Notes on Social Choice Rules

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Contents

1 Social Preference Rules **2**

2 Social Choice Rules **5**

3 Axioms for Social Choices **11**

4 Strategic Manipulation **20**

5 Digression on Monotonicity **26**

6 Manipulation with Ties **27**

7 Manipulation with Lotteries **33**

8 Majority Rule in One Dimension **37**

9 Issue-by-issue Medians **44**

10 Structure-induced Equilibrium **52**

These notes are a brief introduction to the problem of social choice: a group of individuals, with possibly conflicting preferences, must make a collective choice from a given set of alternatives. Sections 1 and 2 provide some preliminaries, including a variety of different possible procedures, "social choice rules," for making such choices. Section 3 gives a number of basic axioms that formalize plausible or desirable properties of social choice rules. Sections 4–8 analyze the main axiom of interest, Non-manipulability, in abstract environments. Section 4 presents the classical Gibbard-Satterthwaite theorem on the impossibility of strategy-proof choice, and Sections 6 and 7 extend the classical result to social choice rules that, respectively, allow for social ties and lotteries over alternatives. Section 8 considers Non-manipulability in the one-dimensional model with single-peaked preferences, while Sections 9 and 10 extend the analysis to multiple dimensions.

1 Social Preference Rules

We consider a set $N = \{1, \ldots, n\}$ of individuals who must make a collective choice from a set A of alternatives. Many definitions below require us to count alternatives, and so they implicitly assume A is finite; in such cases, we leave the assumption of finiteness unstated. We assume each individual has a *weak order* P_i of alternatives, i.e., each P_i is asymmetric and negatively transitive, and individual preferences are summarized by a preference profile $(P_1, \ldots, P_n)$. As such, we can imagine listing the alternatives vertically so that higher alternatives are preferred to lower ones, and we often refer to P_i as a "ranking" or "ordering." Corresponding to each strict preference relation, P_i, are weak preference, R_i, and indifference, I_i, relations defined in the usual way. That is, xR_iy if and only if not yP_ix,[1] and xI_iy if and only if neither xP_iy nor yP_ix.[2] Then R_i is complete and transitive, and I_i is reflexive and symmetric. We say P_i is a *linear order* if it admits no indifferences, i.e., for all distinct x and y, either xP_iy or yP_ix. In this case, viewed as a ranking, no two alternatives occupy the same level.

We view the profile of preferences as a "variable," i.e., we seek a theory of social choice that holds for a range of possible orderings for each individual. We refer to the set of possible preference profiles as the *domain of preferences,* and we usually assume this is fairly large. We say *Unrestricted domain* holds if every profile of weak orders is allowed; we say *Linear domain* holds if the possible preference profiles are just the profiles of linear orders, so no individual can be indifferent between distinct alternatives.

A *social preference rule* generates an asymmetric relation $\mathbf{P}(P_1, \ldots, P_n)$ for every profile $(P_1, \ldots, P_n)$ in the domain of preferences. Formally, we think of $\mathbf{P}$ as a mapping from the preference domain into the set of asymmetric relations on A. Central examples are the following.

Majority rule One alternative, x, is majority preferred to another, y, if more individuals prefer x to y than prefer y to x, i.e.,

$$x\mathbf{P}_M(P_1, \ldots, P_n)y \quad \Leftrightarrow \quad \#\{i \in N \mid xP_iy\} > \#\{i \in N \mid yP_ix\}.$$

Pareto rule One alternative, x, is Pareto preferred to another, y, if everyone weakly prefers x to y and at least some individuals strictly prefer x, i.e.,

$$x\mathbf{P}_P(P_1, \ldots, P_n)y \quad \Leftrightarrow \quad \begin{cases} (1) & \forall i \in N : xR_iy, \text{ and} \\ (2) & \exists j \in N : xP_iy. \end{cases}$$

[1]Equivalently, xP_iy if and only if not yR_ix.
[2]Equivalently, xI_iy if and only if xR_iy and yR_ix.

We define the corresponding weak social preference relations, $\mathbf{R}_M(P_1,\ldots,P_n)$ and $\mathbf{R}_P(P_1,\ldots,P_n)$, along the usual conventions.

Next, we define a family of social preference rules related to the first two. Given an arbitrary quota $q > \frac{n}{2}$, we can define a corresponding rule as follows.

Quota rule One alternative, x, is socially preferred to another, y, if at least q individual strictly prefer x to y, i.e.,

$$x\mathbf{P}_q(P_1,\ldots,P_n)y \quad \Leftrightarrow \quad \#\{i \in N xP_iy\} \geq q.$$

Under Linear domain, majority rule is equivalent to the quota rule with $q = \frac{n+1}{2}$, and the Pareto rule is equivalent to the quota rule with $q = n$. In general, a social preference for majority rule or Pareto rule is implied by a social preference with the corresponding quota rule.

Given any asymmetric preference relation P, we define the *maximal set* of P as

$$\mathscr{M}(P) \quad = \quad \{x \in A \mid \text{there is no } y \in A \text{ such that } yPx\}.$$

That is, an alternative belongs to the maximal set of P, or is simply "maximal," if there is no other alternative preferred to it. At an intuitive level, this produces a set of alternatives that are "best" for P; indeed, if P is an ordering, then $\mathscr{M}(P)$ consists of the alternatives at the top of the ordering. This notion can be applied to a social preference relation to generate a set of viable social choices.

Ultimately, we are concerned with social choices. The socially maximal elements of majority and Pareto rule play central roles in social choice theory; these sets,

$$\mathscr{M}(\mathbf{P}_M(P_1,\ldots,P_n)) \quad \text{and} \quad \mathscr{M}(\mathbf{P}_P(P_1,\ldots,P_n)),$$

are referred to as the *majority core* and *Pareto optimal* alternatives, respectively. When the majority core consists of an alternative x that is strictly majority-preferred to all others, i.e., $x\mathbf{P}_M(P_1,\ldots,P_n)y$ for all $y \neq x$, we refer to that alternative as a *Condorcet winner.* Of course, the well-known Condorcet Paradox, pictured to the right, demonstrates that the majority core may be empty. In contrast, under very mild assumptions (finiteness of A is more than sufficient), Pareto optimal alternatives always exist, and indeed, there is typically a plethora of Pareto optimals. Also, if there is one alternative x such that every other alternative is majority-preferred to it, i.e., $y\mathbf{P}_M(P_1,\ldots,P_n)x$ for all $y \neq x$, then x is a *Condorcet loser.*

P_1	P_2	P_3
a	b	c
b	c	a
c	a	b

It is known that existence of maximal elements hinges on acyclicity properties of the relation P. We say P is *acyclic* if there do not exist a natural number k and alternatives $x_1,\ldots,x_k$ such that $x_1Px_2P\cdots x_kPx_1$. The following result assumes finiteness of the set of alternatives, but the result extends to infinite sets as well, under appropriate compactness and continuity conditions.

Theorem 1.1. *Assume A is finite. If P is acyclic, then $\mathscr{M}(P) \neq \emptyset$.*

Proof. Assume P is acyclic, but suppose there is no maximal element. Choose any $y_1 \in A$. Since y_1 is not maximal, there exists $y_2 \in A$ such that $y_2 P y_1$. Since y_2 is not maximal, there exists $y_3 \in A$ such that $y_3 P y_2$, and so on. Continuing in this way, since A is finite, the sequence $y_1, y_2, y_3, \ldots$ must contain a repetition. Let ℓ and m be such that $\ell < m$ and $y_\ell = y_m$ is the first repetition. Then define $k = m - \ell$ and

$$x_1 = y_m,\ x_2 = y_{m-1},\ \ldots,\ x_{k-1} = y_2,\ x_k = y_{\ell+1}.$$

This contradicts acyclicity of P, as required. □

The acyclicity properties of quota rules are well understood. In the following result, which is due to Ferejohn and Grether (1974), the notation $\lceil x \rceil$ refers to the least integer greater than or equal to x.

Theorem 1.2 (Ferejohn and Grether)**.** *Assume A is finite and Unrestricted or Linear domain, and let $q > \frac{n}{2}$ be a quota. Then $\mathbf{P}_q(P_1, \ldots, P_n)$ is acyclic for all profiles $(P_1, \ldots, P_n)$ if and only if*

$$\left\lceil \frac{n}{n-q} \right\rceil \; > \; \#A.$$

In particular, if $q = n$, then the corresponding quota rule is acyclic regardless of the size of the set of alternatives. Setting $q = \frac{n+1}{2}$, the critical ratio in Theorem 1.2 reduces to

$$\left\lceil \frac{n}{n-q} \right\rceil \; = \; \left\lceil \frac{2n}{n-1} \right\rceil,$$

which is equal to three unless $n = 4$, in which case it is equal to four. Finally, note that the critical ratio is increasing in q, and that for $q = n - 1$, it is n.

One implication of the theorem is that when there are at least three alternatives, majority rule will sometimes produce cycles (unless Linear domain holds, and there are exactly three alternatives and four individuals). Another implication is that whenever there are fewer alternatives than individuals, i.e., $\#A < n$, we can find quotas sufficiently high but short of n that guarantee acyclic social preferences. When there are at least as many alternatives as individuals, i.e., $\#A \geq n$, the only quota rule that avoids cycles is $q = n$.

2 Social Choice Rules

We have to this point assumed individual preferences determine social preferences through a social preference rule, social choices then being determined as the maximal elements of social preference:

$$(P_1, \ldots, P_n) \stackrel{\text{s.p.r.}}{\longrightarrow} \mathbf{P}(P_1, \ldots, P_n) \stackrel{\max}{\longrightarrow} \mathscr{M}(\mathbf{P}(P_1, \ldots, P_n)).$$

A more general approach is to view social choices as determined directly as a function of individual preferences, bypassing social preferences.

A *social choice rule,* $\mathbf{C}$, assigns a nonempty set $\mathbf{C}(P_1, \ldots, P_n) \subseteq A$ to every preference profile, where we interpret this social choice set as consisting of the viable choices for the group as a function of individual preferences. Formally, we view $\mathbf{C}$ as a mapping from the preference domain into the set of all nonempty subsets of alternatives.

The possible social choice rules are endless. Or, to be precise, if there are n individuals, m alternatives, all profiles of linear orders are possible, there are

$$(2^m - 1)^{(m!)^n}$$

social choice rules—and many more if individual indifferences between alternatives are possible. If there are five individuals and five alternatives, that is 31 raised to the power $24,883,200,000$.

As we have seen, one way of generating examples of social choice rules is through social preferences. For each social preference rule we have considered so far, we can try to define a social choice rule as follows: for every profile, we can define

$$\mathbf{C}(P_1, \ldots, P_n) \;\; = \;\; \mathscr{M}(\mathbf{P}(P_1, \ldots, P_n))$$

as the socially maximal alternatives. But, as with majority rule, there may not be any maximal elements, and the social choice set, so-defined, may be empty. This is not an issue if the social preference relation is acyclic; this is implied if $\mathbf{P}(P_1, \ldots, P_n)$ is always transitive; and it is of course implied if $\mathbf{P}(P_1, \ldots, P_n)$ is negatively transitive and therefore forms an ordering of the alternatives.

Recall that majority rule can produce cycles when there are three or more alternatives and three or more individuals; under Linear domain, the exceptional case is when there are exactly three alternatives and four individuals, in which case majority rule is acyclic. Then we can define the majority social choice rule, $\mathbf{C}_m$, by the formula

$$\mathbf{C}_m(P_1, \ldots, P_n) \;\; = \;\; \mathscr{M}(\mathbf{P}_M(P_1, \ldots, P_n)).$$

Using Theorem 1.2, we can also define the more general class of quota social choice rules.

Corollary 2.1. *Assume A is finite and Unrestricted domain, and let $q > \frac{n}{2}$ be a quota. If $\lceil \frac{n}{n-q} \rceil > \#A$, then the mapping $\mathbf{C}_q$ defined by*

$$\mathbf{C}_q(P_1, \ldots, P_n) \quad = \quad \mathscr{M}(\mathbf{P}_q(P_1, \ldots, P_n))$$

is a well-defined social choice rule.

While the above social preferences are based on binary comparisons of alternatives (meaning that to calculate the social preference between two alternatives, we just need to look at individual preferences between those two alternatives), a bevy of social choice rules are defined by first calculating scores—points assigned to each alternative based on their locations in individual rankings—and then choosing the alternatives with the highest score. Social preferences generated in this way inherit properties of natural numbers, and so they automatically produce orderings of the alternatives and circumvent the problem of empty choice sets.

Plurality rule The "plurality score" of an alternative is the number of times it appears at the top of a voter's ranking.

Copeland's rule The "Copeland score" of an alternative x is the number of other alternatives to which it is majority preferred minus the number of alternatives preferred to it; that is,

$$\#\{y \in A \mid x\mathbf{P}_M(P_1, \ldots, P_n)y\} - \#\{y \in A \mid y\mathbf{P}_M(P_1, \ldots, P_n)x\}.$$

Borda's rule For each alternative x and each individual i, define $B_i(x)$ as the number of alternatives below x in P_i minus the number of alternatives above x in P_i; then the "Borda score" of x is

$$B(x) \quad = \quad \sum_{i=1}^{n} B_i(x).$$

Note that when individual preferences are linear orderings, the computation of Borda scores can be simplified. Let $\underline{B}_i(x)$ denote the number of alternatives strictly below x in P_i, and let $\underline{B}(x)$ be the sum of these scores across individuals. It is easy to check that

$$B(x) \quad = \quad 2\underline{B}(x) - n(\#A - 1),$$

so that social preferences are preserved when only alternatives below each x are tabulated.

The second and third of the above rules were proposed by Copeland (1951) and Borda (1781), respectively. Of course, Borda's rule can be generalized by assigning any score to first-place votes, any score to second-place votes, etc., the only restrictions being that the weights are weakly decreasing, and that the weights used to evaluate one individual's ranking are the same as for any other. The members of this larger class are known as *scoring rules.*

There is no need to first define social preferences before specifying a social choice set. The following social choice rules are not naturally formulated in terms of social preferences, but are rather defined from individual preferences directly in terms of simple algorithms.

Plurality runoff Same as plurality unless there are exactly two alternatives with higher plurality scores than all others, and neither is top-ranked by more than half of the voters; then choose the majority preferred of those two alternatives.

Antiplurality rule Count the number of times each alternative appears at the bottom of a voter's ranking; delete the alternatives with the highest "antiplurality score," recalculate scores, again delete the alternatives with highest antiplurality score, and so on — until all alternatives have the same antiplurality score; then choose the remaining alternatives.

Plurality elimination Delete the alternatives with the lowest plurality score, recalculate scores, again delete the lowest scoring alternatives, and so on — until all alternatives have the same score; then choose the remaining alternatives.

While Plurality elimination discards only the alternatives with the lowest score in each round, a reasonable variant would delete the alternatives with below average score in each round.

To help comprehend the magnitude of the social choice problem, consider the following preferences,

16	10	8	8	4	2	2	2	2	1
a	b	c	d	e	e	b	a	c	d
d	e	b	e	b	c	e	e	b	c
c	d	e	c	c	d	c	c	d	e
e	c	d	b	d	b	d	d	e	b
b	a	a	a	a	a	a	b	a	a

where the number above a column indicates the number of individuals with the ranking below. Note that the majority social preference generates a linear ordering of the alternatives in this example, and that alternative e is ranked strictly above all others, i.e., it is the Condorcet winner. Note also that a is the Condorcet loser.

This profile leads to a number of social choices by seemingly reasonable social choice rules all based, in one way or another, on plurality rule.

- The unique plurality choice is a, the Condorcet loser. (In fact, the plurality scores are in reverse order of the majority ranking!)
- The unique plurality runoff choice is b.
- The unique plurality elimination choice is c.
- The unique antiplurality choice is d.

The unique Copeland choice is e, the Condorcet winner. It is striking that every one of the five alternatives in the above example, each is the unique choice of a seemingly reasonable social choice rule.

For the record, Copeland scores are 0, 1, 2, 3, and 4 in the above example, and the choice is e. This is easily seen by constructing the *pairwise comparison matrix,*

	a	b	c	d	e
a	0	18	18	18	18
b	37	0	16	26	22
c	37	39	0	20	27
d	37	29	35	0	27
e	37	33	28	28	0

where each entry registers the number of individuals who prefer the row alternative to the column alternative. Counting only alternatives below each x, the Borda scores of a, b, c, d, and e are 72, 101, 123, 128, and 126, respectively, and the choice is d. (In fact, the Borda score of an alternative is just the sum of entries in its row of the pairwise comparison matrix!) It's no coincidence that Copeland chooses the Condorcet winner.

In the remainder of this section, we consider several more esoteric social choice rules; these rules are of interest in themselves, and they help illustrate the tremendous variety of social choice rules. The first extends Borda's rule, in effect applying it iteratively to narrow down the set of viable choices.

Borda elimination Delete the alternatives with the lowest Borda score, recalculate scores, again delete the lowest scoring alternatives, and so on — until all alternatives have the same score; choose the remaining alternatives.

Borda elimination is proposed by Baldwin (1926) and is a variant of a rule proposed by Nanson (1882), the difference being that the current rule eliminates only the alternatives with the lowest score in each round, whereas Nanson removes the alternatives with below average score. The computation of the Borda elimination winners is in general laborious. Initially, we use the Borda scores to determine that a has the lowest score; we delete that alternative and recalculate to find that b's score is 64, c's is 86, d's is 91, and e's is 89. Since the score of b is lowest, we delete that alternative and recalculate to find that c's score is 47, d's is 62, and e's is 56. Since the score of c is the lowest, we delete that alternative, and then e is the choice from the remaining two alternatives.

We only consider a small fraction of the axioms that have been proposed for social choice rules. In fact, the above rules based on plurality scores do not fair very well over all. Interestingly, Borda elimination does possess some desirable properties; when a Condorcet winner exists, for example, it is the unique choice of Borda elimination, so we could have skipped the lengthy calculations above.

Next, we define a social choice rule, due to Schwartz (1972), based directly on majority preferences.

Top cycle Say "x is indirectly majority preferred to y" if there are alternatives, a, b, etc., satisfying

$$x\mathbf{R}_M(P_1,\ldots,P_n)a\mathbf{R}_M(P_1,\ldots,P_n)b\mathbf{R}_M(P_1,\ldots,P_n)\cdots c\mathbf{R}_M(P_1,\ldots,P_n)y.$$

Then top cycle consists of each alternative that is indirectly majority preferred to every other alternative.

Note that the top cycle is defined in terms of weak majority preferences—this is done to ensure non-emptiness of the set. When the number of individuals is odd and orderings are linear, however, the top cycle can be formulated equivalently in terms of strict majority preferences.

If there is a Condorcet winner, then that alternative is directly majority preferred to all other alternatives, and the top cycle consists of that alternative alone. In contrast to the concept of Condorcet winner, however, the top cycle is non-vacuous extremely generally. If the set of alternatives is finite, for example, then the top cycle is non-empty; thus, it is a well-defined social choice rule. In fact, it is straightforward to show that every alternative in the top cycle is strictly majority preferred to every alternative outside the top cycle; moreover, the top cycle is the smallest set possessing this property.

