A Simple Example of a Finite Stochastic Game with Sequential Moves and No Pure-strategy Stationary Markov Perfect Equilibrium

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Barelli and Duggan (2015) show that in a noisy stochastic game with sequential moves, a pure stationary Markov perfect equilibrium exists; moreover, for every (mixed) stationary Markov perfect equilibrium, there is a pure one that is payoff-equivalent to it. This note provides an example showing that without noise, existence of stationary Markov perfect equilibrium does not generally hold—even in sequential-move games that are otherwise extremely well-behaved. The example is a simple dynamic variant of matching pennies, where two players alternate moves, player 1 wants to match, and player 2 wants to mismatch.

1 Dynamic Matching Pennies

Let the set of states be $S = \{H, T\} \times \{1, 2\}$, where (c, i) indicates that the state of a coin is c, and player i moves. In state (c, i), player i's feasible actions are $\{H, T\}$, and the other player's action set is trivial, i.e., $\{\emptyset\}$. Given action a by the active player i, next period's state is $(a, |i - 3|)$, so that the role of active player alternates between the two players, and the active player directly chooses the coin for the following period.

Payoffs are such that player 1's objective is to match when player 2 is active, with a slight preference $\epsilon > 0$ for heads:

$$u_1(a, c, i) = \begin{cases} 1 + \epsilon & \text{if } a = c = H \text{ and } i = 2, \\ 1 & \text{if } a = c = T \text{ and } i = 2, \\ \epsilon & \text{if } a = H \neq c \text{ and } i = 2, \\ 0 & \text{else.} \end{cases}$$

Player 2's objective is to mismatch when player 1 is active:

$$u_2(a, c, i) = \begin{cases} 1 & \text{if } a \neq c = H \text{ and } i = 1, \\ 0 & \text{else.} \end{cases}$$

Payoffs are discounted by $\delta \in (0, 1)$, as usual.

The focus of the analysis is subgame perfect equilibrium in stationary Markovian strategies (SMPE). Existence of SMPE in mixed strategies is well-known; the subsequent section establishes the existence in pure strategies does not hold in the dynamic matching pennies example.

2 Non-existence of Pure-strategy SMPE

A pure strategy simply specifies whether a player will match (or not) the state of the coin $c \in \{H, T\}$. Each player has the following four pure strategies:

$$\begin{array}{lclcl} MM & : & H \to H & & T \to T \\ MN & : & H \to H & & T \to H \\ NM & : & H \to T & & T \to T \\ NN & : & H \to T & & T \to H. \end{array}$$

For example, MN means that the player matches the coin when $c = H$ but does not match when $c = T$, i.e., she always chooses H.

First, suppose player 2 uses MM. Then player 1's unique best response, regardless of the current state c, is to choose H: then the coin next period is $c' = H$, and player 2 chooses H, giving player 1 a payoff of $1 + \epsilon$ whenever $i = 2$. That is, player 1's unique best response is MN. But then player 2's unique best response is to always choose T, yielding a payoff of one whenever $i = 1$. That is, her best response is NM, a contradiction.

Second, suppose player 2 uses MN, so that she always chooses H. Then player 1's unique best response is to always choose H, which yields a payoff of $1 + \epsilon$ whenever $i = 2$. That is, her best response is MN. Then player 2's unique best response is to always choose T, yielding a payoff of one whenever $i = 1$. That is, her best response is NM, a contradiction.

Third, suppose player 2 uses NM, so that she always chooses T. Then player 1's unique best response is to always choose T, which yields a payoff of one whenever $i = 1$. That is, her best response is NM. Then player 2's unique best response is to always choose H, yielding a payoff of one whenever $i = 1$. That is, her best response is MN, a contradiction.

Fourth, suppose player 2 uses NN. Then player 1's unique best response is to always choose tails, which yields a payoff of ϵ whenever $i = 2$. That is, her best response is NM. Then player 2's unique best response is to always choose H, yielding a payoff of one whenever $i = 1$. That is, her best response is MN, a contradiction.

In conclusion, there is no SMPE in which player 2 uses a pure strategy, and thus all such equilibria involve non-trivial mixing on the part of the player.

References

[1] P. Barelli and J. Duggan (2015) "Extremal Choice Equilibrium with Applications to Large Games, Stochastic Games, and Endogenous Institutions," *Journal of Economic Theory,* 155: 95–130.