LECTURE 8. FILTERED OBJECTS

8.1. Notation. Denote by \( M \) the ordinary category whose objects of \( M \) are pairs \((m, i)\) consisting of an object \( m \in \Delta \) and an element \( i \in m \) and whose morphisms \((n, j) \to (m, i)\) are maps \( \varphi : m \to n \) of \( \Delta \) such that \( j \leq \varphi(i) \). This category comes equipped with a natural projection \( M \to \Delta^{op} \).

The nerve \( NM \) can be endowed with a pair structure by setting

\[
\left( NM \right)_{\uparrow} := NM \times_{\Delta^{op}} iN\Delta^{op}.
\]

Put differently, an edge of \( M \) is ingressive just in case it covers an equivalence of \( \Delta \).

8.2. Construction. Fix a quasicategory \( S \). For any morphism of pairs \( \mathcal{X} \to S \), define the simplicial set \( \mathcal{F}(\mathcal{X}/S) \) as the simplicial set over \( \Delta^{op} \times S \) satisfying the following universal property. We require, for any simplicial set \( K \) and any map \( \sigma : K \to \Delta^{op} \times S \), a bijection

\[
\text{Mor}_{/\Delta^{op} \times S}(K, \mathcal{F}(\mathcal{X}/S)) \cong \text{Mor}_{\text{Set}(2)}/(S, S)(K \times_{\Delta^{op}} NM, K \times_{\Delta^{op}} (NM)_{\uparrow}, (\mathcal{X}, \mathcal{X})_{\uparrow}),
\]

functorial in \( \sigma \). Here, \( (NM, (NM)_{\uparrow}) \) is the pair introduced in 8.1, and the category \( \text{Set}(2) \) is the one defined in ??.

For any object \( m \in \Delta \), write \( \mathcal{F}_m(\mathcal{X}/S) \) for the fiber of the projection \( \mathcal{F}(\mathcal{X}/S) \to \Delta^{op} \times S \) over \( m \). When \( S = \Delta \), write \( \mathcal{F}(\mathcal{X}) \) for \( \mathcal{F}(\mathcal{X}/S) \), and for any object \( m \in \Delta \), write \( \mathcal{F}_m(\mathcal{X}) \) for the fiber of the projection \( \mathcal{F}(\mathcal{X}) \to \Delta^{op} \times m \).

8.3. Definition. Suppose \( \mathcal{X} \to S \) a morphism of pairs. Then the vertices of \( \mathcal{F}(\mathcal{X}/S) \) will be called filtered objects of \( \mathcal{X} \) over \( S \); more specifically, the vertices of \( \mathcal{F}_m(\mathcal{X}/S) \) will be called \( m \)-filtered objects of \( \mathcal{X} \) over \( S \). The edges will be called morphisms of filtered objects of \( \mathcal{X} \) over \( S \).

8.4. Proposition. Suppose \( \mathcal{X} \to S \) a pair cocartesian fibration. Then the structure map \( \mathcal{F}(\mathcal{X}/S) \to \Delta^{op} \times S \) is a cocartesian fibration.

8.4.1. Corollary. For any pair cocartesian fibration \( \mathcal{X} \to S \), the simplicial set \( \mathcal{F}(\mathcal{X}/S) \) is a quasicategory.

8.5. The functor \( \Delta^{op} \times S \to \text{Cat}_{\infty} \) classified by the cocartesian fibration

\[
\mathcal{F}(\mathcal{X}/S) \to \Delta^{op} \times S
\]

assigns to any object \((m, s) \in \Delta^{op} \times S \) the quasicategory \( \text{Fun}_{\text{Pair}_{\infty}}((\Delta^{op})^{2}, \mathcal{F}_s) \) of filtered objects

\[
X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_m
\]

de \( \mathcal{X}_s \) and to any morphism \((\varphi, f) : (m, s) \to (n, t) \) of \( \Delta^{op} \times S \) the functor

\[
\text{Fun}_{\text{Pair}_{\infty}}((\Delta^{op})^{2}, \mathcal{X}_s) \to \text{Fun}_{\text{Pair}_{\infty}}((\Delta^{op})^{2}, \mathcal{X}_t)
\]

that carries the filtered object above to the filtered object

\[
f_{\varphi}(X_{\varphi(0)}) \hookrightarrow f_{\varphi}(X_{\varphi(1)}) \hookrightarrow \cdots \hookrightarrow f_{\varphi}(X_{\varphi(m)})
\]