We refine the idea of the top cycle in the next definition. Related concepts have appeared in Gillies (1959), Fishburn (1977), Miller (1980), and Bordes (1983), but the one used here is due to McKelvey (1986).

Uncovered set Say "x covers y" if (i) x is majority-preferred to y, (ii) for every alternative z, if $z\mathbf{P}_M(P_1,\ldots,P_n)x$, then $z\mathbf{P}_M(P_1,\ldots,P_n)y$, and (iii) for every z, if $y\mathbf{P}_M(P_1,\ldots,P_n)z$, then $x\mathbf{P}_M(P_1,\ldots,P_n)z$. Then uncovered set consists of every alternative that is not covered by any other.

Of course, we can define a "covering" social preference rule, say $\mathbf{P}_C$ using conditions (i)–(iii) above, and then the uncovered set is just the maximal elements

$\mathscr{M}(\mathbf{P}_C(P_1,\ldots,P_n))$. Like the top cycle, if there is a Condorcet winner, then the uncovered set consists of that alternative alone; but the uncovered set is nonempty quite generally, e.g., when the set of alternatives is finite. The next result establishes that the uncovered set is nested between the Copeland choice set and the top cycle.

Proposition 2.2. *Copeland set $\subseteq$ uncovered set $\subseteq$ top cycle.*

Proof. Note that x is uncovered if and only if for all y, it is not the case that y covers x. This implies that for all y, either (i) $x\mathbf{R}_M(P_1,\ldots,P_n)y$, or (ii) there exists z such that $x\mathbf{R}_M(P_1,\ldots,P_n)z\mathbf{P}_M(P_1,\ldots,P_n)y$, or (iii) there exists z such that $x\mathbf{P}_M(P_1,\ldots,P_n)z\mathbf{R}_M(P_1,\ldots,P_n)y$. In all three cases, there exists z (possibly equal to x) such that $x\mathbf{R}_M(P_1,\ldots,P_n)z\mathbf{R}_M(P_1,\ldots,P_n)y$. It follows immediately that x belongs to the top cycle, so the uncovered set is a subset of the top cycle. Now consider any element x of the Copeland choice set, and suppose in order to deduce a contradiction that it does not belong to the uncovered set. Then there is an alternative y that covers x. Note that $y\mathbf{P}_M(P_1,\ldots,P_n)x$, and

$$\begin{aligned}\{z \mid x\mathbf{P}_M(P_1,\ldots,P_n)z\} &\subseteq \{z \mid y\mathbf{P}_M(P_1,\ldots,P_n)z\}\\ \{z \mid z\mathbf{P}_M(P_1,\ldots,P_n)y\} &\subseteq \{z \mid z\mathbf{P}_M(P_1,\ldots,P_n)x\}.\end{aligned}$$

But then

$$\begin{aligned}\#\{z \mid y\mathbf{P}_M(P_1,\ldots,P_n)z\} &> \#\{z \mid x\mathbf{P}_M(P_1,\ldots,P_n)z\}\\ \#\{z \mid z\mathbf{P}_M(P_1,\ldots,P_n)x\} &> \#\{z \mid z\mathbf{P}_M(P_1,\ldots,P_n)y\},\end{aligned}$$

which implies

$$\begin{aligned}&\#\{z \mid x\mathbf{P}_M(P_1,\ldots,P_n)z\} - \#\{z \mid z\mathbf{P}_M(P_1,\ldots,P_n)x\}\\ &\quad > \#\{z \mid y\mathbf{P}_M(P_1,\ldots,P_n)z\} - \#\{z \mid z\mathbf{P}_M(P_1,\ldots,P_n)y\}.\end{aligned}$$

But then the Copeland score of y is greater than the Copeland score of x, a contradiction. Therefore, x belongs to the uncovered set. □

There are many other rules that generalize the notion of Condorcet winner. The next two are due, respectively, to Simpson (1969) and Kramer (1977) and to Black (1958).

Simpson-Kramer (maxmin) rule For each alternative x, find the alternative y that minimizes the number of votes in favor of x; let that minimum number of votes be $m(x)$, and choose every alternative that maximizes $m(x)$.

The Simpson-Kramer scores in the above example are $m(a) = 18$, $m(b) = 16$, $m(c) = 20$, $m(d) = 27$, and $m(e) = 28$, and the choice is e.

Black's rule Choose the alternatives with highest Borda score, unless there is a Condorcet winner, in which case choose it.

Black's rule is not an especially elegant extension of the notion of Condorcet winner; of course, any social choice rule could be modified in this way. Nevertheless, we will see that it does possess fairly nice properties.

In addition to the above, there are a number of social choice rules that produce choice sets based on measurements of "distance" between preference relations. These proximity-based rules are not our focus, and so I list them with very brief, and incomplete, descriptions. For references, consult Slater (1961), Kemeny (1959), Dodgson (1876), and Sertel (1986) and Sertel and Yilmaz (1999).

Slater's rule Find the rankings "closest" to R_M and choose their top-ranked alternatives.

Kemeny's rule Find the rankings that are "closest," on average, to the ranking of each individual, and choose their top-ranked alternatives.

Dodgson's rule For each alternative, find the preference profile "closest" to the actual profile for which it is a Condorcet winner to compute the "Dodgson score," and choose the alternatives with lowest Dodgson score.

Sertel's rule For each alternative x and each individual i, let $m_i(x)$ be the number of alternatives below x in P_i; and for each group G, let $m_G(x) = \min_{i \in G} m_i(x)$ be the score for x in G, and let $m(x)$ be the highest score for x in any majority group. Choose every alternative that maximizes $m(x)$.

In fact, Klamler (2005) notes that the Copeland rule can be viewed in similar terms when the set of alternatives is finite and there are no majority indifferences: then the Copeland score of an alternative x is an inverse measure of the distance between the majority preference relation P_M and the "closest" linear order having x ranked first.

To elaborate on Dodgson's rule, we need to consider the number of preference reversals needed to make an alternative the Condorcet winner. Consider the following preference profile.

a	a	a	a	b	b	b	c	c	c	d	d	d
b	b	b	b	d	d	d	d	a	a	b	b	b
c	c	c	c	c	c	c	a	b	b	c	c	c
d	d	d	d	a	a	a	b	d	d	a	a	a

Here, b needs only one reversal to become a Condorcet winner, but a, c, and d each need four.

3 Axioms for Social Choices

Given the multitude of possible social choice rules, we formulate axioms that embody properties of particular importance, from either normative or positive perspectives.

Our first axiom requires that a social choice rule treat individuals equally, in the sense that the effect of any given ranking on the social choice is the same, regardless of the individual who holds it. Formally, we require that if the rankings of any two individuals are interchanged, then the social choice set remains the same. This is similar in spirit to May's (1952) axiom of the same name, but translated into social choices, rather than social preferences.

Anonymity Consider any profile $(P_1, \ldots, P_n)$ in which i's ranking is $\hat{P}$ and j's ranking is $\tilde{P}$; then the social choice set remains unchanged when i's and j's orderings are interchanged:

$$\mathbf{C}(P_1, \ldots, P_{i-1}, \underbrace{\hat{P}}_{i\text{'s ranking}}, P_{i+1}, \ldots, P_{j-1}, \underbrace{\tilde{P}}_{j\text{'s ranking}}, P_{j+1}, \ldots, P_n)$$

$$= \mathbf{C}(P_1, \ldots, P_{i-1}, \underbrace{\tilde{P}}_{i\text{'s ranking}}, P_{i+1}, \ldots, P_{j-1}, \underbrace{\hat{P}}_{j\text{'s ranking}}, P_{j+1}, \ldots, P_n)$$

Note that the formulation of Anonymity presumes that the domain of preferences is closed with respect to permutations of individual rankings; otherwise, the social choice set may not defined after we switch the rankings of the two individuals in the definition. All of the social choice rules defined thus far satisfy the Anonymity axiom.

The next axiom follows Arrow (1951,1963) in precluding the existence of a dictator, stated in terms of social choices rather than preferences. The idea, similar in spirit to Arrow's axiom of the same name, is that no individual should have absolute power to decide the choice for society.

No-dictator There is no individual i such that for all x and y and every profile $(P_1, \ldots, P_n)$, xP_iy implies $y \notin \mathbf{C}(P_1, \ldots, P_n)$.

An individual i is a *dictator,* in the sense of the axiom, if social choices are restricted to the individual's top-ranked alternatives; if an alternative is not deemed best by i, then it cannot be chosen. Such an individual effectively wields absolute power over social choices. Clearly, Anonymity implies No-dictator as long as there are at least two individuals and the domain of preferences is nontrivial.

Whereas Anonymity requires that the preferences of any two individuals be treated in the same way, the next requires that any two alternatives be treated the same. It is similar in spirit to May's axiom of the same name.

Neutrality Consider any two alternatives x and y, any profile $(P_1, \ldots, P_n)$, and any profile $(P'_1, \ldots, P'_n)$ such that for all i, P_i and P'_i are identical except that x and y are interchanged; then the social choice sets are identical except that x and y

are interchanged:

$$\begin{array}{rcl} \mathbf{C}(P_1,\ldots,P_n)\setminus\{x,y\} & = & \mathbf{C}(P_1',\ldots,P_n')\setminus\{x,y\} \\ \mathbf{C}(P_1,\ldots,P_n)\cap\{x\} & = & \{x,y\}\setminus(\mathbf{C}(P_1',\ldots,P_n')\cap\{y\}) \\ \mathbf{C}(P_1,\ldots,P_n)\cap\{y\} & = & \{x,y\}\setminus(\mathbf{C}(P_1',\ldots,P_n')\cap\{x\}). \end{array}$$

All of the social choice rules defined above satisfy Neutrality.

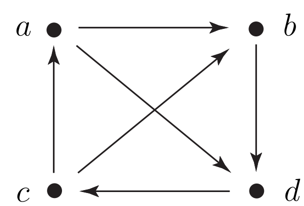
The next axiom parallels Arrow's Pareto axiom. It is distinct from the axiom for social preferences, but the intuition is similar: a group should not choose an alternative if it is possible to make some voters better off while hurting no one.

Pareto For all alternatives x and y and every profile $(P_1,\ldots,P_n)$, if xP_iy for all i, then $y\notin\mathbf{C}(P_1,\ldots,P_n)$.

An implication is that if each P_i is a linear order (with no indifferences), then the social choice set is contained in the set of Pareto optimal alternatives. Another implication is that, if one alternative, say x, is strictly preferred to all others by all individuals, then x must be the social choice.

All but one of the social choice rules we have defined satisfy Pareto: perhaps surprisingly, the top cycle can choose alternatives outside the Pareto optimal set! Given three individuals with preferences over four alternatives below, we construct the digraph for majority rule to compute the top cycle.

P_1	P_2	P_3
c	d	a
a	c	b
b	a	d
d	b	c



Here, the top cycle is the entire set of alternatives, $\{a,b,c,d\}$, despite the fact that every individual strictly prefers a to b.

It may not be obvious that the uncovered set satisfies Pareto, so we establish this fact in the next result; an implication, since Copeland is contained in the uncovered set, is that it also satisfies Pareto.

Proposition 3.1. *The uncovered set satisfies Pareto.*

Proof. Consider any profile $(P_1,\ldots,P_n)$ and any alternatives x and y such that xP_iy for all i. Clearly, $x\mathbf{P}_M(P_1,\ldots,P_n)y$. If $z\mathbf{P}_M(P_1,\ldots,P_n)x$, then defining $G=\{i \mid zR_ix\}$, we must have $\#G>\frac{n}{2}$. For each $i\in G$, we then have zR_ixP_iy, which implies

zP_iy; then a majority of individuals strictly prefer z to y, so $z\mathbf{P}_M(P_1,\ldots,P_n)y$. And if $y\mathbf{P}(P_1,\ldots,P_n)z$, then defining $G = \{i \mid yR_iz\}$, we must have $\#G > \frac{n}{2}$. For each $i \in G$, we then have xP_iyR_iz, which implies xP_iz; then a majority of individuals strictly prefer x to z, so $x\mathbf{P}_M(P_1,\ldots,P_n)z$. We conclude that x covers y, and therefore y is not contained in the uncovered set. □

The following axioms are considerably more selective than the preceding ones. They require that social choices are consistent with majority rule, in the sense of either of two axioms based on ideas due to Condorcet (1785):

Condorcet winner For every profile $(P_1,\ldots,P_n)$, if x is a Condorcet winner, then $\mathbf{C}(P_1,\ldots,P_n) = \{x\}$.

Condorcet loser For every profile $(P_1,\ldots,P_n)$, if x is a Condorcet loser, then $x \notin \mathbf{C}(P_1,\ldots,P_n)$.

All of the plurality-based methods violate the Condorcet winner axiom, and the above example shows that the plurality method actually violates Condorcet loser. It is not hard to see that plurality runoff, plurality elimination, and antiplurality satisfy the latter axiom "in spirit." (If these rules produce a unique choice, then it cannot be a Condorcet loser; but there can be several tied alternatives that include a Condorcet loser.)

Copeland, Simpson-Kramer, and the top cycle all satisfy the Condorcet winner axiom. Copeland and the top cycle satisfy Condorcet loser, but Simpson-Kramer violates it; see below.

P_1	P_2	P_3	P_4	P_5
a	d	c	b	e
b	e	d	c	a
x	a	e	x	b
c	x	x	d	c
d	b	a	e	d
e	c	b	a	x

	a	b	c	d	e	x
a	0	4	3	2	1	3
b	1	0	4	3	2	3
c	2	1	0	4	3	3
d	3	2	1	0	4	3
e	4	3	2	1	0	3
x	2	2	2	2	2	0

Here, x is the unique Simpson-Kramer choice, but it is the Condorcet loser.

Borda's rule violates the Condorcet winner axiom: below, a is a Condorcet winner, but the Borda choice is b.

P_1	P_2	P_3
a	a	b
b	b	c
c	c	d
d	d	a

In fact, Borda's rule does satisfy the second of our Condorcet axioms.

Proposition 3.2. *Borda's rule satisfies Condorcet loser.*

Proof. Let there be m alternatives. Note that the average of the off-diagonal cells in the pairwise comparison matrix is $n/2$. So the average total in each row is $(m-1)(n/2)$. Suppose x is a Condorcet loser. Then the x row has a zero in the x column and entries $< n/2$ in the rest of the row. The total is $< (m-1)(n/2)$, so x has a below average Borda score. So some alternative has a higher Borda score, so x is not chosen. □

The proof of the previous proposition shows that a Condorcet loser always has a below average Borda score, so it must be eliminated at some iteration in the construction of the Borda elimination social choice set. A further implication, perhaps not obvious, is that Borda elimination satisfies the Condorcet winner axiom.

Corollary 3.3. *Borda elimination satisfies the Condorcet winner and Condorcet loser axioms.*

Proof. Suppose x is a Condorcet winner. Then the x row has a zero in the x column and entries $> n/2$ in the rest of the row. The total is $> (m-1)(n/2)$, so x always has an above average Borda score. So some alternative has a lower Borda score, so the algorithm in the definition of Borda elimination does not terminate until a single alternative remains, and x is never eliminated; therefore, it is the unique choice. □

The next axiom formalizes the idea that if an alternative is a viable choice given one preference profile, and then we consider another profile in which it receives more support, it should still be chosen.

Positive association Consider any two profiles $(P_1, \ldots, P_n)$ and $(P'_1, \ldots, P'_n)$ such that x is the unique choice for $(P_1, \ldots, P_n)$; if some voters move x up in their rankings (leaving all other preferences the same), then x is still in the choice set for $(P'_1, \ldots, P'_n)$.

This axiom, which is related to May's axiom of positive responsiveness, is eminently reasonable. It is satisfied by plurality, Borda, Copeland, the top cycle, the uncovered set, Simpson-Kramer, and Black. Perhaps surprisingly, it is violated by plurality runoff, plurality elimination, antiplurality, and Borda elimination.

According to plurality runoff and plurality elimination, the unique choice below is a.

2	6	4	5
b	a	b	c
a	b	c	a
c	c	a	b

Moving a up in the rankings of the first two individuals, c is then chosen, violating positive association.

Now consider antiplurality. The unique choice below is a.

2	3	2
c	a	c
b	b	a
a	c	b

Moving a up to the top of the rankings of the first two individuals, then c is chosen, violating positive association.

Finally, consider Borda elimination. Below, we eliminate c, and the unique choice for Borda elimination is a.

2	4	2	1	4
b	a	b	c	c
a	b	c	b	a
c	c	a	a	b

Moving a up to the top of the rankings of the first two individuals, as below,

2	4	2	1	4
a	a	b	c	c
b	b	c	b	a
c	c	a	a	b

we then eliminate b and choose c.

Next, we consider a different kind of axiom, a weakening of an axiom (called simply "consistency") due to Young (1975), that imposes a consistency condition across choices from different groups. It requires that if an alternative is a viable choice for two separate groups, then it should remain viable when the two groups are combined.

Population consistency Consider any two (disjoint) groups of voters and any preference profiles for each, say $(P_1, \ldots, P_k)$ and $(P_{k+1}, \ldots, P_n)$, with respective social choice sets $\mathbf{C}(P_1, \ldots, P_k)$ and $\mathbf{C}(P_{k+1}, \ldots, P_n)$; let $\mathbf{C}(P_1, \ldots, P_n)$ be the social choice set for the combined profile $(P_1, \ldots, P_n)$; if an alternative x is contained in $\mathbf{C}(P_1, \ldots, P_k)$ and $\mathbf{C}(P_{k+1}, \ldots, P_n)$, then x is in $\mathbf{C}(P_1, \ldots, P_n)$.

Plurality example:

P_1	P_2
a	b
b	a
c	c

$C = \{a, b\}$

P_3	P_4
a	c
b	a
c	b

$C = \{a, c\}$

P_1	P_2	P_3	P_4
a	b	a	c
b	a	b	a
c	c	c	b

$C = \{a\}$

Borda example:

P_1	P_2	P_3
a	b	b
c	a	a
b	c	c

$C = \{a, b\}$

P_4	P_5	P_6
c	c	a
a	a	b
b	b	c

$C = \{a, c\}$

P_1	P_2	P_3	P_4	P_5	P_6
a	b	b	c	c	a
c	a	a	a	a	b
b	c	c	b	b	c

$C = \{a\}$

In fact, the previous two examples are generalizable.

Proposition 3.4. *The plurality and Borda social choice rules satisfy Population consistency.*

Proof. Suppose x is chosen by either method in two groups. Let $S^1(x)$ and $S^2(x)$ be the score of x in the two groups, and $S(x)$ the score in the combined group. Given any other alternative y, we have $S^1(x) \geq S^1(y)$ and $S^2(x) \geq S^2(y)$. Then the scores in the combined group satisfy

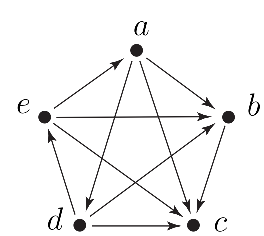
$$S(x) \;=\; S^1(x) + S^2(x) \;\geq\; S^1(y) + S^2(y) \;=\; S(y),$$

so x is chosen in the combined group. □

In fact, the plurality and Borda social choice rules are the only ones we have seen that satisfy population consistency, and the above arguments and examples show that these rules (along with all scoring rules) satisfy a stronger version of Population consistency requiring that if there are choices in common to two disjoint groups, then exactly those alternatives are chosen when the groups are combined.

