, then she is re-elected, and otherwise, she is replaced by the challenger. Given the win set, in the absence of an offer from a lobby group, the incumbent optimally chooses a winning policy if she is moderate, while more extreme politicians may choose their ideal points, foregoing re-election. This policy choice in lieu of a lobby offer is a "default" policy that serves as a reversion point in the

[1] http://www.opensecrets.org/lobby/

[2] See Grossman and Helpman (2002) for a discussion of this issue.

lobbying stage. Anticipating the default policy choice, one of two lobby groups makes an offer to the incumbent that consists of a proposed policy and a monetary payment to the politician. We assume that a liberal politician is lobbied by the left interest group, while conservative politicians are lobbied by the right group; and for simplicity, we assume that the contract, if accepted by the politician, is binding. The proposed policy maximizes the joint surplus of the lobby group and politician, pulling policy choices to the extremes of the policy space; and the monetary payment compensates the politician for moving policy from her default choice.

We establish existence of a simple lobbying equilibrium, in which politicians and voters use stationary strategies and such that choices are optimal at every point in the game. In a simple lobbying equilibrium, each lobby group makes an offer that is monotonic in the politician's type, and politicians are always lobbied to policies that are weakly more extreme than their default choices; thus, lobbying facilitates policy extremism. In general, some politicians who would compromise in the absence of a lobby offer are pulled to a more extreme policy that causes them to lose the election; thus, lobby groups can lead to increased turnover of politicians. Moderate politicians close to the median are offered winning policies but are pulled away from the median voter's ideal point; thus, there is a "flight from the center," as even a politician with ideal point at the median is lobbied away from the median. We show that this wedge between policy outcomes and the median goes to zero as politicians become highly office motivated: if the benefit of office increases, or if politicians become very patient (therefore putting greater weight on the potential stream of office benefits from re-election), then all politicians are lobbied to policies that converge to the median. This dynamic version of the median voter theorem connects our game-theoretic analysis to the social choice theory literature, and it illustrates the attractiveness of the median even in the presence of lobby groups with incentives to pull policy outcomes to the extremes of the policy space.

The median voter result fixes the effectiveness of money and establishes a positive result by letting office incentives become large. Alternatively, to examine the effect of relaxing restrictions on political contributions, we can fix office incentives and let the effectiveness of money become large. Our main result in this connection shows that in this case, all simply lobbying equilibria are polarized, in the sense that all liberal politicians choose the same policy, all conservative politicians choose the same policy, and these two policies are bounded away from each other. A multiplicity of such equilibria can obtain, but we show that the most polarized equilibrium becomes extremist: all politicians choose policies that are arbitrarily close to the extremes of the policy space. We conclude that lobbying can precipitate extreme policy polarization as the role of money in elections grows large, suggesting that caution should be taken in relaxing restrictions on political contributions and pointing to the importance of the further study of the linkages between money and policy.

The analysis of lobbying in this paper contributes to the literature on electoral accountability and builds on the repeated elections framework of Duggan (2000), in which lobby groups are not modeled, and the incumbent politician chooses policy independently in each period. As in our paper, voters observe policy choices but do not observe the preferences of politicians, so that elections are subject to pure adverse selection. Early studies of electoral accountability are Barro

(1973), who studies an electoral model in which there is one type of politician and voters observe policy choices, and Ferejohn (1986), who analyzes a pure moral hazard setting, where policy choices are not perfectly observable. In the pure adverse selection context, closer to our work, Bernhardt et al. (2004) study the effect of term limits; Bernhardt et al. (2009) add partisanship to the model by assuming that challengers can be drawn from different pools, depending on their partisan affiliation; and Bernhardt et al. (2011) add valence to the model, so that a politician's type is composed of two components, valence (which is observed) and her ideal point (which is not). The pure adverse selection model is extended to the multidimensional setting by Banks and Duggan (2008), who establish a dynamic median voter theorem in one dimension. Our median voter result reinforces that of the latter paper by showing that convergence to the median policy obtains even when lobby groups have incentives to pull policy away from the median.[3]

Most of the previous literature on lobbying and elections studies models in which donations from interest groups increase a politician's probability of winning the election.[4] Another branch of the literature uses a common agency approach to study lobbying. Grossman and Helpman (1996) analyze a static model of campaign finance, in which interest groups contribute to political parties to gain influence or serve electoral motives. Grossman and Helpman (1994) uses the common agency framework to explore how special interests affect trade policy. Martimort and Semenov (2007) considers how officeholder decisions are affected, in an election free setting, by contributions from competing lobbying firms that have opposing ideological preferences. These papers do not consider the dynamic incentives inherent in the repeated elections framework. Austen-Smith and Wright (1994) examine how two lobbies may attempt to influence the same politicians to try and off-set one other. We abstract from this possibility by assuming one lobby acts at a time and only attempts to influence politicians on the same side of the aisle. In our context, however, this does not appear to be an onerous assumption, as it captures the empirical regularity that interest groups tend to lobby ideological allies.[5]

Closest to the analysis of this paper is Snyder and Ting (2008), who also study a model of repeated elections with lobbying, but the papers differ in many important ways. First, Snyder and Ting (2008) assume that politicians are purely office motivated, so that they do not face a trade off between policy and re-election. An implication of their assumption is that if two policies fail to gain re-election, then a politician is indifferent between them. This leads to uninteresting stationary subgame equilibria in which the incumbent politician always chooses the ideal point of the interest group and voters always remove the incumbent in favor of a challenger. Such equilibria cannot occur in our model, because politicians care about policy. This means that the optimal policy proposal of a lobby group would depend on the incumbent's type, but then the incumbent's policy choice would reveal her type to voters. But then a politician whose type is close to the median will choose a policy better than the expectation of a challenger, and the incumbent would

[3]We also generalize Banks and Duggan (2008) by allowing partisanship, as in Bernhardt et al. (2009).

[4]See, e.g., Austen-Smith (1987) and Baron (1989).

[5]See, e.g., Bauer et al. (1964), Hojnacki and Kimball (1998), Kollman (1997), Milbrath (1976), and Carpenter et al. (2004).

necessarily be re-elected in equilibrium. Second, and more subtly, politicians differ only in their innate benefit from holding office, and this type is revealed to voters after a politician's first term of office. Thus, once a first-term incumbent is re-elected, the politician no longer has an incentive to signal her type. Third, and perhaps most importantly, Snyder and Ting (2008) assume that lobby groups are short-lived, and that in each period there is a single lobby group with ideal policy drawn independently over time. In contrast, we analyze the influence of two competing lobby groups that persist over time and that care about the future consequences of policy decisions, not just pertaining to the incumbent's re-election chances, but anticipating the ideology of future challengers and the effect of lobbying on future policies.

In Section 2, we describe the model of repeated elections with lobbying. The simple lobbying equilibrium concept is defined in Section 3, and our equilibrium existence and characterization results are provided in Section 4. In Section 5, we examine the policy consequences when the effectiveness of money becomes large. Section 6 concludes, and proofs are contained in the appendix.

2 Repeated Elections with Lobbying

We analyze a game-theoretic model of repeated elections in which the players are ideological lobby groups who exchange money for policy, politicians who have policymaking power, and voters who decide between the incumbent politician and a challenger. Each election cycle corresponds to a period in the model, and elections take place over an infinite horizon. Voters and politicians belong to a continuum $N = [\underline{\theta}, \overline{\theta}]$ of citizen types, and each type θ is associated with an ideal point $x(\theta)$ belonging to the policy space $X = [0, 1]$. We assume $0 \leq \underline{\theta} < \overline{\theta}$, where the first inequality is merely a normalization that will be useful in the sequel. There are two lobby groups, L and R, who for simplicity have the policy preferences of the most extreme citizen types: we equate L with $\underline{\theta}$ and R with $\overline{\theta}$, and we assume that the ideal point of group L is the left-most policy, $x(\underline{\theta}) = 0$, and the ideal point of group R is the right-most policy, $x(\overline{\theta}) = 1$. The distribution of citizen types is given by a density $f(\theta)$ with full support on N and unique median denoted θ_m. We assume that citizen types are private information, so a politician's type is not directly observable by voters, but we assume types are observable by lobby groups. Of course, it may be that types can be inferred by voters from observed behavior in equilibrium.

A citizen's preferences over policy are represented by a utility function that is indexed by her type: let $u_\theta(x)$ denote the utility of a type θ citizen from policy x. Since we equate lobby groups with the extreme types, $u_L(x) = u_{\underline{\theta}}(x)$ is the utility of group L and $u_R(x) = u_{\overline{\theta}}(x)$ is the utility of group R from policy x. We assume that the utility function $u_\theta(x)$ is differentiable and strictly concave with unique maximizer $x(\theta)$. Moreover, to facilitate the analysis, we impose the following general functional form restriction:

$$u_\theta(x) \quad = \quad \theta v(x) - c(x) + k_\theta, \tag{1}$$

where $v' > 0$ and $v'' \geq 0$, and $c' \geq 0$ and $c'' > 0$, and k_θ is a term that is constant in policy but can depend on type; in particular, v is concave and c is strictly convex. Then the ideal point $x(\theta)$ of citizen θ is the unique solution to the first order condition $\theta v'(x) = c'(x)$, and it follows from the implicit function theorem that $x(\theta)$ is differentiable and strictly increasing; in other words, citizen types are ranked in terms of policy preferences, with higher types corresponding to higher ideal points. Therefore, the ideal point $x_m = x(\theta_m)$ is in fact the median of the voters' ideal points. Without loss of generality, we assume the median voter weakly prefers policy $x = 1$ to $x = 0$, i.e., $u_m(1) \geq u_m(0)$, so that the right lobby group is weakly more moderate than the left, in terms of voter preferences.[6]

Along with the choice of policy by the politicians, we allow for the possibility of monetary transfers from lobby groups to politicians. Utility from monetary payments, e.g., the utility of lobby group $G \in \{L, R\}$ from policy y and monetary payment m to the incumbent is $u_G(y) - m$. Similarly, the utility to a type θ politician from entering the contract (y, m) with the group is $u_\theta(y) + \gamma m + \beta$, with the difference that the impact of the monetary payment is multiplied by the parameter $\gamma \geq 0$. This parameter measures the effectiveness of money and will be a central focus of the subsequent analysis. In particular, we view γ as an institutional parameter that summarizes restrictions on lobbying expenditures, including expenditures to cover travel and personal expenses. More generally, γ represents constraints on any expenditures that can (explicitly or implicitly) be linked to policy choices; under this interpretation, γ incorporates regulation of campaign advertising by outside groups, limits on donations by corporations, unions, and other organizations, and rules requiring disclosure of funding sources. We will be interested in the effect of an increase in γ due to weaker restrictions on political expenditures, e.g., in the US system, allowances for 527 committees or political nonprofit 501(c) groups. Note that when $\gamma = 0$, lobbying plays no role in the model, and we obtain the model of Duggan (2000) and the one-dimensional version of Banks and Duggan (2008), which do not permit lobbying, as a special case.

Along with their citizen types, politicians are distinguished by their party affiliation and preference for holding office. We assume there are two parties, where $\pi \in \{0, 1\}$ denotes the party affiliation of a politician. Here, $\pi = 0$ indicates that the politician belongs to the liberal party, and $\pi = 1$ indicates membership in the conservative party; party affiliation will be used in equilibrium by voters to make inferences about the policy preferences of untried challengers. We let $h^\pi(\theta)$ denote the density of citizen types within the pool of candidates in party $\pi = 0, 1$. At this point we leave these densities to be completely general, but in our equilibrium characterizations, we make the weak assumption that the median type θ_m is contained in the support of both challenger densities. In addition, a politician receives a benefit of $\beta \geq 0$ in each period she holds office.

Each period t begins with a politician θ_t, the *incumbent,* who has some partisan affiliation π_t and who holds a political office. If the incumbent is to the left of the median, i.e., $\theta_t < \theta_m$, then the *active group* is lobby group L; if the incumbent is to the right of the median, i.e., $\theta_t < \theta_m$, then it is lobby group R; and if the incumbent is at the median, i.e., $\theta = \theta_m$, then the active group is

[6]A special class of models are those that are symmetric around the median, in which case $x_m = 1/2$ and $u_m(0) = u_m(1)$. Our general formulation allows for asymmetries, and we assume $u_m(1) \geq u_m(0)$ merely to simplify the characterization by eliminating cases that are, by reflection across the median, already covered in the analysis.

determined by a coin flip.[7] In general, given that the incumbent belongs to party π, we may write $G(\theta)$ for the active group given an incumbent with type θ, selecting $G(\theta_m) = R$ without loss of generality in the zero probability event that the politician is the median type. The timing of moves in the first term of office for the politician is as follows:

(1) The active group G_t offers a binding contract (y_t, m_t), where $y_t \in X$ is a policy to be chosen henceforth by the politician and m_t is a monetary payment to be made in period t if the office holder enters the contract; this offer is not observed by voters.

(2) The office holder decides whether to accept or reject the offer, $a_t \in \{0, 1\}$, where $a_t = 1$ indicates acceptance and $a_t = 0$ rejection; this acceptance decision is not observed by voters.

(3) The office holder chooses policy x_t, and this is observed by voters; if the office holder accepts the offer, $a_t = 1$, then she is committed to policy $x_t = y_t$, and otherwise, x_t is unrestricted.

(4) A candidate θ'_t, the *challenger,* is drawn from the density function $h^{1-\pi_t}(\theta)$ for the opposition party to challenge the incumbent; the challenger's type θ'_t is observed by lobby groups but not by voters, and the partisan affiliation of θ'_t is $1 - \pi_t$, and this is observed by lobby groups and voters.

(5) Each voter casts a ballot in a majority-rule election between the incumbent and challenger, with the winner taking office at the beginning of period $t + 1$.

If period t is not the first term of office for the politician, then either the politician has entered a contract (y_s, m_s) with the active lobby group in some previous period $s < t$, in which case she is committed to $x_t = y_s$, or the politician has opted not to engage with the active group, in which case the current period consists of steps (3)–(5).

Before we proceed to discuss information and payoffs in greater detail, three remarks are in order. First, our specification of utilities is a general one that captures the canonical model with *quadratic utility,* in which θ is identified with the ideal point of a citizen and utility is defined as $u_\theta(x) = -(x - \theta)^2$.[8] To see this, we expand this expression as $-x^2 + 2x\theta - \theta^2$, and we then set

$$v(x) = 2x, \;\; c(x) = x^2, \;\; k_\theta = -\theta^2.$$

Second, we model the lobby group's offer as a binding contract in which the payment m_t is made once, and thereafter the politician is committed to y_t. Because of the stationary nature of equilibria analyzed in the sequel, we could as well have modeled interaction between lobbyists and politicians as a series of short-term contracts that hold only for a single period. The current specification, by virtue of reducing interaction to a single exchange, can be viewed as an analytically tractable way of modeling such short-term contracts. Third, we have assumed that an officeholder is lobbied by

[7] The incumbent is at the median with zero probability, so this specification is innocuous; it is used only so that we can specify lobby strategies as functions defined on closed intervals, including the endpoint θ_m.

[8] The functional form in (1) is used by Duggan and Martinelli (2015), who also introduce an exponential specification as a special case. Specifically, the *exponential* functional form is $u_\theta(x) = xe^\theta - e^x + k_\theta$.

the proximate interest group, i.e., liberal politicians deal with group L and conservative politicians deal with group R. This is consistent with the empirical regularity that interest groups tend to lobby ideological allies,[9] but this assumption amounts to an exogenous restriction on behavior that could be modeled as an endogenous choice.[10]

As described above, we assume that a politician's type is private information and not directly observable by voters. Voters do observe the policy choices of the incumbent politician, and thus elections are characterized by pure adverse selection. In contrast, lobby groups have more information, due to greater experience or political connections, and we assume that the active lobby group does observe the type of a first-term politician before making an offer. We assume that lobbying takes place "behind closed doors," so that voters do not observe the offer made by the active group or the acceptance decision of the politician.

All players discount the streams of utility by the common factor $\delta \in [0, 1)$. Given a sequence of offers $(y_1, m_1), (y_2, m_2), \ldots$, a sequence of acceptance decisions $a_1, a_2, \ldots$, and a sequence $x_1, x_2, \ldots$ of policy choices, the discounted sum of per period payoffs of a type θ citizen is given by the expression

$$\sum_{t=1}^{\infty} \delta^{t-1} \Big[a_t u_\theta(y_t) + (1 - a_t) u_\theta(x_t) + I_t(a_t \gamma m_t + \beta) \Big],$$

where $I_t \in \{0, 1\}$ is an indicator variable that takes the value one if the citizen holds office in period t and zero otherwise. In the above, note that the office benefit accrues to the citizen only if she holds office ($I_t = 1$), and she receives the monetary payment only if she holds office and accepts the lobby group's offer ($I_t a_t = 1$). The discounted sum of per period payoffs of lobby group $G \in \{L, R\}$ is

$$\sum_{t=1}^{\infty} \delta^{t-1} \Big[a_t u_G(y_t) + (1 - a_t) u_G(x_t) - I_t a_t m_t \Big],$$

where now $I_t = 1$ indicates that the interest group is active in period t, and $I_t = 0$ indicates it is inactive.

3 Simple Lobbying Equilibrium

The analysis focuses on a selection of perfect Bayesian equilibrium (PBE) of the model of repeated elections with lobbying. It is known that in repeated games, many outcomes can be supported by strategies in which players condition on histories in complex ways; the complexity of these strategies is implausible in models of elections and voting, and we therefore consider strategies that can be described by means of simple behavioral rules. We study equilibria that are stationary, in the sense

[9] See, e.g., Bauer et al. (1964), Hojnacki and Kimball (1998), Kollman (1997), Milbrath (1976), and Carpenter et al. (2004).

