18.06.10: ‘Spaces of vectors’

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Another day, another system of linear equations.

\[
\begin{align*}
0 &= -12x + 11y - 17z; \\
0 &= 2x - y + 9z; \\
0 &= -3x + 4y + 5z.
\end{align*}
\]

Solve it!!
The rows are not linearly independent, so there are infinitely many solutions.

You can reduce these equations to just two: \( 41y = 37x \) and \( 41z = -5x \). In other words, any solution is a multiple of the vector

\[
\begin{pmatrix}
41 \\
37 \\
-5
\end{pmatrix}.
\]

This is the information that the system of linear equations provides.
How infinite is infinite?

We’ve said a few times that a system of linear equations has either 0, 1, or $+\infty$ many solutions.
When the system of linear equations is of the form

$$0 = \sum_{i=1}^{n} a_{1i}x_i,$$

$$0 = \sum_{i=1}^{n} a_{2i}x_i,$$

$$\vdots$$

$$0 = \sum_{i=1}^{n} a_{ni}x_i,$$

we always have the solution $x_1 = x_2 = \cdots = x_n = 0$, so our only two options in this case are 1 or $+\infty$. But we can say more. For example, are all the solutions multiples of a single vector??
Here’s a system of linear equations with infinitely many solutions:

\[
\begin{align*}
0 &= 3x - 2y; \\
0 &= 4y - 5z; \\
0 &= 6x - 5z.
\end{align*}
\]

We reduce to \(3x = 2y\) and \(4y = 5z\), and that’s all the information. The last equation doesn’t actually participate. So any vector that satisfies the system above is a multiple of

\[
\begin{pmatrix}
1 \\
2/3 \\
8/15
\end{pmatrix}.
\]
Here’s a system of linear equations with infinitely many solutions:

\[0 = 4u + 2v + 6x + 3y;\]
\[0 = 2u + 3x;\]
\[0 = 2v + 3y;\]
\[0 = 2u - 4v + 3x - 6y.\]

How close can you come?
You can see straightaway that the equations $2u = -3x$ and $2v = -3y$ completely determine the system. The other equations are just offering the same information. So any vector that satisfies the system above is a linear combination of

\[
\begin{pmatrix}
  3 \\
  0 \\
  -2 \\
  0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  0 \\
  3 \\
  0 \\
  -2
\end{pmatrix}.
\]

This lets us parametrize the solution space!!
What we’re doing when we solve systems of linear equations is finding a basis for the space of solutions.

**Definition.** A subspace $V \subseteq \mathbb{R}^n$ is a collection $V$ of vectors of $\mathbb{R}^n$ such that:

1. for any vectors $\vec{v}, \vec{w} \in V$, the sum $\vec{v} + \vec{w} \in V$;

2. for any real number $r$ and any vectors $\vec{v} \in V$, the scalar multiple $r\vec{v} \in V$. 
Example. For any $m \times n$ matrix $A$, the set

$$\text{ker}(A) := \{ \bar{v} \in \mathbb{R}^n \mid A\bar{v} = \bar{0} \}$$

is a subspace of $\mathbb{R}^n$. This is called the kernel of $A$. This may also be called the space of solutions of the system of linear equations:

$$0 = \sum_{i=1}^{n} a_{1i}x_i;$$

$$\vdots$$

$$0 = \sum_{i=1}^{n} a_{ni}x_i.$$
Definition. A basis of a subspace $V \subseteq \mathbb{R}^n$ is a collection $\{\vec{v}_1, \ldots, \vec{v}_k\}$ of vectors $\vec{v}_i \in V$ such that:

1. the vectors $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent;

2. the vectors $\vec{v}_1, \ldots, \vec{v}_k$ span $V$.

We say that $V$ is $k$-dimensional.
The two conditions in the definition above are complementary. To illustrate, let’s write them this way.

(1) The vectors $\vec{v}_1, \ldots, \vec{v}_k \in V$ are \textit{linearly independent} if and only if, for any vector $\vec{w} \in V$, there exists \textit{at most one} way to write $\vec{w}$ as a linear combination

$$\vec{w} = \sum_{i=1}^{k} \alpha_i \vec{v}_i.$$ 

(2) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ \textit{span} $V$ if and only if, for any vector $\vec{w} \in V$, there exists \textit{at least one} way to write $\vec{w}$ as a linear combination

$$\vec{w} = \sum_{i=1}^{k} \alpha_i \vec{v}_i.$$
(3) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ are a basis of $V$ if and only if, for any vector $\vec{w} \in V$, there exists exactly one way to write $\vec{w}$ as a linear combination

$$\vec{w} = \sum_{i=1}^{k} \alpha_i \vec{v}_i.$$ 

The similarities between these conditions and the conditions of injectivity, surjectivity, and bijectivity are no accident...
Take some vectors $\vec{v}_1, \ldots, \vec{v}_k \in V$ and make them into the columns of an $n \times k$ matrix

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{pmatrix}. $$

Multiplication by $A$ is a map $T_A : \mathbb{R}^k \to V$ that carries $\hat{e}_i$ to $\vec{v}_i$.

(1) The vectors $\vec{v}_1, \ldots, \vec{v}_k \in V$ are linearly independent if and only if $T_A$ is injective.

(2) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ span $V$ if and only if $T_A$ is surjective.

(3) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ are a basis of $V$ if and only if $T_A$ is bijective.