Abbreviated Notes on Social Choice

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1 Binary Relations

1.1 Basics of Relations

A *binary relation* on a set X is a subset, say B, of ordered pairs of elements of X, i.e., $B \subseteq X \times X$. If $(x, y) \in B$, we say "x bears B to y." Usually we write "xBy" instead of "$(x, y) \in B$" and "$xByBz$" instead of "xBy and yBz."

For example, let $X = \{a, b, c\}$, the set $B = \{(a, b), (b, b), (b, c), (c, b)\}$ is a binary relation. Sometimes it can be helpful to represent B by a *directed graph,* in which each element $x \in X$ is represented by a dot (or "node") and each comparison $(x, y) \in B$ is represented by an arrow (or "arc") drawn from x to y.

Common examples of binary relations are the greater-than-or-equal-to relation, $\geq$, and the greater-than relation, $>$, on $\mathbb{R}$. These can both be extended to relations on $\mathbb{R}^d$ as follows: given $x, y \in \mathbb{R}^d$, say $x \geq y$ if and only if, for all i, $x_i \geq y_i$; and say $x > y$ if and only if $x \geq y$ and, for some j, $x_j > y_j$.

For a binary relation B we define the *upper section* and *lower section* at x by

$$\begin{aligned} B(x) &= \{y \in X \mid yBx\} \\ B^{-1}(x) &= \{y \in X \mid xBy\}, \end{aligned}$$

respectively. $B(x)$ consists of the elements that bear B to x, and $B^{-1}(x)$ consists of elements to which x bears B.

Given a binary relation B on X and a set $Y \subseteq X$, the *restriction* of B to Y, denoted $B|_Y$, is a relation on Y defined as follows: for all $x, y \in Y$, $xB|_Y y$ if and only if xBy. That is, $B|_Y = B \cap (Y \times Y)$.

We can also define several "derivative" binary relations from B using the familiar set-theoretic operations:

$$\begin{aligned} xB^{-1}y &\Leftrightarrow yBx \\ x\overline{B}y &\Leftrightarrow \neg xBy. \end{aligned}$$

Graphically, the first of these relations, the *inverse* of B, is obtained by reversing the arcs in the graph of B. The second relation is the *complement* of B, which can be defined in terms of set notation as $\overline{B} = (X \times X) \setminus B$. Note that the order in which we take complements and inverses does not matter, i.e., $(\overline{B})^{-1} = \overline{(B^{-1})}$. Indeed, for all $x, y \in X$, we have

$$\begin{aligned} x(\overline{B})^{-1}y &\Leftrightarrow y\overline{B}x \\ &\Leftrightarrow \neg yBx \\ &\Leftrightarrow \neg xB^{-1}y \\ &\Leftrightarrow x\overline{(B^{-1})}y. \end{aligned}$$

Therefore, we simply write $\overline{B}^{-1}$ for this relation, which we call the *dual* of B. Note that the dual of the dual of B is B itself.

Note also that the notation $B^{-1}(x)$ for the lower section of B at x is the same as the notation for the upper section of B^{-1} at x, and indeed these sets are the same; thus, no ambiguity results.

1.2 Properties of Relations

We first list some useful properties of binary relations. Here x, y, and z range over the set X, and k ranges over the natural numbers 1,2,3. . . .

- *symmetric*: $\forall x, y : xBy \Rightarrow yBx$.
- *reflexive*: $\forall x : xBx$.
- *total*: $\forall x, y : x \neq y \Rightarrow xBy \vee yBx$.
- *complete*: $\forall x, y : xBy \vee yBx$.
- *transitive*: $\forall x, y, z : xByBz \Rightarrow xBz$.
- *acyclic*: $\forall k$, $\forall x_0, \ldots, x_k : x_0Bx_1B \cdots Bx_k \Rightarrow x_k \neq x_0$.
- *irreflexive*: $\forall x : \neg xBx$.
- *anti-symmetric*: $\forall x, y$: $xBy \wedge yBx \Rightarrow x = y$.
- *asymmetric*: $\forall x, y : \neg xByBx$.
- *negatively transitive*: $\forall x, y, z : [\neg xBy \wedge \neg yBz] \Rightarrow \neg xBz$.
- *negatively acyclic*: $\forall k$, $\forall x_0, \ldots, x_k : [\neg x_0Bx_1 \wedge \cdots \wedge \neg x_{k-1}Bx_k] \Rightarrow x_k \neq x_0$.

Note that completeness implies B is reflexive and total. Similarly, asymmetry implies B is irreflexive and anti-symmetric. It is also clear that acyclicity implies asymmetry by setting $k = 2$. Note that transitivity implies that if we can get from x to z in any finite B-steps, then we can get from x to z directly.

Proposition 1.1. *A relation B is transitive if and only if for all k and all $x_0, \ldots, x_k$, the condition $x_0Bx_1B \cdots Bx_k$ implies x_0Bx_k.*

Proof. The "if" direction follows by setting $k = 2$. For the "only if" part, if B is transitive, then the condition "for all $x_0, \ldots, x_k$, $x_0Bx_1B \cdots Bx_k$ implies x_0Bx_k" holds for $k = 1$. Now suppose it holds for arbitrary k, and consider any elements $x_0, \ldots, x_k, x_{k+1}$ such that $x_0Bx_1B \cdots Bx_kBx_{k+1}$. By the induction hypothesis, $x_0Bx_kBx_{k+1}$. Transitivity then implies that x_0Bx_{k+1}. □

The next proposition lists connections among these properties.

Proposition 1.2. *Let B be a binary relation.*

1. *B is asymmetric if and only if it is irreflexive and anti-symmetric.*
2. *If B is acyclic, then it is asymmetric.*
3. *If B is irreflexive and transitive, then it is acyclic.*
4. *If B is anti-symmetric and negatively transitive, then it is transitive.*
5. *B is complete if and only if it is reflexive and total.*
6. *If B is negatively acyclic, then it is complete.*
7. *If B is reflexive and negatively transitive, then it is negatively acyclic.*
8. *If B is total and transitive, then it is negatively transitive.*

The arguments for the above claims in the above proposition are elementary. For example, if we write reflexivity and totalness as

$$\begin{aligned} x = y &\;\Rightarrow\; xBy \vee yBx \\ x \neq y &\;\Rightarrow\; xBy \vee yBx, \end{aligned}$$

we see that their conjunction is equivalent to $xBy \vee yBx$, which is the property of completeness.

Finally, note that the above properties can be paired into dual partnerships, e.g., B is complete if and only if its dual is asymmetric, and B is asymmetric if and only if its dual is complete. The pairings are as follows.

symmetric	symmetric
reflexive	irreflexive
total	anti-symmetric
complete	asymmetric
transitive	negatively transitive
acyclic	negatively acyclic

1.3 Classes of Relations

Table 1 presents some classes of complete relations, where $*$'s denote defining properties and $+$'s, which follow from Proposition 1.2, denote implied ones. Note that each of these classes is termed "weak", which we use to denote completeness.

	weak sub-order	weak quasi-order	weak order	weak linear order
reflexive	+	+	+	+
total	+	+	+	+
complete	*	*	*	*
negatively acyclic	*	+	+	+
negatively transitive		*	+	+
transitive			*	*
anti-symmetric				*

Table 1: Complete relations

By part 4 of Proposition 1.2, negative acyclicity implies completeness. Thus, the condition of completeness used in the definition of weak sub-order is redundant. And by part 4 and 6 of Proposition 1.2, reflexivity and negative transitivity together imply completeness. Therefore, the condition of completeness used in the definitions of weak quasi-order, weak order, and weak linear order can be weakened to reflexivity.

Table 2 presents some classes of asymmetric relations. Note that the term "strict" denotes asymmetry.

	strict sub-order	strict quasi-order	strict order	strict linear order
irreflexive	+	+	+	+
anti-symmetric	+	+	+	+
asymmetric	*	*	*	*
acyclic	*	+	+	+
transitive		*	+	+
negatively transitive			*	*
total				*

Table 2: Asymmetric relations

By part 3 of Proposition 1.2, acyclicity implies asymmetry. Thus, the condition of asymmetry used in the definition of strict sub-order is redundant. And by part 5 of Proposition 1.2, irreflexivity and transitivity together imply asymmetry. Therefore, the condition of asymmetry used in the definitions of strict quasi-order, strict order, and strict linear order can be weakened to irreflexivity.

2 Preferences

2.1 Formalizing Preferences

Let X be a set of *alternatives*. Let P be a *strict preference* relation over X, as in

$$P \;=\; \{(x,y) \in X \times X \mid x \text{ is strictly better than } y\};$$

R be a *weak preference* relation, as in

$$R \;=\; \{(x,y) \in X \times X \mid x \text{ is at least as good as } y\};$$

We will always impose the first three of the following axioms on preference relations, whether preferences of an individual or of a group.

Asymmetry: *Strict preference relation P is asymmetric.*

Completeness: *Weak preference relation R is complete.*

Duality: *Strict and weak preference are dual.*

That is, $P = \overline{R}^{-1}$ and $R = \overline{P}^{-1}$, or in more detail: for all $x, y \in X$, we have

$$xPy \Leftrightarrow \neg yRx \quad \text{and} \quad xRy \Leftrightarrow \neg yPx.$$

For later use, we call such a pairing (P, R) a *dual pair*.

Given strict and weak preference, P and R, and given the above three axioms, we define the *indifference relation* I equivalently as

$$I \;=\; R \cap R^{-1} \;=\; \overline{P} \cap \overline{P}^{-1}.$$

So xIy means each alternative is weakly preferred to the other, or equivalently neither is strictly preferred to the other. It is easy to see that I is reflexive and symmetric.

For individual preferences, we also impose the following axiom.

Individual rationality: *Strict preference P is negatively transitive, and weak preference R is transitive.*

That is, under the latter axiom, R is a weak order and P is a strict order.

The following relationships follow from the discussion of the previous subsection.

Proposition 2.1.

$$\begin{array}{ccccc} P \text{ neg. transitive} & \Rightarrow & P \text{ transitive} & \Rightarrow & P \text{ acyclic} \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ R \text{ transitive} & \Rightarrow & R \text{ neg. transitive} & \Rightarrow & R \text{ neg. acyclic} \end{array}$$

For the case in which preferences order the set of alternatives, transitivity can be stated in a stronger form.

Proposition 2.2. *Assume P is a strict order and R is a weak order. Then for all $x, y, z \in X$, we have*

$$\begin{aligned} xPyRz &\Rightarrow xPz \\ xRyPz &\Rightarrow xPz. \end{aligned}$$

As well, when P and R are orders, it is straightforward to check that the indifference relation I is an equivalence relation (i.e., it is reflexive, symmetric, and transitive), and the collection $\{I(x) \mid x \in X\}$ is a partition of X (each element of the partition is called an *indifference class* or *indifference curve*).

We say $u\colon X \to \mathbb{R}$ is a *utility representation* of P if for all $x, y \in X$, we have xPy if and only if $u(x) > u(y)$. Of course, if P has a utility representation, then it is a strict order; the converse holds if X is finite, and indeed it holds much more generally, but we omit details at this point.

2.2 Metric spaces

Given a set X, a *metric* on X is a function $\rho\colon X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$, (i) $\rho(x, y) \geq 0$, with equality if and only if $x = y$, (ii) $\rho(x, y) = \rho(y, x)$, and (iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Let $B_r(x) = \{y \in X \mid \rho(x, y) < r\}$ be the *open ball of radius r*. Then a set $Y \subseteq X$ is *open* if for all $x \in Y$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq Y$. It is *closed* if it is the complement of an open set. Note that a metric determines a notion of *convergence,* specifically, a sequence $\{x^n\}$ in X converges to $x \in X$ if and only if $\rho(x^n, x) \to 0$.

The collection of open sets generated by a metric is called a *topology.* Some metrics generate the same topology, but in general there are many possible topologies on a set X.

Example: Euclidean space $\mathbb{R}^d$ with the *Euclidean metric,* defined by

$$\rho^e(x, y) = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}.$$

For another example, let $X \subseteq \mathbb{R}^d$ be any subset of Euclidean space, and define the metric ρ as follows: for all $x, y \in X$, $\rho(x,y) = \rho^e(x,y)$. That is, $\rho = \rho^e|_{X \times X}$. In this metric space, it is fairly obvious that a set $Y \subseteq X$ is open if and only if there is a set $G \subseteq \mathbb{R}^d$ that is open in $\mathbb{R}^d$ with the Euclidean metric such that $Y = G \cap X$. As well, a set $Y \subseteq X$ is closed if and only if there is a set $F \subseteq \mathbb{R}^d$ that is closed in $\mathbb{R}^d$ with the Euclidean metric such that $Y = F \cap X$. The topology generated by ρ is called the *relative topology,* and an open subset of X with the relative topology is said to be "relatively open" or "open in X." A sequence $\{x^n\}$ in X converges to $x \in X$ if and only if it converges in the Euclidean metric, since $\rho(x^n, x) = \rho^e(x^n, x)$.

For a final example, let X_1 and X_2 be two metric spaces with metrics ρ_1 and ρ_2, respectively. Then we make the product set $X = X_1 \times X_2$ a metric space by endowing it with the product metric as follows:

$$\rho(x,y) \quad = \quad \rho_1(x_1,y_1) + \rho_2(x_2,y_2),$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. It is fairly obvious that a set $Y \subseteq X$ is open if and only if for all $x = (x_1, x_2) \in Y$, there exist sets $G_1 \subseteq X_1$ and $G_2 \subseteq X_2$ such that G_1 is open in X_1, G_2 is open in X_2, and $x \in G_1 \times G_2 \subseteq Y$. The topology generated by ρ is called the *product topology.* A sequence $\{x^n\}$ in X converges to $x \in X$ if and only if $x_1^n \to x_1$ in X_1 and $x_2^n \to x_2$ in X_2. This form of convergence is called *pointwise convergence.* All of these ideas extend to finite (and even countably infinite) products of sets.

2.3 Continuity

Let X be a subset of $\mathbb{R}^d$ endowed with the relative Euclidean topology. Given preferences P and R, we say that P is *upper semicontinuous (usc)* if for all $x \in X$, $P^{-1}(x)$ is open in X; equivalently, R is *upper semicontinuous* if for all $x \in X$, $R(x)$ is closed in X.

We say that P is *lower semicontinuous (lsc)* if for all $x \in X$, $P(x)$ is open in X; equivalently, R is *lower semicontinuous* if for all $x \in X$, $R^{-1}(x)$ is closed in X.

We say that P (likewise R) is *continuous* if it is both upper semicontinuous and lower semicontinuous.

A stronger condition is that P has *open graph*: for every $(x,y) \in P$, there exist subsets U and V open in X such that $x \in U$, $y \in V$, and $U \times V \subseteq P$, or in other words, the set P is open in the product topology on $X \times X$.

Equivalently, R has *closed graph* if for every sequence $\{(x_m, y_m)\} \in X \times X$ and every $(x,y) \in X \times X$, if $x_m R y_m$ for all m, and $(x_m, y_m) \to (x,y)$, then xRy. That is, R is closed in the product topology on $X \times X$.

When strict and weak preference are orders, continuity is equivalent to open graph of P and closed graph of R.

Proposition 2.3. *Assume that $X \subseteq \mathbb{R}^d$, and that P is a strict order and R is a weak order. Then P is continuous if and only if it has open graph; equivalently, R is continuous if and only if it has closed graph.*

Proof. We prove that continuity implies closed graph of R. Let $\{(x_m, y_m)\}$ be a sequence in $X \times X$ such that $x_m R y_m$ for all m and $(x_m, y_m) \to (x, y)$. Suppose in order to deduce a contradiction that not xRy, or equivalently yPx. I claim that for all m, there exists $k \geq m$ such that yPy_k, for suppose otherwise. Then there exists m such that for all $k \geq m$, we have $y_k R y$. This yields $x_k R y_k R y$ for all $k \geq m$, and by transitivity of R, this implies $x_k R y$ for all $k \geq m$. That is, $x_k \in R(y)$ for all $k \geq m$. Since $R(y)$ is closed, by continuity, we have $x \in R(y)$, or equivalently, xRy, a contradiction. This establishes the claim.

Next, I claim that $P(x) \cap P^{-1}(y) \neq \emptyset$. Indeed, using the first claim, we can choose a subsequence $\{(x_{k_m}, y_{k_m})\}$ such that $(x_{k_m}, y_{k_m}) \to (x, y)$ and for all m, yPy_{k_m}. Since $y_{k_m} \to y \in P(x)$, and since $P(x)$ is open, it follows that for some m, we have $yPy_{k_m}Px$, i.e., $y_{k_m} \in P(x) \cap P^{-1}(y)$. This establishes the claim.

For notational simplicity, we thus choose $z \in P(x) \cap P^{-1}(y)$. In particular, we have $x \in P^{-1}(z)$ and $y \in P(z)$. Since $P^{-1}(z)$ and $P(z)$ are open, and since $(x_m, y_m) \to (x, y)$, it follows that for sufficiently high m, we have $x_m \in P^{-1}(z)$ and $y_m \in P(z)$ for such m. But then $y_m P z P x_m$, and transitivity implies $y_m P x_m$, a contradiction. $\square$

2.4 Convexity

Assuming $X \subseteq \mathbb{R}^d$, we say a strict order P is *convex* if for all $x \in X$, the set $P(x)$ is convex; similarly, the weak order R is *convex* if for all $x \in X$, $R(x)$ is convex. In fact, it can be shown that these conditions are equivalent when (P, R) form a dual pair.

We say P or R are *strictly convex* if for all $x \in X$, all distinct $y, z \in R(x)$, and all $\alpha \in (0, 1)$, we have $\alpha y + (1 - \alpha) z \in P(x)$. Of course, strict convexity implies convexity.

We do not typically invoke the above convexity properties for preferences that do not form an ordering of alternatives, but we do consider the following condition on general asymmetric relations: say P is *semi-convex* if for all $x \in X$, we have $x \notin \text{conv}P(x)$. That is, semi-convexity means that no alternative is contained in the convex hull of alternatives strictly preferred to it.

In case P is a strict order, semi-convexity is equivalent to convexity, but the former condition has wider applications to preferences that fail negative transitivity.

