**PROBLEM SET VIII**

DUE FRIDAY, 25 APRIL

**Exercise 37.** Consider the n-dimensional disk

\[ D^n := \left\{ (x_1, \ldots, x_n) \in \mathbb{A}^n \mid \sum_{i=1}^{n} x_i^2 \leq 1 \right\}. \]

Write \( V_n := \text{vol}(D^n) \). You have already shown that

\[ \lim_{n \to \infty} V_n = 0, \]

but none of you took my hint *not* to write down a formula for \( V_n \). So let’s try this again. This time, do *not* develop a formula for \( V_n \). Instead, show that

\[ \lim_{n \to \infty} \frac{V_{n+1}}{V_n} = 0 \]

by proving the formula

\[ V_{n+1} = V_n \int_{-1}^{1} (1 - z^2)^n \, dz \]

and then demonstrating

\[ \lim_{n \to \infty} \int_{-1}^{1} (1 - z^2)^n \, dz = 0. \]

**Exercise 38.** Consider the \( n \times n \) matrix \( A \) whose \((i,j)\)-th entry \( a_{ij} \) is given by the formula

\[ a_{ij} = \begin{cases} j/i & \text{if } i \text{ divides } j, \\ 0 & \text{if } i \text{ does not divide } j. \end{cases} \]

Compute \( \text{det}(A) \).

**Exercise 39.** Suppose \( m \geq 2 \) an integer. For any distinct positive integers \( r, s \leq n \), and for any nonzero real number \( \alpha \), let \( E^n(\alpha) = (e^n(\alpha)_{ij}) \) denote the \( m \times m \) matrix whose \((i,j)\)-th entry is \( \alpha \) if \((i,j) = (r,s)\), and is the Kronecker symbol otherwise:

\[ e^n(\alpha)_{ij} = \begin{cases} \alpha & \text{if } (i,j) = (r,s) \\ \delta_{ij} & \text{otherwise.} \end{cases} \]
These matrices are called elementary matrices. Show that multiplying a matrix $M$ by an elementary matrix $E^\alpha(r)$ on the right is the same as taking column $r$ of $M$, multiplying it by $\alpha$, and adding it to column $s$. Show that for any matrix $M$, one may multiply $M$ by elementary matrices on both the left and right to obtain a diagonal matrix. Deduce that the determinant $\det: \text{Mat}_{m \times m} \to \mathbb{R}$ is the unique map such that the following conditions hold.

1. For any diagonal matrix $\text{diag} [\lambda_1, \lambda_2, \ldots, \lambda_m]$, one has 
   $$\det \text{diag} [\lambda_1, \lambda_2, \ldots, \lambda_m] = \lambda_1 \lambda_2 \cdots \lambda_m.$$ 

2. For any elementary matrix $E^\alpha(r)$, one has $\det E^\alpha(r) = 1$.

3. For any $M, N \in \text{Mat}_{m \times m}$, one has 
   $$\det(MN) = \det(M) \det(N).$$

**Exercise 40.** The trace $\text{tr} M$ of a square matrix $M$ is the sum of the entries along the main diagonal. Show that for any square matrix $M$, one has 
   $$\exp(\text{tr}(M)) = \det(\exp(M)).$$

**Exercise 41.** Fix vectors $v_1, v_2, \ldots, v_{n-1} \in \mathbb{R}^n$. Show that there exists a unique vector $x \in \mathbb{R}^n$ such that for any $w \in \mathbb{R}^n$, one has 
   $$x \cdot w = \det(v_1, v_2, \ldots, v_{n-1}, w).$$

In this situation, we call $x$ the cross product of $v_1, v_2, \ldots, v_{n-1}$, and we write 
   $$x = v_1 \times v_2 \times \cdots \times v_{n-1}.$$ 

Note that the cross product is a map $(\mathbb{R}^n)^{n-1} \to \mathbb{R}^n$; it does not make sense to speak of the cross product of fewer than $n-1$ vectors!

**Exercise 42.** Show that the cross product is an alternating multilinear map $(\mathbb{R}^n)^{n-1} \to \mathbb{R}^n$, and show that, for any vectors $v_1, v_2, \ldots, v_{n-1} \in \mathbb{R}^n$, one has 
   $$|v_1 \times v_2 \times \cdots \times v_{n-1}| = \sqrt{\det M},$$ 
   where $M$ is the $(n-1) \times (n-1)$ matrix whose $(i, j)$-th entry is $v_i \cdot v_j$. 