We may endow the \( \infty \)-categories \( \mathcal{F}(\mathcal{X}/S) \) of filtered objects with a pair structure in a variety of ways, but we wish to focus on one pair structure that will retain good formal properties when we pass to the subcategory of totally filtered objects.
8.6. Definition. Suppose $\mathcal{C}$ a Waldhausen quasicategory, and consider the functors

$$s: \text{Fun}_{\text{Pair},\infty}^b((\Delta^1)^2, \mathcal{C}) \to \mathcal{C} \quad \text{and} \quad t: \text{Fun}_{\text{Pair},\infty}^b((\Delta^1)^2, \mathcal{C}) \to \mathcal{C}.$$ 

Clearly $s$ is a cocartesian fibration. We may now endow the quasicategory $\text{Fun}_{\text{Pair},\infty}^b((\Delta^1)^2, \mathcal{C})$ with a pair structure by letting $\text{Fun}_{\text{Pair},\infty}^b((\Delta^1)^2, \mathcal{C})_1$ be the smallest subcategory of $\text{Fun}_{\text{Pair},\infty}^b((\Delta^1)^2, \mathcal{C})$ containing the following classes of edges:

(8.6.1) any edge $\varphi$ such that both $s(\varphi)$ is an equivalence and $t(\varphi)$ is ingressive and

(8.6.2) any $s$-cocartesian edge that covers a cofibration.

Now suppose $p: \mathcal{X} \to S$ a Waldhausen cocartesian fibration. We may now endow the quasicategory $\mathcal{F}(\mathcal{X}/S)$ with a pair structure in the following manner. We let $\mathcal{F}(\mathcal{X}/S)_1 \subset \mathcal{F}(\mathcal{X}/S)$ be the smallest subcategory containing all equivalences as well as any edge $\varphi: \Delta^1 \to \mathcal{X}$ that covers a degenerate edge $\text{id}_{(m,s)}$ of $N\Delta^0 \times S$ — whence it can be identified as a functor $\varphi: \Delta^1 \to \text{Fun}_{\text{Pair},\infty}^b((\Delta^m)^2, \mathcal{X})$ — such that for any edge $\eta: \Delta^1 \to \Delta^m$, the edge

$$\Delta^1 \overset{\varphi}{\longrightarrow} \text{Fun}_{\text{Pair},\infty}^b((\Delta^m)^2, \mathcal{X}) \overset{\eta}{\longrightarrow} \text{Fun}_{\text{Pair},\infty}^b((\Delta^1)^2, \mathcal{X})$$

is ingressive in the sense above.

8.7. Proposition. Suppose $p: \mathcal{X} \to S$ a Waldhausen cocartesian fibration. The functor $\mathcal{F}(p): \mathcal{F}(\mathcal{X}/S) \to N\Delta^0 \times S$ is a Waldhausen cocartesian fibration.

Proof. With the structure on $\mathcal{F}(\mathcal{X}/S)$ described above, $\mathcal{F}(p)$ is easily seen to be a pair cocartesian fibration.

We claim that for any vertex $(m, s) \in N\Delta^0 \times S$, the pair $\text{Fun}_{\text{Pair},\infty}^b((\Delta^m)^2, \mathcal{X})$ is a Waldhausen quasicategory. Note that since $\mathcal{X}$ admits a zero object, so does $\text{Fun}_{\text{Pair},\infty}^b((\Delta^m)^2, \mathcal{X})$. For the remaining two axioms, one reduces immediately to the case where $m = 1$.