3 Choice

3.1 Maximal Elements

Given a binary relation B on the set of alternatives X, let $Y \subseteq X$ be any subset of *feasible* alternatives. Define the following subsets of Y:

- The *undominated set* of B in Y is
$$U(Y,B) = \{x \in Y \mid \nexists y \in Y, \text{ s.t. } yBx\}.$$
- The *dominant set* of B in Y is
$$D(Y,B) = \{x \in Y \mid \forall y \in Y,\ xBy\}.$$
- The *maximal set* of B in Y is
$$M(Y,B) = \{x \in Y \mid \forall y \in Y,\ yBx \Rightarrow xBy\}.$$

Given a dual pair of preferences, P and R, consider the problem of choice from a set of available alternatives $Y \subseteq X$. For strict preferences P, we would specify the set of feasible alternatives that are no worse than any other feasible alternative, that is, $U(Y,P)$. For weak preferences R, we would specify the set of dominant alternatives, that is, $D(Y,R)$. It is clear that $M(Y,P) = M(Y,R) = U(Y,P) = D(Y,R)$, which would be the implied choice set of the problem.

Suppose that $Y \subseteq X$ is nonempty and finite and preferences P and R are strict and weak orders, i.e., P is asymmetric and negatively transitive or R is complete and transitive, then clearly $M(Y,R) \neq \emptyset$. The next proposition characterizes the acyclicity conditions in terms of their implications for the nonemptiness of choice sets when $Y \subseteq X$ is finite.

Proposition 3.1. *The strict preference relation P is acyclic if and only if for all nonempty, finite sets $Y \subseteq X$, we have $M(Y,P) \neq \emptyset$.*

Proof. Suppose P is acyclic but $M(Y,P) = \emptyset$. Consider $x_1 \in Y$. Since $x_1 \notin M(Y,P) = U(Y,P)$, there exists $x_2 \in Y$ such that $x_2 P x_1$. Similarly, there exists $x_3 \in Y$ such that $x_3 P x_2$. Continue this argument m times where $m = |Y| + 1$. Then we have $x_m P x_{m-1} P \cdots P x_1$ and there exist $k, \ell \in \{1, \ldots, m\}$ such that $x_k = x_\ell$ and $k > \ell$. But then $x_k P x_{k-1} P \cdots P x_\ell$ is a cycle, a contradiction. Conversely, if P is not acyclic, i.e., there exist $x_0, \ldots, x_k \in Y$ such that $x_0 P x_1 P \cdots P x_k = x_0$. But then $M(\{x_1, \ldots, x_k\}, P) = \emptyset$, a contradiction. □

When X is infinite, two technical conditions play a critical role: compactness of the feasible set and continuity of preferences. The next proposition extends our characterization of acyclicity (Proposition 3.1) to infinite sets.

Proposition 3.2. *Assume $X \subseteq \mathbb{R}^d$, and assume that P is upper semicontinuous. Then P is acyclic if and only if $M(Y, P) \neq \emptyset$ for all nonempty and compact sets $Y \subseteq X$.*

Proof. Suppose P is acyclic but $M(Y, P) = \emptyset$ for some nonempty, compact Y, so for every $x \in Y$, there exists $y \in Y$ such that yPx, i.e., $x \in P^{-1}(y)$. Therefore, $\{P^{-1}(y) : y \in Y\}$ is an open cover of Y. By compactness of Y, there exists $y_1, \ldots, y_m \in Y$, such that $\{P^{-1}(y_k) : j = k, \ldots, m\}$ is a finite subcover. Applying Proposition 3.1 to the set $\{y_1, \ldots, y_m\}$, there exists y_j such that for all $k = 1, \ldots, m$, not $y_k P y_j$. However, y_j is then not in any of $P^{-1}(y_k)$ for $k = 1, \ldots, m$. A contradiction. The converse direction follows from Proposition 3.1, since every finite set is compact. □

For an even more general existence result for maximal elements, we say P satisfies *finite dominance* if for every finite set $Y \subseteq X$, there exists $x \in X$ such that for all $y \in Y$, we have xRy. Note that if P is acyclic, then it automatically satisfies finite dominance: simply choose x to be any element of the maximal set $M(Y, P)$, which is nonempty. To conserve notation, the next result is stated for the case in which every alternative is feasible. The proof proceeds along the lines of the preceding proposition and is omitted.

Proposition 3.3. *Assume $X \subseteq \mathbb{R}^d$ is compact, and assume that P is upper semicontinuous. If P satisfies finite dominance, then $M(X, P) \neq \emptyset$.*

Finally, we state the result that semi-convexity and upper semi-continuity imply finite dominance. The proof uses the KKM theorem and is omitted.

Proposition 3.4. *Assume $X \subseteq \mathbb{R}^d$ is convex. If P is upper semi-continuous and semi-convex, then it satisfies finite dominance.*

Proof. By the KKM Theorem (see Theorem 16.40 in Aliprantis and Border's (1999) *Infinite Dimensional Analysis*), if $K = \text{conv}\{y^1, \ldots, y^m\} \subseteq \mathbb{R}^d$ and $\{Y_1, \ldots, Y_m\}$ is a family of closed subsets of $\mathbb{R}^d$ such that for every $I \subseteq \{1, \ldots, m\}$, we have

$$\text{conv}\{y^j \mid j \in I\} \subseteq \bigcup_{j \in I} Y_j,$$

then $K \cap \bigcap_{j=1}^m Y_j$ is compact and nonempty. See Figure 1. To prove the proposition,

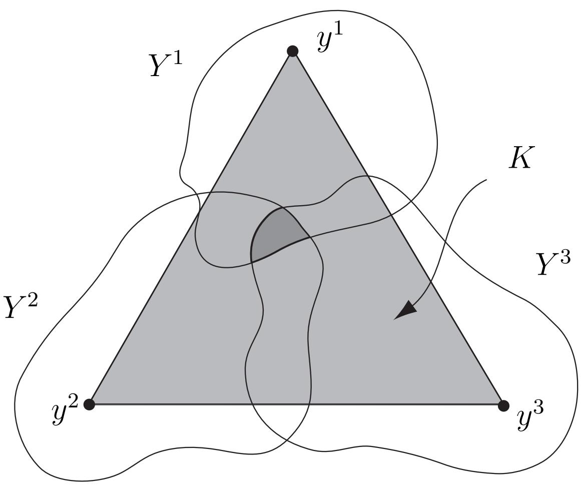


Figure 1: The KKM Theorem

consider any finite subset $\{y^1, \ldots, y^m\}$ of X, and let $K = \text{conv}\{y^1, \ldots, y^m\}$. By upper semicontinuity, $\{R(y^1) \cap K, \ldots, R(y^m) \cap K\}$ is a family of closed subsets of $\mathbb{R}^d$. Take any $I \subseteq \{1, \ldots, m\}$. If it is *not* the case that

$$\text{conv}\{y^j \mid j \in I\} \quad \subseteq \quad \bigcup_{j \in I} (R(y^j) \cap K), \tag{1}$$

then there is some $z \in \text{conv}\{y^j \mid j \in I\}$ such that $z \notin \bigcup_{j \in I}(R(y^j) \cap K)$, and in particular, $z \in \bigcap_{j \in I} P^{-1}(y^j)$. So $y^j \in P(z)$ for all $j \in I$. But then z is a convex combination of strictly preferred alternatives, contradicting semi-convexity. Therefore, (1) holds for every $I \subseteq \{0, 1, \ldots, m\}$, and the KKM Theorem and convexity of X imply $X \cap \bigcap_{j=1}^{m} R(y^j) \neq \emptyset$, fulfilling finite dominance. □

3.2 Top Cycle Set

When maximal elements of a preference relation do not exist, we would like to find alternatives to maximality for the determination of choice sets. In particular, we might at least find reasonable bounds on choices when maximal elements do not exist.

For example, consider the relation $P = \{(a, b), (b, c), (c, a), (a, d), (b, d), (c, d)\}$ on the set $X = \{a, b, c, d\}$. There is no maximal element of this relation, but we can reasonably predict that a choice would belong to the set $\{a, b, c\}$, since each of these elements is preferred to d. Moreover, it is the smallest such set. This set is an example of a "top cycle."

Formally, given a set of alternatives X and a preference relation P, define the *tran-*

sitive closure of P, denoted P^∞, by

$$xP^\infty y \;\Leftrightarrow\; \exists m, \exists x_1, \ldots, x_m : xPx_1Px_2P \cdots Px_{m-1}Px_m = y.$$

It is clear that P^∞ is transitive. In fact, P^∞ is the smallest transitive relation containing P, i.e., if $P \subseteq Q$ and Q is transitive, then $P^\infty \subseteq Q$. However, note that P^∞ could violate asymmetry.

Define the asymmetric part of P^∞, denoted P^*, as

$$xP^*y \;\Leftrightarrow\; [xP^\infty y \text{ and } \neg yP^\infty x].$$

Note that P^* is asymmetric and transitive. That is, P^* is a strict quasi-order, so for all $x, y, z \in X$, we have $xP^*yP^*z \Rightarrow xP^*z$.

The *top cycle* of P, $TC(P)$ is the maximal set of the relation P^*, i.e.,

$$TC(P) \;=\; M(P^*) = U(P^*).$$

Note that $M(P) \subseteq TC(P)$. Even if $M(P) \neq \emptyset$, it maybe the case that the top cycle includes non-maximal elements, i.e., $TC(P) \setminus M(P) \neq \emptyset$, but under some conditions, $M(P) \neq \emptyset$ implies $TC(P) = M(P)$. If P is total, for example, and $M(P) \neq \emptyset$, then $TC(P) = M(P)$.

Given an asymmetric relation P on X, we can partition X into P^∞-components, i.e., cycles that are maximal with respect to set inclusion, and then the induced relation on P^∞-components is a strict quasi-order.

Note that if Y is finite, then $TC(Y, P) \neq \emptyset$. Our last result (Proposition 3.5) on top cycles establishes nonemptiness under the familiar compactness and continuity conditions. Note that there is no acyclicity requirement in the proposition.

Proposition 3.5. *Assume $X \subseteq \mathbb{R}^d$. If X is compact and P is upper semicontinuous, then $TC(P) \neq \emptyset$.*

Proof. If $M(P) \neq \emptyset$, then clearly the top cycle is nonempty, so consider the case $M(P) = \emptyset$. Then for each $x \in X$, there exists $y_x \in X$ such that y_xPx, i.e., $x \in P^{-1}(y_x)$. By upper semicontinuity, $\{P^{-1}(y_x) \mid x \in X\}$ is an open covering of X, and by compactness, this has a finite subcover, say $\{P^{-1}(y_{x_1}), \ldots, P^{-1}(y_{x_m})\}$. Since P^* is a strict quasi-order, we know that $M(\{y_{x_1}, \ldots, y_{x_m}\}, P^*) \neq \emptyset$, so there exists $y \in M(\{y_{x_1}, \ldots, y_{x_m}\}, P^*)$. To see that $y \in TC(P)$, consider any $x \in X$, and suppose that $xP^\infty y$. By construction, there exists $z \in \{y_{x_1}, \ldots, y_{x_m}\}$ such that zPx, and thus $zP^\infty y$. Since y is maximal in $\{y_{x_1}, \ldots, y_{x_m}\}$ according to P^*, it follows that $yP^\infty z$, and therefore $yP^\infty x$, and we conclude that $y \in TC(P)$. □

In the collective choice problems, with more structure than we have here, we will be able to say more about the top cycles. Unfortunately, these sets can be quite large. Therefore, we will consider another alternative to maximality in the next subsection.

3.3 Uncovered Set

An alternative solution to the problem of empty maximal sets is the "uncovered set." Contrary to the top cycles, which are defined in terms of the transitive closure of strict preferences, the uncovered sets are defined in terms of inclusions of upper sections.

Given a dual pair (P, R) on X, define the *covering relation*, C, as

$$xCy \;\Leftrightarrow\; R(x) \subseteq P(y).$$

Note that C is asymmetric and transitive, i.e., it is a strict quasi-order.

The *uncovered set* of P, $UC(P)$, is the maximal set of the relation C, i.e.,

$$UC(P) \;=\; M(C) = U(C).$$

The dual of the covering relation has the following characterization: for all $x, y \in X$,

$$x\overline{C}^{-1}y \;\Leftrightarrow\; \exists z : xRzRy.$$

Since $UC(P) = U(C) = D(\overline{C}^{-1})$, it follows that x belongs to the uncovered set if and only if for every alternative y, either x is directly weakly preferred to y, or it is weakly preferred to y in two steps.

Like the top cycle, we have $M(P) \subseteq UC(P)$, and if P is total and $M(P) \neq \emptyset$, then $UC(P) = M(P)$.

If Y is finite, then $UC(Y, P) \neq \emptyset$. And like the top cycle, this nonemptiness result extends much more generally.

Proposition 3.6. *Assume $X \subseteq \mathbb{R}^d$. If X is compact and P has open graph, then $UC(P) \neq \emptyset$.*

Proof. We have noted that C is a strict quasi-order, so it is a strict sub-order. I claim that it is also upper semi-continuous. To prove this claim, it suffices to show that for each $x \in X$, the set $\overline{C}^{-1}(x)$ is closed. To this end, let $\{y^k\}$ be a sequence in $\overline{C}^{-1}(x)$ that converges to $y \in X$. We must show that $y \in \overline{C}^{-1}(x)$. For each k, there exists $z^k \in X$ such that $y^k R z^k R x$. Since X is compact, there is a subsequence $\{z^{k_\ell}\}$ in $R(x)$ with limit $z \in X$. Moreover, since P is upper semi-continuous, it follows that $R(x)$ is closed, and therefore $z \in R(x)$, i.e., zRx. Finally, note that $y^{k_\ell} R z^{k_\ell}$ for all ℓ, so that closed graph of R implies yRz. Combining these observations, we have $yRzRx$, and therefore $y\overline{C}^{-1}x$, i.e., $y \in \overline{C}^{-1}(x)$, proving the claim. Then Proposition 3.2 implies that C has a maximal element, and we conclude that $UC(P) = U(C) \neq \emptyset$. $\square$

4 Social Choice

4.1 Social Choice Environments

A social choice environment consists of a set of individuals $N = \{1, 2, \ldots, n\}$, a set of alternatives X, and a domain $\mathbf{D}$ that consists of possible profiles $(P_1, \ldots, P_n)$ of individual preferences. We assume that each P_i is asymmetric and negatively transitive, so individual preferences satisfy the Individual Rationality axiom.

Associated with each P_i is a weak preference relation R_i satisfying the Duality Axiom and an indifference relation I_i derived as usual from P_i (or R_i).

To denote the groups strictly preferring, weakly preferring, and indifference between any two alternatives, let

$$\begin{aligned} P(x,y) &= \{i \in N \mid xP_iy\} \\ R(x,y) &= \{i \in N \mid xR_iy\} \\ I(x,y) &= \{i \in N \mid xI_iy\}. \end{aligned}$$

We impose minimal richness on the domain: there exist $x, y \in X$ such that for all $G \subseteq N$, there exists $(P_1, \ldots, P_n) \in \mathbf{D}$ such that $P(x,y) = G$ and $P(y,x) = N \setminus G$. We refer to such a pair $\{x, y\}$ of alternatives as a *free pair.*

Two common examples of domains satisfying this richness condition are:

- *Unrestricted domain,* $\mathbf{U}$: the set of profiles of strict orders on X,
- *Linear domain,* $\mathbf{L}$: the set of profiles of strict linear orders on X.

Letting $\mathbf{A}$ denote the set of asymmetric relations on X, a *social preference rule* (or SPR) is a mapping $\mathbb{P} \colon \mathbf{D} \to \mathbf{A}$, where $\mathbb{P}(P_1, \ldots, P_n)$ is a strict social preference relation. We can interpret the relation $x\mathbb{P}(P_1, \ldots, P_n)y$ as expressing that the group would choose x over y.

Associated with the social strict preference relation are social weak preference and indifference relations, $\mathbb{R}(P_1, \ldots, P_n)$ and $\mathbb{I}(P_1, \ldots, P_n)$, using the usual conventions.

We refer to the maximal set of $\mathbb{P}(P_1, \ldots, P_n)$ as the *core* of $\mathbb{P}$ at $(P_1, \ldots, P_n)$, and we denote it by

$$\mathbb{C}(P_1, \ldots, P_n) \;\; = \;\; M(X, \mathbb{P}(P_1, \ldots, P_n)).$$

By asymmetry of strict social preference, it follows that

$$\mathbb{C}(P_1, \ldots, P_n) \;\; = \;\; U(X, \mathbb{P}(P_1, \ldots, P_n)) \;\; = \;\; D(X, \mathbb{R}(P_1, \ldots, P_n)).$$

4.2 Social Preferences

There are many ways of aggregating individual preferences into social preferences, P and R. Here, we give some common definitions of social preferences, each of which implicitly determines a social preference rule.

- *Simple majority,* P_{SM}: $xP_{SM}y \Leftrightarrow |P(x,y)| > \frac{n}{2}$.
- *Relative majority,* P_{RM}: $xP_{RM}y \Leftrightarrow |P(x,y)| > |P(y,x)|$.
- *Simple Pareto,* P_{SP}: $xP_{SP}y \Leftrightarrow P(x,y) = N$.
- *Relative Pareto,* P_{RP}: $xP_{RP}y \Leftrightarrow R(x,y) = N$ and $P(x,y) \neq \emptyset$.
- *Quota rule,* P_q (with $n/2 < q \leq n$): $xP_qy \Leftrightarrow |P(x,y)| \geq q$.

Note that P_{SM} is asymmetric, and P_{SP} is asymmetric and transitive. Moreover, $P_{SM} = P_{\frac{n+1}{2}}$ and $P_{SP} = P_n$. The condition for simple majority preferences is stronger than that for relative majority preferences. The condition for weak Pareto unanimity preferences is stronger than that for strong Pareto unanimity preferences.

We have defined strict social preferences above. It is straightforward to define the corresponding weak preferences. For example,

$$xR_{SM}y \;\Leftrightarrow\; \neg yP_{SM}x \;\Leftrightarrow\; |R(x,y)| \geq \frac{n}{2}$$

$$xR_{SP}y \;\Leftrightarrow\; R(x,y) \neq \emptyset.$$

Condorcet's paradox demonstrates the possibility of majority preference cycles.

Condorcet's paradox. Let $n = 3$ and $X = \{a, b, c\}$, and consider the following individual preferences

P_1	P_2	P_3
a	b	c
b	c	a
c	a	b

Clearly, these individual preferences generate a majority strict preference cycle,

$$aP_{SM}bP_{SM}cP_{SM}a$$

and moreover the majority core is empty, $M(P_{SM}) = \emptyset$.

This example is easily generalized to more alternatives and more individuals.

