6.1. **Exercise.** Suppose $X$ a quasicategory. Consider the cartesian fibration $s: \mathcal{O}(X) \to X$. Show that an edge $s: \Delta^1 \to \mathcal{O}(X)$ is $s$-cocartesian just in case the corresponding diagram $(\Lambda^k_0)^\perp \to \Delta^1 \times \Delta^1 \to X$ is a colimit. Conclude that $s$ is a cocartesian fibration if and only if $X$ admits all pushout diagrams.

This exercise suggests that cartesian fibrations that are also cocartesian have special significance. This is true!

6.2. **Definition.** Suppose $p: X \to \Delta^1$ a cartesian fibration, and suppose $\alpha: C \xrightarrow{\sim} X_0$ and $\beta: D \xrightarrow{\sim} X_1$ are equivalences of quasicategories. Then a functor $g: D \to C$ is said to be attached to $p$ via $\alpha$ and $\beta$ just in case there is a map $G: D \times \Delta^1 \to X$ over $\Delta^1$ such that the obvious diagram

$$
\begin{array}{c}
D & \xleftarrow{D} & D \times \Delta^1 & \xleftarrow{D} & D \\
\beta \downarrow & & (\alpha \circ g) \downarrow & & \downarrow \\
X_0 & \xrightarrow{g} & X & \xleftarrow{X_1} & \\
\downarrow & & \downarrow & & \\
\Delta^{(0)} & \xrightarrow{\Delta^1} & \Delta^1 & \xleftarrow{\Delta^{(1)}} & \\
\end{array}
$$

commutes, and, moreover, such that for every vertex $x \in D_0$, the edge $G(\{x\} \times \Delta^1)$ is a $p$-cartesian edge of $X$.

We will show now that such attached functors exist and are essentially unique.

6.3. **Proposition.** Suppose $g: D \to C$ a functor between quasicategories.

(6.3.1) There exist a cartesian fibration $p: X \to \Delta^1$ and equivalences of quasicategories $\alpha: C \xrightarrow{\sim} X_0$ and $\beta: D \xrightarrow{\sim} X_1$ such that $g$ is attached to $p$ via $\alpha$ and $\beta$.

(6.3.2) Suppose $p: X \to \Delta^1$ and $p': X' \to \Delta^1$ are cartesian fibrations, and suppose $\varphi: X' \to X$ an equivalence of quasicategories over $\Delta^1$. Then $g$ is attached to $X'$ via equivalences $\alpha': C \xrightarrow{\sim} X_0'$ and $\beta': D \xrightarrow{\sim} X_1'$ if and only if $g$ is attached to $X$ via equivalences $\alpha := \varphi \circ \alpha': C \xrightarrow{\sim} X_0$ and $\beta := \varphi \circ \beta': D \xrightarrow{\sim} X_1$.

(6.3.3) Suppose $p': X' \to \Delta^1$ and $p'': X'' \to \Delta^1$ are cartesian fibrations. Suppose that $g$ is attached to $p'$ via equivalences $\alpha': C \xrightarrow{\sim} X_0'$ and $\beta': D \xrightarrow{\sim} X_1'$, and suppose that $g$ is attached to $p''$ via equivalences $\alpha'': C \xrightarrow{\sim} X_0''$ and $\beta'': D \xrightarrow{\sim} X_1''$. Then there exists a cartesian fibration $p: X \to \Delta^1$, equivalences $\alpha: C \xrightarrow{\sim} X_0$ and $\beta: D \xrightarrow{\sim} X_1$, and equivalences of quasicategories $\varphi: X \xrightarrow{\sim} X'$ and $\psi: X \xrightarrow{\sim} X''$ over $\Delta^1$ such that $g$ is associated to $p$ via $\alpha$ and $\beta$, and the diagrams of equivalences

$$
\begin{array}{c}
C \\
\downarrow \\
X_0' \leftarrow X_0 \xrightarrow{\alpha} X_0'' \\
\downarrow \\
X_1' \leftarrow X_1 \xrightarrow{\beta} X_1'' \\
\end{array}
and
\begin{array}{c}
D \\
\downarrow \\
X_0' \leftarrow X_0 \xrightarrow{\alpha} X_0'' \\
\downarrow \\
X_1' \leftarrow X_1 \xrightarrow{\beta} X_1'' \\
\end{array}
$$

commute.