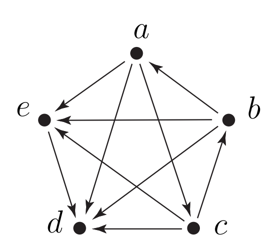
Counterexample for Copeland: consider a group with rankings and majority digraph below.

P_1	P_2	P_3
d	a	e
e	d	b
a	b	c
b	e	a
c	c	d

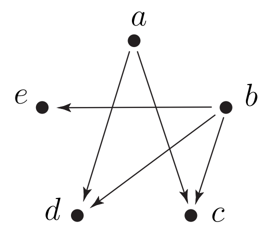


Then the Copeland social choice set is $\{a, d, e\}$. Consider another group with rankings and majority digraph below.

P_4	P_5	P_6
c	a	b
b	c	a
e	b	d
a	e	c
d	d	e



The Copeland social choice set is $\{a, b, c\}$. Only alternative a is common to both Copeland social choice sets. But when we combine these groups and consider the list $(P_1, \ldots, P_6)$ of voter rankings, the majority digraph is below,



and the unique Copeland choice is b! Note that the above example shows that Copeland's rule violates even a much weaker version of population consistency: even if we only required that x is chosen for the combined group when it is the unique element in the intersection of $\mathbf{C}(P_1, \ldots, P_k)$ and $\mathbf{C}(P_{k+1}, \ldots, P_n)$, the axiom would still be violated by Copeland.

The above examples that Borda violates Condorcet winner and that Copeland violates Population consistency have deeper theoretical foundations: Young and Levenglick (1978) show that given any social choice rule satisfying Population consistency for a variable population, there is some number n of individuals such that the rule violates the Condorcet axiom.

Theorem 3.5 (Young and Levenglick)**.** *Assume $\#A \geq 3$ and Linear domain. There is no social choice rule satisfying Condorcet winner and Population consistency.*

Proof. Suppose the social choice rule $\mathbf{C}$ satisfies both Condorcet winner and Population consistency, select any three alternatives a, b, and c, and consider a population of six voters and preferences $(P_1, \ldots, P_6)$ as follows.

P_1	P_2	P_3	P_4	P_5	P_6
a	a	b	b	c	c
b	b	c	c	a	a
c	c	a	a	b	b

Here, other alternatives, if any, may be placed below these three in all individuals' ranking. Some alternative must be chosen, and neglecting the possibility that some alternative outside $\{a, b, c\}$ is chosen (a separate argument is possible in that case), symmetry of the example allows us to assume without loss of generality that $a \in \mathbf{C}(P_1, \ldots, P_6)$. Note that there is a majority preference cycle through the alternatives, and in particular, $c\mathbf{P}_M(P_1, \ldots, P_6)a$. Now consider a separate population of seven voters with preferences below.

P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}
a	a	a	a	c	c	c
c	c	c	c	a	a	a
b	b	b	b	b	b	b

Here, the Condorcet winner is a, and therefore $\mathbf{C}(P_7, \ldots, P_{13}) = \{a\}$. By Population consistency, a must be among the alternatives chosen when the two groups of voters are combined, but it is easily seen that the Condorcet winner given $(P_1, \ldots, P_{13})$ is c. Thus, Condorcet winner implies $\mathbf{C}(P_1, \ldots, P_{13}) = \{c\}$, a contradiction. □

In fact, Population consistency is nearly characteristic of scoring rules, as defined above. Using a somewhat more general definition of "scoring rule," Young (1975) shows that a social choice rule satisfies Anonymity, Neutrality, and Population consistency if and only if it is a scoring rule. Moreover, he shows that Borda's rule is the only rule satisfying Neutrality, Population consistency, and two other axioms he calls "Faithfulness" and the "Cancellation" axiom, which are outside the scope of these notes.

At this point, it may be useful to summarize the analysis of the preceding axioms and social choice rules. See Figure 1, where a solid circle indicates satisfaction of an axiom, and an open circle indicates failure. As a pure matter of tabulating scores, it appears Borda, Copeland, the uncovered set, and Black do well—the distinction being that Borda satisfies Population consistency but not the Condorcet winner axiom, while the situation is reversed for the other rules. Among the latter three, Copeland has an advantage over the uncovered set in that it produces smaller choice sets, while Black's rule has received less attention because its definition has a relatively ad hoc flavor. Of course, axioms favoring the uncovered set or Black's rule could be constructed to improve their axiomatic qualifications. At any rate, Borda and Copeland seem to arise as focal options, and in fact there is some competition between these two rules in the literature.

We end with an axiom that is actually satisfied by *none* of the social choice rules defined so far, at least when there are three or more alternatives and the domain of preferences is fairly large. The axiom demands that the social choice rule choose a single alternative as a function of the individual preference profile, in effect ruling out the possibility of ties.

No-ties: For every profile $(P_1, \ldots, P_n)$, we have $\#\mathbf{C}(P_1, \ldots, P_n) = 1$.

Since no well-known social choice rule satisfies No-ties when the domain of possible preferences is large, it is, technically, a restrictive axiom. But we can modify any given social choice rule to satisfy No-ties by specifying a "tie-breaking rule," which selects exactly one alternative from the social choice set in case there is more than one. A possibility (when alternatives have names) is to choose the alphabetically lowest socially viable alternative. A related solution (when the alternatives are finite in number) is to index alternatives by natural numbers and select the lowest indexed alternative chosen by the rule. This may appear strange, because of the arbitrary nature of the selection. From a normative perspective, this selection is really without

	Anonymity	No dictator	Neutrality	Pareto	Condorcet winner	Condorcet loser	Positive association	Pop. consistency
plurality	●	●	●	●	○	○	●	●
Borda	●	●	●	●	○	●	●	●
Copeland	●	●	●	●	●	●	●	○
plurality runoff	●	●	●	●	○	○	○	○
plurality elimination	●	●	●	●	○	○	○	○
antiplurality	●	●	●	●	○	○	○	○
Borda elimination	●	●	●	●	●	●	○	○
top cycle	●	●	●	○	●	●	●	○
uncovered set	●	●	●	●	●	●	●	○
Simpson-Kramer	●	●	●	●	●	○	●	○
Black	●	●	●	●	●	●	●	○

Figure 1: Standings

consequence: if we view a socially viable alternative as at least as good as all others, then it does not matter (on normative grounds) which one is ultimately chosen. From a positive perspective, however, we do not often see voting rules with biases for or against certain alternatives built in.

A more realistic tie-breaking assumption would be to assume some randomization over elements of the social choice set. We will not consider that kind of tie-breaking rule until later. For now, we will only consider deterministic rules, where given the profile of individual preferences, we can say with certainty which alternative will be chosen.

4 Strategic Manipulation

We now examine an assumption underlying our analysis to this point, namely, that social preferences are determined using the individuals' true preferences. Until now, we have conceived of social choices as depending on individual preferences, but we have not been concerned with how individual preferences are elicited. In practice, an individual's ranking of alternatives is not observable, so the individual must report her own preferences, typically by casting a ballot of some sort. If plurality rule is used, for example, and individual preferences are linear orders, then it is enough that a voter report a single candidate (which we take to be her top-ranked alternative). If Borda rule is used, then each voter must report an entire ranking of alternatives (which we take to be her true ranking).

This leads us to our first consideration of individual incentives. If individual preferences are elicited by voting and these votes are used to make a collective choice, then it may be in the interests of some individuals to misrepresent their preferences. In an election with two major candidates and one minor one, it is common, for example, that a voter who prefers the minor candidate chooses not to "waste her vote," instead voting for her favorite of the two major candidates.[3]

To be concrete, suppose the choice is to be decided by Copeland rule, and suppose there are four individuals and four alternatives, with preferences below.

P_1	P_2	P_3	P_4
w	w	x	z
x	y	y	y
y	z	z	x
z	x	w	w

The pairwise comparison matrix corresponding to these preferences is as follows.

	w	x	y	z
w	0	2	2	2
x	2	0	2	2
y	2	2	0	3
z	2	2	1	0

Thus, y is the unique Copeland winner, and if these individuals report their true preferences, then it will be the Copeland outcome. But if individual 1 instead reports the ranking below,

P'_1
x
w
z
y

then x will have a Copeland score of one, while every other alternative has a score of zero, and x will be the outcome. Thus, since xP_iy, it is possible for the individual to "manipulate" the election to obtain an outcome that is personally preferable but inconsistent with the voting rule used to make the choice.

For an example using Borda elimination, consider the profile $(P_1, \ldots, P_7)$ below.

[3] You can imagine that some voters who voted for Ralph Nader in the 2000 Florida presidential election might regret wasting their vote...

Borda elimination eliminates c, then b, then a, then x, and the unique choice is y.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P'_7
c	a	b	c	y	b	x	x
a	x	y	y	a	x	y	a
b	b	x	x	x	c	a	b
y	c	c	a	b	a	b	c
x	y	a	b	c	y	c	y

But if individual 7 reports P'_7, we eliminate y, then c, then b, then a, and the unique choice is x—which is better for her!

To formalize the idea of manipulation discussed above, let $\mathbf{C}$ be a social choice rule. As we will see, it is crucial for the definition of manipulation that $\mathbf{C}$ satisfies No-ties, so that it selects a unique alternative as a function of the preference profile. In this case, we can view $\mathbf{C}(P_1, \ldots, P_n)$ as a single alternative, rather than a set containing a single alternative. Suppose the preference profile is $(P_1, \ldots, P_n)$, and consider a different ranking P'_i for individual i. If

$$\mathbf{C}(P_1, \ldots, P'_i, \ldots, P_n) \; P_i \; \mathbf{C}(P_1, \ldots, P_i, \ldots, P_n),$$

then i can achieve a preferred outcome by reporting the false preference P'_i rather than the true one P_i. We call this a *manipulation,* or when more detail is called for, a "P_i-to-P'_i-manipulation." We seek to understand the class of social choice rules that are immune to the possibility of such manipulations.

Note that in the above definition of a manipulation, we denoted i's true preference relation by P_i and the false report by P'_i. Of course, we could use any notation, possibly $P_i^{☺}$, to represent i's true preference, and any notation, maybe $P_i^{☹}$, to denote a false report. What is critical is that given a ranking $P_i^{☺}$ for i and rankings $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n$ for the remaining individuals, there is no $P_i^{☹}$ such that

$$\mathbf{C}(P_1, \ldots, P_i^{☹}, \ldots, P_n) \; P_i^{☺} \; \mathbf{C}(P_1, \ldots, P_i^{☺}, \ldots, P_n).$$

The next axiom, also referred to as *strategy-proofness,* precludes the possibility of manipulation for a social choice rule satisfying No-ties.

Non-manipulability There do not exist a profile $(P_1, \ldots, P_n)$, an individual i, and a preference relation P'_i such that

$$\mathbf{C}(P_1, \ldots, P'_i, \ldots, P_n) \; P_i \; \mathbf{C}(P_1, \ldots, P_i, \ldots, P_n).$$

This condition rules out the P_i-to-P'_i manipulation described above. Less obviously, it also rules out a P'_i-to-P_i manipulation. Why? Such a manipulation would yield a profile $(P_1, \ldots, P'_i, \ldots, P_n)$, an individual i, and a relation P_i such that

$$\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n) \; P'_i \; \mathbf{C}(P_1, \ldots, P'_i, \ldots, P_n),$$

which contradicts Non-manipulability: that condition precludes any such profile, individual, and preference relation; the notation we use for those objects (in particular whether P_i has a prime or not) is irrelevant.

The axiom of Non-manipulability cannot technically be applied to the Copeland social choice rule, because it does not generally satisfy No-ties, but Copeland can be combined with a tie-breaking procedure, and the above example shows that it violates the Non-manipulability axiom, regardless of how ties are broken. Furthermore, we have seen that Borda elimination is manipulable when there are seven individuals and five alternatives, again independent of how ties are broken.

The next theorem, which occupies a central role in social choice theory next to Arrow's impossibility theorem, shows that the above examples are extremely general with respect to the number of individuals, the number of alternatives, and the social choice rule, as long as it satisfies No-ties. For such a rule, it establishes the inconsistency of No-dictator, Pareto, and Non-manipulability. Equivalently, if a social choice rule satisfies No-ties, Pareto, and Non-manipulability, then some individual must be a dictator.

Theorem 4.1 (Gibbard-Satterthwaite)**.** *Assume $\#A \geq 3$ and Linear domain. There is no social choice rule satisfying No-ties, No-dictator, Pareto, and Non-manipulability.*

The remainder of this section consists of proving the Gibbard-Satterthwaite theorem. We consider an arbitrary social choice rule $\mathbf{C}$ satisfying No-ties, Pareto, and Non-manipulability, and we prove that it must make some individual a dictator (so $\mathbf{C}$ always chooses her top-ranked alternative). We do so using Arrow's theorem. Specifically, (1) we use $\mathbf{C}$ to define a particular social preference rule, $\mathbf{P}$; (2) we then prove that $\mathbf{P}$ satisfies Arrow's axioms of Pareto, IIA, and Rationality; (3) by Arrow's theorem, the social preference rule $\mathbf{P}$ makes some individual i a dictator (so $\mathbf{P}$ always produces i's ranking); and (4) we show that i is a dictator for the social choice rule $\mathbf{C}$.

To define $\mathbf{P}$, we need some terminology. Given alternatives x and y and a profile $(P_1, \ldots, P_n)$ of linear orderings, we say the profile $(P'_1, \ldots, P'_n)$ is an *xy-twin* if $(P_1, \ldots, P_n)$ if each individual's preference is unchanged across the two profiles, i.e.,

$$\begin{aligned} \{i \in N \mid xP_iy\} &= \{i \in N \mid xP'_iy\} \\ \{i \in N \mid yP_ix\} &= \{i \in N \mid yP'_ix\}. \end{aligned}$$

As well, $(P'_1, \ldots, P'_n)$ is an *xy-lifting* of $(P_1, \ldots, P_n)$ if it is an xy-twin and each individual ranks x and y above all other alternatives, i.e., for all i and all $z \in X \backslash \{x, y\}$, we have xP'_iz and yP'_iz.

We are now ready to define $\mathbf{P}$. Given a profile $(P_1, \ldots, P_n)$ of linear orders and distinct alternatives x and y, we define $x\mathbf{P}(P_1, \ldots, P_n)y$ to hold if and only if x is the choice from every xy-lifting of $(P_1, \ldots, P_n)$, i.e., for every xy-lifting $(P'_1, \ldots, P'_n)$, we have

$$\mathbf{C}(P'_1, \ldots, P'_n) = x.$$

The relation $\mathbf{P}(P_1, \ldots, P_n)$ is asymmetric (since the xy-liftings are the same as the yx-liftings), so this social preference relation is well-defined. To prove that $\mathbf{P}$ satisfies Pareto, consider any x and y and any profile $(P_1, \ldots, P_n)$ such that xP_iy for all i. Let $(P'_1, \ldots, P'_n)$ be any xy-lifting of $(P_1, \ldots, P_n)$, and note that x is top-ranked in each weak order P'_i. By No-ties, $\mathbf{C}$ chooses a single alternative at profile $(P'_1, \ldots, P'_n)$, and by Pareto, the only possibility is $\mathbf{C}(P'_1, \ldots, P'_n) = x$. We conclude that $x\mathbf{P}(P_1, \ldots, P_n)y$, as required.

To prove that $\mathbf{P}$ satisfies IIA, consider any x and y, and any xy-twins $(P_1, \ldots, P_n)$ and $(P'_1, \ldots, P'_n)$. We need to prove that the social preference between x and y is the same for the two profiles. Actually, this is obvious: the two profiles have the same xy-liftings, so the social preference between x and y is the same. For a more detailed argument, we consider three possibilities: (1) $x\mathbf{P}(P_1, \ldots, P_n)y$, (2) $y\mathbf{P}(P_1, \ldots, P_n)x$, or (3) $x\mathbf{I}(P_1, \ldots, P_n)y$. We must argue that the social preference between x and y given $(P'_1, \ldots, P'_n)$ is the same in all three cases. In the first case, consider any xy-lifting $(\overline{P}_1, \ldots, \overline{P}_n)$ of $(P'_1, \ldots, P'_n)$. Note that $(\overline{P}_1, \ldots, \overline{P}_n)$ is also an xy-lifting of $(P_1, \ldots, P_n)$; and $x\mathbf{P}(P_1, \ldots, P_n)y$ implies $\mathbf{C}(\overline{P}_1, \ldots, \overline{P}_n) = x$; and therefore, we have $x\mathbf{P}(P'_1, \ldots, P'_n)y$. In the second case, an analogous argument (with x and y switched) yields $y\mathbf{P}(P'_1, \ldots, P'_n)x$. In the third case, suppose either $x\mathbf{P}(P'_1, \ldots, P'_n)y$ or $y\mathbf{P}(P'_1, \ldots, P'_n)x$. But then analogous arguments (with $(P_1, \ldots, P_n)$ and $(P'_1, \ldots, P'_n)$ switched) would imply $x\mathbf{P}(P_1, \ldots, P_n)y$ or $y\mathbf{P}(P_1, \ldots, P_n)x$, in both cases contradicting $x\mathbf{I}(P_1, \ldots, P_n)y$. We conclude that $x\mathbf{I}(P'_1, \ldots, P'_n)y$, as required.

So far, we have not used the assumption that $\mathbf{C}$ satisfies Non-manipulability. We now use that axiom to establish social Rationality; in fact, we make use only of the following implication of Non-manipulability. First, we say that $\mathbf{C}$ is *monotonic* if for all x, all $(P_1, \ldots, P_n)$, and all $(P'_1, \ldots, P'_n)$ such that $\mathbf{C}(P_1, \ldots, P_n) = x \neq \mathbf{C}(P'_1, \ldots, P'_n)$, there are an individual i and an alternative y such that xP_iy and yP'_ix. Equivalently, if $\mathbf{C}(P_1, \ldots, P_n) = x$, and if $(P'_1, \ldots, P'_n)$ is another profile such that x does not move down relative to any alternative in any individual's ranking, then x is still chosen: $\mathbf{C}(P'_1, \ldots, P'_n) = x$. Thus, monotonicity is similar to the axiom of Positive association introduced above, but it is a more stringent requirement: if x is chosen and x does not move down relative to any alternatives, it must remain the choice, even if individual preferences over alternatives other than x are not the same. It turns out that Non-manipulability entails monotonicity.

