PROBLEM SET VIII
“FUNDAMENTALS”

DUE FRIDAY, 1 NOVEMBER 2013

In this problem set, we will use define some important functions using integrals. Use only the definitions given here in the following exercises.

**Definition.** Define a function \( \arctan : \mathbb{R} \rightarrow \mathbb{R} \) by the following formula:

\[
\arctan(x) = \int_0^x \frac{1}{1 + t^2} dt.
\]

**Exercise 48.** Show that \( \arctan \) is increasing and that for any \( x \in \mathbb{R} \), one has

\[
\arctan(-x) = -\arctan(x) \quad \text{and} \quad |\arctan(x)| \leq |x|.
\]

**Definition.** Now define \( \pi \) as the real number \( 4 \arctan(1) \).

**Exercise 49.** Show that the image \( \arctan(\mathbb{R}) \) is the open interval \((-\pi/2, \pi/2)\).

[Hint: show that for any \( \varepsilon > 0 \), there exists a real number \( N > 0 \) such that for any \( x > N \), one has \( \pi/2 - \varepsilon < \arctan(x) < \pi/2 \).]

**Exercise 50.** Prove that \( \arctan : \mathbb{R} \rightarrow (-\pi/2, \pi/2) \) is a bijection of class \( C^\infty \).

**Definition.** Denote by \( \mathbb{R}_+ \) the ray \((0, +\infty)\). Define a function \( \log : \mathbb{R}_+ \rightarrow \mathbb{R} \) by the formula

\[
\log(x) = \int_1^x \frac{1}{t} dt.
\]

**Exercise 51.** Prove that for any \( x, y \in \mathbb{R}_+ \), one has

\[
\log(xy) = \log(x) + \log(y).
\]

**Exercise 52.** Show that \( \log \) is increasing, and show that the image \( \log(\mathbb{R}_+) \) is \( \mathbb{R} \).

**Exercise 53.** Prove that \( \log : \mathbb{R}_+ \rightarrow \mathbb{R} \) is a bijection of class \( C^\infty \).

**Exercise 54.** Prove that the inverse \( \exp : \mathbb{R} \rightarrow \mathbb{R}_+ \) of the function \( \log \) is a differentiable function such that for any elements \( x, y \in \mathbb{R} \),

\[
\exp'(x) = \exp(x), \quad \exp(0) = 1, \quad \text{and} \quad \exp(x + y) = \exp(x) \exp(y).
\]