**Proof.** To prove the first item, use the cartesian model structure on the category $sSet^+_{/\Delta^1}$ to factor

$$(D^\perp \times (\Delta^1)^\perp) \cup D^\perp \times (\Delta^{(0)})^\perp \xrightarrow{\tilde{C}} (\Delta^1)^\perp$$

as a trivial cofibration followed by a cartesian fibration.

On to the second item. In one direction, one may simply compose the map $D \times \Delta^1 \to X'$ exhibiting the attachment of $g$ to $p'$ with $\varphi$ to obtain a map $D \times \Delta^1 \to X$ exhibiting the attachment of $g$ to $p$. In the other direction, we...
aim to extend a map \( D \times \partial \Delta^1 \to X' \) to a map \( D \times \Delta^1 \to X' \). Since this can be done in \( \text{hSet}_{/\Delta^1}^+ \), and since \( X \) is fibrant and everything is cofibrant therein, the proof is complete.

Finally, for the third item, consider the square

\[
\begin{array}{ccc}
(D \times \Delta^1) \cup (D \times \Delta^0) & \to & X' \\
\downarrow & & \downarrow \\
X & \to & \Delta^1, 
\end{array}
\]

where the vertical map on the left hand side is the trivial cofibration from the first part, and the top map exhibits the attachment of \( g \) to \( p' \). One may prove directly that the top map is an equivalence. Now the lift \( X \to X' \) exists, and the result is proved. \( \square \)

Conversely, we have the following.

6.4. **Proposition.** Suppose \( p : X \to \Delta^1 \) a cartesian fibration. There is a functor \( g : X_1 \to X_0 \) attached to \( p \) via the identities. Moreover, any other functor \( g' : X_1 \to X_0 \) attached to \( p \) via the identities is equivalent to \( g \).

**Proof.** Consider the square

\[
\begin{array}{ccc}
X_1^2 \times (\Delta^1)^2 & \to & X^2 \\
\downarrow & & \downarrow \\
X_0^2 \times (\Delta^1)^2 & \to & (\Delta^1)^2.
\end{array}
\]

The lift exists, and the fiber of this lift over \( \Delta^0 \) is a map \( X_1 \to X_0 \). It’s easy to see that this a functor attached to \( p \) via the identities. We leave the unicity proof as an exercise. \( \square \)

6.5. **Definition.** Suppose \( C \) and \( D \) quasicategories. An **adjunction**

\[
f : C \rightleftarrows D : u
\]

is a map \( p : X \to \Delta^1 \) that is both a cartesian fibration and a cocartesian fibration, along with equivalences \( \alpha : C \to X_0 \) and \( \beta : D \to X_1 \) such that \( u \) is attached to \( p \) via \( \alpha \) and \( \beta \), and \( f^{\text{op}} \) is attached to \( p^{\text{op}} \) via \( \alpha^{\text{op}} \) and \( \beta^{\text{op}} \).

Let us now unpack this to explain how we may extract the **unit** \( \text{id} \to u \circ f \) of an adjunction \( f : C \rightleftarrows D : u \) from this definition. There exist maps \( F : C \times \Delta^1 \to X \) and \( U : D \times \Delta^1 \to X \) such that:

1. \( F((C \times \Delta^0)) = \text{id}_C \);
2. \( F((C \times \Delta^1)) = f \);
3. for every object \( x \in C_0 \), the edge \( F(\{x\} \times \Delta^1) \) is \( p \)-cocartesian;
4. \( U((C \times \Delta^1)) = \text{id}_D \);
5. \( U((C \times \Delta^0)) = u \);
6. for every object \( y \in D_0 \), the edge \( U(\{y\} \times \Delta^1) \) is \( p \)-cartesian.

Now consider the rectangle

\[
\begin{array}{ccc}
C \times \Delta^1 & \to & C \times \Delta^{1,2} & \to & D \times \Delta^{1,2} \\
\downarrow & & \downarrow U & & \downarrow \\
C \times \Delta^{0,2} & \to & X.
\end{array}
\]

This gives us a map \( H' : C \times \Delta_2 \to X \) with the property that for any \( x \in C_0 \), the edge \( H'((x) \times \Delta^{1,2}) \) is \( p \)-cartesian. Hence there is an extension of this map to a map \( H : C \times \Delta^2 \to X \). The unit of the adjunction is then the natural transformation \( \eta := H((C \times \Delta^{0,1})) \). You can prove directly for any \( x \in C_0 \) and \( y \in D_0 \) that the composite

\[
\text{Map}_D(fx, y) \to \text{Map}_C(ufx, uy) \to \text{Map}_C(x, uy)
\]
6.6. **Lemma.** A cartesian fibration \( X \to S \) is also a cocartesian fibration if and only if, for every edge \( s \to t \) of \( S \), an attached functor \( X_t \to X_s \) admits a left adjoint.