Lemma 4.2. *Assume Linear domain. If* $\mathbf{C}$ *satisfies No-ties and Non-manipulability, then it is monotonic.*

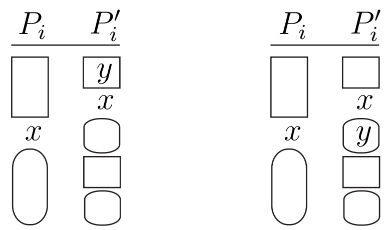


Figure 2: Pair of rankings

Proof. Consider any x, any profile $(P_1, \ldots, P_n)$ with $\mathbf{C}(P_1, \ldots, P_n) = x$, and any profile $(P'_1, \ldots, P'_n)$ such that x moves down relative to no other alternatives in any individual's ranking. To show that $\mathbf{C}(P'_1, \ldots, P'_n) = x$, we argue that the choice remains x when we change the individuals' rankings from P_i to P'_i, one at a time:

$$\begin{aligned}
\mathbf{C}(P_1, P_2, P_3, \ldots, P_n) &= x \\
\mathbf{C}(P'_1, P_2, P_3, \ldots, P_n) &= x \\
\mathbf{C}(P'_1, P'_2, P_3, \ldots, P_n) &= x \\
&\vdots \\
\mathbf{C}(P'_1, P'_2, P'_3, \ldots, P'_n) &= x.
\end{aligned}$$

Technically, we use an induction argument. Clearly, the first equation above holds by assumption. Supposing that

$$\mathbf{C}(P'_1, \ldots, P'_{i-1}, P_i, P_{i+1}, \ldots, P_n) = x,$$

we must show that the equality holds after we replace P_i with P'_i. Otherwise, the social choice after replacement is some alternative $y \neq x$. We will deduce a contradiction in both of the two possible cases: either yP'_ix or xP'_iy. The first case is pictured in the first pair of rankings in Figure 2. Recall that in replacing P_i with P'_i, no alternative below x in P_i moves above it in P'_i, so the "circled" alternatives in P_i remain below x in P'_i, and the "boxed" alternatives in P'_i are above x in P_i. But then yP_ix, and this is a P_i-to-P'_i manipulation, contradicting Non-manipulability. The second case is pictured in the second pair of rankings in Figure 2. Now we have xP'_iy, but then this is a manipulation in the other direction—a P'_i-to-P_i manipulation—again contradicting Non-manipulability. Therefore, $\mathbf{C}$ chooses x after replacing P_i with P'_i, and by induction, $\mathbf{C}(P'_1, \ldots, P'_n) = x$. We conclude that $\mathbf{C}$ is indeed monotonic. □

We now argue that $\mathbf{P}$ satisfies Arrow's social Rationality axiom. Thus, given an arbitrary profile $(P_1, \ldots, P_n)$, we must show that $\mathbf{P}(P_1, \ldots, P_n)$ is negatively transitive. So consider any alternatives, x, y, and z, and assume that $x\mathbf{P}(P_1, \ldots, P_n)z$. Construct a profile $(\overline{P}_1, \ldots, \overline{P}_n)$ by moving x, y, and z to the top of each individual's ranking, preserving their relative rankings for each individual. By Pareto, the social choice is then one of these three alternatives. Note that $\mathbf{C}(\overline{P}_1, \ldots, \overline{P}_n) \neq z$, for suppose otherwise, and consider an arbitrary xz-lifting $(P'_1, \ldots, P'_n)$ of $(P_1, \ldots, P_n)$. Compared to $(\overline{P}_1, \ldots, \overline{P}_n)$, no alternative below z in any $\overline{P}_i$ moves above z in P'_i, so monotonicity implies $\mathbf{C}(P'_1, \ldots, P'_n) = z$, but $x\mathbf{P}(P_1, \ldots, P_n)z$ implies $\mathbf{C}(P'_1, \ldots, P'_n) = x$, a contradiction. Thus, there are two possibilities. First, $\mathbf{C}(\overline{P}_1, \ldots, \overline{P}_n) = x$. Then x must be chosen for all xy-liftings of $(P_1, \ldots, P_n)$. Indeed, this follows directly from monotonicity: consider an arbitrary xy-lifting $(P'_1, \ldots, P'_n)$ of $(P_1, \ldots, P_n)$, and compare this to $(\overline{P}_1, \ldots, \overline{P}_n)$; since no alternative below x in any $\overline{P}_i$ moves above x in P'_i, monotonicity implies $\mathbf{C}(P'_1, \ldots, P'_n) = x$. Then by definition of $\mathbf{P}$, we have $x\mathbf{P}(P_1, \ldots, P_n)y$. Second, $\mathbf{C}(\overline{P}_1, \ldots, \overline{P}_n) = y$, in which case an analogous argument (with y in the role of x and z in the role of y) yields $y\mathbf{P}(P_1, \ldots, P_n)z$. We conclude that $\mathbf{P}$ satisfies Rationality, as required.

We then apply Arrow's theorem to deduce the existence of a dictator, say i, for the social preference rule $\mathbf{P}$, i.e., social preferences always agree with i's preferences. The final step of the argument is to prove that i is a dictator for $\mathbf{C}$, i.e., $\mathbf{C}$ always chooses i's top-ranked alternative. To this end, consider any profile $(P_1, \ldots, P_n)$ and any alternative x that is not top-ranked by i; we will prove that $\mathbf{C}$ cannot choose x. Suppose, to deduce a contradiction, that $\mathbf{C}(P_1, \ldots, P_n) = x$. By assumption, there is an alternative y such that yP_ix, and since i is a dictator for $\mathbf{P}$, we have $y\mathbf{P}(P_1, \ldots, P_n)x$. Letting $(P'_1, \ldots, P'_n)$ be any xy-lifting of $(P_1, \ldots, P_n)$, this implies that $\mathbf{C}(P'_1, \ldots, P'_n) = y$. But no alternative below x in any P_i moves above x in P'_i, and monotonicity implies $\mathbf{C}(P'_1, \ldots, P'_n) = x$, a contradiction. Therefore, i is a dictator for $\mathbf{C}$. We have shown that if a social choice rule satisfies No-ties, Pareto, and Non-manipulability, then it violates Non-dictatorship, completing the proof.

5 Digression on Monotonicity

Recall that a social choice rule $\mathbf{C}$ is *monotonic* if for all x, all $(P_1, \ldots, P_n)$, and all $(P'_1, \ldots, P'_n)$ such that $\mathbf{C}(P_1, \ldots, P_n) = x \neq \mathbf{C}(P'_1, \ldots, P'_n)$, there are an individual i and an alternative y such that xP_iy and yP'_ix. Equivalently, if $\mathbf{C}(P_1, \ldots, P_n) = x$, and if $(P'_1, \ldots, P'_n)$ is another profile such that x does not move down relative to any alternative in any individual's ranking, then x is still chosen: $\mathbf{C}(P'_1, \ldots, P'_n) = x$.

Lemma 4.2 shows that under Linear domain, Non-manipulability implies monotonicity. In fact, a result due to Muller and Satterthwaite (1977) establishes exact equivalence between these two conditions.

Proposition 5.1 (Muller and Satterthwaite)**.** *Assume Linear domain. A social choice rule* $\mathbf{C}$ *satisfies Non-manipulability if and only if it is monotonic.*

One direction if this proposition is already established in Lemma 4.2. For the other, assume $\mathbf{C}$ is monotonic. Consider a profile $(P_1, \ldots, P_n)$, an individual i, and a linear order P'_i, and let $\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n) = x$ and $\mathbf{C}(P_1, \ldots, P'_i, \ldots, P_n) = y$. Suppose, in order to deduce a contradiction, that yP_ix. The argument proceeds by transforming P_i to P'_i in a number of steps. First, construct $\hat{P}_i$ by rearranging all alternatives above x in P_i so that they are ordered as in P'_i, and similarly rearranging all alternatives below x in P_i so that they are ordered as in P'_i. In this step, the ranking of each alternative relative to x is unchanged, so monotonicity implies that $\mathbf{C}(P_1, \ldots, \hat{P}_i, \ldots, P_n) = x$. The linear order $\hat{P}_i$ is now identical to P'_i except that some alternatives above x in $\hat{P}_i$ are below it in P'_i, and some alternatives below x in $\hat{P}_i$ are above it in P'_i. Denote the former set of alternatives by Y and the latter by Z.

Second, construct $\tilde{P}_i$ by moving elements of Y so that they are located below x, and arranging all alternatives below x in $\tilde{P}_i$ so that they are ordered as in P'_i. No alternative that was initially below x has moved above it, so monotonicity implies $\mathbf{C}(P_1, \ldots, \tilde{P}_i, \ldots, P_n) = x$. Now $\tilde{P}_i$ is identical to P'_i except that the alternatives in Z are below x in $\tilde{P}_i$ but above it in P'_i.

Finally, we move from $\tilde{P}_i$ to P'_i by moving the alternatives in Z so that they are located above x, and arranging all alternatives above x so that they are ordered as in P'_i; this gives us P'_i itself. But recall that yP_ix, so in particular, $y \notin Z$, and recall that $\mathbf{C}(P_1, \ldots, P'_i, \ldots, P_n) = y \neq x$. Thus, in moving the reverse direction from P'_i to $\tilde{P}_i$, no alternative moves up relative to y, so monotonicity implies $\mathbf{C}(P_1, \ldots, \tilde{P}_i, \ldots, P_n) = y$, a contradiction.

6 Manipulation with Ties

The Gibbard-Satterthwaite theorem uses the No-ties axiom, despite the fact that all interesting real-world rules do create ties in some situations. We argued above that we could get around this by adding a deterministic tie-breaking rule to any social choice rule, but that approach is not entirely satisfactory; it leaves open the possibility that by choosing non-singleton sets (perhaps very small) in some situations (possibly only a few), we could find social choice rules that preclude the possibility of manipulation in an appropriately defined sense. In fact, plurality rule cannot possibly be manipulated without making or breaking a tie, because moving an alternative to the top of one individual's ranking only increases its plurality score by only one. Adding a tie-breaking procedure to plurality rule, the Gibbard-Satterthwaite theorem implies that the combined rule is manipulable; but it is important to know that the possibility of manipulation is not an artifact of how ties are broken.

Below is an example of what a manipulation might look like when a false report creates a tie.

P_1	P_2	P_3	P_4
a	b	c	c
b	c	a	a
c	a	b	b

Here, c has the highest plurality score and is chosen by that method. Individuals 3 and 4 cannot manipulate, of course, because they are enjoying their favorite alternative. If individual 1 raises b to the top of her ranking, reporting $\begin{smallmatrix} b \\ a \\ c \end{smallmatrix}$, then this changes the set of plausible choices according to Plurality rule from $\{c\}$ to $\{b, c\}$. Here, it seems fairly evident that changing the choice set from $\{c\}$, which is individual 1's worst alternative, to $\{b, c\}$ is profitable; even if the individual believes the outcome will be b with only a small probability, that is better than c for sure.

For an example of a false report breaking a tie, consider the preferences below.

P_1	P_2	P_3
a	b	c
b	c	a
c	a	b

Now the plurality choice set is $\{a, b, c\}$, but individual 1 can report $\begin{smallmatrix} b \\ a \\ c \end{smallmatrix}$ to obtain the choice set $\{b\}$. In contrast to the first example, it is not clear whether this change is profitable for individual 1. Whether b for sure is better than the three-way tie $\{a, b, c\}$ depends on individual 1's assessment of a and c versus b and on the probabilities she assigns to the three alternatives in case of a tie.

It is no longer always unambiguous whether an individual will profit from falsely reporting her ranking, as preferences are defined over alternatives—not sets of alternatives. In fact, whether a change in the social choice set is profitable often depends, as in the latter example, on an individual's assessment of how ties are ultimately resolved. This can be subjective. Even if a tie is supposed to be broken by the toss of a fair coin—an objective event—individuals may have doubts as to whether the lottery will be executed properly. (In real-world elections, one might have concerns about fraud or suspicions that a close election will be decided in court.)

In sum, an individual's judgment of the profitability of a false report depends on her outlook. For simplicity, we focus on two polar extremes: the optimist and the pessimist. When an optimist compares two possible choice sets, C and C', she does not know which alternative in C will be chosen, but she assumes the best; similarly for C'. So C' is better than C for an *optimist* when

$$\begin{bmatrix} \text{best element} \\ \text{in } C' \end{bmatrix} \; P_i \; \begin{bmatrix} \text{best element} \\ \text{in } C \end{bmatrix}.$$

And C' is better than C for a *pessimist* when

$$\begin{bmatrix} \text{worst element} \\ \text{in } C' \end{bmatrix} \; P_i \; \begin{bmatrix} \text{worst element} \\ \text{in } C \end{bmatrix}.$$

Referring to the latter plurality example, recall voter 1's true ranking was $\substack{a\\b\\c}$. Then $\{b\}$ is better than $\{a, b, c\}$ for a pessimist, but not an optimist. Referring to the former plurality example, recall voter 1's true ranking was $\substack{a\\b\\c}$. Then $\{b, c\}$ is better than $\{c\}$ for an optimist, but not a pessimist. In general, suppose the preference profile is $(P_1, \ldots, P_n)$, and consider a different ranking P_i' for individual i. If

$$\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n) P_i \mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)$$

for either an optimist or for a pessimist, then we view this as a possible manipulation.

The next axiom extends the original Non-manipulability axiom to account for the possibility of ties. As before, it precludes the possibility of a manipulation that changes the social choice from one unique alternative to another, preferred unique alternative, but it covers the possibility that a social choice rule generates sets of choices rather than unique alternatives. Given a finite set Y of alternatives and a linear order P, we write "$\max[Y|P]$" and "$\min[Y|P]$" for the top- and bottom-ranked alternatives in Y according to P.

Non-manipulability There do not exist a profile $(P_1, \ldots, P_n)$, an individual i, and a ranking P_i' such that

$$\max[\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)|P_i] P_i \max[\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)|P_i]$$

or

$$\min[\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)|P_i] P_i \min[\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)|P_i]$$

That is, Non-manipulability precludes the possibility that $\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)$ is strictly preferred to $\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)$ given P_i for an optimist or for a pessimist.

Equivalently, the axiom requires that for all profiles $(P_1, \ldots, P_n)$ and all individuals i, we have

$$\max[\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)|P_i] R_i \max[\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)|P_i]$$

and

$$\min[\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)|P_i] R_i \min[\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)|P_i].$$

The new version of the axiom applies to any social choice rule (even those that produce ties), but note that when a social choice rule satisfies the No-ties axiom, the new definition of non-manipulability coincides with the old one. In this sense, it extends the original definition.

Before re-establishing the Gibbard-Satterthwaite theorem, we require a further axiom that demands singleton social choice sets in very specific situations: when all but one individual rank the same alternative above all others.

No-gridlock For every profile $(P_1, \dots, P_n)$ and all alternatives x and y such that 1) every individual ranks x and y above all other alternatives,

$$\#\{i \in N \mid \text{ for all } z \neq x, y,\ xP_iz \text{ and } yP_iz\} \quad = \quad n,$$

and 2) at least $n-1$ individuals rank x first, i.e.,

$$\#\{i \in N \mid xP_iy\} \quad \geq \quad n-1,$$

we have $\#\mathbf{C}(P_1, \dots, P_n) = 1$.

Note that Pareto already demands a unique social choice when all individuals have x atop their rankings, so No-gridlock is to some extent redundant. The idea is that typically, when x is top-ranked by exactly $n-1$ individuals (so Pareto doesn't apply), the unique choice will be x itself; but the axiom allows for the possibility that there are cases in which the one individual who ranks y above x may have the authority to decide between the two alternative. In any event, though the axiom is somewhat restrictive when there are just three individuals, it becomes exceedingly weak when n is large; indeed, it is satisfied by all of the social choice rules we have seen. Finally, the axiom is implied by No-ties, as the latter axiom demands singleton choice sets for all profiles.

We can now extend the Gibbard-Satterthwaite theorem to allow ties. An equivalent statement of the result is that if a social choice rule satisfies No-gridlock, No-dictator, and Pareto, then it violates Non-manipulability, i.e., it creates the possibility of a manipulation in some situations. Since all of the social choice rules we have seen satisfy the three antecedent axioms in this statement, an immediate implication of the theorem is that regardless of the number of alternatives or the number of individuals, the rules are manipulable.

Theorem 6.1. *Assume $\#A \geq 3$ and Linear domain. There is no social choice rule satisfying No-gridlock, No-dictator, Pareto, and Non-manipulability.*

Again, No-ties implies No-gridlock, so this version of Gibbard-Satterthwaite indeed generalizes the original. The proof involves the construction of a social preference rule satisfying certain axioms, as in the proof of the original, but it is somewhat more involved, and we omit it.

There is a noteworthy extension of Theorem 6.1. Rather than Pareto, we can require the following much weaker axiom.

Citizens' sovereignty For each alternative x, there is a profile $(P_1, \ldots, P_n)$ such that $x \in \mathbf{C}(P_1, \ldots, P_n)$.

That is, each alternative is a viable choice for some profile. This is implied by Pareto, for the latter axiom demands that x be chosen whenever it is top-ranked by all individuals. In fact, Pareto then requires that x is the unique choice. Citizens' sovereignty, in contrast, carries no implication of uniqueness and is satisfied by the trivial social choice function that always chooses the entire set of alternatives.

Theorem 6.2 (Duggan and Schwartz). *Assume $\#A \geq 3$ and Linear domain. There is no social choice rule satisfying No-gridlock, No-dictator, Citizens' sovereignty, and Non-manipulability.*

Finally, we note an equivalent formulation of Non-manipulability in terms of expected utility with respect to subjective beliefs. For simplicity, we assume the set of alternatives is finite, though this is not technically required for the theorem. In the following proposition, the set of alternatives is finite, and the *support* of a lottery λ over alternatives consists of every alternative x such that $\lambda(x) > 0$. Given a linear order P, we say a von Neumann-Morgenstern representation u is *compatible* with P if it is a valid ordinal representation, i.e., for all $x, y \in X$, $u(x) > u(y)$ if and only if xPy.