[10] In a symmetric version of the model, we can assume that in a prior stage (0) of her first term of office, the politician decides which lobby group to approach. Assuming that the politician retains a small proportion of rents (which may go to zero) from the exchange, she will indeed choose to approach the closest lobby group.

that the active lobby group's offer depends only on the citizen type of the current politician; the acceptance decision of a politician depends only on the offer by the active group, and her policy choice in lieu of acceptance is independent of the prior history; and each citizen's vote in an election depends only on the policy choice and partisan affiliation of the incumbent in the preceding period.

After formulating the stationarity restriction, we then define our equilibrium concept by imposing the final assumption that voters use retrospective voting strategies with an intuitive form: for each type θ, the type θ voter re-elects an incumbent from party π if and only if the politician's policy choice in the preceding period is greater than or equal to the continuation value of a challenger. This, in turn, implies that the median voter type is a representative voter, i.e., the incumbent is re-elected if and only if she offers the median voter an expected discounted payoff from re-election at least equal to the continuation value of a challenger. If the lobby group's offer is rejected, an office holder whose ideal point is acceptable to the median voter simply chooses that policy; other politicians are faced with a trade off: compromise by choosing the best policy acceptable to the median, or shirk by choosing her ideal point. Finally, the active lobby group makes the most advantageous offer possible, subject to the constraint that the politician receives utility at least equal to the payoff of "going it alone."

Formally, given an incumbent belonging to party π, a strategy of lobby group L is described by mappings $\lambda_L^\pi \colon [\underline{\theta}, \theta_m] \to X$ and $\mu_L^\pi \colon [\underline{\theta}, \theta_m] \to \mathbb{R}$, where $(\lambda_L^\pi(\theta), \mu_L^\pi(\theta))$ is the offer made by the group; that is, L offers to transfer $\mu_L^\pi(\theta)$ to the politician in exchange for the commitment to choose policy $\lambda_L^\pi(\theta)$ thereafter. Similarly, a strategy for group R consists of mappings $\lambda_R^\pi \colon [\theta_m, \overline{\theta}] \to X$ and $\mu_R^\pi \colon [\theta_m, \overline{\theta}] \to \mathbb{R}$, with the same interpretation. A strategy of a type θ politician with partisan affiliation π is represented by mappings $\alpha_\theta^\pi \colon X \times \mathbb{R} \to \{0,1\}$ and $\xi_\theta^\pi \in X$, where $\alpha_\theta^\pi(y,m) = 1$ if and only if the politician accepts the offer (y,m) from the active group, and ξ_θ^π is the default policy chosen by the politician if the offer is rejected. The voting strategy of a type θ citizen is represented by mappings $\nu_\theta^\pi \colon X \to \{0,1\}$, where $\nu_\theta^\pi(x) = 1$ if and only if the type θ citizen votes to re-elect an incumbent from party π who chooses x in the preceding period. In addition to strategies that specify the actions of all players after all histories, because voters do not observe the types of the incumbent and challenger, we must specify a belief system for the voters. This a mapping $\kappa \colon X \times \{0,1\} \to \Delta(N)$, where $\Delta(N)$ is the set of probability distributions over citizen types, and $\kappa(x,\pi)$ represents the voters' beliefs about the type θ of an incumbent from party π following policy choice x in the preceding period. Given that the incumbent is from party π, the voters' beliefs about the challenger's type are of course represented by the prior density $h^{1-\pi}(\theta)$.[11,12]

A strategy profile $\sigma = (\lambda, \mu, \alpha, \xi, \nu)$ is *sequentially rational* given belief system κ if the following conditions are satisfied in every period: (i) for every type θ and party π, neither active lobby group can profitably deviate from $(\lambda_G^\pi(\theta), \mu_G^\pi(\theta))$ to a different offer, (ii) for every type θ, party π, and

[11] In what follows, the calculation of expected payoffs assumes that strategies are jointly measurable: $\lambda_G^\pi(\theta)$, $\mu_G^\pi(\theta)$, and ξ_θ^π are measurable in θ, and $\alpha_\theta^\pi(x,m)$ and $\nu_\theta(x,\pi)$ are measurable in (θ,x). Henceforth, for ease of exposition, technical measurability issues will be suppressed.

[12] To avoid problems in the application of Bayes rule, we restrict attention to strategy profiles such that for each policy x, the set of types θ such that $x \in \{\xi_\theta, \lambda^0_{G(\theta)}(\theta), \lambda^1_{G(\theta)}(\theta)\}$ is finite.

lobby offer (y,m), $\alpha^\pi_\theta(y,m)$ is an optimal response for the politician, (iii) for every type θ and party π, conditional on rejecting the lobby group's offer, a politician cannot profitably deviate from ξ^π_θ to another policy choice, and (iv) for all policy choices x, each type θ voter votes for the candidate who provides the highest expected discounted payoff conditional on their information. The latter condition is equivalent to the assumption of "sincere voting," but it does not assume voter myopia: each voter calculates the expected payoffs from the incumbent and challenger in a sophisticated way, and then she chooses between them optimally.[13] Beliefs κ are *consistent* with σ if for all $x \in X$ and all $\pi \in \{0,1\}$, $\kappa(\cdot|x,\pi)$ is derived via Bayes rule on the path of play determined by σ; if citizens observe a policy that occurs with probability zero under σ, then consistency places no restrictions on beliefs other than stationarity.[14] An assessment $\Psi = (\sigma,\kappa)$ is a *stationary perfect Bayesian equilibrium* if σ is sequentially rational given κ and κ is consistent with σ.

Next, we define several technical concepts that play key roles in the analysis, and we specialize stationary PBE further to impose intuitive restrictions on voting and policy choices. Given an assessment $\Psi = (\sigma,\kappa)$, each voter type θ can calculate the expected discounted payoff, conditional on some policy choice x, from re-electing an incumbent belonging to party π; we denote this by $V^{I,\pi}_\theta(x|\Psi)$. Similarly, let $V^{C,\pi}_\theta(\Psi)$ denote the continuation value from electing the challenger for a citizen type θ, given that the incumbent belongs to π. Note that stationarity implies that $V^{I,\pi}_\theta(x|\Psi)$ and $V^{C,\pi}_\theta(\Psi)$ are constant across time periods. Following Banks and Duggan (2008), we can write the continuation value of the incumbent as the expected utility of the type θ citizen with respect to a particular probability distribution; letting $P^{I,\pi}_x$ denote the *incumbent continuation distribution* following policy x by an incumbent belonging to π, we have

$$V^{I,\pi}_\theta(x|\Psi) \quad = \quad \frac{\mathbb{E}_{P^{I,\pi}_x}[u_\theta(z)]}{1-\delta},$$

where the expectation is with respect to the distribution $P^{I,\pi}_x$ over z. Similarly, the continuation value of a challenger can be written as the expectation of u_θ with respect to the *challenger continuation distribution,* $P^{C,\pi}$, we have

$$V^{C,\pi}_\theta(\Psi) \quad = \quad \frac{\mathbb{E}_{P^{C,\pi}}[u_\theta(z)]}{1-\delta}.$$

Of course, these distributions depend on the underlying assessment Ψ, but importantly, they are independent of citizen type: all citizens, in effect, view the incumbent as the same lottery, and similarly for the challenger.

Furthermore, the functional form in (1) implies that the median voter type θ_m is pivotal in

[13]Condition (iv) is in the spirit of eliminating undominated strategies in voting subgames. In the current context, however, each voter is massless and cannot affect the outcome of an election; a consequence is that voting for the worst of two candidates is not, technically, dominated. Nevertheless, such behavior appears implausible, and it is precluded by condition (iv).

[14]Specifically, our formulation of beliefs incorporates stationarity, in the following sense: after any two histories, if an incumbent from party π chooses policy x, then voters update beliefs in the same way, to $\kappa(\cdot|x,\pi)$. In particular, if a politician deviates by choosing a policy off the path of play, then this does not affect the updating of voter beliefs about future office holders.

majority voting between lotteries,[15] so that a majority of voters strictly prefer the challenger to the incumbent if and only if this is the median voter's preference, i.e., $V_m^{C,\pi}(\Psi) > V_m^{I,\pi}(x|\Psi)$. Conversely, at least half of all voters weakly prefer the incumbent if and only if $V_m^{I,\pi}(x|\Psi) \geq V^{C,\pi}(\Psi)$, and it follows that in a stationary PBE, an incumbent is re-elected following the policy choice x only if $V_m^{I,\pi}(x|\Psi) \geq V_m^{C,\pi}(\Psi)$. Define the *win set,* denoted $W^\pi(\Psi)$, as the set of policy choices such that an incumbent belonging to party π is re-elected following the choice of x. By stationarity, a politician's policy choice is independent of history, so an incumbent who chooses x and is re-elected will continue to choose x and be re-elected. This implies that $V_m^{I,\pi}(x|\Psi) = \frac{u_m(x)}{1-\delta}$, and it can be show that

$$W^\pi(\Psi) \quad \subseteq \quad \left\{ x \in X \mid \frac{u_m(x)}{1-\delta} \;\; \geq \;\; V_m^{C,\pi}(\Psi) \right\}.$$

In words, if an incumbent is re-elected after the choice of policy x, then that policy must provide the median voter a payoff at least equal to the value of a challenger.

It will be useful to define the *dynamic policy utility* of the type θ citizen from policy choice x by an incumbent belonging to party π as

$$U_\theta^\pi(x|\Psi) \quad = \quad \begin{cases} \frac{u_\theta(x)}{1-\delta} & \text{if } x \in W^\pi(\Psi), \\ u_\theta(x) + \delta V_\theta^{C,\pi}(\Psi) & \text{else.} \end{cases}$$

This definition reflects the fact that in a stationary equilibrium, if a policy in the win set is chosen, then it will continue to be chosen in every period thereafter; but if a policy outside the win set is chosen, then it will be in place for just one period, after which a challenger will take office. Similarly, to analyze the optimization problem of a politician, we define the *dynamic office rents* from policy choice x by an incumbent belonging to π as

$$B^\pi(x|\Psi) \quad = \quad \begin{cases} \frac{\beta}{1-\delta} & \text{if } x \in W^\pi(\Psi), \\ \beta & \text{else.} \end{cases}$$

In a stationary equilibrium, if a policy in the win set is chosen, then it will continue to be chosen by the politician, who receives the office benefit β in each period; and if it does not belong to the win set, then the incumbent holds office and receives the office benefit for just one period.

We focus on the concept of *simple lobbying equilibrium,* in which we specialize stationary PBE by strengthening sequential rationality as follows: (i) for every type θ and each party π with active group G, the offer $(\lambda_G^\pi(\theta), \mu_G^\pi(\theta))$ solves

$$\begin{gathered} \max_{(y,m)} U_G^\pi(y|\Psi) - m \\ \text{s.t. } U_\theta^\pi(y|\Psi) + \gamma m + B^\pi(y|\Psi) \; \geq \; U_\theta^\pi(\xi_\theta^\pi(P)|\Psi) + B^\pi(\xi_\theta^\pi(P)|\Psi) \end{gathered}$$

for the active group G; (ii) for every type θ, each party π, and each offer (y, m), we have $\alpha_\theta^\pi(y, m) = 1$

[15] See Duggan (2014) for details of this claim.

if and only if

$$U_\theta^\pi(y|\Psi) + \gamma m + B^\pi(y|\Psi) \quad \geq \quad U_\theta^\pi(\xi_\theta^\pi(P)|\Psi) + B^\pi(\xi_\theta^\pi(P)|\Psi);$$

(iii) for every type θ and each party π, the default policy ξ_θ^π solves

$$\max_x U_\theta^\pi(x|\Psi) + B^\pi(x|\Psi);$$

and (iv) for each party π, the win set is

$$W^\pi(\Psi) \quad = \quad \left\{ x \in X \mid \frac{u_m(x)}{1-\delta} \;\geq\; V_m^{C,\pi}(\Psi) \right\}.$$

That is, in a simple lobbying equilibrium, the active group offers a contract that maximizes its utility, subject to the participation constraint of the politician. The politician accepts the offer if and only if the utility from the contract is at least equal to the utility of declining and choosing the default policy, ξ_θ^π, which maximizes the dynamic utility of the politician. Finally, the incumbent is re-elected if and only if the median voter weakly prefers her policy choice to an untried challenger.

In equilibrium, the active group may offer policies that lead to re-election of the incumbent or to removal of the incumbent. Let

$$\mathcal{E}^\pi(\Psi) \quad = \quad \{\theta \mid \lambda_{G(\theta)}^\pi(\theta) \in W^\pi(\Psi)\}$$

denote the set of politician types from party π such that the active group offers a winning policy, so that the incumbent wins election. Let $E^\pi(\Psi) = x(\mathcal{E}^\pi(\Psi))$ denote the *election set* of ideal points of politician types that are offered a winning policy. Because it plays an important role in the proof of equilibrium existence, we observe that the challenger continuation values $V_\theta^{C,\pi}(\Psi)$ are determined by a system of two recursive equations,

$$\begin{aligned} V_\theta^{C,\pi}(\Psi) \quad = \quad & \int_{\theta' \in \mathcal{E}^{1-\pi}(\Psi)} \frac{u_\theta(\lambda_{G(\theta')}^{1-\pi}(\theta'))}{1-\delta} h^{1-\pi}(\theta') d\theta' \\ & + \int_{\theta' \notin \mathcal{E}^{1-\pi}(\Psi)} \left[u_\theta(\lambda_{G(\theta')}^{1-\pi}(\theta')) + \delta V_\theta^{C,1-\pi}(\Psi) \right] h^{1-\pi}(\theta') d\theta', \end{aligned} \tag{2}$$

for $\pi \in \{0, 1\}$. In words, if an incumbent from party π is replaced by a challenger, then if the challenger's type θ' is such that the active group $G(\theta')$ offers a winning policy, then that politician remains in office forever and chooses the policy agreed to; and if θ' is such that the active group offers a losing policy, then that policy is in place for just one period, after which another challenger takes office. An implication of the contraction mapping theorem is that the continuation values $V_\theta^{C,\pi}(\Psi)$ are, in fact, the unique solution to the system of equations in (2).

We end this section with several comments on simple lobbying equilibrium. First, the active group always makes an offer, but the group can offer the politician's default with no payment,

$(y, m) = (\xi_\theta^\pi, 0)$, effectively choosing to forego lobbying. Second, the politician is assumed to always accept the group's offer when she weakly prefers it to the default, whereas it may seem that acceptance is necessitated by PBE only if this preference is strict. In fact, the restriction is essentially without loss of generality. When the optimal contract (y, m) is distinct from the default $(\xi_\theta^\pi, 0)$, the active group receives positive rents from the exchange, but then it must be that the constraint holds with equality, i.e.,

$$U_\theta^\pi(y|\Psi) + \gamma m + B^\pi(y|\Psi) \quad = \quad U_\theta^\pi(\xi_\theta^\pi|\Psi) + B^\pi(\xi_\theta^\pi|\Psi)$$

and the politician accepts, for otherwise the group could increase its transfer to the politician by a small amount $\epsilon > 0$. The politician then strictly prefers the offer to the default and accepts. But then for small enough $\epsilon > 0$, the contract $(y, m + \epsilon)$ is strictly better for the group than (y, m), contradicting optimality of the latter contract. Third, we have observed that in a stationary equilibrium, every policy in the win set must be at least as good for the median voter as a challenger, and our equilibrium concept is maximally permissive, in the sense that we impose equality in the definition of the win set. Similar to the preceding comment, we assume the politician is re-elected when the median voter is indifferent between the incumbent and challenger, but this is without loss of generality (it is necessitated by existence of an optimal policy) in all but the extreme case in which the win set is a singleton consisting of the median policy. Finally, we note that simple lobbying equilibrium is a selection of stationary PBE in which the decisions of players are described by relatively simple behavioral rules, but players are not prevented from deviating to more complex strategies: given a simple lobbying equilibrium, no player can increase her payoff by deviating to any other different strategy, stationary or otherwise.

4 Equilibrium Politics: Existence and Characterization

The concept of simple lobbying equilibrium, when applied to the dynamic electoral model, identifies policy choices, monetary transfers, and electoral outcomes that are logically consistent with the incentives of politicians, lobbyists, and voters. To begin the analysis of the model, we first establish in Theorem 1 that there is at least one simple lobbying equilibrium.

Theorem 1. *A simple lobbying equilibrium exists.*

The proof, which is provided in the appendix, consists of a fixed point argument; a novel aspect is that the fixed point belongs to the class of potential pairs of challenger continuation distributions, $(P^{C,0}, P^{C,1})$, with the first component summarizing the distribution over future policies when an incumbent from party $\pi = 0$ is replaced by a challenger, and the second summarizing policies when an incumbent from party $\pi = 1$ is replaced. These distributions are a sufficient statistic to compute equilibrium payoffs of all citizen types, and thus from a pair $(P^{C,0}, P^{C,1})$, we can deduce the implied win set, optimal default policy choices of each politician, and the optimal offers by lobby groups. These optimal choices, which are conditioned on $(P^{C,0}, P^{C,1})$, then imply "updated" distributions, denoted $(\tilde{P}^{C,0}, \tilde{P}^{C,1})$, which may or may not be the same as the initial ones. A fixed point is a pair $(P^{*,0}, P^{*,1})$ that is mapped to itself, in this sense, so that optimal voting and policies choices given $(P^{*,0}, P^{*,1})$ in fact generate the same distributions and give us a simply lobbying equilibrium.