Let’s now move along to the study of the sorts of quasicategories that we will input into \( K \)-theory. We will at times have to return to our foundations in what follows, but we’ve got enough now to get started.

6.7. **Definition.** A pair of \( \infty \)-categories \( (C, C_t) \) is an \( \infty \)-category equipped with a subcategory \( C_t \subset C \) that contains \( \iota C \). Morphisms of \( C_t \) will be called ingressive or cofibrations. A functor of pairs \( (\psi, \psi_t) : (C, C_t) \to (D, D_t) \) is a commutative square

\[
\begin{array}{c}
C_t \\
\downarrow \\
C
\end{array} \quad \begin{array}{c}
\psi_t \\
\downarrow \\
\psi
\end{array} \quad \begin{array}{c}
D_t \\
\downarrow \\
D
\end{array}
\]

(6.7.1)

it is said to be strict if the diagram (6.7.1) is a pullback diagram in \( \mathbf{QCat} \).

More generally, a functor \( D \to C \) of quasicategories will be said to exhibit a pair structure on \( C \) if it factors as an equivalence \( D \to E \) followed by an inclusion \( E \to C \) of a subcategory such that \( (C, E) \) is a pair.

6.8. **Exercise.** Show that a functor \( \psi : D \to C \) of quasicategories exhibits a pair structure on \( C \) if and only if it is a homotopy monomorphism in \( \mathbf{QCat} \) that induces an equivalence \( \iota D \to \iota C \).

We write \( \text{Pair}_\infty \) for the full subcategory of \( \mathcal{O}(\mathbf{Cat}_\infty) \) spanned by those functors \( D \to C \) that exhibit a pair structure on \( C \). We may also describe an ordinary category of pairs of \( \infty \)-categories, and declare a functor of pairs \( (\psi, \psi_t) : (C, C_t) \to (D, D_t) \) to be a weak equivalence just in case both \( \psi \) and \( \psi_t \) are equivalences.

6.9. **Example.** Any quasicategory \( C \) can be endowed with a maximal pair structure \((X, X)\) and a minimal pair structure \((X, \iota X)\). If we speak of a quasicategory as a pair without more comment, then we will assume that it is the maximal structure of which we are speaking.

6.10. **Example.** Denote by \( \Lambda_0 \mathcal{Q}^2 \) the pair \((\Lambda_0^2, \Delta^{(0,1)} \sqcup \Delta^{(2)})\):

\[
\begin{array}{c}
0 \\
\downarrow \\
1 \\
\downarrow \\
2
\end{array}
\]

Denote by \( \mathcal{Q}^2 \) the pair \(((\Lambda_0^2)^{\triangleright}, \Delta^{(0,1)} \sqcup \Delta^{(2,\infty)}) \cong (\Delta^1 \times \Delta^1, (\Delta^{(0)} \sqcup \Delta^{(1)}) \times \Delta^1)\):

\[
\begin{array}{c}
0 \\
\downarrow \\
1 \\
\downarrow \\
2 \\
\downarrow \\
\infty
\end{array}
\]

There is an obvious inclusion of pairs \( \Lambda_0 \mathcal{Q}^2 \hookrightarrow \mathcal{Q}^2 \).

Roughly speaking, the cofibrations of a pair will be those along which pushouts are reasonable. To make this more precise, for any pair \((C, C_t)\) be a pair. Then the arrow quasicategory \( \mathbf{O}(C) \) admits a pair structure as well, given by a pullback

\[
\begin{array}{c}
\mathbf{O}_t(C) \\
\downarrow \\
\mathbf{O}_t(C)
\end{array} \quad \begin{array}{c}
\mathbf{O}(C) \\
\downarrow \\
C_t \\
\downarrow \\
C
\end{array}
\]
We are nearly ready to describe the basic inputs of $K$-theory. We only need one more notion.

6.11. **Definition.** A zero object of a quasicategory is an object that is both initial and terminal.

6.12. **Exercise.** Show that a quasicategory has a zero object if and only if it has an initial object $\emptyset$, a terminal object $*$, and an edge $* \to \emptyset$.

Now we have our definition.