Proposition 6.3. *Assume A is finite. The following are equivalent for a social choice rule $\mathbf{C}$, a profile $(P_1, \ldots, P_n)$ of linear orders, an individual i, and an ordering P_i'.*

(a) either

$$\max[\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)|P_i] \ P_i \ \max[\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)|P_i]$$

or

$$\min[\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)|P_i] \ P_i \ \min[\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)|P_i]$$

(b) for every lottery λ with support equal to $\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)$ and every lottery λ' with support equal to $\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)$, there is a von Neumann-Morgenstern representation u compatible with P_i such that

$$\sum_{x \in A} \lambda'(x) u(x) \ > \ \sum_{x \in A} \lambda(x) u(x).$$

Proof. First, suppose (a) holds. Let the best element of $\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)$ according to P_i be x, and let x' be the best element of $\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)$ according to P_i. For now, assume that the latter set is preferred to the former for an optimist, so $x' P_i x$. Now consider any lotteries λ and λ' as in (b). Let $\tilde{u}$ be any ordinal representation of P_i. If it is the case that

$$\sum_{x \in A} \lambda'(x) \tilde{u}(x) \ > \ \sum_{x \in A} \lambda(x) \tilde{u}(x),$$

then set $u = \tilde{u}$. Otherwise, we obtain u from $\tilde{u}$ by adding $\alpha > 0$ to the utility from x' and every alternative ranked above it: for all y, define

$$u(y) \;=\; \begin{cases} \tilde{u}(y) + \alpha & \text{if } yR_ix' \\ u(y) & \text{if } x'P_iy. \end{cases}$$

Choosing α sufficiently high, we fulfill (b). A similar argument applies when the worst element of $\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)$ is preferred according to P_i to the worst element of $\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)$. Now assume (b) holds. Suppose in order to deduce a contradiction that (a) fails. Let λ' be the even chance lottery on $\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)$. To define λ, let x be the top-ranked element of $\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)$ according to P_i, let z be the bottom-ranked element of $\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)$, and let

$$\begin{aligned} \#\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n) &= k \\ \#\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n) &= m. \end{aligned}$$

Define λ so that $\lambda(x) = \frac{m-1}{m}$ and for all $y \in \mathbf{C}(P_1, \ldots, P_i, \ldots, P_n) \setminus \{x\}$, we have $\lambda(y) = \frac{1}{m(k-1)}$. These numbers are positive and sum to one. Furthermore, since (a) fails, this specification is equivalent to (1) moving probability $\frac{1}{m}$ from z, the bottom-ranked alternative in $\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)$, and distributing it equally over the elements of $\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)$ other than x, all of which are at least as good as z, and (2) moving the remaining probability $\frac{m-1}{m}$ from the other alternatives in $\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)$ to x, which is at least as good as those alternatives. It follows that

$$\sum_{x \in A} \lambda(x)u(x) \;\geq\; \sum_{x \in A} \lambda'(x)u(x).$$

This contradicts (b), as required. □

It is instructive to revisit the meaning of the Non-manipulability axiom in light of Proposition 6.3. The axiom requires that for all profiles $(P_1, \ldots, P_n)$, all individuals i, and all linear orders P_i', there exist lotteries λ and λ', with supports equal to $\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)$ and $\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)$, respectively, such that for all von Neumann-Morgenstern representations u compatible with P_i, we have

$$\sum_{x \in A} \lambda(x)u(x) \;\geq\; \sum_{x \in A} \lambda'(x)u(x).$$

So in one sense the axiom is very permissive: given a profile, an individual, and a false report, it just requires that a certain preference restriction hold for at least one pair of lotteries.

7 Manipulation with Lotteries

We now return to the assumption that the social choice rule is single-valued, but now we allow for a profile of preferences to determine a *lottery* over alternatives. We denote by $\mathbf{\Gamma}(P_1,\ldots,P_n)$ the lottery associated with the profile $(P_1,\ldots,P_n)$, and we refer to the mapping $\mathbf{\Gamma}$ as a *stochastic social choice rule* to distinguish it from our original framework. We let $\mathbf{\Gamma}(x|P_1,\ldots,P_n)$ be the probability of alternative x under the lottery $\mathbf{\Gamma}(P_1,\ldots,P_n)$. One way of constructing a stochastic social choice rule is to begin with a non-stochastic rule $\mathbf{C}$, and to then define $\mathbf{\Gamma}(P_1,\ldots,P_n)$ as the even-chance lottery over $\mathbf{C}(P_1,\ldots,P_n)$. It should also be clear that each social choice rule satisfying the No-ties axiom can be put into a one-to-one correspondence with a stochastic social choice rule: the rule $\mathbf{C}$ is associated with $\mathbf{\Gamma}$ such that for all profiles and alternatives, $\mathbf{C}(P_1,\ldots,P_n) = x$ if and only if $\mathbf{\Gamma}(x|P_1,\ldots,P_n) = 1$. Because the lottery thus determined by $\mathbf{\Gamma}$ always puts probability one on a single alternative, we might refer to it as "degenerate."

Again, it is still not always clear what constitutes a manipulation and what does not.

1	2	3
a	b	c
b	a	a
c	c	b

Given the above profile, plurality would choose the set $\{a,b,c\}$, and the associated even-chance stochastic social choice rule chooses each alternative with probability one third. If individual 3 reports a ranking with a at the top, then a is chosen with probability one. Is this change beneficial for the individual? That depends on how she evaluates lotteries.

Assuming the individual has expected utility preferences over lotteries, with von Neumann-Morgenstern representation u, the answer depends on whether

$$\frac{u_3(c)+u_3(b)}{2} \quad < \quad u_3(a)$$

or the reverse inequality holds: if it holds, then the change is beneficial and the individual can manipulate; otherwise, the change is not beneficial, and this is not a manipulation.

This leads to the next definition of the Non-manipulability axiom. As before, the spirit of the axiom is to preclude the possibility, the sort from the preceding example, of a manipulation.

Non-manipulability There do not exist a profile $(P_1,\ldots,P_n)$, an individual i, a linear order P'_i, and a von Neumann-Morgenstern representation u compatible with P_i such that

$$\sum_{x \in A} \mathbf{\Gamma}(x|P_1, \ldots, P_i', \ldots, , P_n)u(x) \;\; > \;\; \sum_{x \in A} \mathbf{\Gamma}(x|P_1, \ldots, P_i, \ldots, P_n)u(x).$$

By Proposition 6.3, this definition extends the concept from the previous section. That axiom requires that for all $(P_1, \ldots, P_n)$, all i, all P_i', and all von Neumann-Morgenstern representations u compatible with P_i, we have

$$\sum_{x \in A} \lambda(x)u(x) \;\; \geq \;\; \sum_{x \in A} \lambda'(x)u(x).$$

Given a profile, an individual, and a false report, the current axiom restricts payoffs from two particular lotteries, $\mathbf{\Gamma}(P_1, \ldots, P_n)$ and $\mathbf{\Gamma}(P_1, \ldots, P_i', \ldots, P_n)$. Compared to the definition from the previous section, which requires that a preference restriction hold for at least one pair of lotteries, the current axiom is more restrictive. This allows us to extend the Gibbard-Satterthwaite theorem to stochastic social choice rules once we extend our earlier axioms. Note that the definitions below, when applied to degenerate stochastic social choice rules, coincide with the definitions of the axioms in the original framework.

As before, the No-dictator axiom stipulates that there is no individual whose top-ranked alternative is always the unique choice.

No-dictator There is no individual i such that for all x and y and every profile $(P_1, \ldots, P_n)$, xP_iy implies $\mathbf{\Gamma}(y|P_1, \ldots, P_n) = 0$.

We can define a Pareto axiom similar to the original…

Pareto For all alternatives x and y and all profiles $(P_1, \ldots, P_n)$, if xP_iy for all i, then $\mathbf{\Gamma}(x|P_1, \ldots, P_n) = 0$.

…and we can weaken it to Citizens' sovereignty.

Citizens' sovereignty For each alternative x, there is a profile $(P_1, \ldots, P_n)$ such that $\mathbf{\Gamma}(x|P_1, \ldots, P_n) > 0$.

Finally, we re-define the No-gridlock axiom in stochastic terms.

No-gridlock For every profile $(P_1, \ldots, P_n)$ and all alternatives x and y such that 1) every individual ranks x and y above all other alternatives,

$$\#\{i \in N \mid \text{ for all } z \neq x, y, \, xP_iz \text{ and } yP_iz\} \;\; = \;\; n,$$

and 2) at at least $n - 1$ individuals rank x first, i.e.,

$$\#\{i \in N \mid xP_iy\} \;\; \geq \;\; n - 1,$$

there exists z satisfying $\mathbf{\Gamma}(z|P_1, \ldots, P_n) = 1$.

Having completed the task of reformulating our axioms, it is a simple matter to extend our generalized version of the Gibbard-Satterthwaite theorem.

Theorem 7.1. *Assume $\#A \geq 3$ and Linear domain. There is no stochastic social choice rule satisfying No-gridlock, No-dictator, Citizens' sovereignty, and Non-manipulability.*

Proof. Consider any stochastic social choice rule $\mathbf{\Gamma}$ that satisfies No-gridlock, No-dictator, and Citizens' sovereignty. Now define the (non-stochastic) social choice rule $\mathbf{C}$ that, for each preference profile, chooses the support set of the lottery determined by $\mathbf{\Gamma}$, i.e.,

$$\mathbf{C}(P_1, \ldots, P_n) \quad = \quad \{x \mid \mathbf{\Gamma}(x|P_1, \ldots, P_n) > 0\}.$$

It is easy to see that $\mathbf{C}$ satisfies the original formulations of No-gridlock, No-dictator, and Citizens' sovereignty in the previous section. By Theorem 6.2, it therefore violates Non-manipulability. By Proposition 6.3, there are a profile $(P_1, \ldots, P_n)$, an individual i, and an ordering P_i' such that: for every lottery λ with support equal to $\mathbf{C}(P_1, \ldots, P_i, \ldots, P_n)$ and every lottery λ' with support equal to $\mathbf{C}(P_1, \ldots, P_i', \ldots, P_n)$, there is a von Neumann-Morgenstern representation u compatible with P_i such that

$$\sum_{x \in A} \lambda'(x) u(x) \quad > \quad \sum_{x \in A} \lambda(x) u(x).$$

In particular, given our construction of $\mathbf{C}$, the above inequality holds for the lotteries $\lambda = \mathbf{\Gamma}(P_1, \ldots, P_i, \ldots, P_n)$ and $\lambda' = \mathbf{\Gamma}(P_1, \ldots, P_i', \ldots, P_n)$. Therefore, $\mathbf{\Gamma}$ violates Non-manipulability, as required. ☐

In fact, more is known than this, for if we strengthen Citizens' sovereignty to Pareto, we can drop the No-gridlock and No-dictator axioms and give a complete characterization of the class of stochastic social choice rules satisfying the Non-manipulability axiom.

Theorem 7.2 (Gibbard)**.** *Assume $\#A \geq 3$ and Linear domain. A stochastic social choice rule satisfies Pareto and Non-manipulability if and only if there exist fixed weights $p_1, \ldots, p_n \geq 0$ with $\sum_{i=1}^n p_i = 1$ such that for all profiles $(P_1, \ldots, P_n)$ and all alternatives x,*

$$\mathbf{\Gamma}(x|P_1, \ldots, P_n) \quad = \quad \sum_{i=1}^{n} p_i I_{G(x)}(i),$$

where $G(x) = \{i \mid \text{ for all } y \neq x,\ xP_iy\}$ is the set of individuals who rank x first and $I_{G(x)}(\cdot)$ is the indicator function for this set.

The previous theorem defines a class of stochastic social choice rules, called *random dictatorships,* parameterized by a vector $(p_1, \ldots, p_n)$ of weights. Since these weights are non-negative and sum to one, we can interpret this vector as a probability distribution over the set of individuals, and we can interpret a random dictatorship rule in terms of a two-step process: first, we randomize over individuals according to the distribution $(p_1, \ldots, p_n)$; and given a realized individual i from the first step, we then select i's top-ranked alternative as the social choice.

From the general perspective of Theorem 7.2, we can see how dictatorship arises once we add the No-gridlock axiom: if a stochastic social choice rule satisfies No-gridlock, Pareto, and Non-manipulability, then it is a random dictatorship; and there cannot be two individuals, say $i = 1, 2$, with positive weights, for then a profile like

P_1	P_2	$\cdots$	P_n
x	y		y
$\vdots$	$\vdots$		$\vdots$
y	x		x

would produce a lottery with $\mathbf{\Gamma}(x|P_1, \ldots, P_n) > 0$ and $\mathbf{\Gamma}(y|P_1, \ldots, P_n) > 0$, contradicting No-gridlock.

We make one last observation in the stochastic social choice framework. Theorem 7.2 gives a very general characterization of non-manipulability, and we just saw how the addition of further structure (namely, the No-gridlock axiom) allowed us to sharpen the conclusion of the theorem from random dictatorship to dictatorship. Suppose we impose a different kind of structure: we say a stochastic social choice rule $\mathbf{\Gamma}$ is *uniform* if the lottery $\mathbf{\Gamma}(P_1, \ldots, P_n)$ is uniform for every preference profile. It turns out that the only uniform rules satisfying Pareto and Non-manipulability are "dual dictatorships," i.e., either the rule is a dictatorship or there are two individuals and for each profile, the rule chooses the top-ranked alternatives of these individuals with equal probability. Note that there is nothing that precludes $i = j$ in the following corollary, i.e., dictatorship is a special case of dual dictatorship. The next result is due to Feldman (1979).

Corollary 7.3 (Feldman)**.** *Assume $\#A \geq 3$ and Linear domain. A stochastic social choice function is uniform and satisfies Pareto and Non-manipulability if and only if it is a random dictatorship and there exist individuals i and j such that $p_i = p_j = \frac{1}{2}$ and for all $k \neq i, j$, $p_k = 0$.*

To see how one direction of the corollary follows from Theorem 7.2, consider a uniform rule $\mathbf{\Gamma}$ that satisfies Pareto and Non-manipulability. It follows from the general theorem that the rule is a random dictatorship, and we can denote the weights by $p_1, \ldots, p_n$. Suppose, contrary to the corollary, that there are three individuals, say

$i = 1, 2, 3$, with positive weights, and consider the following profile, where x, y, and z are any distinct alternatives, and all other alternatives are ranked arbitrarily below these three by all individuals.

P_1	P_2	P_3	$\cdots$	P_n
x	y	y		y
y	z	z		z
z	x	x		x

Since $\mathbf{\Gamma}$ is a random dictatorship, it only puts positive probability on the top-ranked alternatives of the individuals, so only x and y receive positive probability; and since the rule is uniform, the above profile generates the even chance lottery over x and y. But now consider the false report P_2' that ranks z above all other alternatives. Again, since the rule is a random dictatorship and the first three individuals have positive weight, this means that x, y, and z all receive positive probability; and since the rule is uniform, the profile $(P_1, P_2', P_3, \ldots, P_n)$ generates the even chance lottery over x, y, and z. But the von Neumann-Morgenstern representation u with $u_2(y) = 1$, $u_2(z) = .8$, and $u_2(x) = 0$ is compatible with P_2, and the even chance lottery over the three alternatives yields a higher expected utility than the even chance lottery over x and y, i.e.,

$$\frac{1}{3}u_2(x) + \frac{1}{3}u_2(y) + \frac{1}{3}u_2(z) \;=\; .6 \;>\; .5 \;=\; \frac{1}{2}u_2(x) + \frac{1}{2}u_2(y),$$

violating the Non-manipulability axiom.

8 Majority Rule in One Dimension

Majority rule is actually resistant to manipulations. If there are four individuals and two alternatives, as below,

P_1	P_2	P_3	P_4
x	x	y	x
y	y	x	y

then x is the outcome. Clearly, individuals 1, 2, and 4 cannot do better by reporting different preferences, and individual 3 cannot get a better outcome, because she is already voting for y. But, of course, with three or more alternatives, majority rule does not always give us a well-defined choice, because the majority core may be empty.

This observation generalizes.

Theorem 8.1. *Consider any profile $(P_1, \ldots, P_n)$, any individual i, and any weak order P_i', and suppose that x is a Condorcet winner given $(P_1, \ldots, P_i, \ldots, P_n)$ and that y is a Condorcet winner given $(P_1, \ldots, P_i', \ldots, P_n)$. Then it is not that case that yP_ix.*

This result has an immediate corollary for one-dimensional models, where $A \subseteq \mathbb{R}$ and each individual's ordering is single-peaked. Recall that P_i is *single-peaked* if there exists $\hat{x}^i \in A$ such that (i) for all $y \neq \hat{x}^i$, $\hat{x}^i P_i y$, (ii) for all y and z with $y < z < \hat{x}^i$, $z P_i y$, and (iii) for all y and z with $\hat{x}^i < y < z$, $y P_i z$. A special case is when P_i is *Euclidean,* in which case individual i prefers alternatives based only on distance from her ideal point, i.e., $x P_i y$ if and only if $|x - \hat{x}_i| < |y - \hat{x}_i|$. We say *Single-peaked domain* holds if the domain of possible preference profiles consists of all single-peaked weak orders of A, and *Euclidean domain* holds if all profiles of Euclidean preferences are possible. In the latter case, a profile $(P_1, \ldots, P_n)$ of Euclidean preferences is completely summarized by the corresponding profile $(\hat{x}^1, \ldots, \hat{x}^n)$ of ideal points.

Given the profile $(\hat{x}_1, \ldots, \hat{x}_n)$ of ideal points with n odd, we say $\hat{x}_m$ is the *median* ideal point if

$$\#\{i \mid \hat{x}_i < \hat{x}_m\} \;<\; \frac{n+1}{2} \quad \text{and} \quad \#\{i \mid \hat{x}_m < \hat{x}_i\} \;<\; \frac{n+1}{2}.$$

Given this notion, we can define the *median social choice rule,* denoted $\mathbf{C}_m$, on the domain of single-peaked profiles so that $\mathbf{C}_m(P_1, \ldots, P_n)$ is the median of the corresponding profile $(\hat{x}_1, \ldots, \hat{x}_n)$ of ideal points. Of course, Black's theorem implies that the median ideal point is the unique element of the majority core, so the notation $\mathbf{C}_m$ for the median rule is consistent with its usage above.

Corollary 8.2. *Assume $A \subseteq \mathbb{R}$, n is odd, and Single-peaked domain. Then the median social choice rule $\mathbf{C}_m$ satisfies Non-manipulability.*

Much more is known about the social choice rules satisfying Non-manipulability (and some other background conditions) when individual preferences are restricted to be single-peaked. If the weak order P_i is single-peaked, we may let $\hat{x}_i$ denote the unique ideal point of P_i satisfying $\hat{x}_i P_i y$ for all $y \in A$ distinct from $\hat{x}_i$. Then $\max\{\hat{x}_1, \ldots, \hat{x}_n\}$ is the largest of the individual ideal points, and it should be clear that the social choice rule $\mathbf{C}_1$ defined by

$$\mathbf{C}_1(P_1, \ldots, P_n) \;=\; \max\{\hat{x}_1, \ldots, \hat{x}_n\}$$

is also Non-manipulable.

More generally, given the profile $(\hat{x}_1, \ldots, \hat{x}_n)$, we say $\hat{x}_i$ is the *kth order statistic* of $(\hat{x}_1, \ldots, \hat{x}_n)$ if (i) there are less than k ideal points strictly below $\hat{x}_i$, and (ii) there are less than $n - k$ ideal points strictly above $\hat{x}_i$, i.e.,

$$\#\{j \mid \hat{x}_j \;<\; \hat{x}_i\} \;<\; k \quad \text{and} \quad \#\{j \mid \hat{x}_i < \hat{x}_j\} \;<\; n - k.$$

Of course, the median is just the $\frac{n+1}{2}$-order statistic when n is odd. By the same logic, it should be clear that the social choice rule $\mathbf{C}_k$ defined so that $\mathbf{C}_k(P_1, \ldots, P_n)$ is the kth order statistic of $(\hat{x}_1, \ldots, \hat{x}_n)$ also satisfies Non-manipulability.