A byproduct of the proof is a characterization of equilibria in terms of two 6-tuples of cutoff policies, $(\underline{c}^\pi, \underline{e}^\pi, \underline{w}^\pi, \overline{w}^\pi, \overline{e}^\pi, \overline{c}^\pi)$, such that

$$\underline{c}^\pi \;\leq\; \min\{\underline{e}^\pi, \underline{w}^\pi\} \;\leq\; \max\{\underline{e}^\pi, \underline{w}^\pi\} \;\leq\; x_m \;\leq \min\{\overline{w}^\pi, \overline{e}^\pi\} \;\leq\; \max\{\overline{w}^\pi, \overline{e}^\pi\} \;\leq \overline{c}^\pi$$

for $\pi \in \{0, 1\}$. These determine the win set, default policy choices, and lobby offer in the following way. First, the win set is the closed interval $W^\pi(\Psi) = [\underline{w}^\pi, \overline{w}^\pi]$, where by condition (iv) in the definition of simple lobby equilibrium, we have

$$\frac{u_m(\underline{w}^\pi)}{1-\delta} \;=\; V_m^{C,\pi}(\Psi) \quad \text{and} \quad \frac{u_m(\overline{w}^\pi)}{1-\delta} \;\geq\; V_m^{C,\pi}(\Psi), \tag{3}$$

with the latter holding with equality unless $\overline{w}^\pi = 1$.[16] Second, the default policy choice of a type θ politician is to choose the closest policy in the win set to her ideal point $x(\theta)$ if that ideal point is in the *compromise set,*

$$C^\pi(\Psi) \;=\; [\underline{c}^\pi, \overline{c}^\pi],$$

so that, in particular, if $x(\theta) \in W^\pi(\Psi)$, then the politician chooses her ideal point and is re-elected thereafter; and if, e.g., $\overline{w}^\pi < x(\theta) \leq c^\pi$, then she chooses the endpoint $\overline{w}^\pi$ and is subsequently re-elected. And if the politician's ideal point belongs to the *shirk set,*

$$S^\pi(\Psi) \;=\; [0, \underline{c}^\pi) \cup (\overline{c}^\pi, 1],$$

then in the absence of a lobby offer, she shirks by choosing her ideal point and is removed from office. Third, the election set of ideal points of politician types that are offered a winning policy is the closed interval $E^\pi(\Psi) = [\underline{e}^\pi, \overline{e}^\pi]$, and in this case the lobby group's offer maximizes joint surplus for the politician and lobby group, subject to the win set constraint, i.e., $\lambda_G^\pi(\theta)$ solves

$$\max_{y \in W^\pi(\Psi)} u_G(y) + \frac{1}{\gamma} u_\theta(y).$$

And if the ideal point of the politician is outside the interval, i.e., $x(\theta) \notin B^\pi(\Psi)$, then the lobby group's offer maximizes the unconstrained joint surplus, i.e., $\lambda_G^\pi(\theta)$ solves

$$\max_{y \in X} u_G(y) + \frac{1}{\gamma} u_\theta(y),$$

and furthermore the policy offered lies outside the win set, i.e., $\lambda_G^\pi(\theta) \notin W^\pi(\Psi)$. We say a simple

[16]This inference uses the simplifying assumption that $u_m(1) \geq u_m(0)$.

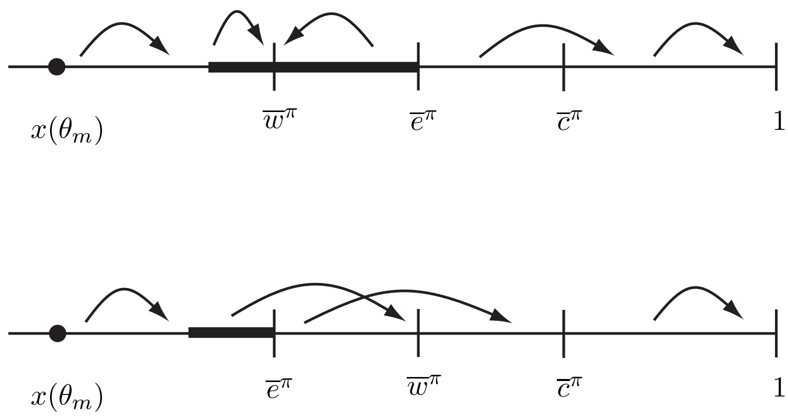

Figure 1: Partitional form of simply lobbying equilibrium

lobbying equilibrium possessing the above structure has the *partitional form.*

In words, politicians with sufficiently moderate ideal points $x(\theta) \in [\underline{e}^\pi, \overline{e}^\pi]$ are lobbied to winning policies, while those who are more ideologically extreme are lobbied to losing policies. Note that because lobby group offers maximize the joint surplus $u_G(y) + \frac{1}{\gamma}u_\theta(y)$, the policy choice of a politician is pulled weakly in the direction of the lobby group's ideal point. For example, a politician whose type $\theta > \theta_m$ is greater than the median is lobbied by group R, and thus her policy choice is moved weakly to the right as a consequence of lobbying. The only types for which this extremization effect does not hold strictly are those in the interval $[\overline{w}^\pi, \overline{e}^\pi]$, who would compromise at the endpoint $\overline{w}^\pi$ in the absence of lobbying and are lobbied to this same policy; for all other types $\theta > \theta_m$, lobbying by group R leads to strictly more extreme policy choices. Similarly, for types $\theta < \theta_m$, lobbying by group L moves politicians to the left. Furthermore, because $\underline{c}^\pi \leq \underline{e}^\pi$ and $\overline{e}^\pi \leq \overline{c}^\pi$, there is no politician type who shirks in the absence of a lobby offer and is lobbied to choose a winning policy. Thus, within equilibrium, the effect of lobbying is to weakly increase the rate of turnover of politicians. This structure is depicted in Figure 1, where we depict the policy choices of conservative politicians in two cases: $\overline{w}^\pi < \overline{e}^\pi$ and $\overline{e}^\pi < \overline{w}^\pi$. Here, white bullets represent several possible ideal points of politicians, and arrows point to lobby offers, represented by black bullets. In the first case, some politicians are lobbied to the right-hand endpoint of the win set and, in so doing, choose a policy that is more moderate than their ideal point; but these politicians compromised by default as well. In all other cases, lobby group R pulls policy choices further to the right than the politician's default choice in the absence of a lobby offer.

The next characterization result establishes that all simple lobbying equilibria have the partitional form.

Theorem 2. *Every simple lobbying equilibrium has the partitional form.*

It is of interest to examine the effect of electoral incentives on policy outcomes in the presence of lobby groups. This issue of policy responsiveness is a central question in the literature on electoral accountability (see Duggan and Martinelli (2015)), and it is known that a positive result holds in

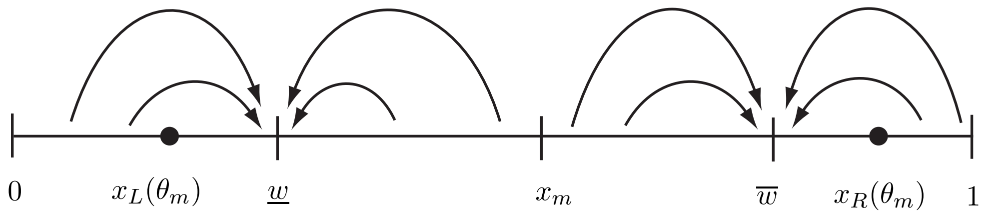

Figure 2: Polarized equilibrium

the model without lobbying: Banks and Duggan (2008) show that when $\delta > 0$ and the office benefit β is sufficiently large, all politician types choose the median policy. In the presence of lobbying, a form of this responsiveness result holds, but we will see that lobbying creates a wedge between policy choices and the median voter's preferences; this wedge holds even when office benefit is arbitrarily large (and re-election incentives are strongest), and it is increasing in the parameter γ, which measures the effectiveness of lobbying.

A simple lobbying equilibrium is *polarized* if there is a single win set $W = [\underline{w}, \overline{w}]$ that is independent of the incumbent's party, lobby group L offers almost all politician types the policy $\underline{w}$, and R offers almost all politician types $\overline{w}$; that is, for all $\pi \in \{0, 1\}$ and for h^π-almost all types θ, $\theta < \theta_m$ implies $\lambda_L^\pi(\theta) = \underline{w}$, and $\theta \geq \theta_m$ implies $\lambda_R^\pi(\theta) = \overline{w}$. In such an equilibrium, all liberal politicians agree to choose the same policy, as do all conservative politicians, and an incumbent's policy choices always ensure re-election; see Figure 2. The length of the win set, $\overline{w} - \underline{w}$, then measures the extent of polarization. Let $x_G(\theta)$ maximize the joint surplus $u_G(x) + \frac{1}{\gamma} u_\theta(x)$ of the lobby group G and the type θ politician, with weight $\frac{1}{\gamma}$ on the politician. We claim that in a polarized equilibrium, it necessarily follows that

$$x_L(\theta_m) \;\leq\; \underline{w} \quad \text{and} \quad \overline{w} \;\leq\; x_R(\theta_m). \tag{4}$$

To see this, note that lobby group R must offer the median politician type the policy $\overline{w}$, and by Theorem 2, this offer is either $x_R(\theta_m)$ or, if $\overline{w} < x_R(\theta_m)$, then it is the endpoint $\overline{w}$. Similarly, group L must offer politician types arbitrarily close to θ_m a policy greater than or equal to $x_L(\theta)$, and by continuity, the inequalities in (4) hold, as claimed.

The next result establishes that if office incentives are high, i.e., $\delta\beta$ is large, then all simple lobbying equilibria are polarized, and the extent of polarization is limited by the ability of the lobby groups to move the median politician type from her ideal point: in fact, the inequalities in (4), along with the median indifference, $u_m(\underline{w}) = u_m(\overline{w})$, are necessary and sufficient for existence of a polarized equilibrium with win set $[\underline{w}, \overline{w}]$. Since $\gamma > 0$, we have

$$x_L(\theta_m) \;<\; x_m \;<\; x_R(\theta_m),$$

and thus there are equilibria in which each type pools on one of two policies, each distinct from the median. In other words, a wedge between policy choices and the median policy is introduced by the possibility of lobbying. For the result we impose the minimal assumption that the median citizen type belongs to the support of the challenger densities, i.e., for all π, there exists $\epsilon > 0$ such that the density h^π is positive on the interval $(x_m - \epsilon, x_m + \epsilon)$.

Theorem 3. *Assume the median type belongs to the support of the challenger densities, and fix γ. When $\delta\beta$ is sufficiently large, every simple lobbying equilibrium is polarized. Furthermore, the most polarized equilibrium is such that one of the inequalities in (4) holds with equality. Finally, there is a polarized equilibrium with win set $[\underline{w}, \overline{w}]$ if and only if $u_m(\underline{w}) = u_m(\overline{w})$ and the inequalities in (4) hold.*

Theorem 3 considers equilibria when, holding all other parameters constant, the office incentive $\delta\beta$ is sufficiently large. A close reading of the proof reveals that if $\delta\beta$ is sufficiently large given a particular value of the effectiveness of money, say $\gamma > 0$, then the result holds when γ decreases to zero. Note that when γ goes to zero, thereby approximating the model with no lobbying, we have $x_G(\theta_m) \to x_m$ for both lobby groups, and we find that in every simple lobbying equilibrium, all politician types choose policies arbitrarily close to the median policy. The next result establishes that when the office incentive is large—even if a small amount of lobbying is possible—electoral incentives lead to policy outcomes close to the median voter's ideal point. For the one-dimensional case, this responsiveness result shows that the dynamic median voter theorem of Banks and Duggan (2008) is robust to the introduction of limited lobbying.[17]

Theorem 4. *Assume the median type belongs to the support of the challenger densities. If $\delta\beta$ is sufficiently large, then as $\gamma \to 0$, every simple lobbying equilibrium is polarized, and the win set $[\underline{w}, \overline{w}]$ in the most polarized equilibrium converges to the median policy, i.e., $\underline{w} \to x_m$ and $\overline{w} \to x_m$.*

This dynamic median voter theorem fixes $\delta\beta$ at a large value, and then establishes a positive result when money has negligible effectiveness. In light of recent relaxation of constraints on spending by interest groups, however, it is of interest to understand the effect of money in the polar situation—where money has a large impact and γ is high. We turn to this question in the next section.

5 Effect of Money on Polarization

We now characterize policy outcomes as the effect of money becomes large. There are different channels through which this effect could be realized in reality: one is through the lifting of restrictions on expenditures by interest groups for personal expenses and travel of politicians; another, perhaps more important, channel is the easing of restrictions on campaign contributions, which can then be used by a politician to secure future gains from office. Though we do not model the role of campaign contributions explicitly, the analysis of the parameter γ, which represents the effectiveness of lobby contributions, can provide some insight into the more general mechanisms

[17] The framework of this paper also allows for partisan challenger pools, which is not allowed by Banks and Duggan (2008).

through which contributions operate; of course, these insights remain to be rigorously investigated in future research.

The analysis of this section focuses on the case in which money becomes very effective, i.e., γ becomes large. In this case, the surplus-maximizing policies converge to the lobby groups' ideal points, i.e., $x_L(\theta_m) \to 0$ and $x_R(\theta_m) \to 1$. A simple corollary of Theorem 3 is that—even if office benefit is arbitrarily high—polarized equilibria exist in which all politician types choose policies at the extremes of the policy space. To obtain this possibility result, we impose the additional weak assumption that the median voter is indifferent between the lobby groups, i.e., $u_m(0) = u_m(1)$. Then for all $\epsilon > 0$, we can choose $\underline{w} \in (0, \epsilon)$ and $\overline{w} \in (1-\epsilon, 1)$ and γ large enough that $x_L(\theta_m) < \underline{w}$ and $x_R(\theta_m) > \overline{w}$; then Theorem 3 implies that when $\delta\beta$ is sufficiently large, there is a polarized equilibrium with win set $[\underline{w}, \overline{w}]$. This result is stated formally next.

Corollary 5. *Assume that the median type belongs to the supports of the challenger densities and that the median voter is indifferent between the lobby groups. Let γ become large, and let $\delta\beta$ be sufficiently large as a function of γ. Then there exist polarized equilibria with win sets $[\underline{w}, \overline{w}]$ such that $\underline{w} \to 0$ and $\overline{w} \to 1$.*

The above corollary assumes that office incentives are high, and moreover it relies on the possibility that $\delta\beta$ can be chosen to be arbitrarily high as γ increases. In this sense, it leaves our understanding of the effect of money incomplete. We say a sequence of simple lobbying equilibria *becomes extremist* if lobby group R offers all type $\theta \geq \theta_m$ politicians policies arbitrarily close to one along the sequence, and lobby group L offers all type $\theta < \theta_m$ politicians policies close to zero: formally, for each $\pi \in \{0, 1\}$, we have $\lambda_R^\pi(\theta_m) \to 1$ and $\lambda_L^\pi(\theta_m) \to 0$. The policies offered may be winning or losing, but an immediate implication is that the median voter's continuation value of a challenger cannot, in the limit, exceed the payoff from the worst policy for the median, i.e.,

$$\limsup V_m^{C,\pi}(\Psi) \quad \leq \quad \frac{u_m(0)}{1-\delta},$$

which implies that either $\overline{w}^\pi \to 1$ or $\underline{w}^\pi \to 0$ (or both). The main result of this section establishes that even when δ and β are held fixed, if the effectiveness of money is sufficiently high, then every simple lobbying equilibrium is polarized, and the most polarized equilibria become extremist, driving policies to the extremes of the policy space.

Theorem 6. *Assume that the median type belongs to the supports of the challenger densities and that the median voter is indifferent between the lobby groups. Fix δ and β. When γ is sufficiently large, every simple lobbying equilibrium is polarized. Furthermore, the most polarized equilibria become extremist.*

Corollary 5 and Theorem 6 impose a small amount of symmetry in the model, in that the median voter is indifferent between the lobby groups. In the complementary case that $u_m(0) < u_m(1)$, money continues to have a polarizing—if less stark—effect on policy outcomes. In the general case, where lobby group R may be advantaged relative to lobby group L, it may be that when the effectiveness of money is high, some simple lobbying equilibria are not polarized and policies offered by group L do not converge to the worst policy, $x = 0$, for the median voter. Nevertheless, if such equilibria exist as γ becomes large, then the median voter's continuation value of a challenger can, in the limit, be no greater than the utility from the right-most extreme policy. Formally, we say a sequence of equilibria *becomes weakly extremist* if

$$\limsup_{\gamma\to\infty} V_m^{C,\pi}(\Psi) \quad \leq \quad \frac{u_m(1)}{1-\delta}$$

for each party $\pi \in \{0,1\}$. Next, we establish that any exceptional equilibria—ones that are excluded by the minimal symmetry in Theorem 6—must become weakly extremist.

Theorem 7. *Assume that the median type belongs to the supports of the challenger densities and that $u_m(0) < u_m(1)$. Fix δ and β. If non-polarized simple lobbying equilibria exist when γ is sufficiently large, then these equilibria become weakly extremist.*

Theorems 6 and 8 establish the potentially polarizing effect of money in elections. As money becomes more effective, lobby groups offer the most extreme winning policies, i.e., equilibria become polarized. Moreover, there exist equilibria supporting arbitrarily extreme policies, with liberal politicians choosing policies close to zero and conservative politicians choosing policies close to one. In such cases, Theorem 3 establishes that there may be multiple polarized equilibria, and not all equilibria will exhibit extremism to this extent. It is possible to show, however, that as money becomes arbitrarily effective, limits of simple lobbying equilibria correspond to equilibria of the model in which office benefit is zero, there are no lobby groups, and there are just two extreme politician types. We define the *dichotomous model* as in Section 2, but now we assume: $\beta = 0$; there are no lobby groups, so that politician choices are given by the default ξ_θ; and given an incumbent from party π, the challenger's type is $\underline{\theta}$ with probability $H^\pi(\theta_m)$ and $\overline{\theta}$ with complementary probability. The statement of the next result considers convergent sequences of win sets, without loss of generality, and it relies on Theorem 6, which implies that equilibria are polarized when γ is high, so that these limits are independent of the incumbent's party.