Generalized Condorcet's paradox. Let $n \geq 3$ (but $n \neq 4$) and $|X| \geq 3$. Partition N into three roughly equal-sized groups G_1, G_2, G_3 (differing in size by at most one, when n is indivisible by three), and consider the following individual preferences.

G_1	G_2	G_3
a	b	$X \setminus \{a,b\}$
b	$X \setminus \{a,b\}$	a
$X \setminus \{a,b\}$	a	b

In this case, similarly, P_{SM} violates acyclicity, and moreover the majority core is empty, $M(P_{SM}) = \emptyset$.

Each of the definitions of social preference above determines an SPR. For example, the simple majority SPR is defined by

$$x\mathbb{P}_{SM}(P_1, \ldots, P_n)y \quad \Leftrightarrow \quad |P(x,y)| > \frac{n}{2},$$

and the quota preference rule $\mathbb{P}_q$ with quota $q \leq n$ is defined by

$$x\mathbb{P}_q(P_1, \ldots, P_n)y \quad \Leftrightarrow \quad |P(x,y)| \geq q.$$

The distinction between simple majority and relative majority (and between simple Pareto and relative Pareto) disappear under Linear domains.

4.3 Simple Rules

Let $\mathcal{G} \subseteq 2^N$ denote a collection of groups. We say $\mathcal{G}$ is

- *monotonic* if for all $G \in \mathcal{G}$ and all $G' \subseteq N$, $G \subseteq G'$ implies $G' \in \mathcal{G}$,
- *proper* if for all $G, G' \in \mathcal{G}$, we have $G \cap G' \neq \emptyset$.
- *strong* if for all $G \subseteq N$, either $G \in \mathcal{G}$ or $N \setminus G \in \mathcal{G}$.

For example, the majority collection $\mathcal{G}^M = \{G \subseteq N \mid |G|\ n/2\}$ is monotonic and proper, and it is also strong if n is odd. The unanimity collection $\mathcal{G}^P = \{N\}$ is monotonic and proper, but it is only strong if $n = 1$.

Given a monotonic collection $\mathcal{G}$, we define the *dual* of $\mathcal{G}$ as

$$\mathcal{G}^* \;=\; \{G \subseteq N \mid N \setminus G \notin \mathcal{G}\}.$$

Then we have:

- $\mathcal{G}^*$ is monotonic,
- $\mathcal{G}^{**} = \mathcal{G}$,
- $\mathcal{G}$ is proper iff $\mathcal{G} \subseteq \mathcal{G}^*$ iff $\mathcal{G}^*$ is strong,
- $\mathcal{G}$ is strong iff $\mathcal{G}^* \subseteq \mathcal{G}$ iff $\mathcal{G}^*$ is proper.

For example, $(\mathcal{G}^M)^* = \{G \subseteq N \mid |G| \geq n/2\}$, $(\mathcal{G}^P)^* = \{G \subseteq N \mid |G| \neq 0\}$.

Given any proper and monotonic collection $\mathcal{G}$, we can define a corresponding SPR $\mathbb{P}_{\mathcal{G}}$ as follows: for all $(P_1, \ldots, P_n) \in \mathbf{D}$ and all $x, y \in X$,

$$x \mathbb{P}_{\mathcal{G}}(P_1, \ldots, P_n) y \quad \Leftrightarrow \quad P(x, y) \in \mathcal{G}.$$

Note that properness ensures that $\mathbb{P}_{\mathcal{G}}$ is well-defined, in the sense that social preferences are asymmetric: if $x\mathbb{P}_{\mathcal{G}}(P_1, \ldots, P_n)y$ and $y\mathbb{P}_{\mathcal{G}}(P_1, \ldots, P_n)x$, then we would have $P(x, y) \in \mathcal{G}$ and $P(y, x) \in \mathcal{G}$, contradicting the assumption that $\mathcal{G}$ is proper. Then the corresponding weak social preference relation $\mathbb{R}_{\mathcal{G}}(P_1, \ldots, P_n)$ satisfies

$$x\mathbb{R}_{\mathcal{G}}(P_1, \ldots, P_n)y \;\Leftrightarrow\; \neg y\mathbb{P}_{\mathcal{G}}(P_1, \ldots, P_n)x \;\Leftrightarrow\; R(x, y) \in \mathcal{G}^*.$$

We say that $\mathcal{G}$ *generates* the SPR $\mathbb{P}_{\mathcal{G}}$.

Alternatively, given an SPR $\mathbb{P}$, we say a group G is *decisive* if for all $(P_1, \ldots, P_n) \in \mathbf{D}$ and all distinct $x, y \in X$, $G \subseteq P(x, y)$ implies $x\mathbb{P}(P_1, \ldots, P_n)y$. A decisive group can always impose a strict preference that is common to its members. Let $\mathcal{D}$ denote the set of decisive groups, and note that $\mathcal{D}$ is proper (by assumption of a free pair) and monotonic.

A group G is *blocking* if there exist $(P_1, \ldots, P_n) \in \mathbf{D}$ and there exist distinct $x, y \in X$, such that $R(x, y) \subseteq G$ and $x\mathbb{R}(P_1, \ldots, P_n)y$. A blocking group can sometimes impose a weak preference that is common to its members. Let $\mathcal{B}$ denote the set of blocking groups.

For example, the collection of decisive groups for simple and relative majority rule is $\mathcal{G}^M$, and the collection of decisive groups for simple and relative Pareto is $\mathcal{G}^P$. The blocking groups are, respectively, $(\mathcal{G}^M)^*$ and $(\mathcal{G}^P)^*$.

Note that:

- $\mathcal{B} = \mathcal{D}^*$, so $\mathcal{B}$ is strong and monotonic,
- every decisive group is blocking, i.e., $\mathcal{D} \subseteq \mathcal{B}$,
- for all $G \in \mathcal{D}$ and all $G' \in \mathcal{B}$, we must have $G \cap G' \neq \emptyset$.

To prove the last claim, suppose the two groups are disjoint, and let $(P_1, \ldots, P_n) \in \mathbf{D}$ and $x, y \in X$ be a profile and distinct alternatives such that $R(x, y) \subseteq G'$ and $x\mathbb{R}(P_1, \ldots, P_n)y$. Then $G \subseteq P(y, x)$, which implies $y\mathbb{P}(P_1, \ldots, P_n)x$, a contradiction.

An SPR $\mathbb{P}$ is *simple* if it is generated by a proper and monotonic collection $\mathcal{G}$, i.e.,

$$\mathbb{P} \;=\; \mathbb{P}_{\mathcal{G}},$$

in which case $\mathcal{G}$ consists precisely of the decisive groups for the SPR, i.e., $\mathcal{G} = \mathcal{D}$.

For example, simple majority, simple Pareto, and all quota rules are simple. More generally, letting $w = (w_1, \ldots, w_n)$, be a vector non-negative weights with $\sum_{i=1}^n w_i = n$ and $n/2 < q \leq n$, we can define the *weighted quota rule* $\mathbb{P}_{w,q}$ so that for all $(P_1, \ldots, P_n) \in \mathbf{D}$ and all $x, y \in X$,

$$x\mathbb{P}_{w,q}(P_1, \ldots, P_n)y \;\;\Leftrightarrow\;\; \sum_{i \in P(x,y)} w_i \;>\; q.$$

This class consists of simple SPRs, but there are simple SPRs that are not weighted quota rules. Note that for a weighted quota rule with $q = \frac{n}{2}$, if the weights are such that for all $G \subseteq N$, $\sum_{i \in G} w_i \neq \frac{n}{2}$, then the collection of decisive groups is strong.

For a simple SPR $\mathbb{P}$, the following hold:

- if there exist $(P_1, \ldots, P_n) \in \mathbf{D}$ and $x, y \in X$ such that $P(x, y) \subseteq G$ and $x\mathbb{P}(P_1, \ldots, P_n)y$, then $G \in \mathcal{D}$,
- if $G \in \mathcal{B}$, then for all $(P_1, \ldots, P_n) \in \mathbf{D}$ and all $x, y \in X$ such that $G \subseteq R(x, y)$, we have $x\mathbb{R}(P_1, \ldots, P_n)y$.
- $\mathcal{D} = \mathcal{B}$ if and only if $\mathcal{D}$ is strong.

To prove one direction of the third claim, assume $\mathcal{D}$ is strong, and consider any blocking group $G \in \mathcal{B}$. By definition, we have $N \setminus G \notin \mathcal{D}$, and since $\mathcal{D}$ is strong, we therefore have $G \in \mathcal{D}$. Of course, it follows that $\mathcal{D} = \mathcal{B}$ if and only if $\mathcal{B}$ is proper.

We also have the following succinct characterization of the strict and weak social preference relations: for all $(P_1, \ldots, P_n) \in \mathbf{D}$,

$$\mathbb{P}(P_1, \ldots, P_n) \;=\; \bigcup_{G \in \mathcal{D}} \bigcap_{i \in G} P_i \quad \text{and} \quad \mathbb{R}(P_1, \ldots, P_n) \;=\; \bigcup_{G \in \mathcal{B}} \bigcap_{i \in G} R_i.$$

Note that simple majority rule, simple Pareto, and in fact all quota rules are simple. Relative majority and relative Pareto are not generally simple.

4.4 Impossibility Theorems

We briefly discuss Arrow's well-known impossibility result for social preference rules, with a focus on Linear domain and simple SPRs. Arrow's impossibility theorem concerns the issue of collective rationality when the preference domain is large and there are at least three alternatives. The result says that under Pareto efficiency and independence of irrelevant alternatives (IIA), rational preferences aggregation implies dictatorship.

Next, we define several conditions on social preference rules.

- **Pareto**: for all $(P_1, \ldots, P_n) \in \mathbf{D}$ and all $x, y \in X$, $P(x, y) = N$ implies $x\mathbb{P}(P_1, \ldots, P_n)y$.
- **Independence of irrelevant alternatives (IIA)**: for all $x, y \in X$ and all $(P_1, \ldots, P_n), (P'_1, \ldots, P'_n) \in \mathbf{D}$ such that

  $$P(x, y) = P'(x, y) \quad \text{and} \quad P(y, x) = P'(y, x),$$

  we have $x\mathbb{P}(P_1, \ldots, P_n)y$ if and only if $x\mathbb{P}(P'_1, \ldots, P'_n)y$.
- **Social rationality**: for all $(P_1, \ldots, P_n) \in \mathbf{D}$, $\mathbb{P}(P_1, \ldots, P_n)$ is negatively transitive.
- **Social quasi-rationality**: for all $(P_1, \ldots, P_n) \in \mathbf{D}$, $\mathbb{P}(P_1, \ldots, P_n)$ is transitive.
- **No dictator**: there does not exist $i \in N$ such that for all $x, y \in X$ and all $(P_1, \ldots, P_n) \in \mathbf{D}$, xP_iy implies $x\mathbb{P}(P_1, \ldots, P_n)y$.
- **No oligarchy**: there does not exist nonempty $G \subseteq N$ such that for all $x, y \in X$ and all $(P_1, \ldots, P_n) \in \mathbf{D}$, (i) for all $i \in G$, xP_iy implies $x\mathbb{R}(P_1, \ldots, P_n)y$, and (ii) $G \subseteq P(x, y)$ implies $x\mathbb{P}(P_1, \ldots, P_n)y$.

IIA means if individual preferences between two alternatives, x and y are the same under two profiles, then the social preference between them should be the same as well. Dictatorship means that there is an individual i such that $\{i\}$ is decisive. We call this individual a *dictator.* A group satisfying (i) and (ii) above is an *oligarchy,* which is a decisive group such that every member can veto a strict social preference.

Note that if an SPR $\mathbb{P}$ is simple, then:

- $\mathbb{P}$ is dictatorial if and only if there exists $i \in N$ such that $\{i\}$ is decisive and $\mathcal{D} = \{G \mid i \in G\}$,
- $\mathbb{P}$ is oligarchical if and only if there exists a group G such that G is decisive and $\mathcal{D} = \{G' \mid G \subseteq G'\}$.

Next, we state a version of Arrow's impossibility theorem for simple SPRs: if a simple SPR satisfies Pareto and Social rationality, then it makes some individual a dictator.

Proposition 4.1. *Assume $|X| \geq 3$ and $\mathbf{L} \subseteq \mathbf{D}$. A simple SPR satisfies Pareto and Social rationality if and only if it is dictatorial.*

Proof. I prove one direction. Suppose there is a simple SPR $\mathbb{P}$ satisfying Pareto, Social rationality, and No dictator. Let $\mathcal{D}$ be the decisive groups of $\mathbb{P}$. By Pareto, for any profile $(P_1, \ldots, P_n)$ with $P(x, y) = N$, we have $x\mathbb{P}(P_1, \ldots, P_n)y$, i.e., $P(x, y) \in \mathcal{D}$. Therefore, $\mathcal{D} \neq \emptyset$. Next, let G be minimal among $\mathcal{D}$ according to set inclusion. Then $|G| \geq 2$ by No dictator. Partition G into nonempty groups, G_1 and G_2, i.e., $G_1 \cap G_2 = \emptyset$ and $G_1 \cup G_2 = G$. Choose distinct alternatives $x, y, z \in X$ and any profile $(P_1, \ldots, P_n)$ such that

G_1	G_2	$N \setminus G$
z	y	x
x	z	y
y	x	z

Since $G_2 \subsetneq G$, we have $G_2 \notin \mathcal{D}$, and it follows that $N \setminus G_2 \in \mathcal{B}$. Therefore, since $R(x, y) = N \setminus G_2$, we have $x\mathbb{R}(P_1, \ldots, P_n)y$. Similarly, we have $N \setminus G_1 \in \mathcal{B}$ and $R(y, z) = N \setminus G_1$, which implies $y\mathbb{R}(P_1, \ldots, P_n)z$. By Social rationality, this implies $x\mathbb{R}(P_1, \ldots, P_n)z$, but $P(z, x) = G \in \mathcal{D}$ implies $z\mathbb{P}(P_1, \ldots, P_n)x$, a contradiction. $\square$

To extend the previous proposition to a version of Arrow's theorem under Linear domain, it suffices to show that Pareto, IIA, and Social quasi-rationality imply that an SPR is simple. Note that the following lemma does not use the full force of the Social rationality axiom, but it does rely on Linear domain.

Lemma 4.2. *Assume $|X| \geq 3$ and Linear domain. If an SPR satisfies Pareto, IIA, and Social quasi-rationality, then it is simple.*

Proof. We need to show that for all $(P_1, \ldots, P_n) \in \mathbf{D}$ and all $x, y \in X$, $x\mathbb{P}(P_1, \ldots, P_n)y$ if and only if $P(x, y) \in \mathcal{D}$. One direction of this statement is clear. For the other, assume that $x\mathbb{P}(P_1, \ldots, P_n)y$. It remains to be argued that $P(x, y)$ is decisive, i.e., for all $(P'_1, \ldots, P'_n)$ and all $a, b \in X$, $P(x, y) \subseteq P(a, b)$ implies $a\mathbb{P}(P_1, \ldots, P_n)b$. To this end, consider any $a, b \in X$ and any profile $(P'_1, \ldots, P'_n)$ such that $P(x, y) \subseteq P'(a, b)$. We must argue that $a\mathbb{P}(P'_1, \ldots, P'_n)b$. To ease notation, let $G = P(x, y)$, and note that by Linear domain, we have $N \setminus G = P(y, x)$. Furthermore, let $H = P'(a, b)$ and $N \setminus H = P'(b, a)$, and note that $G \subseteq H$ by assumption.

Claim 1: for all $(\tilde{P}_1, \ldots, \tilde{P}_n) \in \mathbf{D}$ and all $z \in X \setminus \{x, y\}$ with $G \subseteq \tilde{P}(x, z)$, we have $x\mathbb{P}(\tilde{P}_1, \ldots, \tilde{P}_n)z$. Let $(P^1_1, \ldots, P^1_n)$ be a profile of strict linear orders with preferences over $\{x, y, z\}$ as follows.

G	$\tilde{P}(x,z) \setminus G$	$N \setminus \tilde{P}(x,z)$
x	y	y
y	x	z
z	z	x

Since $P^1(x,y) = G$ and $P^1(y,x) = N \setminus G$ and $x\mathbb{P}(P_1, \ldots, P_n)y$, we have $x\mathbb{P}(P_1^1, \ldots, P_n^1)y$ by IIA. By Pareto, $y\mathbb{P}(P_1^1, \ldots, P_n^1)z$. With $x\mathbb{P}(P_1, \ldots, P_n)y\mathbb{P}(P_1^1, \ldots, P_n^1)z$, Social quasi-rationality implies $x\mathbb{P}(P_1^1, \ldots, P_n^1)z$. The claim then follows by IIA.

Claim 2: for all $(\tilde{P}_1, \ldots, \tilde{P}_n) \in \mathbf{D}$ and all $z \in X \setminus \{a, x\}$ with $H = \tilde{P}(a,z)$, if $x \neq a$, then $a\mathbb{P}(\tilde{P}_1, \ldots, \tilde{P}_n)z$. Let $(P_1^2, \ldots, P_n^2)$ be a profile of strict linear orders with preferences over $\{a, x, z\}$ as follows.

G	$H \setminus G$	$N \setminus H$
a	a	z
x	z	a
z	x	x

By Pareto, $a\mathbb{P}(P_1^2, \ldots, P_n^2)x$. Note that $P^2(x,z) = G$ and $P^2(z,x) = N \setminus G$. If $y = z$, then $x\mathbb{P}(P_1^2, \ldots, P_n^2)z$ follows by IIA; and otherwise, if $y \neq z$, then Claim 1 again implies $x\mathbb{P}(P_1^2, \ldots, P_n^2)z$. With $a\mathbb{P}(P_1^2, \ldots, P_n^2)x\mathbb{P}(P_1^2, \ldots, P_n^2)z$, Social quasi-rationality implies $a\mathbb{P}(P_1^2, \ldots, P_n^2)z$. The claim then follows by IIA.

Claim 3: $a\mathbb{P}(P_1', \ldots, P_n')b$. If $X = \{a, b, x\}$, then $b \in X \setminus \{a, x\}$, and since $|X| \geq 3$, we must have $x \neq a$, and Claim 2 delivers the result. Otherwise, there exists $z \in X \setminus \{a, b, x\}$. If $a = x$ and $z = y$, then $b \in X \setminus \{x, y\}$, and Claim 1 implies $a\mathbb{P}(P_1^3, \ldots, P_n^3)b$. Thus, we assume $a \neq x$ or $z \neq y$. Let $(P_1^3, \ldots, P_n^3) \in \mathbf{D}$ be a profile of strict linear orders with preferences over $\{a, z, b\}$ as follows.

