ANALYSIS I: LAST PROBLEM SET (SNIFF!)

DUE FRIDAY, 6 MAY

Exercise 73. Which subsets of a metric space $X$ can be written as the closure of their interior?

Exercise 74. Suppose $\{W_j\}_{j \in \mathbb{N}}$ a cover of $\mathbb{R}$ (i.e., a collection of closed subsets whose union is $\mathbb{R}$). Show that at least one of the $W_j$ has nonempty interior.

Exercise 75. Suppose $K \subset \mathbb{R}$ a compact subset, and suppose $W \subset \mathbb{R}$ a closed subset, disjoint from $K$. Show that there is a real quantity $\varepsilon > 0$ such that for any $x \in K$ and any $y \in W$, one has $|x - y| > \varepsilon$. Show that this is false if we only assume $K$ closed, or if we assume that both $K$ and $W$ are open.

Exercise 76. Suppose $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ two sequences of real numbers. Which of the following is greater? In each case, give examples when equality obtains, as well as examples when it does not.

\[
\limsup_{n \geq 0} (a_n + b_n) \quad \text{and} \quad \limsup_{n \geq 0} a_n + \limsup_{n \geq 0} b_n;
\]
\[
\limsup_{n \geq 0} (a_n b_n) \quad \text{and} \quad (\limsup_{n \geq 0} a_n)(\limsup_{n \geq 0} b_n);
\]
\[
\limsup_{n \geq 0} a_n^{-1} \quad \text{and} \quad (\limsup_{n \geq 0} a_n)^{-1};
\]
\[
\liminf_{n \geq 0} (a_n + b_n) \quad \text{and} \quad \liminf_{n \geq 0} a_n + \liminf_{n \geq 0} b_n;
\]
\[
\liminf_{n \geq 0} (a_n b_n) \quad \text{and} \quad (\liminf_{n \geq 0} a_n)(\liminf_{n \geq 0} b_n);
\]
\[
\liminf_{n \geq 0} a_n^{-1} \quad \text{and} \quad (\liminf_{n \geq 0} a_n)^{-1}.
\]

Exercise 77. What are the limit points of the sequence $(\cos n)_{n \geq 0}$?

Exercise 78. Let $S$ denote the set
\[
\{1/n \mid n \geq 1 \text{ an integer}\} \cup \{0\}.
\]
Find a sequence $(x_n)_{n \geq 0}$ whose set of limit points is precisely $S$.

Exercise 79. Consider the sequence $(x_n)_{n \geq 1}$ with
\[
x_n := -\log n + \sum_{k=1}^{n} \frac{1}{k}.
\]
Prove that this sequence converges by writing $\gamma := \limsup_{n \geq 1} x_n$ and finding a number $\varepsilon > 0$ such that $|x_n - \gamma| \leq \varepsilon/n$ for any $n \geq 1$.

Exercise 80. Suppose $(a_n)_{n \geq 0}$ a sequence of positive real numbers. Prove that $\sum_{n=0}^{\infty} a_n$ converges only if $\sum_{n=0}^{\infty} a_n^2$ converges. Give an example to show that the positivity condition is necessary.

Exercise 81. Suppose $(a_n)_{n \geq 0}$ a sequence of positive real numbers. Prove that if $\sum_{n=0}^{\infty} a_n$ converges, then
\[
\sum_{n=0}^{\infty} \frac{1}{1 + a_n}
\]
diverges, and
\[
\sum_{n=0}^{\infty} \frac{a_n}{1 + a_n}
\]
converges.
Exercise 82. Does the series
\[
\sum_{j=1}^{\infty} \frac{(2j+3)^{1/2} - (2j)^{1/2}}{j^{1/2}}
\]
converge?

Exercise 83. Suppose \((a_n)_{n \geq 1}\) a sequence of real numbers. For any \(N \geq 1\), let \(z_N\) be the average (arithmetic mean) of \(a_1, a_2, \ldots, a_N\). Show that if \((a_n)_{n \geq 1}\) converges, then \((z_N)_{N \geq 1}\) converges to the same limit. Give a counterexample to the converse.