Theorem 8. *Assume that the median type belongs to the supports of the challenger densities and that the median voter is indifferent between the lobby groups. Fix δ and β, and let γ become large. For every selection of simple lobbying equilibrium with convergent win sets, i.e., $\underline{w}^\pi \to \underset{\sim}{w}$ and $\overline{w}^\pi \to \tilde{w}$, there is a polarized equilibrium of the dichotomous model with win set $[\underset{\sim}{w}, \tilde{w}]$.*

An implication is that equilibria of the dichotomous model provide bounds on simple lobbying equilibria as γ becomes large. In particular, the least polarized equilibrium of the dichotomous model serves as a lower bound on polarization in the original model, and in this equilibrium, at least one extreme type must be indifferent between compromise and shirking. Then the win set $[\underset{\sim}{w}, \tilde{w}]$ must satisfy

$$\frac{u_{\overline{\theta}}(\underset{\sim}{w})}{1-\delta} \quad = \quad u_{\overline{\theta}}(1) + \delta\left[\frac{H^\pi(\theta_m)u_{\overline{\theta}}(\underset{\sim}{w}) + (1-H^\pi(\theta_m))u_{\overline{\theta}}(\tilde{w})}{1-\delta}\right]$$

for some party π, or it must satisfy the corresponding equality for type $\underline{\theta}$. This equality cannot be satisfied when polarization is small, i.e., $\underset{\sim}{w}$ and $\tilde{w}$ are close to x_m, and it follows that even the least polarized equilibria of the original model exhibit non-trivial polarization when money is effective. Thus, when γ is large, all equilibria are polarized, and moreover, the policies delivered by liberal and conservative politicians are bounded away from the median ideal point.

6 Conclusion

We have proposed a model of lobbying in the context of repeated elections and shown that the presence of lobbying implies, under parameters of interest, that political outcomes are polarized, in the sense that all liberal politicians deliver one policy and all conservative politicians deliver another. We find that the centripetal effects of office incentives, found in prior work without lobbying, are robust to the introduction of extreme lobby groups: when office incentives are high and the effectiveness of money is low, equilibrium policies are close to the median. However, money has a centrifugal effect: fixing office incentives, there exist arbitrarily extreme equilibria as money becomes more effective, and a wedge is introduced between liberal and conservative policies choices that bounds equilibrium policies away from the median ideal point.

Our results sound a cautionary note and suggest that restrictions on political contributions, which may be interpreted as an increase in the parameter γ, should be subject to careful analysis. We have provided a baseline model of lobbying in which the mechanism at work is that of policy concession in exchange for sidepayments to politicians. We leave for future work the task of explicitly modeling electoral campaigns in the dynamic setting and tracing the effect of money through campaign financing. We conjecture that the analysis would be largely unaffected, and that the effect of money could indeed be amplified in such a model. In the current framework, lobby groups must take the win set as given, and some politicians may be lobbied to relatively centrist policies because this "win set constraint" is binding. When lobby groups can contribute to political campaigns, however, it may be that they can affect the perceptions of voters and electoral outcomes, effectively enlarging the set of policies leading to re-election, relaxing the win set constraint, and creating the scope for more extreme policies.

A Technical Details

A.1 Existence of Equilibrium: Proof of Theorem 1

The existence proof consists of a fixed point argument, a byproduct of which is a characterization of optimal lobby offers and default policies that give rise to the partitional form; this is used to prove Theorem 2 in the next subsection. The fixed point argument is novel, in that it takes place in the product space of continuation distributions, endowing the space $\Delta(X)$ of probability distributions on X with the topology of weak convergence. Let $P = (P^0, P^1) \in \Delta(X)^2$ be a pair of probability distributions on X that represent continuation lotteries for challengers running against an incumbent from party $\pi \in \{0,1\}$. Mathematically, at this point, these are two arbitrary distributions. The arguments of this subsection construct a particular mapping, $\phi\colon \Delta(X)^2 \to \Delta(X)^2$, from the set of pairs $P = (P^0, P^1)$ into itself. The construction takes place in a number of steps, and along the way we take note of continuity properties that will be critical for the existence proof.

Continuation values: From these, we infer continuation values for challengers, as in

$$V_\theta^{C,\pi}(P) \quad = \quad \frac{\mathbb{E}_{P^\pi}[u_\theta(x)]}{1-\delta}, \tag{5}$$

where the expectation is with respect to the distribution P^π. Note that because u_θ is bounded and continuous, the continuation values $V_\theta^{C,\pi}(P)$ vary continuously as a function of P with the weak topology on $\Delta(X)$. In fact, because the $u_\theta(x)$ is jointly continuous in x and θ, a version of Lebesgue's dominated convergence theorem implies that the continuation value $V_\theta^{C,\pi}(P)$ is jointly continuous as a function of θ and P.

Win set: These continuation values determine a win set via the policies acceptable to the median voter, as in

$$W^\pi(P) \quad = \quad \left\{x \in X \mid \frac{u_m(x)}{1-\delta} \geq V_m^{C,\pi}(P)\right\}.$$

Note that the median policy belongs to the win set, i.e., $x_m \in W^\pi(P)$, and we can write this non-empty interval as $W^\pi(P) = [\underline{w}^\pi(P), \overline{w}^\pi(P)]$. Let $\overline{\theta}_w$ be the unique citizen type with ideal point equal to the greater endpoint, i.e., $x(\overline{\theta}_w) = \overline{w}^\pi(P)$, and let $\underline{\theta}_w$ be such that $x(\underline{\theta}_w) = \underline{w}^\pi(P)$. We will write these as $\overline{\theta}_w^\pi(P)$ and $\underline{\theta}_w^\pi(P)$ to make dependence on π and P explicit. By continuity of $V_m^{C,\pi}(P)$, along with strict concavity of u_m, the endpoints of this interval vary continuously as a function of P, as do the cutoff types. In particular, $W^\pi(P)$, viewed as a function of P, is a continuous correspondence.

Dynamic payoffs: We adapt the above notation for dynamic payoffs as follows: the dynamic policy utility for type θ is

$$U_\theta^\pi(x|P) \quad = \quad \begin{cases} \frac{u_\theta(x)}{1-\delta} & \text{if } x \in W^\pi(P), \\ u_\theta(x) + \delta V_\theta^{C,\pi}(P) & \text{else,} \end{cases}$$

and dynamic office rents are

$$B^\pi(x|P) \quad = \quad \begin{cases} \frac{\beta}{1-\delta} & \text{if } x \in W^\pi(P), \\ \beta & \text{else.} \end{cases}$$

Thus, a policy x belongs to the win set $W^\pi(P)$ if and only if the dynamic payoff from x is at least equal to the continuation value of a challenger for the median voter. Note that these functions are not generally continuous; however, $U_\theta^\pi(x|P)$ is jointly continuous on (x, θ, P) triples such that $x \in W^\pi(P)$, and $B^\pi(x, P)$ is jointly continuous on pairs (x, P) such that $x \in W^\pi(P)$. This joint continuity property will be important in establishing continuity of optimal lobby offers, below.

Default policies: These quantities determine default policy choices $\xi^\pi_\theta(P)$ for each politician type from each party as the solution to the following optimization problem:

$$\max_x U^\pi_\theta(x|P).$$

Of course, the optimal policy choice of a politician type with $x(\theta) \in W^\pi(P)$, or equivalently, $\underline{\theta}^\pi_w(P) \leq \theta \leq \overline{\theta}^\pi_w(P)$, is simply her ideal point. Otherwise, if $x(\theta) > \overline{w}^\pi(P)$, then the politician must choose between compromising at the greater endpoint of the win set or choosing her ideal point and foregoing re-election. Note that the politician weakly prefers to compromise if and only if

$$\frac{u_\theta(\overline{w}^\pi(P)) + \beta}{1-\delta} \quad \geq \quad u_\theta(x(\theta)) + \beta + \delta V^{C,\pi}_\theta(P), \tag{6}$$

where the left-hand side of the inequality represents the type θ politician's discounted payoff from compromising at the nearest winning policy, and the right-hand side is the expected discounted payoff from shirking.

We claim that equality holds in (6) for at most one type $\overline{\theta}_c > \overline{\theta}^\pi_w(P)$, that it holds strictly for θ between $\overline{\theta}^\pi_w(P)$ and $\overline{\theta}_c$, and that the reverse inequality holds strictly for $\theta > \theta_c$. We write the equality as

$$\frac{u_\theta(\overline{w}^\pi(P)) + \beta}{1-\delta} - [u_\theta(x(\theta)) + \beta + \delta V^{C,\pi}_\theta(P)] \quad = \quad 0. \tag{7}$$

The first derivative of the left-hand side with respect to θ is

$$\frac{v(\overline{w}^\pi(P))}{1-\delta} - v(x(\theta)) - \delta \mathbb{E}_{P^\pi}[v(x)],$$

where we use the envelope theorem to neglect the indirect effect of $u_\theta(x(\theta))$ through the ideal point. Note that $v(x(\theta))$ is strictly increasing, so that the left-hand side of (7) is strictly concave. Furthermore, (6) holds strictly for type $\overline{\theta}^\pi_w(P)$. Since the left-hand side of the inequality is strictly concave, it can then hold with equality for at most one type greater than $\overline{\theta}^\pi_w(P)$, and the claim follows. We write $\overline{\theta}^\pi_c(P)$ to make the dependence of this cutoff on P explicit; since (7) is continuous in P, it follows that $\overline{\theta}^\pi_c(P)$ is also a continuous function of P. Let $\overline{c}^\pi(P) = x(\overline{\theta}^\pi(P))$ be the ideal point of the type $\theta > \overline{\theta}^\pi_w(P)$ that is just indifferent between compromising and shirking. A similar analysis for types $\theta < \underline{\theta}^\pi_w(P)$ yields a cutoff $\underline{\theta}^\pi_c(P)$ that determines the willingness to compromise of such types, and we let $\underline{c}^\pi(P)$ be the ideal point of the type $\theta < \underline{\theta}^\pi_w(P)$ that is indifferent between compromise and shirking.

This gives us a partition of the policy space and a characterization of the politicians' optimal policy choices, according to whether a politician's ideal point is winning, or it is not winning but she optimally chooses to compromise her choice in order to gain re-election, or whether her ideal point is losing and she optimally shirks, choosing her ideal point and forgoing re-election. In particular, we define partition

$$\begin{aligned} W^\pi(P) &= [\underline{w}^\pi(P), \overline{w}^\pi(P)] \\ C^\pi(P) &= [\underline{c}^\pi(P), \underline{w}^\pi(P)) \cup (\overline{w}^\pi(P), \overline{c}^\pi(P)] \\ S^\pi(P) &= [0, \underline{c}^\pi(P)) \cup (\overline{c}^\pi(P), 1] \end{aligned}$$

consisting of the sets of ideal points of winners, compromisers, and shirkers, and we specify the politician's default policy choices as

$$\xi^\pi_\theta(P) = \begin{cases} x(\theta) & \text{if } x(\theta) \in [0, \underline{c}^\pi(P)) \\ \underline{w}^\pi(P) & \text{if } x(\theta) \in [\underline{c}^\pi(P), \underline{w}^\pi(P)), \\ x(\theta) & \text{if } x(\theta) \in W^\pi(P) \\ \overline{w}^\pi(P) & \text{if } x(\theta) \in (\overline{w}^\pi(P), \overline{c}^\pi(P)] \\ x(\theta) & \text{if } x(\theta) \in (\overline{c}^\pi(P), 1]. \end{cases}$$

Thus, winners and shirkers optimally choose their ideal points, with winners being re-elected and shirkers being removed from office, whereas compromisers choose the winning policy closest to their ideal point.

Finally, we claim that the maximized value of the politician's objective function,

$$\max_x U^\pi_\theta(x|P) = U^\pi_\theta(\xi^\pi_\theta(P)|P) + B^\pi(\xi^\pi_\theta(P)|P),$$

is jointly continuous as a function of θ and P. Indeed, we can write this as

$$\max\left\{ \max_{x\in W^\pi(P)} \frac{u_\theta(x)+\beta}{1-\delta}, \max_{x\notin W^\pi(P)} u_\theta(x) + V^{C,\pi}_\theta(P) \right\},$$

decomposing the politician's global maximization problem into two smaller ones. Because $W^\pi(P)$ is a continuous correspondence, the theorem of the maximum implies that the maximized value of each smaller problem is jointly continuous in θ and P, and this continuity is preserved by the maximum operation, as claimed.

Lobby offers, part 1: Next, we examine the constrained optimization problem of the active lobby, translated to the current context with arbitrary continuation distributions. For every type θ and each party π with active group G, the optimal offer $(\lambda^\pi_G(\theta), \mu^\pi_G(\theta))$ solves

$$\begin{gathered} \max_{(y,m)} U^\pi_G(y|P) - m \\ \text{s.t. } U^\pi_\theta(y|P) + \gamma m + B^\pi(y|P) \geq U^\pi_\theta(\xi^\pi_\theta(P)|P) + B^\pi(\xi^\pi_\theta(P)|P). \end{gathered}$$

The inequality is the *participation constraint* of the politician. This will be binding at a solution, so we can convert the constrained optimization problem into an unconstrained one by substituting the constraint into the objective function to obtain

$$\max_{(y,m)} U_G^\pi(y|P) + \frac{1}{\gamma}\Big[U_\theta^\pi(y|P) + B^\pi(y|P) - U_\theta^\pi(\xi_\theta^\pi(P)|P) - B^\pi(\xi_\theta^\pi(P)|P)\Big].$$

Because the politician's payoff from the default policy is continuous, as observed above, it follows that the objective function of the lobby group is jointly continuous when restricted to triples (y, θ, P) such that $y \in W^\pi(P)$, and similarly, it is jointly continuous when restricted to triples (y, θ, P) such that $y \notin W^\pi(P)$.

The optimal offer of the lobby group depends on the default policy choice of the politician and thus on the location of the politician's ideal point. We consider the problem of lobby group R when lobbying a politician with type $\theta > \theta_m$ from party $\pi = 1$, as the problems with $\pi = 0$ or with lobby group L are analogous. There are three cases corresponding to the location of the politician's ideal point:

1. $x(\theta) \in W^\pi(P)$
2. $x(\theta) \in C^\pi(P)$
3. $x(\theta) \in S^\pi(P)$.

We examine the lobby group's optimization problem in each case. In each case, we examine the optimal offer of the group in the win set, say y', and the optimal offer outside the win set, say y'', with the global optimum being the preferred of the two offers.

Case 1: Assume $x(\theta) \in W^1(P)$. The lobby group can buy policy $y' \in W$. In this case, the default policy of the politician is $x(\theta)$, and the dynamic rents from y' and $x(\theta)$ are the same. Thus, y' solves

$$\max_y U_R^1(y|P) + \frac{1}{\gamma}U_\theta^1(y|P) - \frac{1}{\gamma}U_\theta^1(x(\theta)|P),$$

where the last term is a constant. In terms of stage utilities, after normalizing by $1-\delta$, the policy $y' \in W$ solves

$$\max_y u_{\overline{\theta}}(y) + \frac{1}{\gamma}u_\theta(y) - \frac{1}{\gamma}u_\theta(x(\theta)).$$

The lobby group can also buy $y'' \notin W^1(P)$. In this case, the dynamic rents from $x(\theta)$ are $\frac{\beta}{1-\delta}$, while the dynamic rents from y'' are β. Then y'' solves

$$\max_y U_R^1(y|P) + \frac{1}{\gamma}U_\theta^1(y|P) - \frac{1}{\gamma}\left(U_\theta^1(x(\theta)|P) + \frac{\delta\beta}{1-\delta}\right).$$

In terms of stage utilities, this is

$$\max_y\ u_{\overline{\theta}}(y) + \delta V_{\overline{\theta}}^{C,1}(P) + \frac{1}{\gamma}\left(u_\theta(y) + \delta V_\theta^{C,1}(P)\right) - \frac{1}{\gamma}\left(\frac{u_\theta(x(\theta))}{1-\delta} + \frac{\delta\beta}{1-\delta}\right).$$

Note that modulo a constant term, the optimal $y \in W^1(P)$ and optimal $y \notin W^1(P)$ maximize the same objective function.

Case 2: Assume $x(\theta) \in C^1(P)$. The lobby group can buy policy $y' \in W$. In this case, the default policy of the politician is $\overline{w}^1(P)$, and the dynamic rents from both y' and the default are $\frac{\beta}{1-\delta}$. Thus, y' solves

$$\max_y U_R^1(y|P) + \frac{1}{\gamma}U_\theta^1(y) - \frac{1}{\gamma}U_\theta^1(\overline{w}^1(P)|P).$$

In terms of stage utilities, after normalizing by $1-\delta$, the policy $y' \in W^1(P)$ solves

$$\max_y u_{\overline{\theta}}(y) + \tfrac{1}{\gamma}u_\theta(y) - \tfrac{1}{\gamma}u_\theta(\overline{w}^1(P)).$$

The lobby group can also buy $y'' \notin W^1(P)$. In this case, the dynamic rents from the default are $\frac{\beta}{1-\delta}$, while the dynamic rents from y'' are β. Then y'' solves

$$\max_y U_R^1(y|P) + \frac{1}{\gamma}U_\theta^1(y|P) - \frac{1}{\gamma}\left(U_\theta^1(\overline{w}^1(P)|P) + \frac{\delta\beta}{1-\delta}\right).$$

In terms of stage utilities, this is

$$\max_y\ u_{\overline{\theta}}(y) + \delta V_{\overline{\theta}}^{C,1}(P) + \frac{1}{\gamma}\left(u_\theta(y) + \delta V_\theta^{C,1}(P)\right) - \frac{1}{\gamma}\left(\frac{u_\theta(\overline{w}^1(P))}{1-\delta} + \frac{\delta\beta}{1-\delta}\right).$$

Again, up to a constant, this is the same objective function as in Case 1.