H	$N \setminus H$
a	z
z	b
b	a

Note that $P^3(a,z) = H$ and $P^3(z,a) = N \setminus H$. In case $a \neq x$, Claim 2 implies $a\mathbb{P}(P_1^3, \ldots, P_n^3)z$; and in case $a = x$ and $z \neq y$, Claim 1 implies that again $a\mathbb{P}(P_1^3, \ldots, P_n^3)z$. By Pareto, $z\mathbb{P}(P_1^3, \ldots, P_n^3)b$. With $a\mathbb{P}(P_1^3, \ldots, P_n^3)z\mathbb{P}(P_1^3, \ldots, P_n^3)b$, Social quasi-rationality implies $a\mathbb{P}(P_1^3, \ldots, P_n^3)b$. The claim then follows by IIA. □

Finally, we state Arrow's theorem as a corollary of Proposition 4.1 and Lemma 4.2.

Corollary 4.3 (Arrow)**.** *Assume $|X| \geq 3$ and Linear domain. Then there is no SPR satisfying Pareto, IIA, Social rationality, and No dictator.*

Put differently, under background conditions, if an SPR satisfies Pareto, IIA, and No dictator, then it violates Social rationality. For example, majority rule (simple or relative) satisfies the first three axioms, so Arrow's theorem implies that there is a preference profile such that the weak majority preference relation is intransitive. Of course, from Condorcet's paradox we know more: there is a profile such that strict majority preference is actually cyclic.

The next result shows that under standard background conditions, if a simple SPR generates transitive strict social preferences, then it is oligarchical.

Proposition 4.4. *Assume* $|X| \geq 3$ *and* $\mathbf{L} \subseteq \mathbf{D}$. *A simple SPR satisfies Pareto and Social quasi-rationality if and only if it is oligarchical.*

Again using Lemma 4.2, we can state an impossibility theorem due to Gibbard.

Corollary 4.5 (Gibbard)**.** *Assume* $|X| \geq 3$ *and Linear domain. Then there is no SPR satisfying Pareto, IIA, Social quasi-rationality, and No oligarchy.*

To characterize the possibility for acyclic preference aggregation, consider an SPR $\mathbb{P}$ satisfying Pareto, and define the *acyclicity index* of $\mathbb{P}$, denoted α, as follows: if no individual belongs to all decisive groups, i.e., $\bigcap \mathcal{D} = \emptyset$, then define

$$\alpha \;=\; \min\left\{ |\mathcal{G}| \;\mid\; \mathcal{G} \subseteq \mathcal{D} \text{ and } \bigcap \mathcal{G} = \emptyset \right\},$$

and if $\bigcap \mathcal{D} \neq \emptyset$, then let assign α a cardinality greater than the set X of alternatives. (If X is finite, then we can let $\alpha = |X| + 1$.) In the latter case, where the decisive groups have nonempty intersection, we say the SPR is *collegial,* and in the former, it is *non-collegial.* In words, when $\mathbb{P}$ is non-collegial, α is the size of the smallest collection of decisive groups having empty intersection.

To state our characterization, we define an additional axiom.

- **Social sub-rationality:** for all $(P_1, \ldots, P_n) \in \mathbf{D}$, $\mathbb{P}(P_1, \ldots, P_n)$ is acyclic.

Proposition 4.6 (Brown; Nakamura)**.** *Assume* $|X| \geq 3$ *and* $\mathbf{L} \subseteq \mathbf{D}$*, and let* $\mathbb{P}$ *be an SPR satisfying Pareto. If* $\mathbb{P}$ *satisfies Social sub-rationality, then* $\alpha > |X|$*. And assuming* $\mathbb{P}$ *is simple, the converse direction holds as well.*

Proof. I prove one direction. Suppose that $\mathbb{P}$ satisfies Pareto and Social sub-rationality but $\alpha \leq |X|$. In particular, $\bigcap \mathcal{D} = \emptyset$, and there exist $G_1, \ldots, G_\alpha \in \mathcal{D}$ such that

$\bigcap_{j=1}^{\alpha} G_j = \emptyset$. Furthermore, because $|X| \geq \alpha$, we can choose distinct $x_1, \ldots, x_\alpha \in X$, and we can specify a profile $(P_1, \ldots, P_n) \in \mathbf{D}$ of preferences satisfying the restrictions below.

$N \setminus G_1$	$G_1 \setminus G_2$	$(G_1 \cap G_2) \setminus G_3$	$\cdots$	$(\bigcap_{j=1}^{k-1} G_j) \setminus G_k$	$\cdots$	$(\bigcap_{j=1}^{\alpha-1} G_j) \setminus G_\alpha$
x_2	x_3	x_4		x_{k+1}		x_1
x_3	x_4	x_5		x_{k+2}		x_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{\alpha-1}$	x_α	x_1		x_{k-2}		$x_{\alpha-2}$
x_α	x_1	x_2		x_{k-1}		$x_{\alpha-1}$
x_1	x_2	x_3		x_k		x_α

But then because each G_k is decisive, we have

$$x_1 \mathbb{P}(P_1, \ldots, P_n) x_2 \mathbb{P}(P_1, \ldots, P_n) x_3 \cdots \mathbb{P}(P_1, \ldots, P_n) x_\alpha \mathbb{P}(P_1, \ldots, P_n) x_1,$$

contradicting Social sub-rationality. □

To apply the preceding result to majority rule, note that the acyclicity index of majority rule is $\alpha = 3$ if $n \geq 3$ but $n \neq 4$; and it is $\alpha = 4$ when $n = 4$. Thus, Proposition 4.6 implies that if $n \geq 3$ but $n \neq 4$, and if $|X| \geq 3$, then the strict majority preference is cyclic for some profiles of preferences; we knew this from Condorcet's paradox, and in fact the profile used in the proof reduces to the Condorcet profile when $n = |X| = 3$. If $n = 4$, then strict majority preference cycles are possible as long as $|X| \geq 4$.

For the record, the acyclicity index of a quota rule with $q < n$ is $\left\lceil \frac{n}{n-q} \right\rceil$, where $\lceil \cdot \rceil$ denotes the integer ceiling.

5 Value Restriction

5.1 Negatively Transitive Strict Preference

Motivated by Arrow's impossibility theorem, we now consider possible restrictions on individual preferences to obtain positive results on social rationality. Given $Y = \{a, b, c\}$ where a, b, c are distinct alternatives, define the set valued mapping $V_i^Y : Y \rightrightarrows \{1, 2, 3\}$ according to

$$V_i^Y(x) \quad = \quad \{|P_i^{-1}(x) \cap Y| + 1, \ldots, |R_i^{-1}(x) \cap Y|\}.$$

We say individual i is *concerned on* Y if there exist $x, y \in Y$ such that $x P_i y$, and otherwise i is *unconcerned on* Y. Let $N(Y)$ denote the set of individuals who are concerned on Y.

We say a preference profile $(P_1, \ldots, P_n)$ is *value restricted* if for every $Y \subseteq X$ with $|Y| = 3$, there exists $x \in Y$ and $r \in \{1, 2, 3\}$ such that for all $i \in N(Y)$, we have $r \notin V_i^Y(x)$. Value-restriction means that for every triple Y, there exists a rank $r \in \{1, 2, 3\}$ and an alternative $x \in Y$ such that x does not attain rank r in Y for any concerned individuals.

For example, letting $n = 3$ and $X = Y = \{a, b, c\}$, then (P_1, P_2, P_3) is value-restricted, but (P_1', P_2', P_3') is not value-restricted.

P_1	P_2	P_3
a	b	c
b	a	a
c	c	b

P_1'	P_2'	P_3'
a	b	c
b	c	a
c	a	b

We say a preference profile $(P_1, \ldots, P_n)$ satisfies *concern over triples* if every individual is concerned on all triples of alternatives, i.e., for every $Y \subseteq X$ with $|Y| = 3$, $N(Y) = N$.

Our first result establishes that under value restriction and concern over triples, if the collection of decisive groups of a simple SPR is strong, then social preferences form an ordering.

Proposition 5.1. *Assume $\mathbb{P}$ is simple and $\mathcal{D}$ is strong, and let $(P_1, \ldots, P_n)$ be value-restricted and satisfy concern over triples. Then $\mathbb{P}(P_1, \ldots, P_n)$ is negatively transitive.*

Proof. Consider any $x, y, z \in X$ such that $x\mathbb{R}(P_1, \ldots, P_n)y\mathbb{R}(P_1, \ldots, P_n)z$, and suppose that $z\mathbb{P}(P_1, \ldots, P_n)x$. Then x, y, z are distinct, and letting $Y = \{x, y, z\}$, concern over triples implies $N(Y) = N$. By definition of blocking group, we have $R(x, y) \in \mathcal{B}$ and $R(y, z) \in \mathcal{B}$, and since $\mathbb{P}$ is simple, we also have $P(z, x) \in \mathcal{D}$. Since $\mathcal{D}$ is strong, we have $\mathcal{B} = \mathcal{D}$. Since $\mathcal{D}$ is proper, it follows that there exist $i, j, k \in N$ such that $i \in R(x, y) \cap R(y, z)$, $j \in R(y, z) \cap P(z, x)$, $k \in P(z, x) \cap R(x, y)$. Therefore, we have xR_iyR_iz, yR_jzP_jx, zP_kxR_ky, and in particular we have $V_i^Y(w) \cup V_j^Y(w) \cup V_k^Y(w) = \{1, 2, 3\}$ for all $w \in Y$, violating value restriction. □

The following two examples show that neither the assumption that $\mathcal{D}$ is strong nor concern over triples can be dropped in the previous result. Consider the majority social preference rule. Dropping the assumption that $\mathcal{D}$ is strong, consider the profile below,

P_1	P_2
c	b
a	c
b	a

and note that $a\mathbb{R}_{SM}(P_1, P_2, P_3)b\mathbb{R}_{SM}(P_1, P_2, P_3)c\mathbb{P}_{SM}(P_1, P_2, P_3)a$. Dropping concern over triples, consider the profile below,

P_1	P_2	P_3
c		b
a	abc	c
b		a

and note that $a\mathbb{R}_{SM}(P_1, P_2, P_3)b\mathbb{R}_{SM}(P_1, P_2, P_3)c\mathbb{P}_{SM}(P_1, P_2, P_3)a$.

The assumption in Proposition 5.1 that the SPR is simple with a strong collection of decisive groups can be weakened: what is needed is that for all $(P_1, \ldots, P_n) \in \mathbf{D}$ and all $x, y \in X$, $x\mathbb{R}(P_1, \ldots, P_n)y$ implies $R(x, y) \in \mathcal{D}$. This is satisfied, for example, by relative majority rule when n is odd, despite the fact that it is not strong. I state the proposition with the stronger assumption to simplify the analysis.

5.2 Transitive Strict Preference

We say a preference profile $(P_1, \ldots, P_n)$ is *weakly value-restricted* if for every $Y \subseteq X$ with $|Y| = 3$, either

(i) there exist $x \in Y$ and $r \in \{1, 2, 3\}$ such that for all $i \in N(Y)$, we have $r \notin V_i^Y(x)$, or

(ii) for all $x \in Y$, there exists $r \in \{1, 2, 3\}$ such that for all $i \in N(Y)$, we have $V_i^Y(x) \neq \{r\}$.

Weak value restriction means that for every triple Y, either individual preferences satisfy the conditions for value restriction, or for each alternative x in the triple, there exists a rank $r \in \{1, 2, 3\}$ such that x does not uniquely attain rank r in Y for any concerned individual.

Our next result establishes that weak value restriction leads to transitivity of strict social preferences for all simple SPRs.

Proposition 5.2. *Assume the SPR $\mathbb{P}$ is simple, and let $(P_1, \ldots, P_n)$ be weakly value-restricted. Then $\mathbb{P}(P_1, \ldots, P_n)$ is transitive.*

Proof. Consider any $x, y, z \in X$ with $x\mathbb{P}(P_1, \ldots, P_n)y\mathbb{P}(P_1, \ldots, P_n)z$, and suppose that $z\mathbb{R}(P_1, \ldots, P_n)x$, which implies $R(z, x) \in \mathcal{B}$. Since $\mathbb{P}$ is simple, we also have $P(x, y) \in \mathcal{D}$ and $P(y, z) \in \mathcal{D}$. Note that there exist $i, j, k \in N$ such that $i \in P(x, y) \cap P(y, z)$, $j \in P(y, z) \cap R(z, x)$, and $k \in R(z, x) \cap P(x, y)$. But then individuals i, j, k are concerned on $Y = \{x, y, z\}$, and we have xP_iyP_iz, yP_jzR_jx, and zR_kxP_ky. In particular, we have $V_i^Y(w) \cup V_j^Y(w) \cup V_k^Y(w) = \{1, 2, 3\}$ for all $w \in Y$. Moreover, $V_i^Y(y) = \{2\}$, $V_j^Y(y) = \{3\}$, and $V_k^Y(y) = \{1\}$, contradicting weak value restriction. □

5.3 Acyclic Strict Preference

A profile $(P_1, \ldots, P_n)$ is *minimally value restricted* if for every finite $Y \subseteq X$,

- if $|Y| = 3$, then there exist $x \in Y$ and $r \in \{1, 2, 3\}$ such that for all $i \in N(Y)$, we have $V_i^Y(x) \neq \{r\}$,
- if $|Y| \geq 4$, then for every enumeration $Y = \{x_1, \ldots, x_m\}$, there exists $\ell \in \{1, \ldots, m\}$ such that either
  - (i) there exist $x \in \{x_\ell, x_{\ell+1}, x_{\ell+2}\}$ and $r \in \{1, 2, 3\}$ such that for all $i \in N(\{x_\ell, x_{\ell+1}, x_{\ell+2}\})$, we have $r \notin V_i^Z(x)$, or
  - (ii) for all $x \in \{x_\ell, x_{\ell+1}, x_{\ell+2}\}$, there exists $r \in \{1, 2, 3\}$ such that for all $i \in N(\{x_\ell, x_{\ell+1}, x_{\ell+2}\})$, we have $V_i^Z(x) \neq \{r\}$,

where addition is understood to be modulo m.

Note that the restriction for triples Y rules out the possibility that the preferences of three individuals over three alternatives form a Condorcet profile, as in the well-known Condorcet paradox. The restriction on preferences over larger subsets mirrors the condition in weak value restriction.

Proposition 5.3. *Assume $\mathbb{P}$ is simple, and let $(P_1, \ldots, P_n)$ be minimally value restricted. Then $\mathbb{P}(P_1, \ldots, P_n)$ is acyclic.*

Proof. Consider any $m \geq 3$ and $x_1, \ldots, x_m$ such that

$$x_1 \mathbb{P}(P_1, \ldots, P_n) x_2 \mathbb{P}(P_1, \ldots, P_n) \cdots x_m \mathbb{P}(P_1, \ldots, P_n) x_1.$$

Furthermore, by selecting a smallest such cycle, we can assume without loss of generality that the m alternatives are distinct. In case $m = 3$, because $\mathbb{P}$ is simple, it follows that $\{P(x_1, x_2), P(x_2, x_3), P(x_3, x_1)\} \subseteq \mathcal{D}$, and since $\mathcal{D}$ is proper, there exist $i, j, k \in N$ such that $x_1 P_i x_2 P_i x_3$, $x_2 P_2 x_3 P_2 x_1$, and $x_3 P_3 x_1 P_3 x_2$, contradicting minimal value restriction. In case $m \geq 4$, note that by choice of a smallest cycle, it follows that for all $\ell \in \{1, \ldots, m\}$, we have

$$x_\ell \mathbb{P}(P_1, \ldots, P_n) x_{\ell+1} \mathbb{P}(P_1, \ldots, P_n) x_{\ell+2} \mathbb{R}(P_1, \ldots, P_n) x_\ell.$$

Indeed, if this were not the case, then we would have $x_\ell \mathbb{P}(P_1, \ldots, P_n) x_{\ell+2}$, but then we could remove $x_{\ell+1}$ to create a smaller cycle. Now consider any $\ell \in \{1, \ldots, m\}$. Since $\mathbb{P}$ is simple, it follows that $P(x_\ell, x_{\ell+1})$ and $P(x_{\ell+1}, x_{\ell+2})$ are decisive, and so there exist i, j, k such that $i \in P(x_\ell, x_{\ell+1}) \cap P(x_{\ell+1}, x_{\ell+2})$, $j \in P(x_{\ell+1}, x_{\ell+2}) \cap R(x_{\ell+2}, x_\ell)$, and $k \in R(x_{\ell+2}, x_\ell) \cap P(x_\ell, x_{\ell+1})$, again contradicting minimal value restriction. □

5.4 Preference Exclusions

The transitivity results of the previous subsection have straightforward generalizations and an extension to acyclicity of social preferences. We say $(P_1, \ldots, P_n)$ satisfies *exclusion condition NT* if for all distinct $x_1, x_2, x_3 \in X$, there do not exist distinct $i, j, k \in N$ such that

(1) $x_1 R_i x_2 R_i x_3$,

(2) $x_2 R_j x_3 P_j x_1$, and

(3) $x_3 P_k x_1 R_k x_2$.

Note that value restriction implies exclusion condition NT.

The proof of Proposition 5.1 immediately yields the following more general result.

Proposition 5.4. *Assume $\mathbb{P}$ is simple and $\mathcal{D}$ is strong, and let $(P_1, \ldots, P_n)$ satisfy exclusion condition NT. Then $\mathbb{P}(P_1, \ldots, P_n)$ is negatively transitive.*

We say $(P_1, \ldots, P_n)$ satisfies *exclusion condition T* if for all distinct $x_1, x_2, x_3 \in X$, there do not exist $i, j, k \in N$ such that

(4) $x_1 P_i x_2 P_i x_3$,

(5) $x_2 P_j x_3 R_j x_1$, and

(6) $x_3 R_k x_1 P_k x_2$.

Note that (4)–(6) imply (1)–(3), after permuting alternatives and individuals, so exclusion condition NT implies exclusion condition T. To see this, suppose that for distinct alternatives x_1, x_2, x_3, there exist individuals i, j, k satisfying (4)–(6). Set

$$\begin{array}{ll} x_1' = x_2 & i' = j \\ x_2' = x_3 & j' = k \\ x_3' = x_1 & k' = i. \end{array}$$

Rewriting (4)–(6), we then have

(4) $x_1' P_{i'} x_2' R_{i'} x_3'$,

(5) $x_2' R_{j'} x_3' P_{j'} x_1'$, and

(6) $x_3' P_{k'} x_1' P_{k'} x_2'$,

implying (1)–(3). Clearly, weak value restriction also implies exclusion condition T.