More generally, fix any vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ of $n-1$ elements of A, and consider a profile $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ of ideal points. Then we can consider the median, say m^*, of $(\hat{x}_1, \ldots, \hat{x}_n, \alpha_1, \ldots, \alpha_{n-1})$. Since the total number of coordinates in $(\hat{x}_1, \ldots, \hat{x}_n, \alpha_1, \ldots, \alpha_{n-1})$ is $2n-1$, m^* is just the nth order statistic of the profile, i.e., m^* is the unique element of $\{\hat{x}_1, \ldots, \hat{x}_n, \alpha_1, \ldots, \alpha_{n-1}\}$ satisfying

$$\begin{aligned} \#\big\{i \mid \hat{x}_i < m^*\big\} + \#\big\{j \mid \alpha_j < m^*\big\} &< n \\ \#\big\{i \mid m^* < \hat{x}_i\big\} + \#\big\{j \mid m^* < \alpha_j\big\} &< n. \end{aligned}$$

Assuming single-peaked domain, we say a social choice rule $\mathbf{C}$ is a *generalized median* rule if there is a vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ of parameters such that for all profiles $(P_1, \ldots, P_n)$, we have

$$\mathbf{C}(P_1, \ldots, P_n) = \mathbf{C}_\alpha(P_1, \ldots, P_n).$$

Let $\mathbf{C}_\alpha$ be the social choice rule defined so that $\mathbf{C}_\alpha(P_1, \ldots, P_n)$ is the median of $(\hat{x}_1, \ldots, \hat{x}_n, \alpha_1, \ldots, \alpha_{n-1})$, where $(\hat{x}_1, \ldots, \hat{x}_n)$ is the profile of ideal points associated with $(P_1, \ldots, P_n)$. In the latter, the median is calculated as if the parameters $\alpha_1, \ldots, \alpha_{n-1}$ are ideal points, even though they do not correspond to actual voters; thus, it is common to refer to them as *phantom ideal points.*

Assume for simplicity that the set A of alternatives is the unit interval, i.e., $A = [0, 1]$. Note that every social choice rule $\mathbf{C}_k$ based on order statistics can be obtained as a rule $\mathbf{C}_\alpha$ based on phantom voters. In particular, we obtain $\mathbf{C}_1$ by specifying $\alpha = (1, 1, \ldots, 1)$, i.e., placing all $n-1$ phantom voters at one. Then the median of $(\hat{x}_1, \ldots, \hat{x}_n, \alpha_1, \ldots, \alpha_{n-1})$ is just the largest individual ideal point. In general, we obtain $\mathbf{C}_k$ by placing $k-1$ phantoms at zero and $n-k$ at one.

It should be clear that every generalized median rule satisfies Anonymity, No-ties, and Pareto. Of course, Anonymity implies No-dictator, except in the trivial case of a single individual. Furthermore, to compute the choice under a generalized median rule, it is only necessary to know the individual ideal points, so they satisfy the following axiom, which presumes single-peaked domain.

Peaks-only For all single-peaked profiles $(P_1, \ldots, P_n)$ and $(P'_1, \ldots, P'_n)$, if P_i and P'_i have the same ideal point for each i, then

$$\mathbf{C}(P_1, \ldots, P_n) = \mathbf{C}(P'_1, \ldots, P'_n).$$

If a social choice rule $\mathbf{C}$ satisfies Peaks-only, then we can suppress the irrelevant information contained in the individual orderings and write it as a function of ideal points alone, e.g., $\mathbf{C}(\hat{x}_1, \ldots, \hat{x}_n) = \mathbf{C}(P_1, \ldots, P_n)$, where $\hat{x}_i$ is the ideal point of P_i for each i. Then Non-manipulability, re-stated for a Peaks-only social choice rule, has the following form.

Non-manipulability There do not exist a profile $(\hat{x}_1, \ldots, \hat{x}_n)$ of ideal points, an individual i, an alternative x'_i, and a single-peaked ordering P_i with ideal point $\hat{x}_i$ such that

$$\mathbf{C}(\hat{x}_1, \ldots, x'_i, \ldots, \hat{x}_n) \, P_i \, \mathbf{C}(\hat{x}_1, \ldots, \hat{x}_i, \ldots, \hat{x}_n).$$

It is straightforward to see that every generalized median rule $\mathbf{C}_\alpha$ satisfies Non-manipulability. Given a profile $(\hat{x}_1, \ldots, \hat{n})$, an individual i cannot manipulate the social choice if the rule chooses her ideal point, so suppose $\hat{x}_i < \mathbf{C}_\alpha(\hat{x}_1, \ldots, \hat{x}_n)$. Then false report $x'_i \leq \mathbf{C}_\alpha(\hat{x}_1, \ldots, \hat{x}_n)$ will not affect the median of the individual and phantom ideal points; and a false report $x'_i > \mathbf{C}_\alpha(\hat{x}_1, \ldots, \hat{x}_n)$ can only move the choice further from i's true ideal point. In any case, the individual cannot alter the social choice in a profitable way.

Remarkably, the generalized median rules are the *only* social choice rules satisfying Pareto, Anonymity, No-ties, and Non-manipulability. The next theorem is proved by Moulin (1981) under the additional assumption of Peaks-only, which was shown to be redundant in later work. The proof here makes use of Peaks-only to simplify the argument.

Theorem 8.3 (Moulin)**.** *Assume $A = [0, 1]$ and Single-peaked domain. A social choice rule $\mathbf{C}$ satisfies Pareto, Anonymity, No-ties, and Non-manipulability if and only if there is a vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in [0, 1]^{n-1}$ such that for all profiles $(P_1, \ldots, P_n)$, we have $\mathbf{C}(P_1, \ldots, P_n) = \mathbf{C}_\alpha(P_1, \ldots, P_n)$.*

The proof begins with a social choice rule $\mathbf{C}$ satisfying Pareto, Anonymity, No-ties, and Non-manipulability, with the additional axiom of Peaks-only assumed for simplicity, and it proceeds to show that $\mathbf{C}$ is a generalized median rule by explicitly constructing phantom ideal points $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ and showing that for all profiles $(P_1, \ldots, P_n)$, we have $\mathbf{C}(P_1, \ldots, P_n) = \mathbf{C}_\alpha(P_1, \ldots, P_n)$. By Peaks-only, we can instead write these social choice rules as functions of ideal points only, i.e., $\mathbf{C}(\hat{x}_1, \ldots, \hat{x}_n) = \mathbf{C}_\alpha(\hat{x}_1, \ldots, \hat{x}_n)$. Before proceeding, we verify the following implication of our axioms.

Claim 1: For every profile $(P_1, \ldots, P_n)$ with ideal points $\hat{x}_1 \leq \cdots \leq \hat{x}_n$ and every individual i such that $\hat{x}_i < \mathbf{C}(\hat{x}_1, \ldots, \hat{x}_n)$, we have

$$\mathbf{C}(\hat{x}_1, \ldots, \hat{x}_n) \quad \leq \quad \mathbf{C}(\underbrace{0, \ldots, 0}_{i \text{ zeroes}}, \underbrace{1, \ldots, 1}_{n-i \text{ ones}}).$$

Consider such a profile and individual, and note that Pareto implies $\mathbf{C}(\hat{x}_1, \ldots, \hat{x}_n) \leq \hat{x}_n$, so it follows that $i < n$. The proof of the claim is in two parts. First, we move

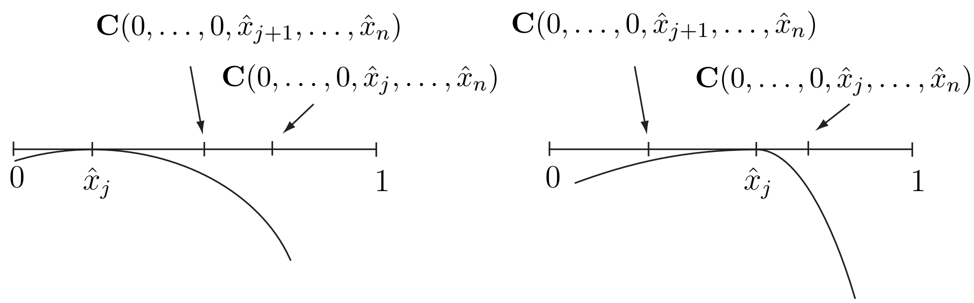


Figure 3: Single-peaked manipulation

the ideal points of individuals $j \leq i$ to zero, one at a time. Given any $j \leq i$, assume that

$$\mathbf{C}(\hat{x}_1, \ldots, \hat{x}_n) \;\leq\; \mathbf{C}(0, \ldots, 0, \hat{x}_j, \ldots, \hat{x}_n).$$

Noting that

$$\hat{x}_j \;\leq\; \hat{x}_i \;\leq\; \mathbf{C}(\hat{x}_1, \ldots, \hat{x}_n) \;\leq\; \mathbf{C}(0, \ldots, 0, \hat{x}_j, \ldots, \hat{x}_n),$$

we now move individual j to zero. If

$$\mathbf{C}(0, \ldots, 0, \hat{x}_{j+1}, \ldots, \hat{x}_n) \;<\; \mathbf{C}(0, \ldots, 0, \hat{x}_j, \ldots, \hat{x}_n),$$

then there is a single-peaked weak order P'_j with ideal point $\hat{x}_j$ such that

$$\mathbf{C}(0, \ldots, 0, \hat{x}_{j+1}, \ldots, \hat{x}_n) \; P'_j \; \mathbf{C}(0, \ldots, 0, \hat{x}_j, \ldots, \hat{x}_n),$$

as in Figure 8. This contradicts Non-manipulability, and we conclude that

$$\mathbf{C}(\hat{x}_1, \ldots, \hat{x}_n) \;\leq\; \mathbf{C}(0, \ldots, 0, \hat{x}_j, \ldots, \hat{x}_n) \;\leq\; \mathbf{C}(0, \ldots, 0, \hat{x}_{j+1}, \ldots, \hat{x}_n).$$

By induction, we have $\mathbf{C}(\hat{x}_1, \ldots, \hat{x}_n) \leq \mathbf{C}(0, \ldots, 0, \hat{x}_{i+1}, \ldots, \hat{x}_n)$. The second part of the proof is to move the ideal points of individuals $j > i$ to one, one at a time. Given any $j > i$, assume that

$$\mathbf{C}(\hat{x}_1, \ldots, \hat{x}_n) \;\leq\; \mathbf{C}(\underbrace{0, \ldots, 0}_{i \text{ zeroes}}, \underbrace{1, \ldots, 1}_{j-i-1 \text{ ones}}, \hat{x}_j, \ldots, \hat{x}_n).$$

We now move individual j to one. If

$$\mathbf{C}(0, \ldots, 0, 1, \ldots, 1, \hat{x}_{j+1}, \ldots, \hat{x}_n) \;<\; \mathbf{C}(0, \ldots, 0, 1, \ldots, 1, \hat{x}_j, \ldots, \hat{x}_n),$$

then for every single-peaked weak order P'_j with ideal point equal to one, we have

$$\mathbf{C}(0, \ldots, 0, 1, \ldots, 1, \hat{x}_j, \ldots, \hat{x}_n) \; P'_j \; \mathbf{C}(0, \ldots, 0, 1, \ldots, 1, \hat{x}_{j+1}, \ldots, \hat{x}_n),$$

contradicting manipulability. We conclude that that

$$\mathbf{C}(\hat{x}_1,\ldots,\hat{x}_n) \le \mathbf{C}(0,\ldots,0,1,\ldots,1,\hat{x}_j,\ldots,\hat{x}_n) \le \mathbf{C}(0,\ldots,0,1,\ldots,1,\hat{x}_{j+1},\ldots,\hat{x}_n).$$

By induction, $\mathbf{C}(\hat{x}_1,\ldots,\hat{x}) \le \mathbf{C}(0,\ldots,0,1,\ldots,1)$, and the claim follows. □

We next construct phantom ideal points $\alpha = (\alpha_1,\ldots,\alpha_{n-1})$ as

$$\begin{aligned} \alpha_1 &= \mathbf{C}(0,\ldots,0,1) \\ \alpha_2 &= \mathbf{C}(0,\ldots,0,1,1) \\ &\vdots \\ \alpha_{n-1} &= \mathbf{C}(0,1,1,\ldots,1). \end{aligned}$$

Note the implication of Claim 1 that $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_{n-1}$. For the remainder of the proof, we consider an arbitrary profile $(P_1,\ldots,P_n)$ of single-peaked weak orders with corresponding ideal points $\hat{x}_1,\ldots,\hat{x}_n$. It suffices to show that $\mathbf{C}(\hat{x}_1,\ldots,\hat{x}_n) = \mathbf{C}_\alpha(\hat{x}_1,\ldots,\hat{x}_n)$ for the above specification of α. To ease notation, define m^* as the median of $(\hat{x}_1,\ldots,\hat{x}_n,\alpha_1,\ldots,\alpha_{n-1})$. Because $\mathbf{C}$ satisfies Anonymity, the social choice is unaffected if we permute individual ideal points, so we assume without loss of generality that $\hat{x}_1 \le \hat{x}_2 \le \cdots \le \hat{x}_n$. The objective of the proof is to show $\mathbf{C}(\hat{x}_1,\ldots,\hat{x}_n) = m^*$, and to deduce a contradiction, suppose not. As the proofs are analogous, we focus on the case that $\mathbf{C}(\hat{x}_1,\ldots,\hat{x}_n) < m^*$.

Claim 2: $m^* \notin \{\alpha_1,\ldots,\alpha_{n-1}\}$. To deduce a contradiction, suppose m^* is located at a phantom ideal point, and let i be the highest index such that $\alpha_i = m^*$ holds. It follows that $\alpha_i < \alpha_j$ for all $j > i$, so there are $n-1-i$ phantom ideal points strictly above α_i. Furthermore, there must be at least $n-i$ individuals with ideal points weakly below α_i, for otherwise there are more than i individuals with ideal points strictly above α_i, which yields

$$\#\{j \mid \alpha_i < \hat{x}_j\} + \#\{j \mid \alpha_i < \alpha_j\} \;>\; i + (n-1-i) \;=\; n-1,$$

contradicting the supposition that $\alpha_i = m^*$. But then we have

$$\alpha_i \;=\; m^* \;<\; \mathbf{C}(\hat{x}_1,\ldots,\hat{x}_n) \;\le\; \mathbf{C}(\underbrace{0,\ldots,0}_{n-i\text{ zeroes}},\underbrace{1,\ldots,1}_{i\text{ ones}}) \;=\; \alpha_i,$$

where we use Claim 1 and the fact that $\hat{x}_{n-i} \le \alpha_i$. This contradiction establishes the claim. □

Claim 3: $m^* \notin \{\hat{x}_1,\ldots,\hat{x}_n\}$. To deduce a contradiction, suppose m^* is located at the ideal point of an individual, and let i be the higest index such that $\alpha_i = \hat{x}_i$. Note that since $m^* < \mathbf{C}(\hat{x}_1,\ldots,\hat{x}_n)$, an implication of Pareto is that $i < n$. It follows that $\hat{x}_i < \hat{x}_j$ for all $j < i$, so there are $n-i$ individuals with ideal points strictly above

$\hat{x}_i$. Furthermore, there must be at least $n-i$ phantom ideal points weakly below $\hat{x}_i$, for otherwise there are more than $i-1$ phantom ideal points strictly above $\hat{x}_i$, which yields

$$\#\{j \mid \hat{x}_i < \hat{x}_j\} + \#\{j \mid \hat{x}_i < \alpha_j\} \;>\; (n-i)+(i-1) \;=\; n-1,$$

contradicting the supposition that $\hat{x}_i = m^*$. Since $i < n$, this implies that $\alpha_{n-i} \leq \hat{x}_i$. But then we have

$$\alpha_{n-i} \;\leq\; \hat{x}_i \;=\; m^* \;<\; \mathbf{C}(\hat{x}_1,\ldots,\hat{x}_n) \;\leq\; \mathbf{C}(\underbrace{0,\ldots,0}_{i \text{ zeroes}},\underbrace{1,\ldots,1}_{n-i \text{ ones}}) \;=\; \alpha_{n-i},$$

where we use Claim 1 and $\alpha_{n-i} \leq \hat{x}_i$. This contradiction establishes the claim. □

Since m^* must belong to $\{\hat{x}_1,\ldots,\hat{x}_n,\alpha_1,\ldots,\alpha_n\}$, we have a contradiction, and we conclude that $\mathbf{C}(\hat{x}_1,\ldots,\hat{x}_n) = m^*$, as required.

Although the set of alternatives is assumed to be the unit interval in Theorem 8.3, this assumption can be relaxed considerably. Of course, nothing in the argument rests on the endpoints being zero and one, so the result holds for arbitrary $A = [a,b]$. The result can be extended to $A = \mathbb{R}$ as well, if we allow for phantom ideal points to take infinite values, $\alpha_j = \pm\infty$. To see this, suppose A is the real line, and let $\mathbf{C}$ be any social choice rule satisfying Moulin's axioms. We can consider any interval $A' = [-c,c]$ with $c > 0$ and the restriction of $\mathbf{C}$ to profiles of single-peaked preferences with ideal points in A', and we can apply Moulin's theorem to obtain phantom ideal points $\alpha_1,\ldots,\alpha_{n-1}$. As we increase c and repeat this process, all of the interior phantoms will remain fixed, with possibly some phantoms remaining at c or $-c$. A phantom who remains at c for all $c > 0$ is fixed at $\alpha_j = \infty$, and one remaining at $-c$ for all $c > 0$ is fixed at $\alpha_j = -\infty$.

Of further note is that fact that the Pareto axiom can be substantially weakened to the requirement that if all individuals possess the same ideal point, then that alternative is the unique choice; in contrast to the Pareto axiom, this axiom relies on the structure of single-peaked preferences.

Unanimity For all single-peaked profiles $(P_1,\ldots,P_n)$ with ideal points $(\hat{x}_1,\ldots,\hat{x}_n)$ and all x, if $\hat{x}_i = x$ for each i, then $\mathbf{C}(P_1,\ldots,P_n) = x$.

Obviously, the Pareto axiom implies Unanimity, for if all individuals possess the same ideal point, then that is the only Pareto optimal alternative, so it must be the unique choice.

Finally, we note that the assumption of Single-peaked domain allows all profiles of single-peaked weak orders, including ones that are discontinuous. Assuming the set A of alternatives is closed, we say P_i is *continuous* if for all $x \in A$, the sets

$R_i(x) = \{y \in A \mid yR_ix\}$ and $R^{-1}(x) = \{y \in A \mid xR_iy\}$ are closed subsets of the real line. This condition formalizes the intuitive idea that if there is a strict preference between two alternatives, then that strict preference should carry over to close enough pairs of alternatives. We say *Continuous single-peaked domain* holds if all profiles of continuous, single-peaked weak orders are possible. The proof of Theorem 8.3 does not make use of discontinuous preferences, so it holds under Continuous single-peaked domain, and in fact it can be formulated on the even smaller Euclidean domain. The next theorem summarizes these extensions.