Exercise 84. Suppose \((a_n)_{n \geq 0}\) and \((b_n)_{n \geq 0}\) two sequences of real numbers. Suppose \(b_0 \geq b_1 \geq b_2 \geq \cdots\), and \(\lim_{n \to \infty} b_n = 0\). Show that if the partial sums \(A_N = \sum_{j=0}^{N} a_j\) form a bounded sequence, then the series
\[
\sum_{k=0}^{\infty} a_k b_k
\]
converges.

Exercise 85. Suppose \(f\) a real-valued function on \(\mathbb{R}\). If \(A, B \subset \mathbb{R}\) are disjoint, does it follow that \(f(A)\) and \(f(B)\) are disjoint? How about \(f^{-1}(A)\) and \(f^{-1}(B)\)?

Exercise 86. Give an example of a continuous function \(f : X \to Y\) and a connected subset \(E \subset Y\) such that \(f^{-1}(E)\) is not connected. Give an example of a continuous function \(g : X \to Y\) and a compact subset \(K \subset Y\) such that \(g^{-1}(K)\) is not compact.

Exercise 87. Suppose \(X\) a metric space, and suppose \(S \subset X\). Define a real valued function \(f\) on \(X\) by the formula
\[
f(x) := \inf \{|d(s, x)| \mid s \in S\}.
\]
Is \(f\) uniformly continuous?

Exercise 88. Suppose \(E, F \subset \mathbb{R}\) two disjoint closed subsets. Show that there is a real-valued continuous function \(f\) on \(\mathbb{R}\) such that \(f^{-1}(0) = E\) and \(f^{-1}(1) = F\).

Exercise 89. Suppose \(K, L \subset \mathbb{R}^n\). Write
\[
K + L := \{x + y \mid x \in K \text{ and } y \in L\}.
\]
If \(K\) and \(L\) are compact, does it follow that \(K + L\) is compact as well?

Exercise 90. Suppose \(f\) a real-valued function on an open interval \(I \subset \mathbb{R}\). Suppose that \(f\) is convex, in the sense that for any \(x, y \in I\) and any \(t \in [0, 1]\), then
\[
f((1-t)x + ty) \leq (1-t)f(x) + tf(y).
\]
Prove that \(f\) is continuous.

Exercise 91. Suppose \(f\) a real-valued function that is continuous on \([0, \infty)\) and differentiable on \((0, \infty)\). Show that if \(f(0) = 0\) and \(|f'(x)| \leq |f(x)|\) for any \(x > 0\), then \(f = 0\).

Exercise 92. Suppose \(f\) a differentiable real-valued function on an open interval \(I \subset \mathbb{R}\). Show that \(f'\) is continuous if and only if the inverse image of any point is closed.

Exercise 93. Suppose \(f\) a differentiable real-valued function on \(\mathbb{R}\), and suppose \(f'\) never vanishes. Show that \(f\) is a homeomorphism, show that \(f^{-1}\) is differentiable, and find an expression for \((f^{-1})'\).

Exercise 94. Show that the improper integral
\[
\int_{0}^{\infty} \frac{\sin x}{x}
\]
exists; that is, show that the limits
\[
\lim_{b \to 0^+} \lim_{t \to 0^+} \int_{t}^{b} \frac{\sin x}{x} \quad \text{and} \quad \lim_{t \to 0^-} \lim_{b \to 0^+} \int_{t}^{b} \frac{\sin x}{x}
\]
exist and are equal.
Exercise 95. Must the composition of two Riemann integrable functions \([a, b] \rightarrow [a, b]\) be Riemann integrable?