The proof of Proposition 5.2 immediately yields the following more general result.

Proposition 5.5. *Assume the SPR $\mathbb{P}$ is simple, and let $(P_1, \ldots, P_n)$ satisfy exclusion condition T. Then $\mathbb{P}(P_1, \ldots, P_n)$ is transitive.*

To consider acyclicity, we say $(P_1, \ldots, P_n)$ satisfies *exclusion condition A* if for all natural numbers $m \geq 3$ and all distinct $x_1, \ldots, x_m \in X$, we have the following:

- if $m = 3$, then there do not exist $i, j, k \in N$ such that
  - (i) $x_1 P_i x_2 P_i x_3$,
  - (ii) $x_2 P_j x_3 P_j x_1$, and
  - (iii) $x_3 P_k x_1 P_k x_2$,
- if $m \geq 4$, then there exists $\ell \in \{1, \ldots, m\}$ such that there do not exist $i, j, k \in N$ such that
  - (iv) $x_\ell P_i x_{\ell+1} P_i x_{\ell+2}$,
  - (v) $x_{\ell+1} P_j x_{\ell+2} R_j x_\ell$, and
  - (vi) $x_{\ell+2} R_k x_\ell P_k x_{\ell+1}$,

  where addition to subscripts is understood to be modulo m.

The proof of Proposition 5.3 immediately yields the following more general result.

Proposition 5.6. *Assume $\mathbb{P}$ is simple, and let $(P_1, \ldots, P_n)$ satisfy preference exclusion A. Then $\mathbb{P}(P_1, \ldots, P_n)$ is acyclic.*

6 Single-peakedness

We have discussed the impossibility results under broad domain restrictions (e.g., unrestricted domain or linear domain) and some positive results under value restrictions on individual preferences. We now investigate a preference restriction that arises naturally in the unidimensional model, namely single-peakedness.

Let $\leqslant$ be a weak linear order of the set of alternatives. We say $(P_1, \ldots, P_n)$ is *single-peaked* with respect to $\leqslant$ if for each $i \in N$, there exists $\hat{x}_i \in X$, the *ideal point* of i, such that

(1) $\forall y \in X \setminus \{\hat{x}_i\}$, $\hat{x}_i P_i y$;

(2) $\forall y, z \in X$, $\hat{x}_i < y < z \;\Rightarrow\; y P_i z$;

(3) $\forall y, z \in X$, $y < z < \hat{x}_i \;\Rightarrow\; z P_i y$.

Let $\mathbf{S}_{\leqslant}$ denote the domain of single-peaked preference profiles with respect to $\leqslant$. Let

$$\mathbf{S} = \bigcup \{\mathbf{S}_{\leqslant} \mid \leqslant \text{ is a linear order on } X\}$$

denote the *single-peaked domain*.

For example, the following preference profile

P_1	P_2	P_3
b	a	c
a	c	a
c	b	b

is single-peaked with respect to the order $b < a < c$. However, it is not single-peaked with respect to $a < b < c$.

Note that the ordering may not be unique, even up to reversals; for example, the profile

P_1	P_2	P_3
a	a	b
b	b	a
c	c	c

is single-peaked with respect to $c < b < a$ and $c < a < b$ (and the reverse of these orderings). It is also clear from Condorcet's paradox that there may be no ordering on X such that a preference profile is single-peaked.

Proposition 6.1. *Assume $X \subseteq \mathbb{R}$ is convex. Let $(P_1, \ldots, P_n)$ be such that for all $i \in N$, there is a utility representation $u_i \colon X \to \mathbb{R}$ of P_i. Then $(P_1, \ldots, P_n)$ is single-peaked with respect to the usual less than-or-equal-to order $\leq$ if and only if for all $i \in N$,*

(a) u_i has a maximizer, and

(b) u_i is strictly quasi-concave.

Proof. Consider any individual i, and assume (a) and (b) hold. Then u_i has a unique maximizer, say $\hat{x}_i$, as in the part (1) of the definition of single-peakedness. To prove part (2), consider $y, z \in X$ such that $\hat{x}_i < y < z$. By strict quasi-concavity, we have

$$u_i(y) > \min\{u_i(\hat{x}_i), u_i(z)\} = u_i(z)$$

so yP_iz. Part (3) follows similarly, as required. For the other direction, assume P_i is single-peaked. To prove (a) and (b), note that (1) implies (a) immediately. Consider $x < y$ and $z \in (x, y)$. We want to show $u_i(z) > \min\{u_i(x), u_i(y)\}$. Assume without loss of generality that $u_i(x) \geq u_i(y)$. To show zP_iy, note that by single-peakedness, we have $\hat{x}_i < y$. In case $z < \hat{x}_i$, then we have $x < z < \hat{x}_i$, and single-peakedness implies zP_ixR_iy, and thus zP_iy. And in case $\hat{x}_i \leq z$, then we have $\hat{x}^i \leq z < y$, and single-peakedness implies zP_iy again, as required. ∎

6.1 Transitivity Properties

The next result shows the relation between single-peakedness and value restriction.

Proposition 6.2. *If $(P_1, \ldots, P_n)$ is single-peaked, then it is value-restricted and satisfies concern over triples.*

Proof. Let $(P_1, \ldots, P_n)$ be single-peaked with respect to a linear order $\leqslant$. Consider $Y = \{x, y, z\}$, where x, y, z are distinct, and assume without loss of generality that $x < y < z$. Then for all $i \in N$, we have $1 \notin V_i^Y(y)$, i.e., y cannot be bottom ranked. Thus, value restriction holds. □

That value restriction is strictly weaker than single-peakedness is demonstrated by the following profile, which is value-restricted but not single-peaked with respect to any ordering of alternatives.

P_1	P_2	P_3
a	b	a
b	c	c
c	a	b

The following corollaries are immediate.

Corollary 6.3. *Assume $\mathbb{P}$ is simple and $\mathcal{D}$ is strong, and let $(P_1, \ldots, P_n)$ be single-peaked. Then $\mathbb{P}(P_1, \ldots, P_n)$ is negatively transitive.*

Corollary 6.4. *Assume $\mathbb{P}$ is simple, and let $(P_1, \ldots, P_n)$ be single-peaked. Then $\mathbb{P}(P_1, \ldots, P_n)$ is transitive.*

The above corollaries imply that there exists a SPR $\mathbb{P}$ mapping from $\mathbf{S}$ into the set of asymmetric relations satisfying IIA, Pareto, No dictator, and Social rationality, as long as $n \geq 3$. When n is odd, for example, simple majority rule suffices.

Corollary 6.5. *Assume X is finite and $\mathbb{P}$ is simple, and let $(P_1, \ldots, P_n)$ is single-peaked. Then the maximal set $M(\mathbb{P}(P_1, \ldots, P_n))$ is nonempty.*

6.2 Core Characterization

Recall that the maximal set of $\mathbb{P}(P_1, \ldots, P_n)$ is the *core* of $\mathbb{P}$ at $(P_1, \ldots, P_n)$, and that it is denoted by

$$\mathbb{C}(P_1, \ldots, P_n) \quad = \quad M(\mathbb{P}(P_1, \ldots, P_n)).$$

Now we derive a characterization of the core, known as the "median voter theorem," under single-peakedness for a large class of SPRs.

Let $(P_1, \ldots, P_n)$ be single-peaked with respect to $\leqslant$, and let $\mathbb{P}$ be a simple SPR. Define

$$N^+_{\leqslant}(x) = \{i \in N \mid x < \hat{x}_i\}$$

$$N^-_{\leqslant}(x) = \{i \in N \mid \hat{x}_i < x\},$$

where we suppress dependence on the preference profile. This gives us the groups of individuals with ideal points above and below, respectively, the alternative x.

We say x is a *median alternative* with respect to the profile $(P_1, \ldots, P_n)$ if

$$N^+_{\leqslant}(x) \notin \mathcal{D} \quad \text{and} \quad N^-_{\leqslant}(x) \notin \mathcal{D}.$$

Denote the set of medians with respect to $(\hat{x}_1, \ldots, \hat{x}_n)$ by $\mu_{\leqslant}(P_1, \ldots, P_n)$.

In fact, $\mu_{\leqslant}(P_1, \ldots, P_n)$ is independent of $\leqslant$, i.e., if $(P_1, \ldots, P_n)$ is single-peaked with respect $\leqslant$ and $\leqslant'$, then $\mu_{\leqslant}(P_1, \ldots, P_n) = \mu_{\leqslant'}(P_1, \ldots, P_n)$. So we write $\mu(P_1, \ldots, P_n)$ as the set of medians with respect to $(P_1, \ldots, P_n)$.

Our next result, the median voter theorem, gives an exact characterization of the core given a profile of single-peaked preferences. An implication is that the core of every simple SPR is nonempty when individual preferences are single-peaked, regardless of the size of the set of alternatives.

Proposition 6.6 (Median voter theorem)**.** *Assume $\mathbb{P}$ is simple, and let $(P_1, \ldots, P_n)$ be single-peaked. Then $\mathbb{C}(P_1, \ldots, P_n) = \mu(P_1, \ldots, P_n)$.*

Proof. To show $\mathbb{C}(P_1, \ldots, P_n) \subseteq \mu(P_1, \ldots, P_n)$, consider $x \in \mathbb{C}(P_1, \ldots, P_n)$, and suppose that $x \notin \mu(P_1, \ldots, P_n)$. Assume without loss of generality that $N^+_{\leqslant}(x) \in \mathcal{D}$. Letting $j = \arg\min\{\hat{x}_i \mid i \in N^+_{\leqslant}(x)\}$, single-peakedness implies that $N^+_{\leqslant}(x) \subseteq P(\hat{x}_j, x)$. Since $N^+_{\leqslant}(x)$ is decisive, we have $\hat{x}_j \mathbb{P}(P_1, \ldots, P_n) x$, contradicting $x \in \mathbb{C}(P_1, \ldots, P_n)$. To show $\mu(P_1, \ldots, P_n) \subseteq \mathbb{C}(P_1, \ldots, P_n)$, consider $x \in \mu(P_1, \ldots, P_n)$, and suppose that $x \notin \mu(P_1, \ldots, P_n)$, i.e., there exists $y \in X$ such that $y \mathbb{P}(P_1, \ldots, P_n) x$. Assume without loss of generality that $x < y$. Since $\mathbb{P}$ is simple, we have $P(y, x) \in \mathcal{D}$. By single-peakedness, $P(y, x) \subseteq N^+_{\leqslant}(x)$. Therefore, $N^+_{\leqslant}(x) \in \mathcal{D}$, which contradicts $x \in \mu(P_1, \ldots, P_n)$. □

To unpack the implications of the median voter theorem, and to better understand the structure of the set of medians, it is useful to define the groups

$$\begin{aligned} G^+ &= \left\{i \in N \mid N^+_{\leqslant}(\hat{x}_i) \notin \mathcal{D}\right\} \\ G^- &= \left\{i \in N \mid N^-_{\leqslant}(\hat{x}_i) \notin \mathcal{D}\right\}, \end{aligned}$$

where we again suppress dependence on the preference profile. Note that $G^+ \neq \emptyset$, since letting i have the greatest ideal point according to $\leqslant$, we have $i \in G^+$. Similarly, $G^- \neq \emptyset$.

Let $i^+ \in \arg\min\{\hat{x}_i \mid i \in G^+\}$ and $i^- \in \arg\max\{\hat{x}_i \mid i \in G^-\}$, where the minimum and maximum are with respect to $\leqslant$. Note that these groups are "comprehensive," in the sense that if $\hat{x}_{i^+} < \hat{x}_i$, then $i \in G^+$; similarly, if $\hat{x}_i < \hat{x}_{i^-}$, then $i \in G^-$.

I claim that if $\mathbb{P}$ satisfies Pareto, then G^+ and G^- are decisive. To see this, note that G^+ is decisive if $G^+ = N$. If $G^+ \subsetneqq N$, then let i be a solution to $\max_{i \in N \setminus G^+} \hat{x}_i$, so that $G^+ = N^+_{\leqslant}(\hat{x}_i) \in \mathcal{D}$. A similar argument shows that $G^- \in \mathcal{D}$.

Therefore, if $\mathbb{P}$ satisfies Pareto, then since $\mathcal{D}$ is proper, we have $G^+ \cap G^- \neq \emptyset$, and this implies $\hat{x}_{i^+} \leqslant \hat{x}_{i^-}$. Next, we characterize the set of median alternatives of the simple rules satisfying Pareto.

Proposition 6.7. *Assume $\mathbb{P}$ is simple and satisfies Pareto, and let $(P_1, \ldots, P_n)$ be single-peaked with respect to $\leq$. Then*

$$\mu(P_1, \ldots, P_n) \;=\; \{x \in X \mid \hat{x}_{i^+} \leqslant x \leqslant \hat{x}_{i^-}\},$$

and $\mu(P_1, \ldots, P_n) \neq \emptyset$.

Proof. Consider any $x \in \mu(P_1, \ldots, P_n)$. If it is not true that $\hat{x}_{i^+} \leqslant x \leqslant \hat{x}_{i^-}$, then assume without loss of generality that $x < \hat{x}_{i^+}$. But then $G^+ \subseteq N^+_{\leqslant}(x)$, which implies $N^+_{\leqslant}(x) \in \mathcal{D}$ and contradicts $x \in \mu(P_1, \ldots, P_n)$. Next, consider $x \in X$ such that $\hat{x}_{i^+} \leqslant x \leqslant \hat{x}_{i^-}$. We want to show $x \in \mu(P_1, \ldots, P_n)$. If not, then assume without loss of generality that $N^+_{\leqslant}(x) \in \mathcal{D}$. However, since $N^+_{\leqslant}(x) \subseteq N^+_{\leqslant}(\hat{x}_{i^+})$, then $N^+_{\leqslant}(\hat{x}_{i^+}) \in \mathcal{D}$, contradicting $i^+ \in G^+$. Thus, the median alternatives contain $\hat{x}_{i^+}$ and $\hat{x}_{i^-}$, so the set of median alternatives is nonempty. □

Let us consider the median alternatives in some special cases. For a quota rule $\mathbb{P}_q$ with quota q satisfying $n/2 < q \leq n$, we have $x \in \mu(P_1, \ldots, P_n)$ if and only if $|N^+_{\leqslant}(x)| < q$ and $|N^-_{\leqslant}(x)| < q$. Ordering the ideal points $\hat{x}_1 < \cdots < \hat{x}_n$, the medians are

$$\mu(P_1, \ldots, P_n) = \{x \in X \mid \hat{x}_{n-q+1} \leqslant x \leqslant \hat{x}_q\}.$$

When $q = \lceil \frac{n+1}{2} \rceil$, i.e., P is simple majority rule, there are two cases. In case n is odd, there is only one majority rule core alternative, which is the median of the ideal points, i.e., $\mu(P_1,\ldots,P_n)=\{\hat{x}_{\frac{n+1}{2}}\}$. In case n is even, we have $\mu(P_1,\ldots,P_n)=\{x\in X\mid \hat{x}_{\frac{n}{2}}\leqslant x\leqslant \hat{x}_{\frac{n}{2}+1}\}$.

The median voter theorem is attributed to Hotelling, Downs, and Black. The uniqueness of the majority rule core when n is odd can be obtained more generally, as the next result shows. In fact, as long as the collection of decisive groups is strong, the core must consist of the ideal point of some individual, the "median voter."

Corollary 6.8. *Assume $\mathbb{P}$ is simple and $\mathcal{D}$ is strong, and let $(P_1,\ldots,P_n)$ be single-peaked. Then there exists $k\in N$ such that $\mu(P_1,\ldots,P_n)=\{\hat{x}_k\}$. Moreover, for all $y\in X\setminus\{\hat{x}_k\}$, we have $\hat{x}_k\mathbb{P}(P_1,\ldots,P_n)y$.*

Proof. Since $\mathcal{D}$ is strong, it follows that $N\in\mathcal{D}$, i.e., the SPR satisfies Pareto. For the first part, it then suffices to show $\hat{x}_{i^+}=\hat{x}_{i^-}$, for suppose not. Then $\hat{x}_{i^+}<\hat{x}_{i^-}$, so $\{i\in N\mid \hat{x}_{i^-}\leq\hat{x}_i\}\subseteq N^+_{\leqslant}(\hat{x}_{i^+})$, which implies $\{i\in N\mid \hat{x}_{i^-}\leq\hat{x}_i\}\notin\mathcal{D}$. Since $\mathcal{D}$ is strong, we then have $\{i\in N\mid \hat{x}_i<\hat{x}_{i^-}\}=N^-_{\leqslant}(\hat{x}_{i^-})\in\mathcal{D}$, a contradiction. Setting $k=i^+$, we are done. For the second part, consider any $y\in X\setminus\{\hat{x}_k\}$, and assume without loss of generality that $\hat{x}_k<y$. From the first part of the proof, we have $\hat{x}_k=\hat{x}_{i^+}$, so that $\{i\in N\mid \hat{x}_k<\hat{x}_i\}$ is not decisive. Since $\mathcal{D}$ is strong, we then have $\{i\in N\mid \hat{x}_i\leq\hat{x}_k\}\in\mathcal{D}$, and single-peakedness implies that each member in that group strictly prefers $\hat{x}_k$ to y. Therefore, $\hat{x}_k\mathbb{P}(P_1,\ldots,P_n)y$, as required. □

The previous results show that under certain conditions, there is a unique alternative at the top of the social preference ordering. But we have not explored the nature of the social ordering. In particular, is it true that the social preference ordering is the same as the core voter's preference ordering? The answer is "no," in general, but after briefly considering acyclicity next, we will consider an order restriction condition that yields a different answer.

6.3 Weak Single-peakedness

Let $\leqslant$ be a weak linear order of the set of alternatives. We say $(P_1,\ldots,P_n)$ is *single-peaked* with respect to $\leqslant$ if for each $i\in N$, there exists $\tilde{x}_i\in X$ such that

(1) $\forall y\in X\setminus\{\tilde{x}_i\}$, $\tilde{x}_iR_iy$;

(2) $\forall y,z\in X$, $\tilde{x}_i<y<z\ \Rightarrow\ yR_iz$;

(3) $\forall y,z\in X$, $y<z<\tilde{x}_i\ \Rightarrow\ zR_iy$.