Case 3: Assume $x(\theta) \in S^1(P)$. The lobby group can buy policy $y' \in W^1(P)$. In this case, the default policy of the politician is $x(\theta)$, the dynamic rents from y' are $\frac{\beta}{1-\delta}$, and the dynamic rents from the default are β. Thus, y' solves

$$\max_y U_R^1(y|P) + \frac{1}{\gamma}U_\theta^1(y|P) - \frac{1}{\gamma}\left(U_\theta^1(x(\theta)|P) - \frac{\delta\beta}{1-\delta}.\right)$$

In terms of stage utilities, after normalizing by $1-\delta$, the policy y' solves

$$\max_y u_{\overline{\theta}}(y) + \frac{1}{\gamma}u_\theta(y) - \frac{1}{\gamma}\left((1-\delta)(u_\theta(x(\theta)) + \delta V_\theta^{C,1}(P)) - \delta\beta\right).$$

The lobby group can also buy policy $y'' \notin W^1(P)$. Then the dynamic rents from y'' and the default are both β, and y'' solves

$$\max_y U_R^1(y|P) + \frac{1}{\gamma}U_\theta^1(y|P) - \frac{1}{\gamma}U_\theta^1(x(\theta)|P).$$

In terms of stage utilities, this is

$$\max_y\ u_{\overline{\theta}}(y) + \delta V_{\overline{\theta}}^{C,1}(P) + \frac{1}{\gamma}\left(u_\theta(y) + \delta V_\theta^{C,1}(P)\right) - \frac{1}{\gamma}\left(u_\theta(x(\theta)) + \delta V_\theta^{C,1}(P))\right)$$

Again, up to a constant, this is the same objective function as above.

Interim conclusions: We conclude that every policy $y' \in W^\pi(P)$ that is optimal for lobby group R subject to being in the win set solves

$$\max_{y \in W^\pi(P)} u_{\overline{\theta}}(y) + \frac{1}{\gamma} u_\theta(y),$$

and every policy $y'' \notin W^\pi(P)$ that is optimal subject to not being in the win set maximizes the same objective function,

$$\max_{y \notin W^\pi(P)} u_{\overline{\theta}}(y) + \frac{1}{\gamma} u_\theta(y).$$

That is, the optimal policy in each region maximizes the joint surplus function, $u_{\overline{\theta}}(y) + \frac{1}{\gamma} u_\theta(y)$, where the weight on the politician's utility decreases with the effectiveness of money in elections. This objective function is strictly concave, so there is exactly one policy that is optimal subject to the constraint that the policy is winning; henceforth, to bring out the dependence on the politician's type and party and on the given probability distributions, we denote this by $y^\pi_w(\theta|P)$. Similarly, ignoring boundary issues at $\overline{w}^\pi(P)$ (which do not affect the analysis), there is exactly one policy that is optimal subject to the constraint that the policy is losing; we denote this by $y^\pi_\ell(\theta|P)$. Recall that the restriction of $U^\pi_R(x|P)$ and $U^\pi_\theta(x|P)$ to triples (x, θ, P) such that $x \in W^\pi(P)$ is jointly continuous in θ and P. Because $y^\pi_w(\theta|P)$ is uniquely defined and $W^\pi(P)$ is a continuous correspondence, the theorem of the maximum therefore implies that $y^\pi_w(\theta|P)$ is a jointly continuous function of θ and P. Likewise, the lobby group's objective function is jointly continuous on the triples with $x \notin W^\pi(P)$, and $y^\pi_\ell(\theta|P)$ is jointly continuous in θ and P. Of course, symmetric conclusions hold for lobby group L. Let

$$\mathcal{E}^\pi(P) \quad = \quad \{\theta \mid \lambda^\pi_{G(\theta)}(\theta|P) \in W^\pi(P)\}$$

denote the set of politician types from party π such that the active group offers a winning policy given the challenger distributions P, so that the incumbent wins election.

The question that remains is which is better for the lobby group—offering the optimal winning policy $y^\pi_w(\theta|P)$ or the optimal losing policy $y^\pi_\ell(\theta|P)$—and this depends on the constant term.

To win or not to win: Is it better to lobby for a policy that will ensure re-election of the politician, or to lobby for a policy that is best in the short run? That depends on the constant terms in the above analysis. It is clearly optimal to lobby for a winning policy if the constraint $y \in W^\pi(P)$ is not binding at the optimal winning policy $y^\pi_w(\theta|P)$, i.e., the policy $y^\pi_w(\theta|P)$ is the unconstrained maximizer of $u_{\overline{\theta}}(y) + \frac{1}{\gamma} u_\theta(x)$. Indeed, this is so because the lobby group can obtain the maximizer of joint surplus without having to compensate the politician for the loss of office benefit. Let $\overline{\theta}^\pi_w(P)$ be the unique type θ such that the ideal point of the joint surplus function $u_{\overline{\theta}}(y) + \frac{1}{\gamma} u_{\overline{\theta}^\pi_w(P)}(y)$ is exactly the right-hand endpoint of the win set, $\overline{w}^\pi(P)$. That is, $\overline{\theta}^\pi_w(P)$ is the highest politician type such that the constraint that $y^\pi_w(\theta|P)$ belongs to the win set is not binding.

Of course, for higher politician types $\theta > \overline{\theta}_w^\pi(P)$, the policy that maximizes surplus subject to being in the win set is just $\overline{w}^\pi(P)$. Note that it is uniquely optimal for the lobby group to offer the winning policy $y_w^\pi(\theta|P) < \overline{w}^\pi(P)$ from every lower type $\theta < \overline{\theta}_w^\pi(P)$. Thus, we define the optimal policy offer

$$\lambda_R^\pi(\theta|P) \quad = \quad y_w^\pi(\theta|P)$$

for all θ with $\theta_m < \theta \leq \overline{\theta}_w^\pi(P)$, with transfer $\mu_R^\pi(\theta|P)$ determined by the politician's default policy choice through the participation constraint, i.e.,

$$\mu_R^\pi(\theta|P) \quad = \quad \frac{1}{\gamma}\Big[U_\theta^\pi(\xi_\theta^\pi(P)|P) + B^\pi(\xi_\theta^\pi(P)|P) - U_\theta^\pi(y_w^\pi(\theta|P)|P) - B^\pi(y_w^\pi(\theta|P)|P)\Big],$$

so that the optimal offer gives the politician her reservation value of choosing the default policy, $\xi_\theta^\pi(P)$.

The optimal offer to types $\theta > \overline{\theta}_w^\pi(P)$ is more complex. In what follows, we define

$$\begin{aligned}\Phi_R^\pi(\theta|P) \quad = \quad & U_R^\pi(y_w^\pi(\theta|P)|P) + \frac{1}{\gamma}\Big[U_\theta^\pi(y_w^\pi(\theta|P)|P) + B^\pi(y_w^\pi(\theta|P)|P)\Big] \\ & - U_R^\pi(y_\ell^\pi(\theta|P)|P) - \frac{1}{\gamma}\Big[U_\theta^\pi(y_\ell^\pi(\theta|P)|P) + B^\pi(y_\ell^\pi(\theta|P)|P)\Big]\end{aligned}$$

to be the lobby group's payoff from the optimal winning policy minus the payoff from the optimal losing policy. Note that lobby group R strictly prefers to lobby the type $\overline{\theta}_w^\pi(P)$ politician to a winning policy, so that $\Phi_R^\pi(\overline{\theta}_w^\pi(P)|P) > 0$. And since the maximizer of joint surplus $u_{\overline{\theta}}(y) + \frac{1}{\gamma}u_\theta(y)$ is greater than $\overline{w}^\pi(P)$ for all $\theta > \overline{\theta}_w^\pi(P)$, the optimal policy subject to belonging to the win set for such types is the right-hand endpoint of the win set, i.e., $y_w^\pi(\theta|P) = \overline{w}^\pi(P)$, as noted above. Furthermore, recall that the restriction of $U_R^\pi(x|P)$ and $U_\theta^\pi(x|P)$ to triples (x, θ, P) such that $x \in W^\pi(P)$ is jointly continuous in θ and P, as is the restriction to triples with $x \notin W^\pi(P)$. Because $y_w^\pi(\theta|P)$ and $y_\ell^\pi(\theta|P)$ are jointly continuous in θ and P, we conclude that $\Phi_R^\pi(\theta|P)$ is jointly continuous in θ and P.

Next, we analyze the optimal lobby offer in the three cases above, corresponding to the location of the politician's ideal point. In Case 1, $x(\theta) \in W^\pi(P)$. The lobby group's payoff from the optimal winning policy $y_w^\pi(\theta|P)$ is

$$\frac{1}{1-\delta}\Big[u_{\overline{\theta}}(y_w^\pi(\theta|P)) + \frac{1}{\gamma}u_\theta(y_w^\pi(\theta|P)) - \frac{1}{\gamma}u_\theta(x(\theta))\Big],$$

and the payoff from the optimal losing policy $y_\ell^\pi(\theta|P)$ is

$$u_{\overline{\theta}}(y_\ell^\pi(\theta|P)) + \delta V_{\overline{\theta}}^{C,\pi}(P) + \frac{1}{\gamma}\Big(u_\theta(y_\ell^\pi(\theta|P)) + \delta V_\theta^{C,\pi}(P)\Big) - \frac{1}{\gamma}\Big(\frac{u_\theta(x(\theta))}{1-\delta} + \frac{\delta\beta}{1-\delta}\Big),$$

and $\Phi_R^\pi(\theta|P)$ is the former payoff minus the latter. As mentioned above, when $y_w^\pi(\theta|P) < \overline{w}^\pi(P)$, it is optimal for the lobby group to offer this winning policy. For $\theta > \overline{\theta}^\pi_w(P)$, we have $y^\pi_w(\theta|P) = \overline{w}^\pi(P)$, and using the envelope theorem, we have

$$\begin{aligned}\frac{d}{d\theta}\Phi^\pi_R(\theta|P) &= \frac{1}{\gamma}\left[\frac{\partial}{\partial\theta}\frac{u_\theta(\overline{w}^\pi(P))}{1-\delta} - \frac{\partial}{\partial\theta}u_\theta(y^\pi_\ell(\theta|P)) - \delta\frac{d}{d\theta}V^{C,\pi}_\theta(P)\right] \\ &= \frac{1}{\gamma}\left[\frac{v(\overline{w}^\pi(P))}{1-\delta} - \left(v(y^\pi_\ell(\theta|P)) + \delta\frac{\mathbb{E}_{P^\pi}[v(x)]}{1-\delta}\right)\right],\end{aligned}$$

where the partial derivative is with respect to the subscript θ, holding lobby offers fixed. Note that $y^\pi_\ell(\theta|P)$ is strictly increasing in θ, so $\Phi^\pi_R(\theta|P)$ is strictly concave in $\theta > \overline{\theta}^\pi_w(P)$.

In Case 2, $x(\theta) \in C^\pi(P)$. The lobby group's payoff from the optimal winning policy $y^\pi_w(\theta|P)$ is

$$\frac{1}{1-\delta}\left[u_{\overline{\theta}}(y^\pi_w(\theta|P)) + \frac{1}{\gamma}u_\theta(y^\pi_w(\theta|P)) - \frac{1}{\gamma}u_\theta(\overline{w}^\pi(P))\right],$$

and the payoff from the optimal losing policy $y^\pi_\ell(\theta|P)$ is

$$u_{\overline{\theta}}(y^\pi_\ell(\theta|P)) + \delta V^{C,\pi}_{\overline{\theta}}(P) + \frac{1}{\gamma}\left(u_\theta(y^\pi_\ell(\theta|P)) + \delta V^{C,\pi}_\theta(P)\right) - \frac{1}{\gamma}\left(\frac{u_\theta(\overline{w}^\pi(P))}{1-\delta} + \frac{\delta\beta}{1-\delta}\right).$$

Note that the difference, $\Phi^\pi_R(\theta|P)$, has exactly the same form as in Case 1. Again, it is strictly concave in $\theta > \overline{\theta}^\pi_w(P)$.

In Case 3, $x(\theta) \in S^\pi(P)$. The lobby group's payoff from the optimal winning policy $y^\pi_w(\theta|P)$ is

$$\frac{1}{1-\delta}\left(u_{\overline{\theta}}(y^\pi_w(\theta|P)) + \frac{1}{\gamma}u_\theta(y^\pi_w(\theta|P))\right) - \frac{1}{\gamma}\left(u_\theta(x(\theta)) + \delta V^{C,\pi}_\theta(P)) - \frac{\delta\beta}{1-\delta}\right),$$

and the payoff from the optimal losing policy $y^\pi_\ell(\theta|P)$ is

$$u_{\overline{\theta}}(y^\pi_\ell(\theta|P)) + \delta V^{C,\pi}_{\overline{\theta}}(P) + \frac{1}{\gamma}\left(u_\theta(y^\pi_\ell(\theta|P)) + \delta V^{C,\pi}_\theta(P)\right) - \frac{1}{\gamma}\left(u_\theta(x(\theta)) + \delta V^{C,\pi}_\theta(P)\right).$$

Again, the difference $\Phi^\pi_R(\theta|P)$ has the same form as in Cases 1 and 2, so it is strictly concave in $\theta > \overline{\theta}^\pi_w(P)$.

Lobby offers, part 2: Recall that lobby R strictly prefers to offer the optimal winning policy for the type $\overline{\theta}^\pi_w(P)$ politician, so that $\Phi(\overline{\theta}^\pi_w(P)|P) > 0$. By strict concavity of $\Phi^\pi_R(\theta|P)$ in θ, it follows that that indifference between the optimal winning and optimal losing policies, i.e., $\Phi^\pi_R(\theta|P) = 0$, holds for at most one type $\overline{\theta}_e$, that $\Phi^\pi_R(\theta|P) > 0$ holds for all θ between $\overline{\theta}^\pi_w(P)$ and $\overline{\theta}_e$, and that $\Phi^\pi_R(\theta|P) < 0$ for all $\theta > \overline{\theta}_e$. We henceforth write $\overline{\theta}^\pi_e(P)$ to make dependence of this cutoff on P explicit. Importantly, since $\overline{\theta}^\pi_e(P)$ is uniquely defined and $\Phi^\pi_R(\theta|P)$ is jointly continuous in θ and P, it follows that $\overline{\theta}^\pi_e(P)$ is also a continuous function of P. A similar analysis holds for types $\theta < \underline{\theta}^\pi_w(P)$, yielding cutoff $\underline{\theta}^\pi_e(P)$ that is also jointly continuous.

With this background, we define the optimal offer of lobby group R as

$$\lambda_R^1(\theta|P) \quad = \quad \begin{cases} y_w^\pi(\theta|P) & \text{if } \theta \leq \overline{\theta}_w^\pi(P), \\ \overline{w}^\pi(P) & \text{if } \theta_w^\pi(P) < \theta \leq \overline{\theta}_e^\pi(P), \\ y_\ell^\pi(\theta|P) & \text{if } \overline{\theta}_e^\pi(P) < \theta. \end{cases}$$

In the above definition, there is some redundancy, as the first and second cases could be collapsed into one. When the politician is not too extreme, the lobby group offers the optimal winning policy, thereby obviating the need to compensate the politician for lost office benefit at not too great a cost in terms of policy. When the politician is more extreme, the lobby group buys even more extreme policy: the policy outcome maximizes the joint surplus function $u_{\overline{\theta}}(y) + \frac{1}{\gamma}u_\theta(y)$, pulling policy from the ideal point of the politician further to the right of the political spectrum. Monetary transfers are determined by the politician's participation constraint, as above. Note that $\Phi_R^\pi(\theta|P) = 0$ holds for at most one type, $\overline{\theta}_e^\pi(P)$, which is continuous in P. Continuity of the optimal winning policy $y_w^\pi(\theta|P)$ and the optimal losing policy $y_\ell^\pi(\theta|P)$ then imply that $\lambda_R^\pi(\theta|P)$ is jointly continuous in θ and P at all $\theta \neq \overline{\theta}_e^\pi(P)$; similarly, $\lambda_L^\pi(\theta|P)$ is jointly continuous in θ and P at all $\theta \neq \underline{\theta}_e^\pi(P)$. This continuity property will play a crucial role in the fixed point argument below.

Updating probability distributions: We now update the continuation distributions, P^0 and P^1, fixed at the beginning of the argument. Technically, we define two probability measures, $\tilde{P}^\pi$, $\pi \in \{0,1\}$, where for every measurable set $Z \subseteq X$, $\tilde{P}^\pi(Z)$ represents the probability (discounted appropriately over time) of a policy in the set Z conditional on replacing an incumbent from party π with an untried challenger. In doing so, we use the above analysis to update P^π in the period immediately after a challenger is elected, and we then use the original distribution $P^{1-\pi}$ to evaluate the probability mass in Z owing to future policy choices if the challenger is removed from office after her first term.