Now, to see that pushouts along cofibrations exist, one may note that cofibrations of $\text{Fun}_{\text{Pair},\infty}^b((\Delta^1)^2, \mathcal{X})$ are in particular cofibrations of $\text{O}(\mathcal{C})$, for which the existence of pushouts is clear. Second, to see that a pushout of a cofibration is again a cofibration, it suffices to see that a pushout of any edge of either of the classes (8.6.1) or (8.6.2) is of the same class. For the class (8.6.1), this follows from the fact that pushouts in $\text{Fun}_{\text{Pair},\infty}^b((\Delta^1)^2, \mathcal{X})$ are computed pointwise. A pushout of a morphism of the class (8.6.2) is a cube

$$X: (\Delta^1)^2 \times (\Delta^1)^2 \times (\Delta^1)^2 \to \mathcal{X},$$

in which the faces

$$X((\Delta^1)^2 \times (\Delta^1)^2 \times \Delta^0), \quad X((\Delta^1)^2 \times \Delta^0 \times (\Delta^1)^2), \quad \text{and} \quad X((\Delta^1)^2 \times \Delta^0 \times (\Delta^1)^2)$$

are all pushouts. By Quetzalcoatl, the face $X((\Delta^1)^2 \times (\Delta^1)^2 \times (\Delta^1)^2)$ must be a pushout as well; this is precisely the claim that the pushout is $s$-cocartesian.

For any $m \in \Delta$ and any edge $f: s \to t$ of $S$, since the functor $f_{[\Gamma]}^*: \mathcal{X} \to \mathcal{X}$ is exact, it follows directly that the functor $f_{[\Gamma]}^*: \text{Fun}_{\text{Pair},\infty}^b((\Delta^m)^2, \mathcal{X}) \to \text{Fun}_{\text{Pair},\infty}^b((\Delta^m)^2, \mathcal{X})$ is exact as well. Now for any fixed vertex $s \in S_0$ and any simplicial operator $\varphi: m \to m$ of $\Delta$, the functor $\varphi_{[\Gamma]}^*: \text{Fun}_{\text{Pair},\infty}^b((\Delta^m)^2, \mathcal{X}) \to \text{Fun}_{\text{Pair},\infty}^b((\Delta^n)^2, \mathcal{X})$ visibly carries cofibrations to cofibrations, and it preserves zero objects as well as any pushouts that exist, since limits and colimits are formed pointwise.

8.7.1. Corollary. The assignment $(\mathcal{X}/S) \mapsto \mathcal{F}(\mathcal{X}/S)$ defines a functor

$$\mathcal{F}: \text{Wald}_{\infty}^{\text{cocart}} \to \text{Wald}_{\infty}^{\text{cocart}}$$

covering the endofunctor $S: N\Delta^0 \times S \to \text{Cat}_{\infty}$.

Now we wish to isolate a certain class of filtered object.

8.8. Definition. Suppose $(\mathcal{X}/S)$ a Waldhausen cocartesian fibration. A filtered object $X: (\Delta^m)^2 \to \mathcal{X}$ will be said to be totally filtered if $X_0$ is a zero object of some fiber $\mathcal{X}$. 


8.9. **Notation.** Suppose $(\mathcal{X}/S)$ a Waldhausen cocartesian fibration. Denote by $\mathcal{F}(\mathcal{X}/S)$ the full subcategory of $\mathcal{F}(\mathcal{X}/S)$ spanned by the totally filtered objects, and for any object $m$ of $\Delta$, write $\mathcal{F}_m(\mathcal{X}/S)$ for the fiber of $\mathcal{F}(\mathcal{X}/S) \rightarrow N\Delta^{op}$ over $m \in \Delta$. When $S = \Delta^0$, write $\mathcal{F}(\mathcal{X})$ for $\mathcal{F}(\mathcal{X}/S)$, and for any object $m$ of $\Delta$, write $\mathcal{F}_m$ for the fiber of $\mathcal{F}(\mathcal{X}) \rightarrow N\Delta^{op}$ over $m \in \Delta$.

8.10. **Proposition.** Suppose $(\mathcal{X}/S)$ a Waldhausen cocartesian fibration. For any integer $m \geq 0$, the 0-th face map defines an equivalence of $\infty$-categories $\mathcal{F}_{1+m}(\mathcal{X}/S) \rightarrow \mathcal{F}_m(\mathcal{X}/S)$, and the map $\mathcal{F}_0(\mathcal{X}/S) \rightarrow S$ is an equivalence.

We may lift the pair structure on $\mathcal{F}_m(\mathcal{X}/S)$ along this equivalence to obtain a pair structure on $\mathcal{F}_{1+m}(\mathcal{X}/S)$. One sees that the inclusion

$$J_m : \mathcal{F}_m(\mathcal{X}/S) \hookrightarrow \mathcal{F}_m(\mathcal{X}/S) \simeq \mathcal{F}_{1+m}(\mathcal{X}/S)$$

is a strict functor of pairs, and we deduce the following.