Proposition 6.9. *Assume $X\subseteq\mathbb{R}$ is convex. Let $(P_1,\ldots,P_n)$ be such that for all $i\in N$, there is a utility representation $u_i\colon X\to\mathbb{R}$ of P_i. Then $(P_1,\ldots,P_n)$ is weakly single-peaked with respect to the usual less than-or-equal-to order $\leq$ if and only if for all $i\in N$,* *(a) u_i has a maximizer, and*

(b) u_i is quasi-concave.

Weak single-peakedness implies exclusion condition A, which is a key sufficient condition for acyclicity of social preferences.

Proposition 6.10. *If $(P_1, \ldots, P_n)$ is weakly single-peaked, then it satisfies exclusion condition A.*

Proof. Let $(P_1, \ldots, P_n)$ be weakly single-peaked with respect to $\leqslant$. Consider any $m \geq 3$ and any distinct $x_1, \ldots, x_m \in X$. In case $m = 3$, suppose there exist $i, j, k \in N$ such that $x_1 P_i x_2 P_i x_3$, $x_2 P_j x_3 P_2 x_1$, and $x_3 P_k x_1 P_3 x_2$. Assume without loss of generality that $x_1 < x_2 < x_3$. By weak single-peakedness, $x_1 P_k x_2$ implies that $\tilde{x}_k < x_2$, but similarly $x_3 P_k x_2$ implies that $x_2 < \tilde{x}_k$, a contradiction. In case $m \geq 4$, I claim that there exists $\ell \in \{1, \ldots, m\}$ such that one of the following holds (addition is modulo m):

(a) $x_\ell < x_{\ell+1} < x_{\ell+2}$,

(b) $x_{\ell+2} < x_{\ell+1} < x_\ell$,

(c) $x_\ell < x_{\ell+2} < x_{\ell+1}$,

(d) $x_{\ell+1} < x_{\ell+2} < x_\ell$.

Note that (a) and (b) are symmetric, as are (c) and (d). To prove this claim, suppose otherwise, and without loss of generality assume $x_1 < x_2$. Then it must be that $x_3 < x_1 < x_2$, and similarly $x_3 < x_1 < x_2 < x_4$, and so on. Finally, we have either

$$x_{m-1} < \cdots < x_3 < x_1 < x_2 < x_4 < \cdots < x_m.$$

or

$$x_m < \cdots < x_3 < x_1 < x_2 < x_4 < \cdots < x_{m-1}.$$

Setting $\ell = m - 1$, we then obtain either (c) or (d), respectively. This contradiction establishes the claim. Thus, we may select ℓ such that one of (a)–(d) hold. To verify exclusion condition A, suppose there exist $i, j, k \in N$ such that $x_\ell P_i x_{\ell+1} P_i x_{\ell+2}$, $x_{\ell+1} P_j x_{\ell+2} R_j x_\ell$, and $x_{\ell+2} R_k x_\ell P_k x_{\ell+1}$. In case (a) holds, by weak single-peakedness, $x_{\ell+2} P_k x_{\ell+1}$ implies $x_{\ell+1} < \tilde{x}_k$, but similarly $x_\ell P_k x_{\ell+1}$ implies $\tilde{x}_k < x_{\ell+1}$, a contradiction. A contradiction is derived analogously in case (b) holds. In case (c) holds, by weak single-peakedness, $x_\ell P_i x_{\ell+2}$ implies $\tilde{x}_i < x_{\ell+2}$, but similarly $x_{\ell+1} P_i x_{\ell+2}$ implies $x_{\ell+2} < \tilde{x}_i$, a contradiction. A contradiction is derived analogously in case (d) holds. We conclude that for the selected ℓ, in each possible case (a)–(d), there do not exist i, j, k whose preferences over $\{x_\ell, x_{\ell+1}, x_{\ell+2}\}$ are given by (iv)–(vi) in the definition of exclusion condition A, as required. □

It follows that for a simple SPR, weak single-peakedness is sufficient for acyclicity.

Corollary 6.11. *Assume* $\mathbb{P}$ *is simple, and let* $(P_1, \ldots, P_n)$ *be weakly single-peaked. Then* $\mathbb{P}(P_1, \ldots, P_n)$ *is acyclic.*

7 Order Restriction

Recall that in the previous section, we assume there is a linear order on the set of alternatives and each individual's preferences are single-peaked with respect to that order. Now we consider a different restriction on preferences. In particular, we consider restrictions with respect to an order on the set of agents.

Given a weak linear order $\leqslant$ of N, we say $(P_1, \ldots, P_n)$ is *order-restricted* with respect to $\leqslant$ if for all $x, y \in X$, either

$$P(x,y) < I(x,y) < P(y,x),$$

or

$$P(y,x) < I(x,y) < P(x,y),$$

where $G < H$ means that for all $i \in G$ and all $j \in H$, $i < j$. In other words, individuals can be ordered so that for all x and y, either (i) the first $|P(x,y)|$ individuals in the order strictly prefer x to y, the last $|P(y,x)|$ individuals strictly prefer y to x, and the $n - |P(x,y)| - |P(y,x)|$ individuals in the middle are indifferent, or (ii) the arrangement of preferences is the reverse of this.

The following example shows that order restriction does not imply single-peakedness. Consider the profile

P_1	P_2	P_3
c	a	a
b	c	b
a	b	c

Here, individual preferences are order-restricted with respect to $1 < 2 < 3$, but not single-peaked.

The next simple example shows that the opposite nesting does not hold either. Consider the preferences below, which are single-peaked with respect to $a < b < c < d$.

P_1	P_2	P_3
b	c	d
c	b	c
d	a	b
a	d	a

When $1 < 2 < 3$, individuals 1 and 3 prefer d to a, while 2 prefers a to d; when $1 < 3 < 2$, individuals 1 and 2 prefer c to d, while 3 prefers d to c; when $2 < 1 < 3$, individuals 2 and 3 prefer c to b, while 1 prefers b to c. The opposite orders are similarly problematic.

Let $\mathbf{O}_{\leqslant}$ be the set of preference profiles that are order-restricted with respect to $\leqslant$. Let $\mathbf{O} = \cup_{\leqslant}\mathbf{O}_{\leqslant}$.

For example, let $X = \mathbb{R}^2_+$, suppose individuals have monotone and convex preferences, then the preference profile is order-restricted with respect to the rank to marginal rate of substitution.

7.1 Transitivity Properties

Order restriction implies value restriction. A partial proof of the following proposition can be found in Rothstein (1990), where the concept of order restriction is introduced.

Proposition 7.1. *If $(P_1, \ldots, P_n)$ is order-restricted, then it is value-restricted.*

Proof. Let $(P_1, \ldots, P_n)$ be order-restricted with respect to $\leqslant$, and assume without loss of generality that $1 < 2 < \cdots < n$. Consider any $a, b, c \in X$, and suppose that for all $x \in Y = \{a, b, c\}$ and all $r \in \{1, 2, 3\}$, there exists $i \in N(Y)$ such that $r \in V_i^Y(x)$. Since individuals who are indifferent among alternatives in Y play no role in the analysis, assume for simplicity that $N(Y) = N$. Since individual 1 is concerned over Y, we can assume that aP_1b, that $3 \in V_1^Y(a)$, and that $1 \in V_1^Y(b)$. Then aR_1cR_1b. Since a and b attain all values in Y, by supposition, it cannot be that all individuals have preferences over Y identical to 1's. Let j be the lowest indexed individual such that $P_j|_Y \neq P_1|_Y$. Then P_j must be characterized by a preference reversal among alternatives in Y that involve c.

We first consider the possible preference reversals between c and a, depending on the initial position of c relative to b. The first five cases involve c moving up relative to a. Case 1: $cP_{j-1}b$ and cP_ja. Then $V_i^Y(c) \subseteq \{2, 3\}$ for all $i = 1, \ldots, j-1$, and by order restriction, we have cP_ia for all $i = j, \ldots, n$. Therefore, $V_i^Y(c) \subseteq \{2, 3\}$ for all $i = j, \ldots, n$, which implies $1 \notin \bigcup_{i \in N(Y)} V_i^Y(c)$, so c does not attain value 1. Case 2: $cI_{j-1}b$ and cP_jaR_jb. By order restriction, we have cP_ib for all $i = j, \ldots, n$, and therefore b does not attain value 3. Case 3: $cI_{j-1}b$, cP_ja, and bP_ja. By order restriction, a does not attain a value of 2. Case 4: $aP_{j-1}c$ and bP_jcI_ja. By order restriction, c does not attain value 3. Case 5: $aP_{j-1}c$ and cI_jaP_jb. Since c attains value 1, there exists $k > j$ such that bP_kcI_ka, and by order restriction and the assumption that all agents are concerned, we can assume cI_iaP_ib for $i = j, \ldots, k-1$. But then b does not attain value 2. Case 6: $cI_{j-1}a$ and aP_jc. By order restriction, a does not attain value 1.

The remaining possibilities concern preferences reversals between c and b. Note, however, that since π is order restricted, so is the profile in which each individual i's preference is the inverse of R_i. Then a preference reversal between c and b in the initial profile is analogous to a reversal between c and a in the inverse profile, and the previous argument can be applied to deduce a contradiction in all possible cases. We conclude that $(P_1, \ldots, P_n)$ is value-restricted, as required. □

That value restriction does not imply order restriction is demonstrated above: since single-peakedness does not imply order restriction, neither does the weaker condition of value restriction.

Two corollaries follow immediately from our results on value restriction, but we will see the first can be improved using the extra structure of order restriction.

Corollary 7.2. *Assume $\mathbb{P}$ is simple and $\mathcal{D}$ is strong, and let $(P_1, \ldots, P_n)$ be order-restricted and satisfy concern over triples. Then $\mathbb{P}(P_1, \ldots, P_n)$ is negatively transitive.*

Corollary 7.3. *Assume $\mathbb{P}$ is simple, and let $(P_1, \ldots, P_n)$ be order-restricted. Then $\mathbb{P}(P_1, \ldots, P_n)$ is transitive.*

Although value restriction does not generally imply negative transitivity unless concern over triples is assumed, order restriction allows us to state a strengthened version of Corollary 7.2 that drops that background assumption.

Proposition 7.4. *Assume $\mathbb{P}$ is simple and $\mathcal{D}$ is strong, and let $(P_1, \ldots, P_n)$ be order-restricted. Then $\mathbb{P}(P_1, \ldots, P_n)$ is negatively transitive.*

Proof. Assume without loss of generality that $(P_1, \ldots, P_n)$ is order restricted with respect to $1 < 2 < \cdots < n$, and consider any alternatives $x, y, z \in X$ such that $x\mathbb{R}(P_1, \ldots, P_n)y\mathbb{R}(P_1, \ldots, P_n)z$. Then $R(x, y), R(y, z) \in \mathcal{B}$, and since $\mathbb{P}$ is simple and $\mathcal{D}$ is strong, we have $\mathcal{B} = \mathcal{D}$. Thus, $R(x, y), R(y, z) \in \mathcal{D}$, and by order restriction we can assume $R(x, y) = \{1, 2, \ldots, k\}$ for some k. If $R(y, z) = \{1, \ldots, \ell\}$ for some ℓ, then either $R(x, y) \cap R(y, z)$ equals $R(x, y)$ or it equals $R(y, z)$. In either case, $R(x, y) \cap R(y, z) \in \mathcal{D}$. Then for all $i \in R(x, y) \cap R(y, z)$, we have xR_iyR_iz, which implies xR_iz, and therefore $x\mathbb{R}(P_1, \ldots, P_n)z$. Otherwise, $R(y, z) = \{\ell, \ell+1, \ldots, n\}$ for some ℓ. If $z\mathbb{P}(P_1, \ldots, P_n)x$, then because $\mathbb{P}$ is simple, we have $P(z, x) \in \mathcal{P}$, and by the above argument, we have either $R(x, y) \cap P(z, x) \in \mathcal{D}$ or $R(y, z) \cap P(z, x) \in \mathcal{D}$. In the latter case, we have yR_izP_ix, and therefore yP_ix, for all $i \in R(y, z) \cap P(z, x)$, but this implies $y\mathbb{P}(P_1, \ldots, P_n)x$, a contradiction; and in the former case, we have zP_ixR_iy, and therefore zP_iy, for all $i \in R(x, y) \cap P(z, x)$, but this implies $z\mathbb{P}(P_1, \ldots, P_n)y$, a contradiction. We conclude that $x\mathbb{R}(P_1, \ldots, P_n)z$, as required. □

To see that $\mathcal{D}$ strong is needed for Proposition 7.4, even when concern over triples is satisfied, consider two individuals, majority voting over three alternatives, and individual preferences as follows: aP_1bP_1c and cP_2aP_2b. Then $b\mathbb{R}_{SM}(P_1,P_2)c\mathbb{R}_{SM}(P_1,P_2)a$ but $a\mathbb{P}(P_1,P_2)b$.

7.2 Representative Voters

Similar to the discussion of single-peakedness, assume $(P_1,\ldots,P_n)$ is order-restricted with respect to $\leq$, and define the following groups:

$$N^+_{\leqslant}(j) = \{i \in N \mid j < i\}$$

$$N^-_{\leqslant}(j) = \{i \in N \mid i < j\}.$$

We say $i \in N$ is a *median voter* with respect to the profile $(P_1,\ldots,P_n)$ and $\leq$ if

$$N^+_{\leqslant}(i) \notin \mathcal{D} \quad \text{and} \quad N^-_{\leqslant}(i) \notin \mathcal{D},$$

and we let $\nu_{\leqslant}(P_1,\ldots,P_n)$ be the set of such individuals. Note that similar to the single-peaked case, if $\mathcal{D}$ is strong, then $\nu_{\leqslant}(P_1,\ldots,P_n)$ consists of a single individual.

In contrast to the case of median alternatives with single-peaked preferences, the set of median voters under order restriction does in general depend on the ordering of individuals.

P_1	P_2	P_3
a	a	a
b	b	b
c	c	c

The above profile is order-restricted with respect to every linear order of individuals, and assuming simple majority rule, for each such linear order, the "middle" individual is the unique median voter.

The next proposition gives a partial characterization of the core for order-restricted preferences. An implication is that when, in addition to being order restricted, individuals have distinct ideal points, the set of median voters is defined independently of the ordering of individuals.

Proposition 7.5. *Assume $\mathbb{P}$ is simple, and let $(P_1,\ldots,P_n)$ be order restricted with respect to $\leqslant$ and such that each individual i has a unique ideal point $\hat{x}_i \in X$ satisfying $\hat{x}_i \neq \hat{x}_j$ for $i \neq j$. Then $i \in \nu_{\leqslant}(P_1,\ldots,P_n)$ if and only if $\hat{x}_i \in \mathbb{C}(P_1,\ldots,P_n)$.*

Proof. First, suppose $i \in \nu(P_1,\ldots,P_n)$ but $\hat{x}_i \notin \mathbb{C}(P_1,\ldots,P_n)$. Then there exists $y \in X$ such that $y\mathbb{P}(P_1,\ldots,P_n)\hat{x}_i$. Since $\mathbb{P}$ is simple, it follows that $P(y,\hat{x}_i) \in \mathcal{D}$. By order restriction, we can assume without loss of generality that $\{i\} < P(y,\hat{x}_i)$. We then have $P(y,\hat{x}_i) \subseteq N^+_{\leqslant}(i)$, which implies $N^+_{\leqslant}(i) \in \mathcal{D}$, contradicting $i \in \nu(P_1,\ldots,P_n)$.

Next, suppose $\hat{x}_i \in \mathbb{C}(P_1, \ldots, P_n)$ but $i \notin \nu(P_1, \ldots, P_n)$. Assume without loss of generality that $N^+_\leqslant(i) \in \mathcal{D}$, and let $j = \min\{k \mid i < k\}$, so that $\hat{x}_i P_i \hat{x}_j$ and $\hat{x}_j P_j \hat{x}_i$. By order restriction, this implies $N^+_\leqslant(i) \subseteq P(\hat{x}_j, \hat{x}_i)$, but then $\hat{x}_j \mathbb{P}(P_1, \ldots, P_n)\hat{x}_i$, contradicting $\hat{x}_i \in \mathbb{C}(P_1, \ldots, P_n)$. □

Note that the above result is not a complete characterization, because there might be other alternatives belonging to the core. For example, let $n = 2$ and consider simple majority rule, $\mathbb{P}_{SM}$. Consider the profile

P_1	P_2
a	b
c	c
b	a

Then the core is $\mathbb{C}_{SM}(P_1, P_2, P_3) = \{a, b, c\}$.

The next result relates social preferences to the preferences of the median voters. It establishes that when $\mathcal{D}$ is strong, the median voter is "representative," in the sense that social preferences exactly coincide with hers.

Proposition 7.6 (Representative voter theorem)**.** *Assume $\mathbb{P}$ is simple, let $(P_1, \ldots, P_n)$ be order-restricted with respect to $\leqslant$, and let $k \in \nu_\leqslant(P_1, \ldots, P_n)$. Then for all $x, y \in X$,*

1. *$x\mathbb{P}(P_1, \ldots, P_n)y$ implies xP_ky, and*
2. *if in addition $\mathcal{D}$ is strong, then xP_ky implies $x\mathbb{P}(P_1, \ldots, P_n)y$.*

Proof. To prove part 1, assume $x\mathbb{P}(P_1, \ldots, P_n)y$. Since $\mathbb{P}$ is simple, we have $P(x, y) \in \mathcal{D}$. And since k is a median voter, we cannot have $P(x, y) \subseteq N^+_\leqslant(k)$ or $P(x, y) \subseteq N^-_\leqslant(k)$. Thus, there exist $i, j \in P(x, y)$ such that $i < k < j$, and then order restriction implies xP_ky. To prove part 2, assume $\mathcal{D}$ is strong and xP_ky, but suppose $y\mathbb{R}(P_1, \ldots, P_n)x$. Then $P(x, y) \notin \mathcal{D}$, and since $\mathcal{D}$ is strong, we then have $R(y, x) \in \mathcal{D}$. Note that xP_ky implies $k \notin R(y, x)$, and because k is a median voter, there exist $i, j \in R(y, x)$ such that $i < k < j$. By order restriction, we have yR_kx, a contradiction. We conclude that $x\mathbb{P}(P_1, \ldots, P_n)y$, as required. □

To unpack the implications of the representative voter theorem, it is useful to define the groups

$$H^+ = \{i \in N \mid N^+_\leqslant(i) \notin \mathcal{D}\}$$
$$H^- = \{i \in N \mid N^-_\leqslant(i) \notin \mathcal{D}\},$$

where min and max refer to $\leqslant$. Note that both H^+ and H^- are nonempty. Thus, we can define $i^+ = \min\{j \mid j \in H^+\}$ and $i^- = \min\{j \mid j \in H^-\}$. Note that $i^+ \leqslant i^-$.