As the analysis for party $\pi = 1$ is analogous, we define the updated probability $\tilde{P}^0(Z)$ here. If an incumbent from party $\pi = 0$ is removed from office, then she is replaced by a challenger from party $\pi = 1$. Define the measures $Q_w^1(\cdot|P)$ and $Q_\ell^1(\cdot|P)$ on policies so that for all measurable $Z \subseteq X$,

$$\begin{aligned} Q_w^1(Z|P) &= \int_{\theta<\theta_m:\lambda_L^1(\theta)\in Z\cap W^1(P)} h^1(\theta)d\theta + \int_{\theta\geq\theta_m:\lambda_R^1(\theta)\in Z\cap W^1(P)} h^1(\theta)d\theta \\ Q_\ell^1(Z|P) &= \int_{\theta<\theta_m:\lambda_L^1(\theta)\in Z\setminus W^1(P)} h^1(\theta)d\theta + \int_{\theta\geq\theta_m:\lambda_R^1(\theta)\in Z\setminus W^1(P)} h^1(\theta)d\theta, \end{aligned}$$

where $Q_w^1(Z|P)$ represents probability mass on winning policies and $Q_\ell^1(Z|P)$ will be used to assign probability mass to losing policies. Define the updated distribution as

$$\tilde{P}^0 \quad = \quad Q_w^1(\cdot|P) + (1-\delta)Q_\ell^1(\cdot|P) + \delta Q_\ell^1(X|P)P^1. \tag{8}$$

In words, we allocate probability mass to Z for each type that is lobbied to a winning policy in the set, because those politicians stay in office and choose that policy thereafter. We also allocate probability to Z for losing types that choose a policy in the set, but now discounted, because such a politician holds office for only one term, and subsequently an amount of probability is allocated via the challenger distribution P^1. Of course, the updated distribution $\tilde{P}^1$ has the same form,

$$\tilde{P}^1 \quad = \quad Q^0_w(\cdot|P) + (1-\delta)Q^0_\ell(\cdot|P) + \delta Q^0_\ell(X|P)P^0, \tag{9}$$

where now future probability mass is allocated according to via the challenger distribution P^0.

Fixed point argument: In the above analysis, we have constructed a mapping $\phi\colon \Delta(X)^2 \to \Delta(X)^2$ as follows: for each $P = (P^0, P^1) \in \Delta(X)^2$, we define

$$\phi(P) \quad = \quad (\tilde{P}^0, \tilde{P}^1)$$

to consist of the updated continuation distributions, revised to account for optimal lobbying, policy choice, and voting given P, as in (8) and (9). Thus far, the distributions $P = (P^0, P^1)$ have been fixed arbitrarily, and so the analysis has abstracted from equilibrium considerations. The final step will be to use the mapping ϕ to obtain a fixed point, i.e., a pair P of distributions such that $\phi(P) = P$, which will yield a simple lobbying equilibrium.

The mathematical technology used in the proof is Schauder's fixed point theorem, which requires the following: $\Delta(X)^2$ must be a non-empty, convex, compact subset of a locally Hausdorff linear space, and the mapping ϕ must be continuous. The first half of this requirement is delivered by well-known properties of the set of Borel probability measures on a compact subset of finite-dimensional Euclidean space, endowed with the topology of weak convergence. The crux of the proof is, therefore, to demonstrate that the mapping ϕ is continuous.

To this end, we consider a convergent sequence $\{P^m\}$ of pairs $P^m = (P^{0,m}, P^{1,m})$ in $\Delta(X)^2$. Letting $P = (P^0, P^1)$ denote the limit of this sequence, we then have $P^{0,m} \to P^0$ and $P^{1,m} \to P^1$ weakly. We must show that $\phi(P^m) \to \phi(P)$, where convergence in each component is in the sense of weak convergence. We write the values of ϕ along the sequence as $\phi(P^m) = (\tilde{P}^{0,m}, \tilde{P}^{1,m})$ and the value at P as $\phi(P) = (\tilde{P}^0, \tilde{P}^1)$. Then we must show that $\tilde{P}^{0,m} \to \tilde{P}^0$ and $\tilde{P}^{1,m} \to \tilde{P}^1$ weakly. Because the argument for the latter case is analogous, we will argue for the former, i.e., $\tilde{P}^{0,m} \to \tilde{P}^0$ weakly. For this, it suffices to consider any closed set $Z \subseteq X$ and to show that

$$\limsup \tilde{P}^{0,m}(Z) \quad \leq \quad \tilde{P}^0(Z).$$

Since $P^{1,m} \to P^1$ weakly, it follows that $\limsup P^{1,m}(Z) \leq P^1(Z)$. Thus, it suffices to show that $Q^1_w(Z|P^m) \to Q^1_w(Z|P)$ and $Q^1_\ell(Z|P^m) \to Q^1_\ell(Z|P)$.

Define the indicator functions $I\colon N \to \mathbb{R}$ and $I^m\colon N \to \mathbb{R}$ by

$$I_w(\theta) \quad = \quad \begin{cases} 1 & \text{if } \theta < \theta_m \text{ and } \lambda^1_L(\theta|P) \in Z \cap W^1(P) \\ 1 & \text{if } \theta \geq \theta_m \text{ and } \lambda^1_R(\theta|P) \in Z \cap W^1(P), \\ 0 & \text{else,} \end{cases}$$

and

$$I_w^m(\theta) \;=\; \begin{cases} 1 & \text{if } \theta < \theta_m \text{ and } \lambda_L^1(\theta|P^m) \in Z \cap W^1(P^m) \\ 1 & \text{if } \theta \geq \theta_m \text{ and } \lambda_R^1(\theta|P^m) \in Z \cap W^1(P^m), \\ 0 & \text{else,} \end{cases}$$

$m = 1, 2, \ldots$. Then we can write

$$Q_w^1(Z|P) \;=\; \int_N I_w(\theta) h^1(\theta) d\theta \quad \text{and} \quad Q_w^1(Z|P^m) \;=\; \int_N I_w^m(\theta) h^1(\theta) d\theta.$$

Now, consider any $\theta \in N$ such that $\theta \neq \overline{\theta}_w^1(P)$, so that the lobby group R has a strict preference to lobby the type θ politician to a winning policy, $\lambda_R^1(\theta|P) \in W^1(P)$, or a losing policy $\lambda_R^1(\theta|P) \notin W^1(P)$. In either case, continuity of the lobby group's optimal offer implies $\lambda_R^1(\theta|P^m) \to \lambda_R^1(\theta|P)$. Thus, if $\lambda_R^1(\theta|P)$ is winning, then $\lambda_R^1(\theta|P^m)$ is winning for high enough m; and, if $\lambda_R^1(\theta|P) \notin W^1(P)$, then $\lambda_R^1(\theta|P^m) \notin W^1(P^m)$ for sufficiently high m. Likewise, for all $\theta \neq \underline{\theta}_w^1(P)$, $\lambda_L^1(\theta|P^m)$ is winning for sufficiently high m if $\lambda_L^1(\theta|P)$ is winning, and $\lambda_L^1(\theta|P^m)$ is losing for sufficiently high m if $\lambda_L^1(\theta|P)$ is.

We have established that the functions I_w^m converge pointwise almost everywhere to the function I_w. By Lebesgue's dominated convergence theorem, we conclude that the integrals converge, and thus $Q_w^1(Z|P^m) \to Q_w^1(Z|P)$. An analogous argument, defining I_ℓ and I_ℓ^m using the set losing policies, $X \setminus W^1(P)$ and $X \setminus W^1(P^m)$ respectively, establishes that $Q_\ell^1(Z|P^m) \to Q_\ell^1(Z|P)$. Therefore, ϕ is a continuous map from $\Delta(X)^2$ into itself, and Schauder's fixed point theorem implies the existence of a fixed point P^*, i.e., a pair $P^* = (P^{*,0}, P^{*,1})$ such that $\phi(P^*) = P^*$. More explicitly, we have

$$P^{*,\pi} \;=\; Q_w^{1-\pi}(\cdot|P^*) + (1-\delta) Q_\ell^{1-\pi}(\cdot|P^*) + \delta Q_\ell^{1-\pi}(X|P^*) P^{*,1-\pi}. \tag{10}$$

for $\pi \in \{0, 1\}$.

Existence of equilibrium: Given the fixed point $P^* = (P^{*,0}, P^{*,1})$ of the mapping ϕ, we define the assessment $\Psi = (\sigma, \kappa)$ so that the strategy profile $\sigma = (\lambda, \mu, \alpha, \xi, \nu)$ is such that all citizens use the optimal strategies derived above given P^*, and κ is derived from Bayes rule when possible. That is, for every type θ and party π with active group G, we have $\lambda_G^\pi(\theta) = \lambda_G^\pi(\theta|P^*)$; each politician type θ from each party π accepts (y, m) if and only if

$$U_\theta^\pi(y|P^*) + \gamma m + B^\pi(y|P^*) \;\;\geq\;\; U_\theta^\pi(\xi_\theta^\pi(P^*)|P^*) + B^\pi(\xi_\theta^\pi(P^*)|P^*);$$

for each type θ and party π, the politician's default choice is $\xi_\theta^\pi = \xi_\theta^\pi(P^*)$; and the win set is $W^\pi(\Psi) = W^\pi(P^*)$. The belief system κ is pinned down by Bayes rule unless the policy chosen by the incumbent is off the path of play. These policies comprise two intervals, $[\underline{c}^\pi(P^*), \underline{w}^\pi(P^*))$ and $(\overline{w}^\pi(P^*), \overline{c}^\pi(P^*)]$, as politicians with ideal points in these regions are expected, in equilibrium, to compromise. Following such out-of-equilibrium policy choices, if the set of shirkers is nonempty, then we specify beliefs so that voters place probability one on the incumbent being a type θ such that $x(\theta) \in S^{\pi}(P^*)$. When $\delta > 0$ and β is sufficiently large, it may be that there are no shirkers, i.e., $\underline{c}^{\pi}(P^*) = 0$ and $\overline{c}^{\pi}(P^*) = 1$, in which case some type is minimally acceptable: either the type $\underline{\theta}$ politician chooses $\underline{w}^{\pi}(P^*)$ or the type $\overline{\theta}$ politician chooses $\overline{w}^{\pi}(P^*)$ or both. The median voter is indifferent between re-electing a minimally acceptable incumbent or replacing her with a challenger, and we specify voters' beliefs following out of equilibrium deviations so that they believe the incumbent is minimally acceptable with probability one.

To check that this constitutes a simply lobbying equilibrium, it suffices to show that the induced continuation values in (5) are in fact the challenger continuation values determined by σ. To confirm that $V_{\theta}^{C,\pi}(P^*) = \frac{\mathbb{E}_{P^*,0}[u_\theta(x)]}{1-\delta}$ is correctly specified for $\pi \in \{0,1\}$, we integrate $u_\theta(x)$ with respect to $P^{*,\pi}$ from (10) to find

$$\begin{aligned}\frac{\mathbb{E}_{P^*,\pi}[u_\theta(x)]}{1-\delta} &= \int_{\theta' \in \mathcal{E}^{1-\pi}(P^*)} \frac{u_\theta(\lambda_{G(\theta')}^{1-\pi}(\theta'))}{1-\delta} h^{1-\pi}(\theta')d\theta' \\ &+ \int_{\theta' \notin \mathcal{E}^{1-\pi}(P^*)} \left[u_\theta(\lambda_{G(\theta')}^{1-\pi}(\theta')) + \delta \frac{\mathbb{E}_{P^*,1-\pi}[u_\theta(x)]}{1-\delta}\right] h^{1-\pi}(\theta')d\theta',\end{aligned}$$

where we use a change of variables to integrate with respect to the density $h^{1-\pi}(\theta')$, rather than the distribution $Q^{1-\pi}(\cdot|P^*)$. Since $\mathcal{E}^{\pi}(P^*) = \mathcal{E}^{\pi}(\Psi)$, it follows that $V_{\theta}^{C,0}(P^*)$ satisfies the recursive system in (2) that uniquely identifies the challenger continuation values determined by the strategy profile σ, and we conclude that $V_{\theta}^{C,\pi}(\Psi) = V_{\theta}^{C,\pi}(P^*)$ for each $\pi \in \{0,1\}$.

A.2 Partitional Form: Proof of Theorem 2

Let $\Psi = (\sigma, \mu)$ be any simple lobbying equilibrium, and let $P^{C,0}$ and $P^{C,1}$ be the challenger continuation distributions given an incumbent from party $\pi = 0, 1$. The existence proof in the preceding subsection takes an arbitrary such pair of distributions as given and shows that the induced win set is an interval $[\underline{w}^{\pi}, \overline{w}^{\pi}]$; that optimal default policies are determined by compromising cutoffs $\underline{c}^{\pi}$ and $\overline{c}^{\pi}$ such that a politician whose ideal point belongs to the interval $[\underline{c}^{\pi}, \overline{c}^{\pi}]$ chooses the winning policy closest to her ideal point; and that optimal lobby offers are determined by cutoff types $\underline{\theta}_e^{\pi}$ and $\overline{\theta}_e^{\pi}$ such that group R offers a winning policy to any type $\theta \in [\theta_m, \overline{\theta}_e^{\pi}]$ and a losing policy to types $\theta > \overline{\theta}_e^{\pi}$, and symmetrically for group L. Letting $\overline{e}^{\pi} = x(\overline{\theta}_e^{\pi})$ and $\underline{e}^{\pi} = x(\underline{\theta}_e^{\pi})$, we see that the equilibrium is pinned down by 6-tuples $(\underline{c}^{\pi}, \underline{e}^{\pi}, \underline{w}^{\pi}, \overline{w}^{\pi}, \overline{e}^{\pi}, \overline{c}^{\pi})$, $\pi \in \{0,1\}$. Moreover, by construction we have $\underline{c}^{\pi} \leq \underline{w}^{\pi}$ and $\overline{w}^{\pi} \leq \overline{c}^{\pi}$. Thus, to conclude that the equilibrium has the partitional form, it suffices to show that $\underline{c}^{\pi} \leq \underline{e}^{\pi}$ and $\overline{e}^{\pi} \leq \overline{c}^{\pi}$. That is, we must show that it is never the case that a politician would shirk by default but is lobbied to a winning policy.

We focus on the inequality $\overline{e}^{\pi} \leq \overline{c}^{\pi}$, as the argument for the remaining inequality is analogous. It suffices to show, in turn, that the type $\overline{\theta}_e^{\pi}$ politician strictly prefers to compromise in lieu of a lobby offer, i.e.,

$$\frac{u_{\overline{\theta}_e^\pi}(\overline{w}^\pi) + \beta}{1-\delta} \quad \geq \quad u_{\overline{\theta}_e^\pi}(x(\theta)) + \beta + \delta V^{C,\pi}_{\overline{\theta}_e^\pi}(\Psi).$$

Suppose toward a contradiction that the above inequality does not hold. Note that $x(\overline{\theta}_e^\pi)$ is losing. Recall from the proof of Theorem 1 that the active group is just indifferent between lobbying the type $\overline{\theta}_e^\pi$ politician to the best winning policy and lobbying the politician to the surplus maximizing policy. That is, $\Phi_R^\pi(\overline{\theta}_e^\pi) = 0$, or expanding, we have

$$\begin{aligned} & U_R^\pi(\overline{w}^\pi|\Psi) + \frac{1}{\gamma}\left[U_{\overline{\theta}}^\pi(\overline{w}^\pi|\Psi) + B^\pi(\overline{w}^\pi|\Psi)\right] \\ = \quad & U_R^\pi(y_\ell^\pi(\overline{\theta}_e^\pi)|\Psi) + \frac{1}{\gamma}\left[U_{\overline{\theta}}^\pi(y_\ell^\pi(\overline{\theta}_e^\pi)|\Psi) + B^\pi(y_\ell^\pi(\overline{\theta}_e^\pi)|\Psi)\right]. \end{aligned}$$

Expanding further, this is

$$\begin{aligned} & \frac{u_{\overline{\theta}}(\overline{w}^\pi)}{1-\delta} + \frac{1}{\gamma}\left[\frac{u_{\overline{\theta}_e^\pi}(\overline{w}^\pi) + \beta}{1-\delta}\right] \\ = \quad & u_{\overline{\theta}}(y_\ell^\pi(\overline{\theta}_e^\pi)) + \delta V^{C,\pi}_{\overline{\theta}}(\Psi) + \frac{1}{\gamma}\left[u_{\overline{\theta}_e^\pi}(y_\ell^\pi(\overline{\theta}_e^\pi)) + \delta V^{C,\pi}_{\overline{\theta}_e^\pi}(\Psi) + \beta\right]. \end{aligned}$$

Since $x(\overline{\theta}_e^\pi)$ is losing and $y_\ell^\pi(\overline{\theta}_e^\pi)$ maximizes joint surplus over losing policies, this implies

$$\begin{aligned} & \frac{u_{\overline{\theta}}(\overline{w}^\pi)}{1-\delta} + \frac{1}{\gamma}\left[\frac{u_{\overline{\theta}_e^\pi}(\overline{w}^\pi) + \beta}{1-\delta}\right] \\ \geq \quad & u_{\overline{\theta}}(x(\overline{\theta}_e^\pi)) + \delta V^{C,\pi}_{\overline{\theta}}(\Psi) + \frac{1}{\gamma}\left[u_{\overline{\theta}_e^\pi}(x(\overline{\theta}_e^\pi)) + \delta V^{C,\pi}_{\overline{\theta}_e^\pi}(\Psi) + \beta\right]. \end{aligned}$$

By supposition, the type $\overline{\theta}_e^\pi$ politician strictly prefers to shirk in the absence of a lobby offer, i.e.,

$$\frac{u_{\overline{\theta}_e^\pi}(\overline{w}^\pi) + \beta}{1-\delta} \quad < \quad u_{\overline{\theta}_e^\pi}(x(\overline{\theta}_e^\pi)) + \beta + \delta V^{C,\pi}_{\overline{\theta}_e^\pi}(\Psi), \tag{11}$$

and therefore we have

$$\frac{u_{\overline{\theta}}(\overline{w}^\pi)}{1-\delta} + \frac{1}{\gamma}\left[\frac{u_{\overline{\theta}_e^\pi}(\overline{w}^\pi) + \beta}{1-\delta}\right] \quad > \quad u_{\overline{\theta}}(x(\overline{\theta}_e^\pi)) + \delta V^{C,\pi}_{\overline{\theta}}(\Psi) + \frac{1}{\gamma}\left[\frac{u_{\overline{\theta}_e^\pi}(\overline{w}^\pi) + \beta}{1-\delta}\right].$$

Canceling terms, this becomes

$$\frac{u_{\overline{\theta}}(\overline{w}^\pi)}{1-\delta} \quad > \quad u_{\overline{\theta}}(x(\overline{\theta}_e^\pi)) + \delta V^{C,\pi}_{\overline{\theta}}(\Psi). \tag{12}$$

For any type θ, define

$$\Gamma(\theta) \quad = \quad \frac{u_\theta(\overline{w}^\pi)}{1-\delta} - [u_\theta(x(\overline{\theta}_e^\pi)) + \delta V_\theta^{C,\pi}(\Psi)]$$

as the net benefit to the type θ politician from compromising at $\overline{w}^\pi$ rather than shirking by choosing the policy $x(\overline{\theta}_e^\pi)$. The derivative of this function is

$$\frac{d}{d\theta}\Gamma(\theta) \quad = \quad \frac{v(\overline{w}^\pi)}{1-\delta} - v(x(\overline{\theta}_e^\pi)) - \delta \mathbb{E}_{P^\pi}[v(x)],$$

which is independent of θ, i.e., Γ is affine linear. But clearly $\Gamma(\overline{\theta}_w^\pi) > 0$, and (11) implies $\Gamma(\overline{\theta}_e^\pi) < 0$, and (12) implies $\Gamma(\overline{\theta}) > 0$, a contradiction. We conclude that $\overline{e}^\pi \leq \overline{c}^\pi$, and thus the equilibrium has the partitional form, as required.