8.10.1. **Corollary.** Suppose $\mathcal{X} \rightarrow S$ a Waldhausen cocartesian fibration. For any integer $m \geq 0$, the 0-th face map defines an essentially surjective functor $\mathcal{F}_{1+m}(\mathcal{X}/S) \rightarrow \mathcal{F}_m(\mathcal{X}/S)$ that is a left inverse in $b\text{Pair}_\infty$ to the inclusion $J_m$.

We now aim to show that for any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, the functor $\mathcal{F}(\mathcal{X}/S) \rightarrow N\Delta^{op} \times S$ is a Waldhausen cocartesian fibration. For this purpose, it is convenient to study the mapping cylinder $\mathcal{M}(\mathcal{X}/S)$ of the functor $\mathcal{F}(\mathcal{X}/S) \rightarrow \mathcal{F}(\mathcal{X}/S)$.

8.11. **Notation.** For any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, write $\mathcal{M}(\mathcal{X}/S)$ for the full subcategory of $\Delta^1 \times \mathcal{F}(\mathcal{X}/S)$ spanned by those pairs $(i, X)$ such that $X$ is totally filtered if $i = 1$. This quasicategory comes equipped with an inner fibration

$$\mathcal{M}(\mathcal{X}/S) \rightarrow \Delta^1 \times N\Delta^{op} \times S.$$

Let $\mathcal{M}(\mathcal{X}/S)_i \subset \mathcal{M}(\mathcal{X}/S)$ be the subcategory whose edges are maps $(i, X) \rightarrow (j, Y)$ such that $i = j$ and $X \rightarrow Y$ is a cofibration of $\mathcal{F}(\mathcal{X}/S)$.

Our first lemma is obvious by construction.

8.12. **Lemma.** For any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, the natural projection $\mathcal{M}(\mathcal{X}/S) \rightarrow \Delta^1$ is a pair cartesian fibration.

Our next lemma, however, is subtler.

8.13. **Lemma.** For any Waldhausen cocartesian fibration $\mathcal{X} \rightarrow S$, the natural projection $\mathcal{M}(\mathcal{X}/S) \rightarrow \Delta^1$ is a pair cocartesian fibration.

**Proof.** It suffices to show that for any vertex $(m, s) \in (N\Delta^{op} \times S)_0$, the inner fibration

$$q : \mathcal{M}_m(\mathcal{X}) \rightarrow \Delta^1$$

is a pair cocartesian fibration. Note that an edge $X \rightarrow Y$ of $\mathcal{M}_m(\mathcal{X})$ covering the nondegenerate edge $\sigma$ of $\Delta^1$ is $q$-cocartesian if and only if it is an initial object of the fiber $\mathcal{M}_m(\mathcal{X})_{X/} \times_{\Delta^1/} \{\sigma\}$. If $m = 0$, then the map

$$\mathcal{M}_0(\mathcal{X})_{X/} \rightarrow \Delta^1_{X/}$$

is a trivial fibration, so the fiber over $\sigma$ is a contractible Kan complex. Let us now induct on $m$; assume that $m > 0$ and that the functor $p : \mathcal{M}_{m-1}(\mathcal{X}) \rightarrow \Delta^1$ is a cocartesian fibration. It is easy to see that the inclusion $\{0, 1, \ldots, m-1\} \hookrightarrow m$ induces an inner fibration $p : \mathcal{M}_m(\mathcal{X}) \rightarrow \mathcal{M}_{m-1}(\mathcal{X})$ such that $q = p \circ p$. It suffices to observe that for any object $X$ of $\mathcal{M}_m(\mathcal{X})$ and any $p$-cocartesian edge $\eta : \rho(X) \rightarrow Y'$ covering $\sigma$, there exists a $\rho$-cocartesian edge $X \rightarrow Y$ of $\mathcal{M}_m(\mathcal{X})$ covering $\eta$. 

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We now show that \( q \) is a pair cocartesian fibration. Suppose

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
\]

is a square of \( \mathcal{M}(\mathcal{X}_t) \) in which \( X' \longrightarrow X \) and \( Y' \longrightarrow Y \) are \( q \)-cocartesian morphisms and \( X \longrightarrow Y \) is a fibration.