Theorem 8.4. *Assume $A \subseteq \mathbb{R}$ is convex and closed, and assume either Euclidean domain or Continuous single-peaked domain. Then a social choice rule $\mathbf{C}$ satisfies Unanimity, Anonymity, No-ties, and Non-manipulability if and only if there exists $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in [A \cup \{-\infty, \infty\}]^{n-1}$ such that for all profiles $(P_1, \ldots, P_n)$, we have $\mathbf{C}(P_1, \ldots, P_n) = \mathbf{C}_\alpha(P_1, \ldots, P_n)$.*

9 Issue-by-issue Medians

Assume $A \subseteq \mathbb{R}^d$, let $x = (x_1, \ldots, x_d) \in A$, and let P_i be a weak order on A. Denote by x_{-j} the $(d-1)$-tuple consisting of coordinates of x other than x_j. Then fixing x_{-j}, we can consider the preferences induced by P_i on the jth coordinate. Specifically, let $A^j(x) = \{(y_j, x_{-j}) \in A \mid y_j \in \mathbb{R}\}$ be the possible values of the jth coordinate with x_{-j} fixed, and define the weak order $P_i^j(x)$ on $A^j(x)$ as follows: for all $\alpha, \beta \in A^j(x)$,

$$\alpha \; P_i^j(x) \; \beta \quad \Leftrightarrow \quad (\alpha, x_{-j}) \; P_i \; (\beta, x_{-j}).$$

When $A = [0,1]^d$, it follows that for all $x \in A$, we have $A^1(x) = \cdots = A^d(x) = [0,1]$. In this case, we say P_i is *separable* if for all $x, y \in A$ and all $j = 1, \ldots, d$, we have $P_i^j(x) = P_i^j(y)$; that is, a weak order is separable if the induced preferences on any coordinate are independent of the other coordinates. In this case, we adopt the notational convention that P_i^j denotes the preferences of individual i along the jth dimension, i.e., for all $x \in A$, $P_i^j = P_i^j(x)$. A special case is when P_i is *weighted Euclidean,* i.e., there exist an ideal point $\hat{x}^i \in A$ and weights $a_j^i > 0$, $j = 1, \ldots, d$, such that for all $x, y \in A$, we have xP_iy if and only if

$$\sum_{j=1}^{d} a_j^i (x_j - \hat{x}_j^i)^2 \;<\; \sum_{j=1}^{d} a_j^i (y_j - \hat{x}_j^i)^2.$$

Weighted Euclidean preferences are separable, with elliptical indifference curves that are oriented along the axes; a further special case (obtained by setting weights $a_j^i = 1$) are *Euclidean* preferences, in which case xP_iy if and only if $||x - \hat{x}_i|| < ||y - \hat{x}_i||$.

Henceforth, assume that A is closed and convex and that each individual i has a *spatial* preference relation P_i in the sense that P_i is a weak order such that

- P_i admits a unique *ideal point* $\hat{x}^i \in A$ such that for all $y \in A \setminus \{\hat{x}^i\}$, we have $\hat{x}^i P_i y$,

- P_i is *continuous,* i.e., for all $x \in A$, the sets $R(x) = \{y \in A \mid yRx\}$ and $R^{-1}(x) = \{y \in A \mid xRy\}$ are closed subsets of $\mathbb{R}^d$,

- P_i is *strictly convex,* i.e., for all distinct $x, y \in A$ with xR_iy and all $\alpha \in (0, 1)$, we have $\alpha x + (1 - \alpha)yP_iy$.

Again, a special case is that in which P_i is *Euclidean,* i.e., xP_iy if and only if $||x-\hat{x}^i|| < ||y - \hat{x}^i||$. Say that *Spatial domain* holds if every profile of spatial weak orders is possible; and when $A = [0, 1]^d$, say that *Separable domain* holds if every profile of continuous, strictly convex, separable preferences is possible.

Assuming n odd, $A = [0, 1]^d$, and continuous, strictly convex, separable preferences $(P_1, \dots, P_n)$ with ideal points $(\hat{x}^1, \dots, \hat{x}^n)$, we can focus on the jth coordinate of the ideal points and define $x_j^{m_j}$ as the median of the projected ideal points $(\hat{x}_j^1, \dots, \hat{x}_j^n)$; this belongs to some individual m_j, who may depend on the coordinate j. Then we define the *issue-by-issue median* as the vector

$$(\hat{x}_1^{m_1}, \dots, \hat{x}_d^{m_d})$$

of medians on all dimensions. By this procedure, we can define the issue-by-issue social choice rule, denoted $\mathbf{C}_m$, on the Separable domain. Note that when $d = 1$, this rule reduces to the usual median social choice rule, so our notation is consistent with the above analysis for one dimension.

It is clear that because the median rule is Non-manipulable in one dimension, by Corollary 8.2, separability of individual preferences implies that the issue-by-issue median rule satisfies Non-manipulability.

Corollary 9.1. *Assume $A = [0, 1]^d$, n is odd, and Separable domain. Then the issue-by-issue median social choice rule satisfies Non-manipulability.*

What is less clear is whether the issue-by-issue median satisfies Pareto. In fact, this issue is somewhat complex, for it depends on the dimensionality of the set of alternatives. When $d = 1$, the rule reduces to the median rule $\mathbf{C}_m$, which satisfies Pareto optimality. Less obviously, when $d = 2$, the rule $\mathbf{C}_\alpha$ still satisfies Pareto. To see this, consider Figure 4, where we assume $m_1 = 1$ and $m_2 = 2$, i.e., individual 1 is the median on the first coordinate and individual 2 is the median on the second, so the issue-by-issue median is $x = (x_1^1, x_2^2)$. Their ideal points are placed without loss of generality, and we claim that there must be some individual whose ideal point belongs to the shaded gray region (including the boundary and x), for suppose otherwise. Since x_1 is a median on the first dimension it must be that the number of individuals

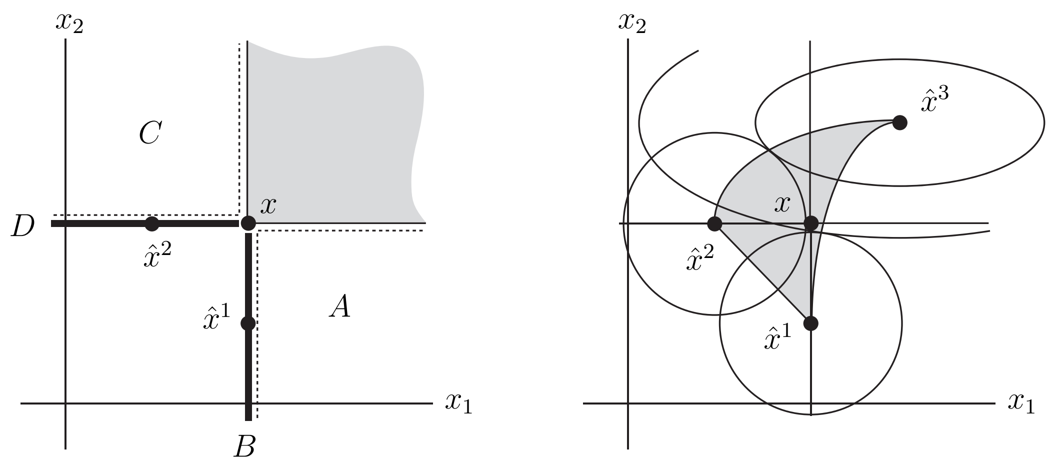


Figure 4: Issue-by-issue Pareto optimality

with ideal points in regions A (the open rectangle) and B (the dark line) exceeds $\frac{n}{2}$; likewise, because x_2 is a median on the second dimension, the number of individuals with ideal points in regions C and D must exceed $\frac{n}{2}$. But $A \cup B$ and $C \cup D$ are disjoint, so they cannot both contain a majority of individuals, a contradiction. Thus, there is an individual in the gray region. Then, with the assumption of separable preferences, it follows that x is Pareto optimal.

Proposition 9.2. *Assume $A = [0,1]^2$, n is odd, and Separable domain. If $d \in \{1,2\}$, then the issue-by-issue median rule $\mathbf{C}_m$ satisfies Pareto.*

When the dimensionality of the set of alternatives is three or more, Pareto optimality is lost, even if individual preferences are restricted to be Euclidean. Consider Figure 5, where three individuals have ideal points located at the unit coordinate vectors, and the issue-by-issue median is the origin, $x = 0$. We can, nevertheless, extend the weaker Unanimity axiom to multiple dimensions with Spatial domain: the form of the axiom is exactly as in the previous section, now viewing ideal points $\hat{x}^i$ as vectors in $A \subseteq \mathbb{R}^d$. It is then clear that the issue-by-issue median rule satisfies the Unanimity axiom, regardless of the number of dimensions.

As in one dimension, we can extend the idea of an issue-by-issue median by introducing phantom ideal points. Now let $\alpha^j = (\alpha_1^j, \ldots, \alpha_d^j)$ denote a phantom ideal point (now a vector), and let $\alpha = (\alpha^1, \ldots, \alpha^{n-1})$ denote a vector of $n-1$ phantom ideal points. Then the *issue-by-issue generalized median* given α is $(\hat{x}_1^{m_1}, \ldots, \hat{x}_d^{m_d})$, where each $x_j^{m_j}$ is equal to the median of $(\hat{x}_j^1, \ldots, \hat{x}_j^n, \alpha_j^1, \ldots, \alpha_j^{n-1})$. We let $\mathbf{C}_\alpha$ denote the social choice rule so-defined. Again, it is clear that every issue-by-issue generalized median satisfies Non-manipulability and Unanimity on the Separable domain of profiles of continuous, strictly convex, separable weak orders.

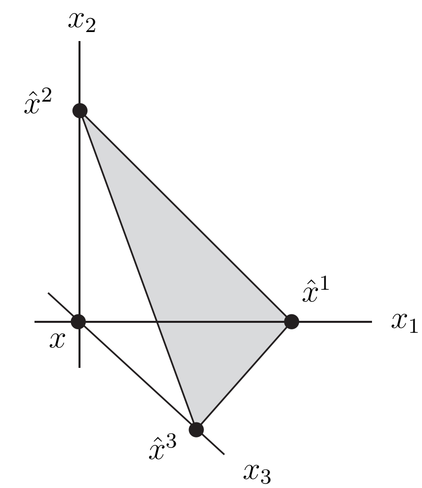


Figure 5: Issue-by-issue Pareto suboptimality

Remarkably, the issue-by-issue generalized medians are the only rules satisfying Unanimity, Anonymity, No-Ties, and Non-manipulability under Separable domain. In one dimension, Separable domain reduces to Continuous single-peaked domain, so Theorem 9.3 implies Theorem 8.4 (with a bounded set of alternatives) as a special case; it is closely related to a result of Border and Jordan (1983) that assumes A the entire Euclidean space, rather than a square.

Theorem 9.3 (Border and Jordan)**.** *Assume $A = [0,1]^d$ and Separable domain. A social choice rule $\mathbf{C}$ satisfies Unanimity, Anonymity, No-ties, and Non-manipulability if and only if there is a vector $\alpha = (\alpha^1, \ldots, \alpha^{n-1}) \in [0,1]^{(n-1)d}$ such that for all profiles $(P_1, \ldots, P_n)$, we have $\mathbf{C}(P_1, \ldots, P_n) = \mathbf{C}_\alpha(P_1, \ldots, P_n)$.*

The "if" direction follows directly from the assumption of Separable domain by familiar arguments. The proof of the "only if" direction consists of an application of Theorem 8.3 and proceeds in a number of steps. The argument here is a graphical one simplified by focussing on the two-dimensional case.

Claim 1: Consider a profile $(P_1, \ldots, P_n)$ with ideal points $(\hat{x}^1, \ldots, \hat{x}^n)$ such that the ideal points differ in only one dimension. In Figure 6, the ideal points differ only along the horizontal dimension. The claim is that the social choice $x = \mathbf{C}(P_1, \ldots, P_n)$ must lie on the line between the two most extreme ideal points; this would follow immediately from Pareto, but it is not a direct implication of the Unanimity axiom. Suppose that x lies below the line containing the individual ideal points, as in Figure 6. Consider any individual i, and replace the weak order P_i by P_i' with ideal point y directly above x and indifference contours as in the figure. By Non-manipulability, it must be that the resulting choice, say x', lies on the indifference curve of P_i through

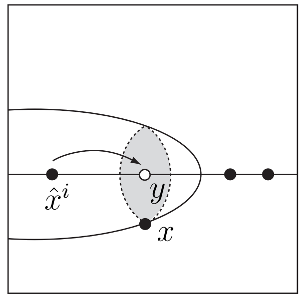


Figure 6: Claim 1 of proof of Theorem 9.3

x or outside it; else, this would be a P_i-to-P_i' manipulation. Similarly, x' must lie on the indifference curve (dashed) of P_i' through x or in the strict upper contour set at x (shaded); else, this would be a P_i'-to-P_i manipulation. But the only alternative satisfying both of these conditions is x itself: $x' = x$. Repeating this for each individual, we have $\mathbf{C}(P_1', \ldots, P_n') = x$, but all individuals have ideal point y in the profile $(P_1', \ldots, P_n')$, so Unanimity implies that $\mathbf{C}(P_1', \ldots, P_n') = y \neq x$, a contradiction.

Claim 2: For each $z \in A$ and each $j = 1, \ldots, d$, we define a social choice rule $\mathbf{C}_j^z$ for a one-dimensional environment with $A' = [0, 1]$ under Continuous single-peaked domain as follows. Given any profile $(P_1', \ldots, P_n')$ of continuous, single-peaked weak orders in the one-dimensional environment, we extend each P_i' to a separable, continuous, strictly convex weak order P_i on $A = [0, 1]^d$ as follows. Since P_i' is a continuous, single-peaked weak order on A', it admits a unique ideal point $\hat{x}_i'$, and there is an alternative $\alpha^* \in A'$ such that either $\alpha^* I_i' 1$ or $0 I_i' \alpha^*$. For simplicity, we describe the construction for the latter case assuming two dimensions. For each $\alpha \in [\alpha^*, \hat{x}_i']$, we draw indifference "curves" from $(\alpha, 0)$ at a 45 degree angle with a peak over $\hat{x}_i'$ and a trough below it; we then find the corresponding alternative $\beta \in [\hat{x}_i', 1]$ and connect the peak and trough to $(\beta, 0)$. For $\alpha < \alpha^*$, we continue this construction, preserving the shape of the indifference curve through α^*. In words, we extend P_i' to a separable, continuous, strictly convex weak order on A such that i's preferences on the jth dimension are given by P_i', and her ideal point on all other dimensions $k \neq j$ is z_k. Then the alternative $x = \mathbf{C}(P_1, \ldots, P_n)$ selected by $\mathbf{C}$ for this preference profile lies on the line through z parallel to the jth axis, by Claim 1, and we define the corresponding choice for the one-dimensional model as the jth coordinate of this choice: $\mathbf{C}_j^z(P_1', \ldots, P_n') = x_j$. See Figure 7. (Technically, the preference relation P_i so-defined is not strictly convex, but we can obtain strict convexity by adding some curvature to the indifference curves described above.)

Claim 3: We note that the social choice rule $\mathbf{C}_j^z$ defined in Claim 2 satisfies the Non-manipulability axiom: if there were a profile $(P_1', \ldots, P_n')$ and an individual i who could manipulate by reporting $\tilde{P}_i'$, then given the associated profile $(P_1, \ldots, P_n)$ over $A = [0, 1]^d$ constructed in Claim 2, individual i could manipulate $\mathbf{C}$ by reporting

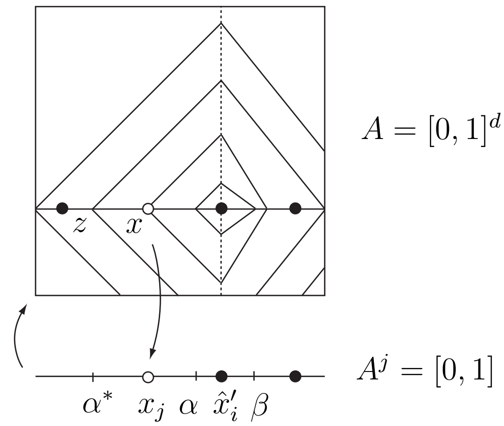


Figure 7: Claim 2 of proof of Theorem 9.3

the weak order $\tilde{P}_i$ associated with $\tilde{P}'_i$. Thus, $\mathbf{C}^z_j$ satisfies Non-manipulability, and it clearly satisfies Unanimity, Anonymity, and No-ties. Then Theorem 8.4 implies that $\mathbf{C}^z_j$ is a generalized majority rule with vector $\alpha(z,j) = (\alpha_1(z,j), \ldots, \alpha_{n-1}(z,j))$ of phantom ideal points. As a consequence, we can write it as a function of individual ideal points in A', as in $\mathbf{C}^z_j(\hat{x}'_1, \ldots, \hat{x}'_n)$.

Claim 4: The social choice rule $\mathbf{C}^z_j$ is independent of how we extend $(P'_1, \ldots, P'_n)$ to the square $A = [0,1]^d$. That is, let $(\tilde{P}_1, \ldots, \tilde{P}_n)$ be any profile over A such that each individual i's preferences on the jth dimension are given by P'_i, with ideal point z_k on dimensions $k \neq j$ (but it is not necessarily the associated profile $(P_1, \ldots, P_n)$ constructed in Claim 2). Then $\mathbf{C}(\tilde{P}_1, \ldots, \tilde{P}'_n) = \mathbf{C}(P_1, \ldots, P_n)$. For suppose otherwise, as depicted in Figure 8, where the alternative chosen for $(P_1, \ldots, P_n)$ is x, and the alternative chosen when individual i reports $\tilde{P}_i$ is $\tilde{x} \neq x$. By Claim 1, $\tilde{x}$ must lie on the same line through z, and Non-manipulability then implies that individual i is indifferent between x and $\tilde{x}$ given both P_i or $\tilde{P}_i$. But now suppose i reports a weak order P^*_i as in the figure. By construction of $\mathbf{C}^z_j$ and the fact that $\mathbf{C}^z_j$ is a generalized majority rule, the choice must be x when P^*_i is reported, but then we have a P^*_i-toP'_i manipulation: if individual i's true preference is P^*_i, then reporting the truth leads to the choice of x, but reporting $\tilde{P}_i$ leads to the preferred alternative $\tilde{x}$.