A.3 Polarization and the Dynamic Median Voter Theorem: Proof of Theorems 3 and 4

We prove Theorem 3 in four steps and complete the proof of Theorem 4 in a fifth.

Step 1: We first show that when $\delta\beta$ is high, each lobby group offers only winning policies, i.e., for all π and all θ, $\lambda_{G(\theta)}^\pi(\theta) \in W^\pi(\Psi)$. Note that in a simple lobbying equilibrium, the median ideal point is always winning: $x_m \in W^\pi(\Psi)$. Thus, in the absence of a lobby offer, a politician's optimal payoff from compromise can be no worse than the payoff from choosing the median ideal point, and so the net benefit of compromise for a type θ politician from party π is at least equal to

$$\begin{aligned} & \frac{u_\theta(x_m)+\beta}{1-\delta} - [u_\theta(x(\theta)) + \beta + \delta V_\theta^{C,\pi}(\Psi)] \\ \geq \quad & \left[\frac{u_\theta(x_m)}{1-\delta} + \frac{\delta\beta}{2(1-\delta)} - \frac{u_\theta(x(\theta))}{1-\delta}\right] + \frac{\delta\beta}{2(1-\delta)}, \end{aligned}$$

where we use the fact that the continuation value of a challenger must be less than the politician's ideal payoff. Note that we can choose $\delta\beta > 2(u_\theta(x(\theta)) - u_\theta(x_m))$ for all types θ to make the first term in brackets positive and the second term arbitrarily large. In particular, the default policy choice of every politician type is winning when office incentives are sufficiently high.

Now consider the problem of the active lobby group $G = G(\theta)$ when office incentives are high, and suppose the lobby group's offer $\lambda_G^\pi(\theta)$ is not winning. Then the participation constraint of the politician requires that the group compensate the politician for losing. Letting ξ_θ^π denote the default policy of the politician, the payment to the politician must be

$$\begin{aligned} & \frac{1}{\gamma}\left[\frac{u_\theta(\xi_\theta^\pi)+\beta}{1-\delta} - [u_\theta(x(\theta)) + \beta + \delta V_\theta^{C,\pi}(\Psi)]\right] \\ \geq \quad & \frac{1}{\gamma}\left[\frac{u_\theta(x_m)}{1-\delta} + \frac{\delta\beta}{2(1-\delta)} - \frac{u_\theta(x(\theta))}{1-\delta}\right] + \frac{\delta\beta}{2\gamma(1-\delta)}, \end{aligned}$$

where we use the fact that the default policy is winning and $u_\theta(\xi_\theta^\pi) \geq u_\theta(x_m)$. Of course, the offer

$\lambda^{\pi}_{G}(\theta)$ cannot be better for the lobby group than its ideal point, x_G, and the continuation value of a challenger must be less than the group's ideal payoff. Thus, the equilibrium payoff to the lobby group from offering $\lambda^{\pi}_{G}(\theta)$ is less than or equal to

$$\begin{aligned}\text{lobby}_{equil} &= \frac{u_G(x_G)}{1-\delta} - \frac{\delta\beta}{4\gamma(1-\delta)} - \frac{1}{\gamma}\left[\frac{u_\theta(x_m)}{1-\delta} + \frac{\delta\beta}{2(1-\delta)} - \frac{u_\theta(x(\theta))}{1-\delta}\right] \\ &\quad - \frac{\delta\beta}{4\gamma(1-\delta)}.\end{aligned}$$

But the lobby group could instead offer the default policy ξ^{π}_{θ}, along with a payment of zero, effectively choosing not to lobby. Since the default policy is closer to x_G than the median policy, the lobby group's payoff from this deviation is no less than

$$\text{lobby}_{dev} = \frac{u_G(x_m)}{1-\delta}.$$

Comparing these payoffs, we find

$$\begin{aligned}\text{lobby}_{equil} - \text{lobby}_{dev} &= \left[\frac{u_G(x_G)}{1-\delta} - \frac{\delta\beta}{4\gamma(1-\delta)} - \frac{u_G(x_m)}{1-\delta}\right] \\ &\quad - \frac{1}{\gamma}\left[\frac{u_\theta(x_m)}{1-\delta} + \frac{\delta\beta}{2(1-\delta)} - \frac{u_\theta(x(\theta))}{1-\delta}\right] - \frac{\delta\beta}{4\gamma(1-\delta)}.\end{aligned}$$

Choosing $\delta\beta$ to also satisfy $\delta\beta > 4\gamma(u_G(x_G) - u_G(x_m))$, the first term in brackets above is negative, the second term in brackets is positive, and the third term can be made arbitrarily large in magnitude. This implies that the lobby group can profitably deviate by refraining from lobbying, avoiding the necessity of compensating the politician for lost office benefit, but this is impossible in equilibrium. We conclude that when office incentives are sufficiently high, lobby groups always offer winning policies. Importantly for the proof of Theorem 4, when $\delta\beta$ is chosen in the above manner for a given value $\gamma > 0$, these conclusions carry over to all smaller $\gamma' \in (0, 1)$. This completes the first step.

Step 2: Next, we argue that the challenger continuation value for the median voter is independent of the incumbent's party. Recall that the win set consists of the policies x that give the median voter a payoff at least equal to her continuation value of a challenger, i.e., $\frac{u_m(x)}{1-\delta} \geq V^{C,\pi}_m(\Psi)$. We have shown that the challenger is always offered a winning policy and remains in office thereafter, regardless of her type, so that the median voter's continuation value of a challenger takes the simple form

$$V^{C,\pi}_m(\Psi) = \frac{1}{1-\delta}\int_N u_m(\lambda^{1-\pi}_{G(\theta)}(\theta))h^{1-\pi}(\theta)d\theta.$$

Since $u_m(\lambda^{1-\pi}_{G(\theta)}(\theta)) \geq (1-\delta)V^{C,1-\pi}_m(\Psi)$ for all θ, this implies that $V^{C,\pi}_m(\Psi) \geq V^{C,1-\pi}_m(\Psi)$. A symmetric argument for the opposite inequality then yields $V^{C,\pi}_m(\Psi) = V^{C,1-\pi}_m(\Psi)$, and we henceforth drop the incumbent's party from the superscript, writing simply $V^{C}_m(\Psi)$ for the continuation value of a challenger.

Step 3: Next, we show that with probability one, only the endpoints, $\underline{w}^{\pi}$ and $\overline{w}^{\pi}$, are offered in equilibrium, that median indifference holds, and that the win set is independent of the incumbent's party. Since $u_m(\lambda^{1-\pi}_{G(\theta)}(\theta)) \geq V^C_m(\Psi)$ holds for $h^{1-\pi}$-almost all θ and

$$V^C_m(\Psi) = \frac{1}{1-\delta}\int_N u_m(\lambda^{1-\pi}_{G(\theta)}(\theta))h^{1-\pi}(\theta)d\theta,$$

we conclude that in fact the equality

$$V^C_m(\Psi) = \frac{u_m(\lambda^{1-\pi}_{G(\theta)}(\theta))}{1-\delta}$$

holds for $h^{1-\pi}$-almost all types θ. In particular, using (3), we have

$$\frac{u_m(\lambda^{1-\pi}_L(\theta))}{1-\delta} = V^C_m(\Psi) = \frac{u_m(\underline{w}^{1-\pi})}{1-\delta}$$

for $h^{1-\pi}$-almost all $\theta < \theta_m$, and thus $\lambda^{1-\pi}_L(\theta) = \underline{w}^{1-\pi}$. For $h^{1-\pi}$-almost all $\theta \geq \theta_m$, we have

$$\frac{u_m(\lambda^{1-\pi}_R(\theta))}{1-\delta} = V^C_m(\Psi) \leq \frac{u_m(\overline{w}^{1-\pi})}{1-\delta}$$

By (3), the latter inequality can hold strictly only if $\overline{w}^{\pi} = 1$, but $x(\theta_m) \leq \lambda^{1-\pi}_R(\theta) \leq 1$, so we cannot have $u_m(\lambda^{1-\pi}_R(\theta)) < u_m(1)$. We conclude that

$$\frac{u_m(\lambda^{1-\pi}_R(\theta))}{1-\delta} = V^C_m(\Psi) = \frac{u_m(\overline{w}^{1-\pi})}{1-\delta},$$

and thus for $h^{1-\pi}$-almost all $\theta \geq \theta_m$, we have $\lambda^{1-\pi}_R(\theta) = \overline{w}^{1-\pi}$.

An analogous argument for party π shows that for h^{π}-almost all types $\theta < \theta_m$, we have

$$\frac{u_m(\lambda^{\pi}_L(\theta))}{1-\delta} = V^C_m(\Psi) = \frac{u_m(\underline{w}^{\pi})}{1-\delta},$$

so that $\lambda^{\pi}_L(\theta) = \underline{w}^{\pi}$, and for h^{π}-almost all types $\theta \geq \theta_m$, we have

$$\frac{u_m(\lambda^{\pi}_R(\theta))}{1-\delta} = V^C_m(\Psi) = \frac{u_m(\overline{w}^{\pi})}{1-\delta},$$

so that $\lambda^{\pi}_R(\theta) = \overline{w}^{\pi}$. Thus, with probability one, lobby groups offer only the endpoints of the win set. Finally, the assumption that the median type belongs to the support of the challenger densities allows us to conclude that $u_m(\underline{w}^{\pi}) = u_m(\overline{w}^{\pi})$ for each $\pi \in \{0, 1\}$, so that median indifference holds, and that in fact $\underline{w}^{\pi} = \underline{w}^{1-\pi}$ and $\overline{w}^{\pi} = \overline{w}^{1-\pi}$, so that the win set is independent of π. This establishes that when $\delta\beta$ is sufficiently large, every simple lobbying equilibrium is polarized.

Step 4: To complete the proof of Theorem 3, assume without loss of generality that the inequality $u_m(x_L(\theta_m)) \geq u_m(x_R(\theta_m))$ holds, i.e., the policy that maximizes joint surplus with group R is at least as good for the median voter as the policy that maximizes joint surplus with L.[18] Let $\underline{w}$ and $\overline{w}$ satisfy $\underline{w} < x_m < \overline{w}$, $x_L(\theta_m) \leq \underline{w}$, $\overline{w} \leq x_R(\theta_m)$, and $u_m(\underline{w}) = u_m(\overline{w})$. Choosing $\delta\beta$ as above, we specify σ so that the default policy of each type θ of politician is the closest policy in the win set to the ideal point $x(\theta)$. Specify lobby offers so that for all $\theta \leq \theta_m$, group L offers $\underline{w}$ and compensates the politician for choosing $\underline{w}$ rather than ξ_θ; and for all $\theta \geq \theta_m$, group R offers $\overline{w}$ and compensates the politician. And we specify the win set as $W = [\underline{w}, \overline{w}]$. Beliefs μ are specified as in the proof of existence. This specification satisfies the conditions for simple lobbying equilibrium, and in particular, the optimal offer for each lobby group is to pull every politician type to the endpoint of the win set closest to the ideal point x_G of the group, and the median voter is indifferent between re-electing an incumbent who chooses $\underline{w}$ or $\overline{w}$ and replacing the incumbent with a challenger. The most polarized equilibrium is given by the choice of $\underline{w}$ and $\overline{w}$ such that $\underline{w} < \theta_m < \overline{w}$ and $u_m(\underline{w}) = u_m(\overline{w})$ and such that one (or both) of the inequalities $x_L(\theta_m) \leq \underline{w}$ and $\overline{w} \leq x_R(\theta_m)$ bind. This completes the proof of Theorem 3.

Step 5: Theorem 4 follows by the fact that the equilibrium conditions are maintained as $\gamma \to 0$, and because the surplus-maximizing policies $x_L(\theta_m)$ and $x_R(\theta_m)$ converge to the median policy as $\gamma \to 0$. Thus, the inequalities in (4) at policies approaching the median, which implies $\underline{w} \to x_m$ and $\overline{w} \to x_m$, as required.

A.4 Extremely Polarized Equilibria: Proofs of Theorems 6 and 8

Let γ become large, and consider any selection of simple lobbying equilibria. We go to a subsequence such that the win set given an incumbent from each party π converges: $\underline{w}^\pi \to \underset{\sim}{w}^\pi$ and $\overline{w}^\pi \to \tilde{w}^\pi$. To show that equilibria become polarized, we consider one party π and begin by observing that by (3), there are three cases: (i) $\underset{\sim}{w}^\pi = 0$ and $\tilde{w}^\pi = 1$, or (ii) $\underset{\sim}{w}^\pi > 0$ and $\tilde{w}^\pi < 1$, or (iii) $\underset{\sim}{w}^\pi > 0$ and $\tilde{w}^\pi = 1$. Note that for every politician type $\theta \geq \theta_m$, the maximizer of joint surplus with lobby group R goes to one as γ becomes large, and likewise for types $\theta < \theta_m$ and group L, i.e., $x_R(\theta) \to 1$ and $x_L(\theta) \to 0$. Furthermore, in case (iii), we have

$$u_m(0) \;<\; u_m(\underset{\sim}{w}^\pi) \;\leq\; u_m(\tilde{w}^\pi) \;=\; u_m(1),$$

so that the median voter is not indifferent between the lobby groups. Thus, the proof of Theorem 6 considers only cases (i) and (ii), while Theorem 8 is proved following case (iii).

Case (i): $\underset{\sim}{w}^\pi = 0$ and $\tilde{w}^\pi = 1$. Note that lobby offers then converge to the extremes of the

[18] Since we have established median indifference, $u_m(\underline{w}) = u_m(\overline{w})$, our assumption that $u_m(1) \geq u_m(0)$ does not affect the analysis, so this hypothesis is indeed without loss of generality for this argument.

policy space: $\lambda_L^\pi(\theta_m) \to 0$ and $\lambda_R^\pi(\theta_m) \to 1$. From (2), we have

$$\begin{aligned}
\frac{u_m(\underline{w}^\pi)}{1-\delta} \;=\; V_m^{C,\pi}(\Psi) \;&\geq\; \int_{\theta' \in \mathcal{E}^{1-\pi}(\Psi)} \frac{u_m(\underline{w}^{1-\pi})}{1-\delta} h^{1-\pi}(\theta') d\theta' \\
&\quad + \int_{\theta' \notin \mathcal{E}^{1-\pi}(\Psi)} \left[u_m(0) + \delta V_m^{C,1-\pi}(\Psi)\right] h^{1-\pi}(\theta') d\theta' \\
&= \int_{\theta' \in \mathcal{E}^{1-\pi}(\Psi)} V_m^{C,1-\pi}(\Psi) h^{1-\pi}(\theta') d\theta' \\
&\quad + \int_{\theta' \notin \mathcal{E}^{1-\pi}(\Psi)} \left[u_m(0) + \delta V_m^{C,1-\pi}(\Psi)\right] h^{1-\pi}(\theta') d\theta',
\end{aligned}$$

where we use $V_m^{C,1-\pi}(\Psi) = \frac{u_m(\underline{w}^{1-\pi})}{1-\delta}$ in the equality. Since $u_m(\underline{w}^\pi) \to u_m(0)$, the above inequality yields

$$\frac{u_m(0)}{1-\delta} \;=\; \lim_{\gamma\to\infty} \frac{u_m(\underline{w}^{1-\pi})}{1-\delta} \;=\; \lim_{\gamma\to\infty} V_m^{C,1-\pi}(\Psi),$$

which implies that $\underset{\sim}{w}^{1-\pi} = 0$ and $\tilde{w}^{1-\pi} = 1$. By an analogous argument, we find that

$$\lim_{\gamma\to\infty} V_m^{C,\pi}(\Psi) \;=\; \frac{u_m(0)}{1-\delta}.$$

Furthermore, this implies that lobby offers to incumbents from party $1-\pi$ converge to the extreme policies: $\lambda_L^{1-\pi}(\theta_m) \to 0$ and $\lambda_R^{1-\pi}(\theta_m) \to 1$.

