PROBLEM SET IV

DUE FRIDAY, 7 MARCH

For the next three problems, consider $\mathbb{R}^4$ (with standard basis $(e_0, e_1, e_2, e_3)$), equipped with the bilinear form $\eta$ given by the matrix

$$H = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},$$

so that $\eta(v, w) = v^T H w$. Consider the set $L$ of linear transformations $M : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ such that for any $v, w \in \mathbb{R}^4$,

$$\eta(M v, M w) = \eta(v, w).$$

This is called the Lorentz group. Its elements are called Lorentz transformations.

**Exercise 16.** Show that all Lorentz transformations are invertible.

**Exercise 17.** Check that for any $3 \times 3$ orthogonal matrix $R$ (i.e., a matrix $R$ such that $R^T R = \text{id}$), the block matrix

$$\begin{pmatrix}
1 & 0 \\
0 & R \\
\end{pmatrix},$$

is a Lorentz transformation.

**Exercise 18.** Write $\phi := \tanh^{-1}(u)$. Prove that the matrix

$$\Lambda_{\phi} = \begin{pmatrix}
\cosh(\phi) & -\sinh(\phi) & 0 & 0 \\
-\sinh(\phi) & \cosh(\phi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},$$

is a Lorentz transformation. (This is the Lorentz boost in the positive $x$-direction at speed $u$.)
Recall that the vector space $\text{Lin}(\mathbb{R}^m, \mathbb{R}^n)$ of linear maps $\mathbb{R}^m \to \mathbb{R}^n$ is isomorphic to the vector space $\text{Mat}_{n \times m}(\mathbb{R}) \cong \mathbb{R}^{mn}$ of $n \times m$ matrices. So we may speak of open sets of $\text{Lin}(\mathbb{R}^m, \mathbb{R}^n)$.

**Exercise 19.** Show that the map

$$K : \text{Lin}(\mathbb{R}^m, \mathbb{R}^n) \times \text{Lin}(\mathbb{R}^k, \mathbb{R}^m) \to \text{Lin}(\mathbb{R}^k, \mathbb{R}^n)$$

given by composition (equivalently, the map $\mathbb{R}^{mn+km} \cong \text{Mat}_{n \times m}(\mathbb{R}) \times \text{Mat}_{m \times k}(\mathbb{R}) \to \text{Mat}_{n \times k}(\mathbb{R}) \cong \mathbb{R}^{kn}$ given by matrix multiplication) is continuous.

**Exercise 20.** Consider the subset $\text{GL}(\mathbb{R}^n) \subset \text{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ of the isomorphisms $\mathbb{R}^n \to \mathbb{R}^n$. (Equivalently, you may consider the subset $\text{GL}_n(\mathbb{R}) \subset \text{Mat}_{n \times n}(\mathbb{R})$ of invertible $n \times n$ matrices.) Show that $\text{GL}(\mathbb{R}^n)$ is an open set.

**Exercise 21.** List all the vector subspaces of $\mathbb{R}^m$ that are open subsets. List all the vector subspaces of $\mathbb{R}^m$ that are closed subsets. Justify your answers.

**Exercise 22.** Consider the map $g : \mathbb{R}^2 \to \mathbb{R}$ given by the formula

$$g(x, y) := \begin{cases} \frac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

For any unit vector $v \in \mathbb{R}^2$, compute $\lim_{\varepsilon \to 0} g(\varepsilon v)$. Show that $g$ is not continuous at the origin. [In the first semester, we learned that a function $\mathbb{R} \to \mathbb{R}$ was continuous at a point $a \in \mathbb{R}$ just in case the left-hand limit and the right-hand limit both exist and are equal to the value of the function. Do things look different here?]

**Exercise 23.** Consider the unit $m$-sphere in $\mathbb{R}^{m+1}$:

$$S^m = \left\{ (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} \mid \sum_{i=0}^m x_i^2 = 1 \right\}.$$

Write down a homeomorphism $\mathbb{R}^m \to S^m - \{(1, 0, \ldots, 0)\}$.

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1Recall that we may define an isomorphism $M : \text{Lin}(\mathbb{R}^m, \mathbb{R}^n) \to \text{Mat}_{n \times m}(\mathbb{R})$ by $M(\phi) := (\phi(e_1), \phi(e_2), \ldots, \phi(e_m))$, where we think of the vectors $\phi(e_j)$ arranged vertically.