6.13. **Definition.** A Waldhausen $\infty$-category $(\mathcal{C}, \mathcal{C}_1)$ is a pair of essentially small $\infty$-categories such that the following axioms hold.

1. The $\infty$-category $\mathcal{C}$ contains a zero object.
2. For any zero object $0$, any morphism $0 \to X$ is ingressive.
3. The source functor $\mathcal{C}_1 \to \mathcal{C}$ is a cocartesian fibration, and an edge $\eta \in \mathcal{C}_1(\mathcal{C})$ is $s$-cocartesian only if $t(\eta)$ is ingressive.

Call a functor of pairs $\psi: \mathcal{C} \to \mathcal{D}$ between two Waldhausen $\infty$-categories exact if it satisfies the following conditions.

1. The underlying functor $\psi$ carries zero objects of $\mathcal{C}$ to zero objects of $\mathcal{D}$.
2. In the diagram

$$
\begin{array}{ccc}
\mathcal{C}_1(\mathcal{C}) & \to & \mathcal{C}_1(\mathcal{D}) \\
\downarrow^s \psi & & \downarrow^s \psi \\
\mathcal{C}_1 & \to & \mathcal{D}_1,
\end{array}
$$

the functor $\mathcal{C}_1(\psi)$ sends $s_\mathcal{C}$-cocartesian edges to $s_\mathcal{D}$-cocartesian edges.

A Waldhausen subcategory of a Waldhausen $\infty$-category $\mathcal{C}$ is a subpair $\mathcal{D} \subset \mathcal{C}$ such that $\mathcal{D}$ is a Waldhausen $\infty$-category, and the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ is exact.

The conditions demanded of a Waldhausen $\infty$-category $(\mathcal{C}, \mathcal{C}_1)$ can be rephrased in the following manner: there is a zero object in $\mathcal{C}$ that is initial in $\mathcal{C}_1$, and pushouts of cofibrations exist and are cofibrations. This last point can again be rephrased as the condition that the morphism of pairs $\Lambda_0 \mathcal{D}^2 \to \mathcal{D}^2$ (6.10) induces an equivalence of $\infty$-categories

$$
\text{Colim}(\mathcal{D}^2, \mathcal{C}) \times_{\text{Fun}(\mathcal{D}^2, \mathcal{C})} \text{FunPair}_\infty(\mathcal{D}^2, \mathcal{C}) \to \text{FunPair}_\infty(\Lambda_0 \mathcal{D}^2, \mathcal{C}).
$$

An exact functor $\mathcal{C} \to \mathcal{D}$ is now one that preserves cofibrations, zero objects, and pushouts along cofibrations.

6.14. **Example.** There are many examples of Waldhausen $\infty$-categories; here’s a short list.

1. The ordinary category of pointed finite sets. The cofibrations are just monomorphisms.
2. The nerve of the relative category of pointed finite CW complexes, with its maximal pair structure.
3. The category of finitely generated projective $R$-modules with “admissible monomorphisms,” i.e., monomorphisms whose cokernels are projective.
4. The nerve of the relative category of perfect complexes with its maximal pair structure.

Right away, we can prove Waldhausen’s approximation theorem in this context.

6.15. **Theorem** (Approximation). Suppose $\mathcal{C}$ and $\mathcal{D}$ two $\infty$-categories that become Waldhausen $\infty$-categories when equipped with the maximal pair structure. Then an exact functor $\psi: \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it induces an equivalence of homotopy categories $b_\mathcal{C} \cong b_\mathcal{D}$.

**Proof.** Since $\mathcal{C}$ and $\mathcal{D}$ admit all finite colimits and since $\psi$ preserves them, it follows that $\psi$ preserves the tensor product with any finite space. Thus for any positive integer $n$ and any morphism $\eta: X \to Y$ of $\mathcal{C}$, the map

$$
\pi_n(\text{Map}_\mathcal{C}(X, Y), \eta) \to \pi_n(\text{Map}_\mathcal{D}(\psi(X), \psi(Y)), \psi(\eta))
$$
can be identified with the fiber of the bijection

\[ \pi_0 \text{Map}_\varphi(X \otimes S^n, Y) \longrightarrow \pi_0 \text{Map}_\varphi(\psi(X) \otimes S^n, \psi(Y)) \]

over \([\psi(\eta)] \in \pi_0 \text{Map}_\varphi(\psi(X), \psi(Y))\). Since this is a bijection, \(\psi\) is fully faithful, hence an equivalence. \(\square\)