Now, consider the choice of default policy by an extreme politician type $\theta \in \{\underline{\theta}, \overline{\theta}\}$. By the above arguments, we have shown that

$$\lim_{\gamma\to\infty} V_\theta^{C,\pi}(\Psi) \;=\; H^{1-\pi}(\theta_m) u_\theta(0) + (1 - H^{1-\pi}(\theta_m)) u_\theta(1) \;<\; \frac{u_\theta(x(\theta))}{1-\delta},$$

reflecting the fact that if an incumbent from party π is replaced by a challenger from party $1-\pi$, then she is lobbied to policies close to zero by lobby group L and to policies close to one by lobby group R. Then we have for type $\overline{\theta}$,

$$\lim_{\gamma\to\infty} \frac{u_{\overline{\theta}}(\overline{w}^\pi) + \beta}{1-\delta} \;=\; \lim_{\gamma\to\infty} \frac{u_{\overline{\theta}}(x(\overline{\theta})) + \beta}{1-\delta} \;>\; \lim_{\gamma\to\infty} V_{\overline{\theta}}^{C,\pi}(\Psi),$$

so that inequality (6) holds strictly for γ sufficiently large (substituting the equilibrium challenger distributions for P). Using an analogous argument for type $\underline{\theta}$, we conclude that when γ is sufficiently large, all politician types are either winners or compromisers, i.e., $\underline{c}^\pi = 0$ and $\overline{c}^\pi = 1$, and by a symmetric argument, $\underline{c}^{1-\pi} = 0$ and $\overline{c}^{1-\pi} = 1$.

Referring to the analysis of group R's optimal lobby offer in the proof of Theorem 1 in Subsection

A.1, recall that the lobby group's payoff from the optimal winning policy is

$$\frac{1}{1-\delta}\left[u_{\overline{\theta}}(y_w^\pi(\theta)) + \frac{1}{\gamma}u_\theta(y_w^\pi(\theta)) - \frac{1}{\gamma}u_\theta(x(\theta))\right],$$

where the optimal offer $y_w^\pi(\theta)$ is evaluated at the equilibrium challenger distributions, and the payoff from the optimal losing policy is

$$u_{\overline{\theta}}(y_\ell^\pi(\theta)) + \delta V_{\overline{\theta}}^{C,\pi}(\Psi) + \frac{1}{\gamma}\left(u_\theta(y_\ell^\pi(\theta)) + \delta V_\theta^{C,\pi}(\Psi)\right) - \frac{1}{\gamma}\left(\frac{u_\theta(x(\theta))}{1-\delta} + \frac{\delta\beta}{1-\delta}\right).$$

Also, recall that the sign of the difference, $\Phi_R^\pi(\theta)$, determines whether the group offers a winning or losing policy. This difference is at least

$$\frac{1}{1-\delta}\left[u_{\overline{\theta}}(y_w^\pi(\theta)) + \frac{1}{\gamma}u_\theta(y_w^\pi(\theta))\right] - u_{\overline{\theta}}(y_\ell^\pi(\theta)) - \frac{1}{\gamma}u_\theta(y_\ell^\pi(\theta)) - \delta\left[V_{\overline{\theta}}^{C,\pi}(\Psi) + \delta\frac{1}{\gamma}V_\theta^{C,\pi}(\Psi)\right].$$

Since $\lim y_w^\pi(\theta) = \lim y_\ell^\pi(\theta) = 1$ as $\gamma \to \infty$, this quantity converges uniformly in θ to

$$\delta\left[\frac{u_{\overline{\theta}}(1)}{1-\delta} - (H^{1-\pi}(\theta_m)u_{\overline{\theta}}(0) + (1-H^{1-\pi}(\theta_m))u_{\overline{\theta}}(1))\right],$$

and this limit is strictly positive. Therefore, for γ sufficiently high, lobby group R offers winning policies to every type, and similarly, lobby group L also offers only winning policies; moreover, the same analysis shows that the lobby groups offer only winning policies to incumbents from part $1-\pi$.

Now we can invoke the argument in the proof of Theorem 3 in Subsection A.3. The first step of that argument establishes that the lobby groups offer only winning policies, and the remaining steps leverage that insight to show that simple lobbying equilibria are polarized. The arguments from Steps 2 and 3 apply here, and we conclude that every simply lobbying equilibrium is polarized.

Case (ii): $\underset{\sim}{w}^\pi > 0$ and $\tilde{w}^\pi < 1$. For γ large, if group R offers a winning policy, then the win set constraint binds, and we have $\lambda_R^\pi(\theta) = \overline{w}^\pi$, and losing offers coincide with $x_R(\theta)$ and converge uniformly to one. Likewise, if L offers a winning policy, then it is $\underline{w}^\pi$, and all losing offers, $x_L(\theta)$, converge to zero. Moreover, since $\overline{w}^\pi < 1$ for γ large, (3) implies $u_m(\underline{w}^\pi) = u_m(\overline{w}^\pi)$ in equilibrium. Then the median voter's continuation value of a challenger given an incumbent from party $1-\pi$

satisfies

$$
\begin{aligned}
V_m^{C,1-\pi}(\Psi) &= \int_{\theta'\in\mathcal{E}^\pi(\Psi)} \frac{u_m(\underline{w}^\pi)}{1-\delta} h^\pi(\theta')d\theta' \\
&\quad + \int_{\theta'\notin\mathcal{E}^\pi(\Psi)} [u_m(x_{G(\theta')}(\theta')) + \delta V_m^{C,\pi}(\Psi)] h^\pi(\theta')d\theta' \\
&\le \int_{\theta'\in\mathcal{E}^\pi(\Psi)} \frac{u_m(\underline{w}^\pi)}{1-\delta} h^\pi(\theta')d\theta' \\
&\quad + \int_{\theta'\notin\mathcal{E}^\pi(\Psi)} [u_m(\underline{w}^\pi) + \delta V_m^{C,\pi}(\Psi)] h^\pi(\theta')d\theta' \\
&= V_m^{C,\pi}(\Psi),
\end{aligned}
\tag{13}
$$

where we use (2), translated to party $1-\pi$ and $V_m^{C,\pi}(\Psi) = \frac{u_m(\underline{w}^\pi)}{1-\delta}$.

Since $V_m^{C,1-\pi}(\Psi) = u_m(\underline{w}^{1-\pi})$, we have deduced that

$$
u_m(\overline{w}^{1-\pi}) \;\ge\; u_m(\underline{w}^{1-\pi}) \;\ge\; u_m(\underline{w}^\pi) \;=\; u_m(\overline{w}^\pi) \;>\; u_m(1).
$$

Taking limits, we see that $\underset{\sim}{w}^{1-\pi} > 0$ and $\tilde{w}^{1-\pi} < 1$. Now, by an analogous argument to that above, we deduce that the opposite inequality, $V_m^{C,\pi}(\Psi) \ge V_m^{C,1-\pi}(\Psi)$, holds, and we conclude that the continuation value of a challenger is independent of party, i.e., $V_m^{C,\pi}(\Psi) = V_m^{C,1-\pi}(\Psi)$. Thus, we have in fact

$$
u_m(\overline{w}^\pi) \;=\; u_m(\underline{w}^\pi) \;=\; u_m(\underline{w}^{1-\pi}) \;=\; u_m(\overline{w}^{1-\pi})
$$

for γ large, and it follows that the win set is independent of the incumbent's party; for the remainder of this case, we write it as $[\underline{w}, \overline{w}]$. Substituting into the equality in (13), we then have

$$
\begin{aligned}
u_m(\underline{w}^\pi) &= \int_{\theta'\in\mathcal{E}^\pi(\Psi)} \frac{u_m(\underline{w}^\pi)}{1-\delta} h^\pi(\theta')d\theta' \\
&\quad + \int_{\theta'\notin\mathcal{E}^\pi(\Psi)} \left[u_m(x_{G(\theta')}(\theta')) + \delta\frac{u_m(\underline{w}^\pi)}{1-\delta}\right] h^\pi(\theta')d\theta'.
\end{aligned}
$$

Note that for all $\theta' \notin \mathcal{E}^\pi(\Psi)$, we have $u_m(\underline{w}^\pi) > u_m(x_{G(\theta')}(\theta'))$, and thus the above inequality implies that $\mathcal{E}^\pi(\Psi) = X$, i.e., lobby group L offers almost all politician types the policy $\underline{w}$, and R offers almost all politician types $\overline{w}$. A symmetric argument holds for party $1-\pi$, and we conclude that the equilibrium is polarized.

To complete the proof of Theorem 6, we must show that the most polarized equilibria become extremist; the result does not follow from Corollary 5, because the latter result requires an arbitrarily high choice of $\delta\beta$. To this end, we now fix δ and β and choose any $\epsilon > 0$, and we demonstrate that for sufficiently high γ, there is a polarized equilibrium with win set $[\underline{w}, \overline{w}]$ satisfying $\underline{w} \in (0, \epsilon)$ and $\overline{w} \in (1-\epsilon, 1)$. We specify σ so that the default policy of each type θ of politician is the closest policy in the win set to the ideal point $x(\theta)$. Specify lobby offers so that for all $\theta \le \theta_m$, group L offers $\underline{w}$ and compensates the politician for choosing $\underline{w}$ rather than ξ_θ; and for all $\theta \ge \theta_m$, group R offers $\overline{w}$ and compensates the politician. We specify the win set as $W = [\underline{w}, \overline{w}]$ so that $u_m(\underline{w}) = u_m(\overline{w})$, where $\underline{w}$ and $\overline{w}$ will be chosen close to zero and one, respectively. Beliefs μ are specified as in the proof of existence. To see that this specification satisfies the conditions for simple lobbying equilibrium, we must verify the optimality conditions for default policies and lobby offers.

First, we must show that if the endpoints of the win set are sufficiently extreme, then the default policy choice of every politician type θ is compromising, i.e., $\overline{c}^{\pi} = 1$ and $\underline{c}^{\pi} = 0$ for each party. Thus, it suffices to show that the extreme politician types, $\overline{\theta}$ and $\underline{\theta}$, prefer to compromise. By (6), this holds for type $\overline{\theta}$ if

$$\frac{u_{\overline{\theta}}(\overline{w})}{1-\delta} \quad > \quad u_{\overline{\theta}}(1) + \delta V_{\overline{\theta}}^{C,\pi}(\Psi). \tag{14}$$

By the above construction, the continuation value of a challenger is the expected payoff from a lottery with support $\underline{w}$ and $\overline{w}$, and thus we have $\frac{u_{\overline{\theta}}(\overline{w})}{1-\delta} > V_{\overline{\theta}}^{C,\pi}(\Psi)$. It follows that the desired inequality holds, and that the type $\overline{\theta}$ politician compromises, as long as $\overline{w}$ is close enough to one. Similarly, the type $\underline{\theta}$ politician prefers to compromise if $\underline{w}$ is close enough to zero, as required. Note that this incentive to compromise holds regardless of the magnitude of γ, and that the continuation value $V_{\overline{\theta}}^{C,\pi}(\Psi)$ is determined by the win set only; in particular, it is independent of γ.

Next, we must show that for the choice of win set above, if γ is sufficiently large, then the optimal lobby offers are indeed the endpoints of the win set. Of course, we have $x_L(\theta_m) \to 0$ and $x_R(\theta_m) \to 1$ as $\gamma \to \infty$. Thus, we can choose γ such that for all $\theta \leq \theta_m$, we have $x_L(\theta) \in [0, \underline{w})$; and such that for all $\theta \geq \theta_m$, we have $x_R(\theta) \in (\overline{w}, 1]$. In turn, this implies that if optimal lobby offers are winning, then the lobby groups offer the endpoints of the win set. Referring to the analysis of group R's optimal lobby offer in the proof of Theorem 1 in Subsection A.1, recall that the lobby group offers a winning policy if

$$\frac{1}{1-\delta}\left[u_{\overline{\theta}}(\overline{w}) + \frac{1}{\gamma}u_{\theta}(\overline{w})\right] - u_{\overline{\theta}}(x_R(\theta)) - \frac{1}{\gamma}u_{\theta}(x_R(\theta)) - \delta\left[V_{\overline{\theta}}^{C,\pi}(\Psi) + \delta\frac{1}{\gamma}V_{\theta}^{C,\pi}(\Psi)\right]$$

is positive. Taking the limit as $\gamma \to \infty$, this expression has limit infimum greater than or equal to

$$\frac{u_{\overline{\theta}}(\overline{w})}{1-\delta} - u_{\overline{\theta}}(1) - \delta V_{\overline{\theta}}^{C,\pi}(\Psi) \quad > \quad 0,$$

where the inequality follows from (14), and we use the fact that the continuation value of a challenger is constant in γ.

We conclude that given an interval with endpoints close to the extreme policies (and such that the median voter is indifferent between the endpoints), if the effectiveness of money is sufficiently large, then there is a polarized simple lobbying equilibrium with win set equal to that interval. Since the most polarized equilibrium is at least as extreme, this completes the proof of Theorem 6.

Case (iii): $\underset{\sim}{w}^{\pi} > 0$ and $\tilde{w}^{\pi} = 1$. To prove Theorem 8, we must argue that in this case, every sequence of non-polarized simple lobbying equilibria become weakly extremist. To this end, note that

$$\lim_{\gamma\to\infty} V_m^{C,\pi}(\Psi) \;=\; \lim_{\gamma\to\infty} \frac{u_m(\underline{w}^\pi)}{1-\delta} \;\leq\; \lim_{\gamma\to\infty} \frac{u_m(\overline{w}^\pi)}{1-\delta} \;=\; \frac{u_m(1)}{1-\delta}.$$

Furthermore, we have $\underset{\sim}{w}^{1-\pi} > 0$ and $\tilde{w}^{1-\pi} = 1$, for otherwise the arguments for cases (i) and (ii) would imply that the equilibrium is polarized. And then the inequalities above deliver the inequality $\lim_{\gamma\to\infty} V_m^{C,1-\pi}(\Psi) \leq \frac{u_m(1)}{1-\delta}$, as required.

B Limits of Polarized Equilibria: Proof of Theorem 8

Let γ become large, and consider any selection of simple lobbying equilibria with convergent win sets, i.e., for each party π, we have $\underline{w}^\pi \to \underset{\sim}{w}$ and $\overline{w}^\pi \to \tilde{w}$, where we use the fact that by Theorem 6, equilibria are polarized for γ sufficiently high. We claim that in the dichotomous model, it is a simple lobbying equilibrium for the type $\underline{\theta}$ politician to choose $\underset{\sim}{w}$ and the type $\overline{\theta}$ politician to choose $\tilde{w}$, with win set $[\underset{\sim}{w}, \tilde{w}]$; voter beliefs following off path policies may assign arbitrary probability to types $\underline{\theta}$ and $\overline{\theta}$. The challenger continuation values determined by this specification are given by

$$\tilde{V}_\theta^C \;=\; \frac{H^{1-\pi}(\theta_m)u_\theta(\underset{\sim}{w}) + (1-H^{1-\pi}(\theta_m))u_\theta(\tilde{w})}{1-\delta}.$$

Thus, we have

$$V_\theta^{C,\pi}(\Psi) \;=\; \frac{H^{1-\pi}(\theta_m)u_\theta(\underline{w}) + (1-H^{1-\pi}(\theta_m))u_\theta(\overline{w})}{1-\delta} \;\to\; \tilde{V}_\theta^C$$

for each party π. In particular, since $u_m(\underline{w}^\pi) = V_m^{C,\pi}(\Psi) = u_m(\overline{w}^\pi)$ in equilibrium, it follows that

$$u_m(\underset{\sim}{w}) \;=\; \tilde{V}_m^C \;=\; u_m(\tilde{w}),$$

so that the median voter is indifferent between the endpoints $\underset{\sim}{w}$ and $\tilde{w}$ and a challenger in the dichotomous model. In particular, the equilibrium condition on the win set $[\underset{\sim}{w}, \tilde{w}]$ holds. We must verify that policy choices in the dichotomous model are optimal. It suffices to show that the type $\overline{\theta}$ politician cannot gain by deviating to $x(\overline{\theta}) = 1$, as an analogous argument applies to the type $\underline{\theta}$ politician. Thus, we must show

$$\frac{u_{\overline{\theta}}(\tilde{w})}{1-\delta} \;\geq\; u_{\overline{\theta}}(1) + \delta\tilde{V}_{\overline{\theta}}^C, \tag{15}$$

where the left-hand side of the inequality is the discounted payoff from compromising to $\overline{w}$, and the right-hand side is the payoff from shirking. As in the proof of Theorem 1, let $\Phi_R^\pi(\theta_m)$ be lobby group R's payoff from the optimal winning policy minus the payoff from the optimal losing policy, given politician type θ_m. Since equilibria are polarized for γ sufficiently large, $\Phi_R^\pi(\theta_m)$ takes the simple form

$$\frac{1}{1-\delta}\left[u_{\overline{\theta}}(\overline{w}) + \frac{1}{\gamma}u_\theta(\overline{w}) - \frac{1}{\gamma}u_\theta(x(\theta_m))\right],$$

minus

$$u_{\overline{\theta}}(x_R(\theta_m)) + \delta V_{\overline{\theta}}^{C,\pi}(P) + \frac{1}{\gamma}\left(u_\theta(x_R(\theta_m)) + \delta V_m^{C,\pi}(\Psi)\right) - \frac{1}{\gamma}\left(\frac{u_m(x_m)}{1-\delta} + \frac{\delta\beta}{1-\delta}\right),$$

and moreover $\Phi_R^\pi(\theta_m) \geq 0$. Taking the limit as $\gamma \to \infty$ and using $x_R(\theta_m) \to 0$, we conclude that

$$\frac{u_{\overline{\theta}}(\tilde{w})}{1-\delta} - u_{\overline{\theta}}(1) - \delta\tilde{V}_{\overline{\theta}}^C \;\; = \;\; \lim_{\gamma\to\infty} \Phi_R^\pi(\theta_m) \;\; \geq \;\; 0,$$

which delivers (15), as required.

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