Moreover, if $\mathbb{P}$ satisfies Pareto, then H and H^+ are decisive, so $H^+ \cap H^- \neq \emptyset$, and thus $i^+ \leq i^-$.

Next, we characterize the set of median voters.

Proposition 7.7. *Assume $\mathbb{P}$ is simple and satisfies Pareto, and let $(P_1, \ldots, P_n)$ be single-peaked with respect to $\leq$. Then*

$$\nu_{\leq}(P_1, \ldots, P_n) \;\; = \;\; \{i \in i^+ \leq i \leq i^-\},$$

and $\nu_{\leq}(P_1, \ldots, P_n) \neq \emptyset$.

Proof. Consider any $i \in \nu_{\leq}(P_1, \ldots, P_n)$. If it is not true that $i^+ \leq i \leq i^-$, then assume without loss of generality that $i < i^+$. But then $H^+ \subseteq N^+_{\leq}(i)$, which implies $N^+_{\leq}(i) \in \mathcal{D}$ and contradicts $i \in nu_{\leq}(P_1, \ldots, P_n)$. Next, consider $i \in N$ such that $i^+ \leq i \leq i^-$. To show $i \in \nu_{\leq}(P_1, \ldots, P_n)$, suppose that $N^+_{\leq}(i) \in \mathcal{D}$. But since $N^+_{\leq}(i) \subseteq N^+_{\leq}(i^+)$, we have $N^+_{\leq}(i^+) \in \mathcal{D}$, a contradiction. Thus, i^+ and i^- are median voters, so the set of median voters is nonempty. $\square$

Assuming a profile is order with respect to the natural ordering of individuals (in order of their indices), then for the case of a quota rule with $n/2 < q \leq n$, we have $\nu_{\leq}(P_1, \ldots, P_n) = \{n - q + 1, \ldots, q\}$. Finally, for majority rule, the unique median voter is individual $\frac{n+1}{2}$ when n is odd, and the median voters are $\frac{n}{2}$ and $\frac{n}{2} + 1$ when n is even.

A simple corollary of Propositions 7.5 and 7.6 gives us a complete characterization of the core when individuals have distinct ideal points and the collection of decisive groups is strong.

Corollary 7.8. *Assume $\mathbb{P}$ is simple and $\mathcal{D}$ is strong, and let $(P_1, \ldots, P_n)$ be order restricted and such that each individual i has a unique ideal point $\hat{x}_i \in X$ satisfying $\hat{x}_i \neq \hat{x}_j$ for $i \neq j$. Then there is a unique median voter k independent of the ordering of individuals, $\mathbb{C}(P_1, \ldots, P_n) = \{\hat{x}_k\}$, and for all $y \in X \backslash \{\hat{x}_k\}$, we have $\hat{x}_k \mathbb{P}(P_1, \ldots, P_n) y$.*

7.3 Special Cases

The next result provides sufficient conditions for a representative voter that are met in many modeling applications.

Proposition 7.9. *Assume $X \subseteq \mathbb{R}$, $\mathbb{P}$ is simple, and $\mathcal{D}$ is strong. Moreover, assume that each individual i has ideal point $\hat{x}_i$ and that there exists $v \colon \mathbb{R} \to \mathbb{R}$ such that v is strictly quasi-concave and for all $i \in N$, $u_i(x) = v(x - \hat{x}_i)$ is a utility representation* *of P_i. Let $k \in \nu_{\leqslant}(P_1, \ldots, P_n)$, where the weak linear order $\leqslant$ is such that for all $i, j \in N$, $\hat{x}_i < \hat{x}_j$ implies $i < j$. Then for all $x, y \in X$, we have $x\mathbb{P}(P_1, \ldots, P_n)y$ if and only if xP_ky.*

Note that the restrictions on utility functions in the preceding result imply order restriction. One commonly used functional form is quadratic, i.e., $v(x) = -x^2$.

As a second application, we assume $A \subseteq \mathbb{R}$ is a set of pure alternatives and X is the set of lotteries on A,[1] and we consider a profile $(P_1, \ldots, P_n)$ of preferences over X such that each individual i has a quadratic von Neumann-Morgenstern representation $v_i \colon A \to \mathbb{R}$ defined by $v_i(a) = -(a - \hat{a}_i)^2$ for all $a \in A$, where $\hat{a}_i$ is the ideal pure alternative for individual i. We denote lotteries by $\lambda, \lambda' \in X$. By mean-variance analysis, given any $\lambda, \lambda' \in X$, we have $\lambda P_i \lambda'$ if and only if $E_\lambda[v_i(a)] > E_{\lambda'}[v_i(a)]$, where

$$E_\lambda[v_i(a)] = v_i(E_\lambda[a]) - V_\lambda[a] = -E_\lambda[a]^2 + 2\hat{a}_i E_\lambda[a] - \hat{a}_i^2 - V_\lambda[a].$$

Then the difference in expected utilities is

$$E_\lambda[v_i(a)] - E_{\lambda'}[v_i(a)] \quad = \quad 2\hat{a}_i\big(E_\lambda[a] - E_{\lambda'}[a]\big),$$

which is monotonic in $\hat{a}_i$. It follows that the profile $(P_1, \ldots, P_n)$ is order restricted with respect to the ordering of ideal pure alternatives; technically, we can use any linear order such that for all $i, j \in N$, $\hat{a}_i < \hat{a}_j$ implies $i < j$. Thus, if $\mathbb{P}$ is simple and $\mathcal{D}$ is strong, and if k is a median voter, then we have $\lambda\mathbb{P}(P_1, \ldots, P_n)\lambda'$ if and only if $\lambda P_k \lambda'$.

By the same argument, we can be somewhat more general with respect to the functional form. Now let A be a general set of pure alternatives, and assume that each P_i has preferences over lotteries given by a von Neumann-Morgenstern representation such that

$$v_i(a) \quad = \quad \alpha_i w(a) - c(a) + \beta_i,$$

where $w \colon A \to \mathbb{R}$ and $c \colon A \to \mathbb{R}$ are functions that are constant across individuals, and $\alpha_i, \beta_i \in \mathbb{R}$ are parameters than can vary across individuals. Then the difference in expected utilities is

$$E_\lambda[v_i(a)] - E_{\lambda'}[v_i(a)] \quad = \quad \alpha_i\big(E_\lambda[w(a)] - E_{\lambda'}[w(a)]\big) - \big(E_\lambda[c(a)] - E_{\lambda'}[c(a)]\big),$$

which is monotonic in α_i, and so $(P_1, \ldots, P_n)$ is order restricted with respect to the ordering of the α_i parameters.

[1] We do not delve into measure-theoretic technicalities. The reader can avoid these by interpreting "lottery" as a distribution with countable support.

Proposition 7.10. *Assume $\mathbb{P}$ is simple and $\mathcal{D}$ is strong. Let X be the set of lotteries on a set A of pure alternatives, and assume that there exist mappings $w\colon A \to \mathbb{R}$ and $c\colon A \to \mathbb{R}$ and parameters $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \ldots, n$, such that each P_i has a von Neumann-Morgenstern representation of the form*

$$v_i(a) \;=\; \alpha_i w(a) - c(a) + \beta_i.$$

Let $k \in \nu_{\leqslant}(P_1, \ldots, P_n)$, where the weak linear order $\leqslant$ is such that for all $i, j \in N$, $\alpha_i < \alpha_j$ implies $i < j$. Then for all $\lambda, \lambda' \in X$, we have $\lambda \mathbb{P}(P_1, \ldots, P_n) \lambda'$ if and only if $x P_k y$.

Note that the quadratic case is obtained by setting $w(a) = 2a$, $c(a) = a^2$, $\alpha_i = \hat{a}_i$, and $\beta_i = -\hat{a}_i^2$.

8 Multi-dimensional Spatial Model

We now impose the restriction that X is a subset of d-dimensional Euclidean space and consider the core, top cycle, and uncovered set. Our main interest here will be on the multi-dimensional model, with $d \geq 2$. We generalize the single-peakedness idea to spatial models and study the implications. While some ideas from the unidimensional model extend to multiple dimensions, most of the results do not. In particular, we explore conditions under which the majority core is nonempty, and we describe properties of top cycle and uncovered sets.

We typically impose restrictions on individual preferences that arise naturally from the Euclidean structure of the set of alternatives. We commonly assume each P_i is continuous, and when X is convex, we often assume P_i is convex or even strictly convex. These properties have intuitive formulations in terms of utility functions.

Proposition 8.1 (Debreu)**.** *Assume $X \subseteq \mathbb{R}^d$. Then P_i is continuous if and only if it has a continuous utility representation $u_i\colon X \to \mathbb{R}$.*

Proposition 8.2. *Assume $X \subseteq \mathbb{R}^d$ is convex, and assume that P_i has a utility representation $u_i\colon X \to \mathbb{R}$. Then*

- *P_i is convex if and only if u_i is quasi-concave, and*
- *P_i is strictly convex if and only if u_i is strictly quasi-concave.*

We say P_i is *Euclidean* if there exists $\hat{x}^i \in X$ such that for all $x, y \in X$, $x P_i y$ if and only if $||x - \hat{x}^i|| < ||y - \hat{x}^i||$. Such a preference relation has the obvious utility representations $u_i(x) = -\|x - \hat{x}_i\|$ or $u_i(x) = -\|x - \hat{x}^i\|^2$. We refer to the latter at *quadratic utility.* Euclidean preferences are well-behaved, in the sense that they are continuous and strictly convex.

In this section, we consider a simple SPR $\mathbb{P}$ and fix a profile $(P_1, \ldots, P_n)$ of individual preferences. To conserve notation, we let $P = \mathbb{P}(P_1, \ldots, P_n)$ denote the social preference relation generated by $\mathbb{P}$ at the profile $(P_1, \ldots, P_n)$. As usual, $\mathcal{D}$ and $\mathcal{B}$ denote the decisive and blocking groups of $\mathbb{P}$. Note that since the SPR is simple, the strict social preference relation has the simple form

$$P \;=\; \bigcup_{G \in \mathcal{D}} \bigcap_{i \in G} P_i,$$

and the associated weak social preference relation, R, has the form

$$R \;=\; \bigcup_{G \in \mathcal{B}} \bigcap_{i \in G} R_i.$$

Of course, if $\mathcal{D}$ is strong, then $\mathcal{D} = \mathcal{B}$.

An implication is that for all $x \in X$, the strict and weak upper sections of the social preferences are

$$P(x) = \bigcup_{G \in \mathcal{D}} \bigcap_{i \in G} P_i \quad \text{and} \quad R(x) = \bigcup_{G \in \mathcal{B}} \bigcap_{i \in G} R_i(x).$$

These sets are depicted in Figure 2, where there are three individuals with Euclidean preferences and ideal points arranged in a triangle and we assume simple majority rule. The strict upper section $P(x)$ is the lightly shaded area consisting of the three lenses around x, and the weak upper section $R(x)$ is the shaded area together with the dark boundary.

8.1 Continuity and Convexity of Social Preferences

Social preferences inherit continuity properties of individual preferences.

Proposition 8.3. *Assume $X \subseteq \mathbb{R}^d$. If each P_i is continuous, then P has open graph.*

Proof. Assuming each P_i is continuous, Proposition 2.3 implies that P_i has open graph, i.e., P_i is an open subset of $X \times X$. For each $G \subseteq N$, it follows that the finite intersection $\bigcap_{i \in G} P_i$ is open, and therefore the union

$$P \;=\; \bigcup_{G \in \mathcal{D}} \bigcap_{i \in G} P_i$$

is open. □

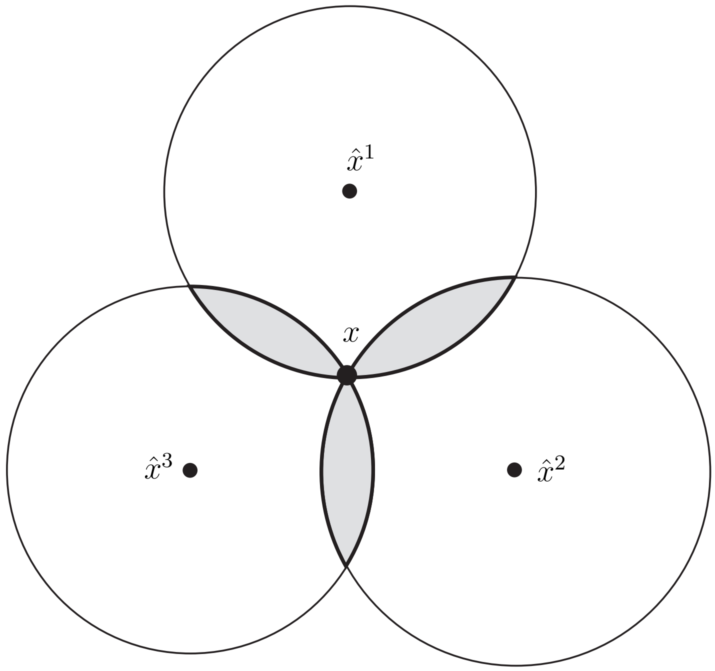


Figure 2: Strict and weak upper sections

This immediately yields a corollary on non-emptiness of the core in the one-dimensional model.

Corollary 8.4. *Assume that $X \subseteq \mathbb{R}$ is compact and convex, and assume that each P_i is continuous and convex. Then $M(P) \neq \emptyset$.*

Proof. By Proposition 8.1, continuity of P_i implies that P_i has a continuous utility representation u_i, and since X is compact, u_i has at least one maximizer, say $\tilde{x}^i$. By Proposition 8.2, u_i is quasi-concave. By Proposition 6.9, it follows that each P_i is weakly single-peaked with respect to $\leq$. By Corollary 6.11, it follows that P is acyclic. Finally, since Proposition 8.3 implies that P is upper semi-continuous, Proposition 3.2 implies that $M(P) \neq \emptyset$. □

Social preferences do not generally inherit convexity of individual preferences, although we can state some more limited convexity properties of social preferences. We begin with the following lemma, which is self-evident.

Lemma 8.5. *Assume $X \subseteq \mathbb{R}^d$ is convex, and assume that for each i, P_i is convex. Then for all $G \subseteq N$, all $x \in X$, and all $\alpha \in (0,1)$, we have:*

(i) for all $y \in \bigcap_{i \in G} R_i(x)$, we have $\alpha x + (1-\alpha)y \in \bigcap_{i \in G} R_i(x)$,

(ii) if each P_i is strictly convex, then for all $y \in \bigcap_{i \in G} R_i(x)$ with $y \neq x$, we have $\alpha x + (1-\alpha)y \in \bigcap_{i \in G} P_i(x)$.

We say P is *star-shaped* if for all $x \in X$, all $y \in P(x)$, and all $\alpha \in (0,1)$, we have $\alpha x + (1-\alpha)y \in P(x)$, and it is *strictly star-shaped* if for all $x \in X$, all $y \in R(x) \setminus \{x\}$, and all $\alpha \in (0,1)$, we have $\alpha x + (1-\alpha)y \in P(x)$. The following result on star-shapeness of social preferences is useful.

Proposition 8.6. *Assume $X \subseteq \mathbb{R}^d$ is convex. If each P_i is strictly convex, then P is star-shaped, and if $\mathcal{D}$ is strong, then P is strictly star-shaped.*

Proof. We prove the second part of the proposition. Consider any $x \in X$, any $y \in R(x) \setminus \{x\}$, and any $\alpha \in (0,1)$. Since yRx, we have $R(y,x) \in \mathcal{B}$. And since P is generated by a simple SPR and $\mathcal{D}$ is strong, we have $\mathcal{B} = \mathcal{D}$. Thus, $R(y,x) \in \mathcal{D}$. Part (ii) of Lemma 8.5 implies that for each $i \in R(y,x)$, we have $\alpha x + (1-\alpha)y P_i x$, and therefore $\alpha x + (1-\alpha)y P x$, as required. □

A more useful convexity property for purposes of obtaining maximal elements is semi-convexity. Let α denote the acyclicity index of $\mathbb{P}$, and recall that when the SPR is non-collegial, i.e., $\bigcap \mathcal{D} = \emptyset$, this is the size of the smallest collection of decisive groups having empty intersection.

Proposition 8.7. *Assume that $X \subseteq \mathbb{R}^d$ is convex and that each P_i is convex. If $\bigcap \mathcal{D} = \emptyset$ and $d+1 < \alpha$, or if $\bigcap \mathcal{D} \neq \emptyset$, then P is semi-convex.*

Proof. Consider any $x \in X$, and suppose that $x \in \mathrm{conv}P(x)$. By Caratheodory's theorem, there exist $y^0, y^1, \ldots, y^d \in P(x)$ with $y^j \in P(x)$ for each $j = 0, 1, \ldots, d$ and such that $x \in \mathrm{conv}\{y^0, \ldots, y^d\}$. Since P is generated by a simple SPR, it follows that for each j, $P(y^j, x) \in \mathcal{D}$. Setting $G_j = P(y^j, x)$, there are two cases. In case $\bigcap \mathcal{D} \neq \emptyset$, we have $\bigcap_{j=0}^d G_j \neq \emptyset$. In case $\bigcap \mathcal{D} = \emptyset$, then we have $d+1 < \alpha$, and thus $\bigcap_{j=0}^d G_j \neq \emptyset$. Therefore, in both cases, we can choose $i \in \bigcap_{j=0}^d G_j$. But then we have $y^j \in P_i(x)$ for each $j = 0, 1, \ldots, d$, and using convexity of P_i, we have $x \in \mathrm{conv}\{y^0, \ldots, y^d\} \subseteq P_i(x)$, which implies $x P_i x$, a contradiction. □

For the special case of majority rule, if $n \geq 3$ but $n \neq 4$, we have $\alpha = 3$, and then the preceding proposition implies semi-convexity of social preferences as long as $d = 1$. When $n = 4$, semi-convexity obtains as long as $d = 1, 2$.