Claim 5: For all $z, w \in A = [0,1]^d$, all coordinates $j = 1, \ldots, d$, and all profiles $(\hat{x}'_1, \ldots, \hat{x}'_n)$ of ideal points on $A' = [0,1]$, we have $\mathbf{C}^z_j(\hat{x}'_1, \ldots, \hat{x}'_n) = \mathbf{C}^w_j(\hat{x}'_1, \ldots, \hat{x}'_n)$. In words, if individual ideal points line up on one dimension j and are identical on all other dimensions $k \neq j$, and we then translate the ideal points on dimensions $k \neq j$ (basically shifting the line through the square), the jth coordinate of the social choice does not change. Suppose not, in order to deduce a contradiction. In Figure 9, for example, individual ideal points $(\hat{x}^1, \hat{x}^2, \hat{x}^3)$ initially fall on the line through z parallel to the horizontal axis, and the social choice is x, but when the ideal points

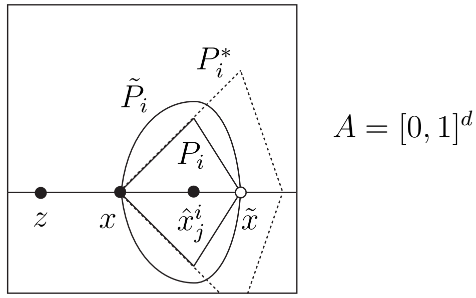


Figure 8: Claim 4 of proof of Theorem 9.3

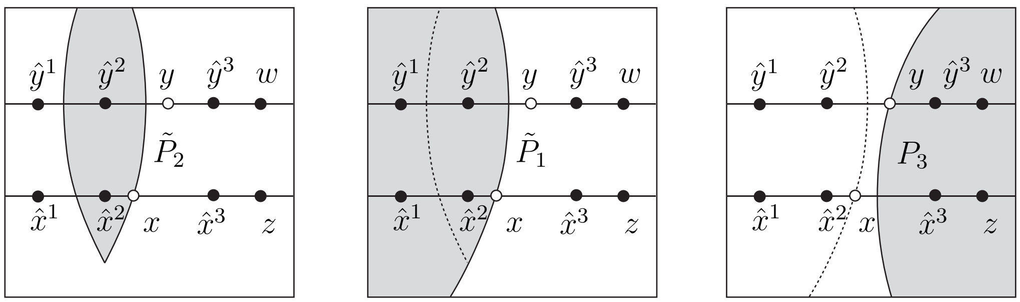


Figure 9: Claim 5 of proof of Theorem 9.3

are translated to the line through w, the choice is $y \neq x$. By Claim 4, we can assume without loss of generality that individual 3's ordering P_3 has the form depicted in the third panel of the figure, with weak upper contour set through y indicated by the shaded region. In the first panel, we change individual 2's weak order from P_2 (with ideal point $\hat{y}^2$) to $\tilde{P}_2$ (with ideal point $\hat{x}^2$); the weak upper contour set of $\tilde{P}_2$ through x is indicated by the shaded region. By Non-manipulability, the resulting social choice must lie in this shaded region; else, we would have a $\tilde{P}_2$-to-P_2 manipulation. In the second panel, we change P_1 (with ideal point $\hat{x}^1$) to $\tilde{P}_1$ (with ideal point $\hat{y}^1$), again shading the weak upper contour set through x. By Non-manipulability, the social choice must again remain in this shaded region. Finally, we observe that the social choice for profile $(\tilde{P}_1, \tilde{P}_2, P_3)$ lies outside the weak upper contour set of P_3 through y, and when individual 3 reports $\tilde{P}_3$, the choice moves to the preferred alternative y; thus, we have a P_3-to-$\tilde{P}_3$ manipulation, a contradiction. As a consequence of this claim, the location of the phantom ideal points is independent of z, i.e., $\alpha(z, j) = \alpha(w, j)$, so we can write $\alpha(j) = (\alpha_j^1, \ldots, \alpha_j^{n-1})$ for the vector of phantom ideal points on the jth dimension; furthermore, we may write $\mathbf{C}_j(\hat{x}'_1, \ldots, \hat{x}'_n)$ (no longer depending on z) for the one-dimensional social choice rule on the jth coordinate.

Claim 6: For all coordinates $j = 1, \ldots, d$ and all profiles $(P_1, \ldots, P_n)$ on $A = [0,1]^d$ with ideal points $(\hat{x}^1, \ldots, \hat{x}^n)$, the jth coordinate of the social choice is $\mathbf{C}_j(\hat{x}_j^1, \ldots, \hat{x}_j^n)$.

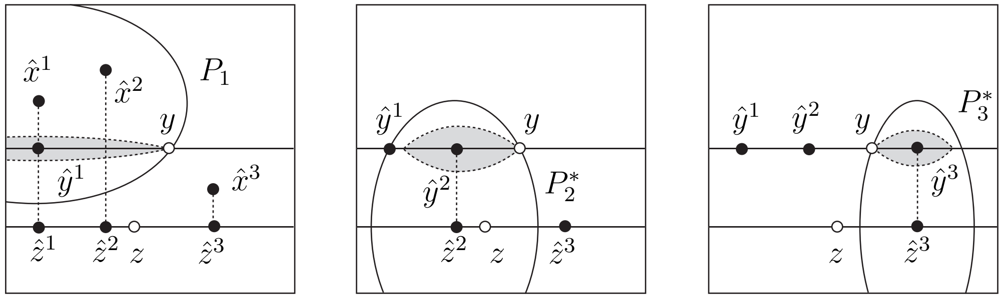


Figure 10: Claim 6 of proof of Theorem 9.3

Thus, the jth coordinate of the social choice is determined by $\mathbf{C}_j$ even when the ideal points of the individuals are not the same on all other dimensions. In Figure 10, for example, we consider a profile (P_1, P_2, P_3) with ideal points $(\hat{x}^1, \hat{x}^2, \hat{x}^3)$. Now, preserving the individuals' ideal points on the horizontal dimension, we consider preferences $(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3)$ such that the corresponding ideal points line up as in $(\hat{z}^1, \hat{z}^2, \hat{z}^3)$, so $\hat{z}^i_1 = \hat{x}^i_1$ for all i. For the latter ideal points, the social choice is some alternative $z = \mathbf{C}_1(\hat{x}^1_1, \hat{x}^2_1, \hat{x}^3_1)$. Now change $\tilde{P}_1$ to P_1, let the social choice be y, and suppose that $y_1 \neq z_1$. Next, change individual 1's weak order to P^*_1 with ideal point $\hat{y}^1$ and weak upper contour set at y shaded in the first panel of the figure. By Non-manipulability, the social choice must remain y. Then change $\tilde{P}_2$ to P^*_2 with ideal point $\hat{y}^2$ and weak upper contour set at y shaded in the second panel. By Non-manipulability, the social choice remains y. We similarly transform individual 3's preferences, and again the choice remains y. But now we have a profile (P^*_1, P^*_2, P^*_3) with ideal points lining up on the horizontal dimension as in $(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3)$, but the first coordinate of the choice given (P^*_1, P^*_2, P^*_3) is different than that given (P^*_1, P^*_2, P^*_3), contradicting Claim 5. Thus, we must have $y_1 = z_1$. Now, after having changed individual 1's weak order to P_1, we change individual 2's weak order from $\tilde{P}_2$ to P_2, and a similar argument shows that the first coordinate of the resulting choice remains z_1. Repeating this for individual 3, we find that the first coordinate remains z_1, and we conclude that the first coordinate of $\mathbf{C}(P_1, P_2, P_3)$ is indeed $\mathbf{C}_1(\hat{x}^1_1, \hat{x}^2_1, \hat{x}^3_1)$.

Claim 7: Finally, it follows that for an arbitrary profile $(P_1, \ldots, P_n)$ with ideal points $(\hat{x}^1, \ldots, \hat{x}^n)$, we have

$$\mathbf{C}(P_1, \ldots, P_n) \;\; = \;\; (\mathbf{C}_1(\hat{x}_1, \ldots, \hat{x}^n_1), \ldots, \mathbf{C}_d(\hat{x}^1_d, \ldots, \hat{x}^n_d)).$$

For each $j = 1, \ldots, n-1$, let $\alpha^j = (\alpha^j_1, \ldots, \alpha^j_d)$ be the vector consisting of the jth phantom ideal point on each dimension $1, \ldots, d$. Then the above is equivalent to

$$\mathbf{C}(P_1, \ldots, P_n) \;\; = \;\; \mathbf{C}_\alpha(P_1, \ldots, P_n),$$

as required.

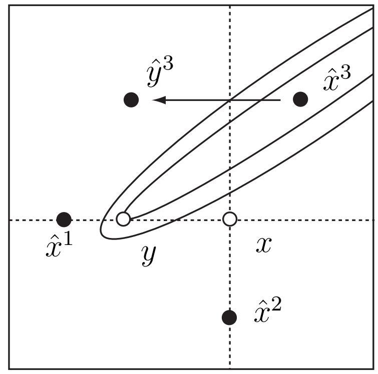


Figure 11: Manipulating the issue-by-issue median

10 Structure-induced Equilibrium

What are the possibilities for non-manipulable social choice when the domain of preferences includes non-separable preferences, as in Spatial domain? Given any social choice rule **C** defined for a domain of preferences larger than Separable domain, we can always consider the restriction $\mathbf{C}'$ to Separable domain; if **C** satisfies Non-manipulability, then the restriction will satisfy the axiom as well, and we can apply Theorem 9.3 to $\mathbf{C}'$ to conclude that it is an issue-by-issue generalized median rule. So at issue is how we might extend an issue-by-issue generalized median to the larger Spatial domain.

Given a profile $(P_1, \ldots, P_n)$ of spatial preferences, we can of course consider the corresponding profile $(\hat{x}^1, \ldots, \hat{x}^n)$ of ideal points, and we can simply specify the social choice as the median on each dimension. This is certainly the simplest way of extending the issue-by-issue median rule $\mathbf{C}_m$ to the Spatial domain. But this extension is easily seen to violate Non-manipulability. In Figure 11, suppose the true profile of ideal points is $(\hat{x}^1, \hat{x}^2, \hat{x}^3)$, so the issue-by-issue median is x. But suppose that individual 3 has non-separable preferences with indifference curves as pictured. Then the individual can report preferences with the ideal point $\hat{y}^3$ and move the issue-by-issue median to the preferred alternative y, violating Non-manipulability.

This is perhaps expected, because when separability is violated, the jth coordinate of individual's ideal point is relatively uninformative about the individual's preferences on the jth dimension. In Figure 11, for example, suppose that individuals 1 and 2 have Euclidean preferences. Then if we consider the horizontal line through individual 1's ideal point, the best alternative on that line for individual 1 is $\hat{x}^1$ and the best for 2 is x, but the best alternative for 3 on the line is y. Thus, the issue-by-issue median does not belong to the majority core when alternatives are restricted to the line, giving individual 3 the opportunity to manipulate. In other words, if we extend the concept of issue-by-issue median to Spatial domain in the simplest way, we do

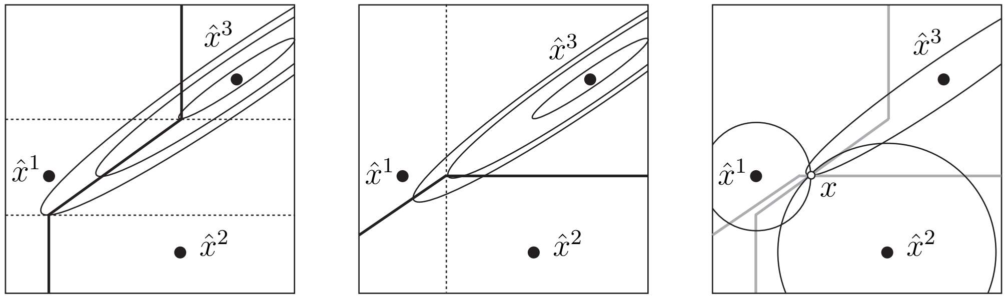


Figure 12: Structure-induced equilibrium

not preserve stability of social choices when restricted to movements parallel to the axes.

We next investigate a more sophisticated extension of the issue-by-issue median that avoids this problem. Recall that when $A = [0,1]^d$ is a square and $x \in A$, the relation $P_i^j(x)$ on $[0,1]$ represents i's preferences on the line through x parallel to the jth axis: for all $\alpha, \beta \in [0,1]$, $\alpha P_i^j(x)\beta$ if and only if $(\alpha, x_{-j})P_i(\beta, x_{-j})$. Assuming n is odd, we say z is a *median in direction j relative to x* if z lies on the line through x parallel to the jth axis, and there is no alternative on this line majority-preferred to it, i.e., $z_{-j} = x_{-j}$ and there is no $\alpha \in [0,1]$ with

$$\#\{i \in N \mid \alpha P_i^j(x)x_j\} \quad > \quad \#\{i \in N \mid x_j P_i^j(x)\alpha\}.$$

Under spatial domain, the preference relation $P_i^j(x)$ on $[0,1]$ is single-peaked, and it follows that there is a unique median in each direction j relative to each x.

Then x is a *structure-induced equilibrium* if it is a median in each direction $j = 1, \ldots, d$ relative to itself. This captures the idea that x is stable with respect to moves parallel to the axes. When individual preferences are separable, the concepts of structure-induced equilibrium and issue-by-issue median coincide. To compute the structure-induced equilibria when $d = 2$, we can draw a horizontal line, find the median on this line, and then trace the path of medians as the line is varied up and down. In Figure 12, this is the piece-wise linear path drawn with a thick line in the first panel. We do the same for vertical lines, tracing the path of medians as we vary the line to the left and right; this is the thick line in the second panel. These lines must intersect at least once, and every intersection is a structure-induced equilibrium. In the third panel of Figure 12, the unique structure-induced equilibrium is x, where the two paths of medians intersect.

In general, in higher dimensions, it is not obvious that structure-induced equilibrium will always exist. The following result is due to Shepsle (1979) and provides a positive resolution to this issue.

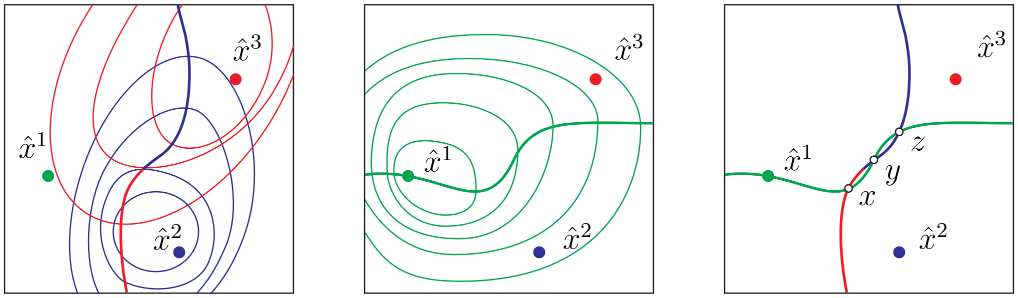


Figure 13: Multiple structure-induced equilibria

Theorem 10.1 (Shepsle)**.** *Assume n is odd, $A = [0, 1]^d$, and let $(P_1, \ldots, P_n)$ be a profile of spatial preferences. There is a structure-induced equilibrium.*

Proof. For each $x \in A$ and each direction $j = 1, \ldots, d$, let $\mu_j(x)$ be the unique median in direction j relative to x. It is straightforward to show that the mapping $\mu_j : A \to [0, 1]$ so-defined is continuous. Furthermore, for each $x \in A$, we have $\mu(x) = (\mu_1(x), \ldots, \mu_d(x)) \in A$. Therefore, Brower's fixed point theorem implies that there exists $x^* \in A$ such that $x^* = \mu(x^*)$, and x^* is a structure-induced equilibrium. □

Thus, we can define the structure-induced equilibrium social choice rule, $\mathbf{C}_{sie}$, on the domain of profiles of spatial preferences. We have not touched on uniqueness of structure-induced equilibrium. In Figure 12, it is unique, but this is not a general property, for the assumption of spatial preferences does not preclude the possibility that the paths of directional medians "bend backward" and intersect more than once. In Figure 13, for example, three individuals have non-separable preferences. When we consider horizontal lines, individual 2's ideal point is the median in the horizontal direction for lines at the top of the square, while individual 3 is the median for lines at the bottom of the square. The path of medians in the horizontal direction is traced out in the first panel of the figure, with the blue portion indicating where individual 2 is median and the red portion indicating where individual 3 is median. Considering vertical lines, individual 1's ideal point is always the median in the vertical direction, and the path of medians in the vertical direction is traced in green in the second panel. In the third panel, we see that these paths intersect at three alternatives, x, y, and z, and each of these alternatives is therefore a structure-induced equilibrium. We conclude that the structure-induced equilibrium rule $\mathbf{C}_{sie}$ violates the No-ties axiom.

The concept of structure-induced equilibrium conveys stability with respect to changes in a single coordinate that is not guaranteed by the issue-by-issue median. Finally, we return to the issue of manipulability raised at the beginning of this section. Since the structure-induced equilibrium social choice rule violates No-ties, the standard definition of Non-manipulability does not technically apply, but it is straightforward to show that despite the desirable stability properties of the structure-induced median

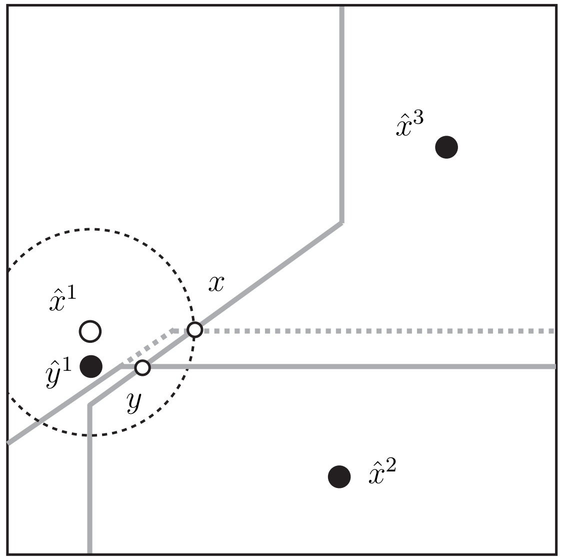


Figure 14: Manipulating the structure-induced equilibrium

rule, it is not immune to the possibility of manipulation. In Figure 14, we return to the example of Figure 12, where the true preferences of the agents determine x as the unique structure-induced equilibrium. Note, however, that if individual 1 reports Euclidean preferences with ideal point $\hat{y}^1$, instead of $\hat{x}^1$, this changes the path of medians in the vertical direction from the dashed gray line in Figure 14 to the solid one. This leads to a unique structure-induced equilibrium y, moving the choice from x to the preferred alternative y. Thus, we have an example (that does not involve ties) showing the structure-induced median rule is vulnerable to manipulation.

Our final result establishes that there are no social choice rules that satisfy Anonymity, Unanimity, No-ties, and Non-manipulability under Spatial domain. In fact, the impossibility result persists even if we weaken Anonymity to the minimal requirement of No-dictator. A version of the result when the set of alternatives is the entire Euclidean space is given by Border and Jordan (1983), while the result here (which is stated for the compact square $A = [0, 1]^d$) is proven by Zhou (1991).

Theorem 10.2. *Assume $d \geq 2$, $A = [0, 1]^d$, and Spatial domain. There is no social choice rule satisfying Unanimity, No-dictator, No-ties, and Non-manipulability.*

In fact, although this impossibility result is stated for Spatial domain, Zhou (1991) shows that the result carries over even if we assume the smaller domain consisting of profiles of "generalized Euclidean preferences," which are similar to weighted Euclidean but that we allow for elliptical indifference contours that are not oriented along the axes. Formally, P_i is *generalized Euclidean* if there exist an ideal point

$\hat{x}^i \in A$ and weights $\{a^i_{j,k} \mid j,k = 1,\ldots,d\}$ with $a_{j,k} = a_{k,j} \geq 0$ for all j and k and such that xP_iy if and only if

$$\sum_{j=1}^{d}\sum_{k=1}^{d} a^i_{j,k}(x_j - \hat{x}^i_j)(x_k - \hat{x}^i_k) \; < \; \sum_{j=1}^{d}\sum_{k=1}^{d} a^i_{j,k}(y_j - \hat{x}^i_j)(y_k - \hat{x}^i_k).$$

Thus, in multiple dimensions, in the absence of strong restrictions on individual preferences (such as separability), the possibility of strategic manipulation established in the Gibbard-Satterthwaite theorem is unavoidable. In light of this fact, we are forced to accept strategic behavior as an inherent aspect of collective choice — and politics in particular — and the problem then becomes understanding and predicting the outcomes of strategic incentives.

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