We aim to show that for any edge \( \eta: \Delta^{(p,q)} \longrightarrow \Delta^m \), the morphism \( X'|\Delta^{(p,q)} \longrightarrow Y'|\Delta^{(p,q)} \) is a fibration. For this, we may factor \( X \longrightarrow Y \) as

\[
X \longrightarrow Z \longrightarrow Y,
\]

where \( Z|\Delta^{(0,\ldots,p)} = Y|\Delta^{(0,\ldots,p)} \), and for any \( r > p \), the edge \( X|\Delta^{(p,r)} \longrightarrow Z|\Delta^{(p,r)} \) is cocartesian. Now choose a cocartesian morphism \( Z' \longrightarrow Z \) as well. The proof is now completed by the following observations.

(8.13.1) Since the morphism \( X|\Delta^{(p,q)} \longrightarrow Z|\Delta^{(p,q)} \) is of type (8.6.2), it follows by Quetzalcoatl \( X'|\Delta^{(p,q)} \longrightarrow Z'|\Delta^{(p,q)} \) is of type (8.6.2) as well.

(8.13.2) The morphism \( Z|\Delta^{(p,q)} \longrightarrow Y|\Delta^{(p,q)} \) is of type (8.6.1) and the morphism \( Z'_p \longrightarrow X'_p \) is an equivalence; so again by Quetzalcoatl, the morphism \( Z'|\Delta^{(p,q)} \longrightarrow Y'|\Delta^{(p,q)} \) is of type (8.6.1).

Together, these lemmas exhibit, for any Waldhausen cocartesian fibration \( \mathcal{X} \longrightarrow S \), an adjunction

\[
F: \mathcal{I}(\mathcal{X}/S) \rightleftarrows \mathcal{J}(\mathcal{X}/S): J
\]

er over \( N\Delta^{op} \times S \) in which both \( F \) and \( J \) are functors of pairs.

8.14. Theorem. Suppose \( \mathcal{X} \longrightarrow S \) a Waldhausen cocartesian fibration. Then the functor \( \mathcal{I}(\mathcal{X}/S) \longrightarrow N\Delta^{op} \times S \) is a Waldhausen cocartesian fibration.

Proof. We first show that the functor \( \mathcal{I}(\mathcal{X}/S) \longrightarrow N\Delta^{op} \times S \) is a cocartesian fibration by proving the stronger assertion that the inner fibration

\[
p: \mathcal{M}(\mathcal{X}/S) \longrightarrow \Delta^1 \times N\Delta^{op} \times S
\]

is a cocartesian fibration. By 8.7, the map

\[
(8.14.1) \quad \Delta^{(0)} \times_{\Delta^1} \mathcal{M}(\mathcal{X}/S) \longrightarrow \Delta^{(0)} \times N\Delta^{op} \times S
\]

is a cocartesian fibration. By 8.13, for any vertex \( (m,s) \in (N\Delta^{op} \times S)_0 \), the map

\[
(8.14.2) \quad \mathcal{M}(\mathcal{X}/S) \times_{N\Delta^{op} \times S} \{ (m,s) \} \longrightarrow \Delta^1 \times \{ (m,s) \}
\]

is a cocartesian fibration. For any \( m \in \Delta \), the map

\[
(\Delta^{(1)} \times \{ m \}) \times_{\Delta^1 \times N\Delta^{op}} \mathcal{M}(\mathcal{X}/S) \longrightarrow \Delta^{(1)} \times \{ m \} \times S
\]

is easily seen to be a cocartesian fibration. Now one may complete the proof that \( p \) is a cocartesian fibration by showing that for any vertex \( s \in S_0 \), any simplicial operator \( \varphi: n \longrightarrow m \), and any totally \( m \)-filtered object \( X \) of \( \mathcal{X}_t \), then there exists a \( p \)-cartesian morphism \( \{ 1, X \} \longrightarrow \{ 1, Y \} \) of \( \mathcal{I}(\mathcal{X}/S) \) covering \( (id_1, \varphi, id_s) \). [Exercise!]