8.2 Non-emptiness of the Core

We have one result on non-emptiness of the core for the one-dimensional model that relies on weak single-peakedness and acyclicity of social preferences. In multiple dimensions, acyclicity no longer holds, even for majority rule with n odd and Euclidean preferences. In Figure 3, for example, there are three voters with Euclidean preferences and a majority preference cycle through a, b, and c. In fact, you can check that the presence of such cycles is critical in this example, because the majority core is empty.

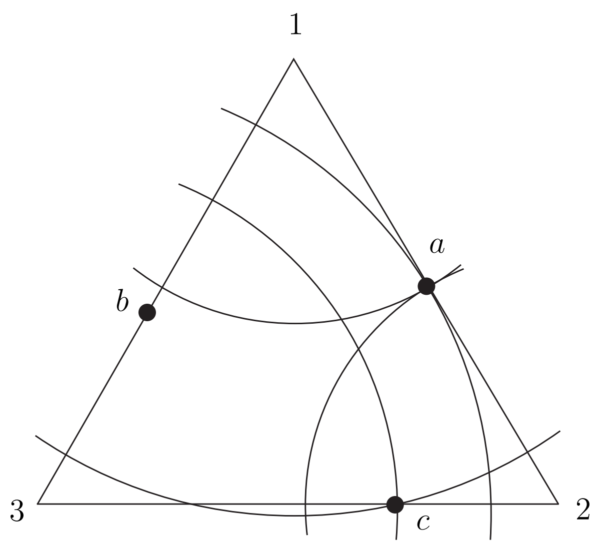


Figure 3: Majority preference cycle

The main result on general existence of maximal elements, which uses our conditions for semi-convexity in the previous subsection, is stated next.

Proposition 8.8. *Assume that $X \subseteq \mathbb{R}^d$ is compact and convex and each P_i is continuous and convex. If $\bigcap \mathcal{D} = \emptyset$ and $d+1 < \alpha$, or if $\bigcap \mathcal{D} \neq \emptyset$, then $\mathbb{C}(P_1, \ldots, P_n) \neq \emptyset$.*

Proof. By Propositions 8.3 and 8.7, P is continuous and semi-convex. By Proposition 3.4, P satisfies finite dominance, and therefore Proposition 3.3 implies $M(P) \neq \emptyset$, as required. □

For majority rule with $n \geq 3$ and $n \geq 4$, the proposition delivers maximal elements when $d = 1$, which we know already from Corollary 8.4. But in the n even case, the proposition implies nonemptiness of the majority core when $d = 1, 2$, and more generally for a quota rule with quota $q < n$, it delivers maximal elements as long as $d + 1 < \left\lceil \frac{n}{n-q} \right\rceil$.

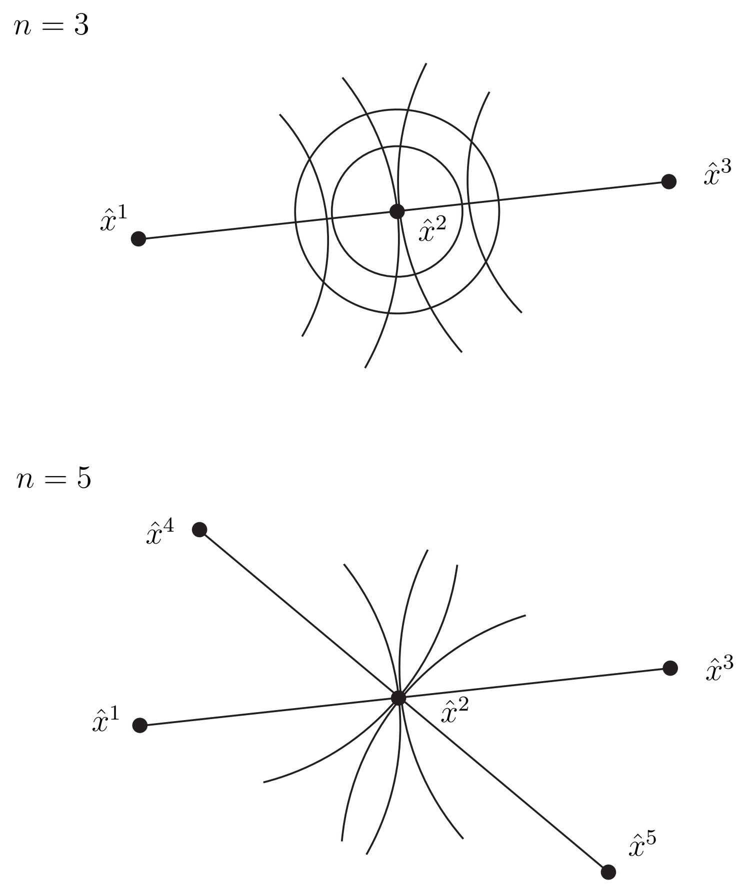


Figure 4: Nonempty core with $d + 1 \geq \alpha$.

In fact, the dimensionality cutoff in Proposition 8.8 is tight.

Proposition 8.9. *Assume that $X \subseteq \mathbb{R}^d$ has nonempty interior in $\mathbb{R}^d$. If $\bigcap \mathcal{D} = \emptyset$ and $d + 1 \geq \alpha$, then there is a profile $(P_1, \ldots, P_n)$ of Euclidean preferences such that $\mathbb{C}(P_1, \ldots, P_n) = \emptyset$.*

Thus, when the dimensionality of the set of alternatives exceeds the cutoff in Proposition 8.8, the core may be empty, but it is not necessarily. When $d = 2$, for example, Figure 4 demonstrates for $n = 3$ and $n = 5$ that the majority core may be nonempty; in fact, in both cases, the ideal point of individual 2 is the unique majority core point.

The next proposition provides sufficient conditions under which the core, if nonempty, consists of a unique alternative that is strictly socially preferred to all other alternatives.

Proposition 8.10. *Assume that $X \subseteq \mathbb{R}^d$ is convex, that $\mathcal{D}$ is strong, and that each P_i is strictly convex. If $M(P) \neq \emptyset$, then there exists $x^* \in X$ such that $M(P) = \{x^*\}$, and for all $y \in X \setminus \{x^*\}$, we have xPy.*

Proof. Consider any $x \in M(P)$ and any $y \in X \setminus \{x\}$, and suppose that yRx. By Proposition 8.6, P is strictly star-shaped, but then $\frac{1}{2}x + \frac{1}{2}yPx$, a contradiction. Therefore, we have xPy for all $y \in X \setminus \{x\}$, so $M(P) = \{x\}$. Setting $x^* = x$, we are done. □

The next result provides sufficient conditions under which each core alternative is maximal for some individual. We say a differentiable utility representation $u_i \colon X \to \mathbb{R}$ is *pseudo-concave* if for all $x, y \in X$, $u_i(y) > u_i(x)$ implies $\nabla u_i(x) \cdot (y - x) > 0$. We say that u_i is *strictly pseudo-concave* if for all $x, y \in X$ with $x \neq y$, $u_i(x) \geq u_i(y)$ implies $\nabla u_i(x) \cdot (y - x) > 0$. When X is convex, strict pseudo-concavity of u_i implies strict convexity of P_i.

Proposition 8.11. *Assume that $X \subseteq \mathbb{R}^d$, that $\mathcal{D}$ is strong, and that each P_i has a pseudo-concave utility representation. Let $x^* \in intX$. If $x^* \in M(P)$, then there exists $k \in N$ such that $x^* \in M(P_k)$.*

Proof. Assume $x^* \in \text{int}X$ belongs to the core, and for each i, let u_i be a pseudo-concave utility representation. To deduce a contradiction, suppose that for all i, there exists $y \in X$ such that yP_ix. By pseudo-concavity, it follows that $\nabla u_i(x^*) \neq 0$ for all i. Then there is a vector $p \neq 0$ such that $\nabla u_i(x^*) \cdot p \neq 0$ for all $i \in N$. Since $\mathcal{D}$ is strong, either $\{i \mid \nabla u_i(x^*) \cdot p > 0\} \in \mathcal{D}$ or $\{i \mid \nabla u_i(x^*) \cdot p < 0\} \in \mathcal{D}$. Assume the former, and let $G = \{i \mid \nabla u_i(x^*) \cdot p > 0\}$. Define $x^\epsilon = x^* + \epsilon p$, and note that for small enough $\epsilon > 0$, we have $x^\epsilon \in X$ and for all $i \in G$, $u_i(x^\epsilon) > u_i(x^*)$. Therefore, $x_\epsilon P x^*$, a contradiction. □

Combining the last two propositions, we have the following corollary.

Corollary 8.12. *Assume that $X \subseteq \mathbb{R}^d$ is convex, that $\mathcal{D}$ is strong, and that each P_i has a strictly pseudo-concave utility representation. Let $x^* \in intX$. If $x^* \in M(P)$, then there exists $k \in N$ such that x^* is the unique maximal element of P_i, $M(P) = \{x^*\}$, and for all $y \in X \setminus \{x^*\}$, we have x^*Py.*

8.3 Core Symmetry Conditions

In Figure 4, individual ideal points are aligned in such a way to suggest that further necessary conditions on core alternatives might be deduced. In this subsection, we focus on majority rule with n odd, and we prove that strong symmetry conditions on individual gradients must be satisfied at every core alternative.

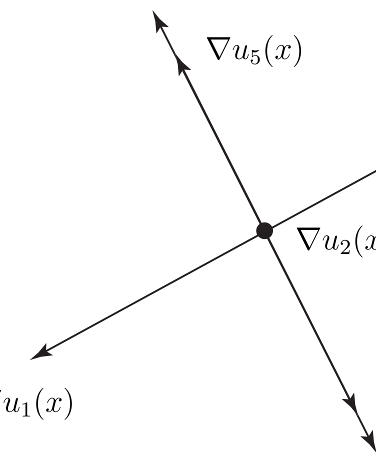


Figure 5: Radial symmetry

In Figure 4, while the majority core is nonempty, it exhibits three properties: (i) the core alternative is the ideal point of an individual, (ii) it is strictly majority preferred to every other alternative, and (iii) individual ideal points are matched against each other in a very precise way. Corollary 8.12 generalizes the first two properties, and the next result (Proposition 8.13), due to Charles Plott, extends the third.

Let $(P_1, \ldots, P_n)$ be a profile such that each P_i has a differentiable utility representation u_i. We say that *radial symmetry* holds at $x \in X$ if for every direction $p \in \mathbb{R}^d$ with $||p|| = 1$, we have

$$|\{i \in N \mid \exists \alpha > 0 : \nabla u_i(x) = \alpha p\}| = |\{i \in N \mid \exists \alpha > 0 : \nabla u_i(x) = -\alpha p\}|.$$

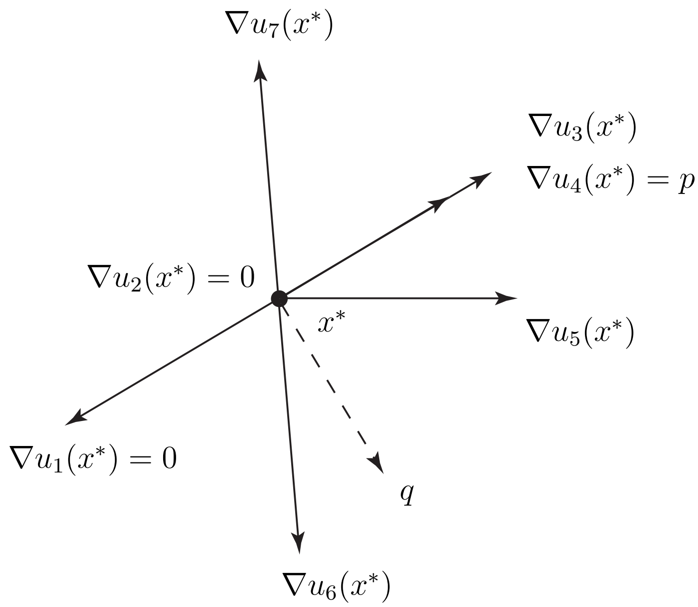
This condition is illustrated in Figure 5.

Plott's theorem establishes that radial symmetry is a very general necessary condition for alternatives belonging to the majority core.

Proposition 8.13 (Plott)**.** *Assume that $X \subseteq \mathbb{R}^d$ and n is odd. Let $x^* \in intX$, and let $(P_1, \ldots, P_n)$ be such that each P_i has different utility representation u_i. Assume that for all $i, j \in N$, $\nabla u_i(x^*) = \nabla u_j(x^*) = 0$ implies $i = j$. If x^* belongs to the majority core, then*

(i) there is some $k \in N$ such that $\nabla u_k(x^*) = 0$, and

(ii) radial symmetry holds at x^*.



$G_1 = \{3, 4\}$ and $G_2 = \{1\}$

Figure 6: A helpful picture

Proof. Part (i) follows from Proposition 8.11. Suppose part (ii) is false, i.e., there exists some $p \in \mathbb{R}^d$ with $||p|| = 1$ such that

$$|\{i \in N \mid \exists \alpha > 0 : \nabla u_i(x) = \alpha p\}| \neq |\{i \in N \mid \exists \alpha > 0 : \nabla u_i(x) = -\alpha p\}| .$$

Call the first group G_1 and the second G_2. Note that $p \neq 0$, and therefore $k \notin G_1 \cup G_2$. Without loss of generality, suppose $|G_1| > |G_2|$. Choose a non-zero vector $q \in \mathbb{R}^d$ such that

$$\{i \in N \mid \nabla u_i(x^*) \cdot q = 0\} = G_1 \cup G_2 \cup \{k\}.$$

See Figure 6. Let

$$H_1 = \{i \in N \mid \nabla u_i(x^*) \cdot q > 0\} \quad \text{and} \quad H_2 = \{i \in N \mid \nabla u_i(x^*) \cdot q < 0\}.$$

Then $\{G_1, G_2, \{k\}, H_1, H_2\}$ partitions N. Assume without loss of generality that $|H_1| \geq |H_2|$. Since $|G_1| \geq |G_2| + 1$, we have

$$|G_1| + |H_1| \geq |G_2| + 1 + |H_2| \geq n - |G_1| - |H_1|.$$

Therefore, $|G_1| + |H_1| \geq n/2$, and hence $G_1 \cup H_1$ is a majority. Now choose $\delta > 0$ small enough such that for all $i \in H_1$, we have

$$\nabla u_i(x^*) \cdot (q + \delta p) > 0,$$

and note that for all $i \in G_1$, we have

$$\nabla u_i(x^*) \cdot (q + \delta p) = \delta \nabla u_i(x^*) \cdot p > 0.$$

Let $x^\epsilon = x^* + \epsilon(q + \delta p)$. Then for $\epsilon > 0$ small enough, we have $x^\epsilon \in X$ and for all $i \in G_1 \cup H_1$, $u_i(x_\epsilon) > u_i(x^*)$. But then $x^\epsilon \mathbb{P}_{SM}(P_1, \ldots, P_n)x^*$, a contradiction. □

The implications of Plott's theorem are that in multiple dimensions ($d \geq 2$), the necessary conditions that must be satisfied by a core alternative are exceedingly restrictive—so much so that for "almost all" profiles of preferences, the majority core is empty. In other words, when n is odd and $d \geq 2$, the majority core is generically empty. When n is even, the core is nonempty for $d = 1, 2$, but when $d \geq 3$, we are again faced with generic emptiness of the majority core.

Although the preceding analysis has focused on majority rule, the message is more general. As long as $\bigcap \mathcal{D} = \emptyset$, there is a dimensionality cutoff such that for d exceeding that level, the core is generically empty.

8.4 Remarks on Top Cycle and Uncovered Set

When the core is empty, we may consider alternatives to maximality, such as the top cycle or uncovered set defined using the social preference relation, P.

Proposition 8.14. *Assume $X \subseteq \mathbb{R}^d$ is compact and each P_i is continuous. Then $TC(P)$ and $UC(P)$ are nonempty.*

Proof. Proposition 8.3 implies that P has open graph, and then by Propositions 3.5 and 3.6, we have $TC(P) \neq \emptyset$ and $UC(P) \neq \emptyset$. □

An advantage of both the top cycle and uncovered set is that fairly generally, when the core is non-empty, they coincide with the core. This follows immediately from Proposition 8.10, which provides conditions such that if the core is non-empty, it consists of a single alternative that is strictly socially preferred to every other alternative.

Proposition 8.15. *Assume that $X \subseteq \mathbb{R}^d$ is convex, that $\mathcal{D}$ is strong, and that each P_i is strictly convex. If $M(P) \neq \emptyset$, then $TC(P) = UC(P) = M(P)$.*

A serious drawback of the top cycle generated by majority rule is that when n is odd, X is convex with non-empty interior, and $d \geq 3$, it generically contains the entire set of alternatives in its closure. In fact, if we begin with a profile $(P_1, \ldots, P_n)$ satisfying the conditions of Proposition 8.15 and such that the core is nonempty, arbitrarily small perturbations of individual preferences can (and almost surely will) lead to the core being empty and the top cycle filling the entire set of alternatives.

Relative to the top cycle, the uncovered set is better behaved, in the sense of having better continuity properties. For example, if we begin with a profile $(P_1, \ldots, P_n)$

satisfying the conditions of Proposition 8.15 and such that the core is nonempty, arbitrarily small perturbations of individual preferences can (and almost surely will) lead to the core being empty, but the uncovered set will expand only slightly. More generally, the uncovered set can be computed in different examples, and it often looks like a "blob" that is centrally located among the ideal points of individuals.

In fact, when the collection of decisive groups is strong, as is the case with majority rule and n odd, the uncovered alternatives are Pareto optimal.

Proposition 8.16. *Assume that $X \subseteq \mathbb{R}^d$, and that $\mathcal{D}$ is strong. Then the uncovered set is contained in the set of weakly Pareto optimal alternatives, i.e., for all $x \in UC(P)$, there does not exist $y \in X$ such that $P(y,x) = N$.*

Proof. Consider any $x \in UC(P)$, and suppose there exists $y \in X$ such that $P(y,x) = N$. Consider any $z \in R(y)$. Then $R(z,y) \in \mathcal{B}$, and since $\mathbb{P}$ is simple and $\mathcal{D}$ is strong, we have $\mathcal{B} = \mathcal{D}$. Thus, we have $R(z,y) \in \mathcal{D}$, and for all $i \in R(z,y)$, we have $zR_i yP_i x$, which implies $zP_i x$. This implies that zPx, and thus $R(y) \subseteq P(x)$. But then yCx, a contradiction. We conclude that x is weakly Pareto optimal, as required. □