From 8.10 and 8.7 it follows that the fibers of \( \mathcal{I}(\mathcal{X}/S) \longrightarrow N\Delta^{op} \times S \) are all Waldhausen \( \infty \)-categories. For any \( m \in \Delta \) and any edge \( f: s \longrightarrow t \) of \( S \), the functor \( j_{m,s}: \mathcal{I}_m(\mathcal{X}_t) \longrightarrow \mathcal{I}_s(\mathcal{X}_t) \) is exact, whence it follows by 8.10 that the functor

\[
f_{\varphi,1}: \mathcal{I}_m(\mathcal{X}_t) \simeq \text{Fun}_{\text{pair}}^b((\Delta^{m-1})^2, \mathcal{X}_t) \longrightarrow \text{Fun}_{\text{pair}}^b((\Delta^{m-1})^2, \mathcal{X}_t) \simeq \mathcal{I}_m(\mathcal{X}_t)
\]

is exact, just as in the proof of 8.7. Now for any fixed vertex \( s \in S_0 \) and any simplicial operator \( \varphi: n \longrightarrow m \) of \( \Delta \), the functor \( \varphi_{\varphi,1}: \mathcal{I}_m(\mathcal{X}_t) \longrightarrow \mathcal{I}_n(\mathcal{X}_t) \) is by construction the composite

\[
\mathcal{I}_m(\mathcal{X}_t) \xrightarrow{j_{m,s}} \mathcal{I}_m(\mathcal{X}_t) \xrightarrow{\varphi_{\varphi,1}} \mathcal{I}_n(\mathcal{X}_t) \xrightarrow{f_{\varphi,1}} \mathcal{I}_n(\mathcal{X}_t),
\]

and as \( \varphi_{\varphi,1} \) is an exact functor (8.7), we are reduced to checking that the functors \( j_{m,s} \) and \( f_{\varphi,1} \) are each exact functors.
For this, it is clear that \( f_{m,j} \) and \( f_{n,i} \) each carry zero objects to zero objects, and as \( F_{n,i} \) is a left adjoint, it preserves any pushout squares that exist in \( \mathcal{F}_n(\mathcal{X}) \). Moreover, a pushout square in \( \mathcal{F}_m(\mathcal{X}) \) is nothing more than a pushout square in \( \mathcal{F}_m(\mathcal{X}) \) of totally \( m \)-filtered objects; hence \( f_{m,j} \) preserves pushouts along cofibrations. \( \square \)

For any Waldhausen cocartesian fibration \( \mathcal{X} \to S \), write \( \mathcal{F}(\mathcal{X}): N\Delta^{op} \times S \to \text{Wald}_\infty \) for the functor classified by the Waldhausen cocartesian fibration \( \mathcal{I}(\mathcal{X}/S) \to N\Delta^{op} \times S \), and write \( \mathcal{T}(\mathcal{X}): N\Delta^{op} \times S \to \text{Wald}_\infty \) for the functor classified by the Waldhausen cocartesian fibration \( \mathcal{I}(\mathcal{X}/S) \to N\Delta^{op} \times S \). An instant consequence of the construction of the functoriality of \( \mathcal{I} \) in the proof above is the following.

8.14.1 Corollary. The functors \( F_m: \mathcal{F}_m(\mathcal{X}/S) \to \mathcal{I}_m(\mathcal{X}/S) \) assemble to a natural transformation

\[
F: \mathcal{F}(\mathcal{X}) \to \mathcal{T}(\mathcal{X}).
\]

Note, however, that it is not the case that the functors \( f_m \) assemble to a natural transformation of this kind.

8.15. For any Waldhausen cocartesian fibration \( \mathcal{X} \to S \), the functor \( N\Delta^{op} \times S \to \text{Wald}_\infty \) classified by the Waldhausen cocartesian fibration \( \mathcal{I}(\mathcal{X}/S) \to N\Delta^{op} \times S \) assigns to any object \((m,s)\) the quasicategory of totally filtered objects

\[
0 \simeq X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_m
\]

of \( \mathcal{X} \). For any morphism \((\varphi, f): (m, s) \to (n, t)\) of \( N\Delta^{op} \times S \), the induced functor carries the totally filtered object \( X \) above to a representative of the totally filtered object

\[
0 \simeq f_\ast(X(0)/X(0)) \leftarrow f_\ast(X(1)/X(0)) \leftarrow \cdots \leftarrow f_\ast(X(n)/X(0)).
\]

Giving a definition directly in this style requires solving certain homotopy-coherence problems. Waldhausen showed that solving these coherence problems amounts to making compatible choices of successive quotients. We have chosen instead to avoid these issues by means of the theory of fibrations.

The assignments

\[
(\mathcal{X}/S) \to (\mathcal{F}(\mathcal{X}/S)/(N\Delta^{op} \times S)) \quad \text{and} \quad (\mathcal{X}/S) \to (\mathcal{I}(\mathcal{X}/S)/(N\Delta^{op} \times S))
\]

define endofunctors of \( \text{Wald}_{\infty}^{\text{cocart}} \) over the endofunctor \( S \to N\Delta^{op} \times S \) of \( \text{Cat}_\infty \). We may descend these functors to endofunctors of the quasicategory of virtual Waldhausen \( \infty \)-categories.

8.16. Lemma. The functors \( \text{Wald}_{\infty} \to \text{Wald}_{\infty}^{\text{cocart}}/(N\Delta^{op}) \) given by

\[
\mathcal{C} \leftarrow (\mathcal{F}(\mathcal{C})/(N\Delta^{op})) \quad \text{and} \quad \mathcal{C} \leftarrow (\mathcal{I}(\mathcal{C})/(N\Delta^{op}))
\]

preserve filtered colimits.

8.17. Construction. One may compose the functors

\[
\mathcal{F}: \text{Wald}_{\infty} \to \text{Wald}_{\infty}^{\text{cocart}}/(N\Delta^{op}) \quad \text{and} \quad \mathcal{I}: \text{Wald}_{\infty} \to \text{Wald}_{\infty}^{\text{cocart}}/(N\Delta^{op})
\]

with the functor \( \cdot |_{N\Delta^{op}} \): the results are models for the functors \( \text{Wald}_{\infty} \to \text{VWald}_{\infty} \) that assign to any Waldhausen quasicategory the geometric realizations of the simplicial virtual Waldhausen \( \infty \)-categories \( \mathcal{F}_s(\mathcal{C}) \) and \( \mathcal{I}_s(\mathcal{C}) \). In particular, these composites are \( \omega \)-continuous functors \( \text{Wald}_{\infty} \to \text{VWald}_{\infty} \), whence one obtains essentially unique endofunctors \( \mathcal{F} \) and \( \mathcal{I} \) of \( \text{VWald}_{\infty} \) that preserve sifted colimits such that the squares

\[
\begin{array}{ccc}
\text{Wald}_{\infty} & \xrightarrow{\mathcal{F}} & \text{Wald}_{\infty}^{\text{cocart}}/(N\Delta^{op}) \\
\downarrow j & \downarrow \cdot |_{N\Delta^{op}} & \downarrow j \\
\text{VWald}_{\infty} & \xrightarrow{\mathcal{F}} & \text{VWald}_{\infty}
\end{array}
\]

\[
\begin{array}{ccc}
\text{Wald}_{\infty} & \xrightarrow{\mathcal{I}} & \text{Wald}_{\infty}^{\text{cocart}}/(N\Delta^{op}) \\
\downarrow j & \downarrow \cdot |_{N\Delta^{op}} & \downarrow j \\
\text{VWald}_{\infty} & \xrightarrow{\mathcal{I}} & \text{VWald}_{\infty}
\end{array}
\]

commute via a specified homotopy.
8.18. Now the natural transformation $F$ from Cor. 8.14.1 also descends to a natural transformation $F: \mathcal{F} \to \mathcal{I}$ of endofunctors of $\mathbf{VWald}_\infty$.

As it happens, the functor $F: \mathbf{VWald}_\infty \to \mathbf{VWald}_\infty$ is not particularly exciting:

8.19. **Proposition.** For any virtual Waldhausen quasicategory $\mathcal{X}$, the virtual Waldhausen quasicategory $\mathcal{F}(\mathcal{X})$ is terminal.

**Proof.** For any Waldhausen quasicategory $\mathcal{C}$, the virtual Waldhausen quasicategory $|\mathcal{F}(\mathcal{C})|_{\mathcal{N}\Delta^\mathcal{C}}$ is by definition a functor $\mathbf{Wald}_\infty \to \mathbf{Kan}$ that assigns to any compact Waldhausen quasicategory $\mathcal{Y}$ the geometric realization of the simplicial space

$$m \mapsto \mathbf{Wald}_\infty^A(\mathcal{Y}, \mathcal{F}_m(\mathcal{C})).$$

By 8.10, this is the path space of the simplicial space $m \mapsto \mathbf{Wald}_\infty^A(\mathcal{Y}, \mathcal{I}_m(\mathcal{C}))$. □

This result permits us to regard the virtual Waldhausen quasicategory $\mathcal{F}(\mathcal{X})$ as a cone on the virtual Waldhausen quasicategory $\mathcal{X}$. With this perspective, we will view the induced morphism $F: \mathcal{F}(\mathcal{X}) \to \mathcal{I}(\mathcal{X})$ induced by the functor $F$ as a suitable “quotient” of $\mathcal{F}(\mathcal{X})$ that identifies $\mathcal{I}(\mathcal{X})$ as a suspension of $\mathcal{X}$ in a suitable quasicategory. We shall return to this point in the next section.

The essential unicity of the extensions $\mathcal{F}$ and $\mathcal{I}$ to $\mathbf{VWald}_\infty$ now implies the following.

8.20. **Proposition.** If $S$ is a small sifted quasicategory, then the squares

$$
\begin{array}{ccc}
\mathbf{Wald}_\infty^\text{cocont} & \longrightarrow & \mathbf{Wald}_\infty^\text{cocont} \\
\mathcal{F} \downarrow & & \downarrow \mathcal{F} \\
\mathbf{VWald}_\infty & \longrightarrow & \mathbf{VWald}_\infty
\end{array}
\quad
\begin{array}{ccc}
\mathbf{Wald}_\infty^\text{cocont} & \longrightarrow & \mathbf{Wald}_\infty^\text{cocont} \\
\mathcal{I} \downarrow & & \downarrow \mathcal{I} \\
\mathbf{VWald}_\infty & \longrightarrow & \mathbf{VWald}_\infty
\end{array}
$$

commute via a specified homotopy.

Of course this is no surprise for $\mathcal{F}: \mathbf{VWald}_\infty \to \mathbf{VWald}_\infty$, as we have already seen that $\mathcal{F}$ is an essentially constant functor whose value at any object is terminal.

8.21. **Construction.** Suppose $\mathcal{C}$ a Waldhausen category. For any integer $m \geq 0$, one has, corresponding to the unique edge $m \to 0$ of $\Delta$, exact functors

$$E_m: \mathcal{C} \simeq \mathcal{F}_0(\mathcal{C}) \to \mathcal{F}_m(\mathcal{C}) \quad \text{and} \quad E'_m: 0 \simeq \mathcal{I}_0(\mathcal{C}) \to \mathcal{I}_m(\mathcal{C}).$$

We have a commutative square

$$
\begin{array}{ccc}
\mathcal{F}_0(\mathcal{C}) & \longrightarrow & \mathcal{F}_m(\mathcal{C}) \\
E_m \downarrow & & \downarrow \mathcal{E}_m \\
\mathcal{F}_m(\mathcal{C}) & \longrightarrow & \mathcal{I}_m(\mathcal{C})
\end{array}
$$

The functor $E_m$ may equivalently be described as the $m$-th component of the counit of an adjunction

$$C: \mathbf{Wald}_\infty \rightleftarrows \mathbf{Wald}_\infty^\text{cocont} / \mathcal{N}\Delta^\mathcal{C}: R,$$

where $C$ is the “constant” functor $\mathcal{C} \hookrightarrow \mathcal{C} \times \mathcal{N}\Delta^\mathcal{C}$; hence $E_m$ is natural in $\mathcal{C}$ and functorial in $m \in \mathcal{N}\Delta^\mathcal{C}$. Consequently there is an induced natural transformation $E: \id \to \mathcal{F}$ of endofunctors of $\mathbf{VWald}_\infty$, and the square above descends to a square of natural transformations

$$
\begin{array}{ccc}
\id & \longrightarrow & 0 \\
E \downarrow & & \downarrow \\
\mathcal{F} & \longrightarrow & \mathcal{I},
\end{array}
$$

where $0$ is the essentially constant endofunctor of $\mathbf{VWald}_\infty$ whose value at any object is a zero object.