Exercise 96. Suppose \(f\) a real-valued function on \([a, b]\). For any \(x \in [a, b]\), define the total variation
\[
V_{f[a, b]} := \sup \left( \sum_{j=0}^{k} |f(p_j) - f(p_{j-1})| \mid a = p_0 \leq p_1 \leq \cdots \leq p_k = b \right).
\]

Show that if \(f\) is continuously differentiable on \([a, b]\), then
\[
V_{f[a, b]} = \int_a^b |f'(x)| \, dx.
\]

Exercise 97. Suppose \(f\) a continuous real-valued function on the closed interval \([0, 1]\) with \(f \geq 0\). Show that
\[
\lim_{n \to \infty} \left( \int_0^1 f(x)^n \, dx \right)^{1/n} = \sup_{x \in [0, 1]} f(x).
\]

Exercise 98. Consider the function \(\alpha(x) = x - \lfloor x \rfloor\). Compute
\[
\int_0^5 x \, dx.
\]

Exercise 99. Suppose \(f \in C^1([0, 2\pi])\) such that \(f(0) = f(2\pi)\) and \(f'(0) = f'(2\pi)\). For any natural number \(n\), define
\[
\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin(nx) \, dx.
\]

Show that the series
\[
\sum_{n=1}^{\infty} |\hat{f}(n)|^2
\]
converges.

Exercise 100. A real-valued function \(f\) on \([a, b]\) is said to be polygonal if is continuous, and if there exists a partition \(a = p_0 \leq p_1 \leq \cdots \leq p_n = b\) such that \(f\) is affine on each \([p_i, p_{i+1}]\). Show that the set of polygonal functions is dense in the space of continuous functions on \([a, b]\).

Exercise 101. Show that sin and cos are not polynomials.

Exercise 102. Prove that if \(f\) is a continuous real-valued function on \([a, b]\) such that for any polynomial \(p\), one has
\[
\int_a^b f(x) p(x) \, dx = 0,
\]
then \(f = 0\).

Exercise 103. Prove that the sequence of real-valued functions \((f_j)_{j \geq 0}\) on \(\mathbb{R}\) defined by \(f_j(x) = \sin(jx)\) has no convergent subsequence.

Exercise 104. Suppose \((f_j)_{j \geq 0}\) a pointwise convergent sequence of real-valued functions on a compact space \(K\) such that for any \(j \geq 0\) and any \(x, y \in K\), one has
\[
d(f_j(x), f_j(y)) \leq d(x, y).
\]

Show that \((f_j)_{j \geq 0}\) converges uniformly.

Exercise 105. Prove that Euler’s constant \(e\) is transcendental in the following manner. Suppose \(a_0 + a_1 e + \cdots + a_m x e^m = 0\) for some integer coefficients \(a_i\) with \(a_0 \neq 0\). For any prime number \(p\), write
\[
g(x) := \frac{x^{p-1}(x-1)^p(x-2)^p\cdots(x-m)^p}{(p-1)!},
\]
and set
\[ G(x) := \sum_{k=0}^{mp+p-1} g^{(k)}(x). \]

Check that
\[ |g(x)| < \frac{m^{mp+p-1}}{(p-1)!}, \quad \frac{d}{dx}(e^{-x}G(x)) = -e^{-x}g(x), \quad \text{and} \quad a_j \int_0^j e^{-x}g(x) \, dx = a_jG(0) - a_je^{-j}G(j). \]

Deduce that
\[ -\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x}g(x) \, dx = -\sum_{j=0}^{m} \sum_{i=0}^{mp+p-1} a_j g^{(i)}(j). \]

Now show that \( g^{(i)}(j) \) is always an integer, and show that it is divisible by \( p \) unless \( j \neq 0 \) and \( i \neq p - 1 \). If \( p > m \), show that \( g^{(p-1)}(0) \) is not divisible by \( p \). Finally, show that if \( p \) is sufficiently large, then
\[ -\sum_{j=0}^{m} \sum_{i=0}^{mp+p-1} a_j g^{(i)}(j) \]
is nonzero, yet
\[ \left| \sum_{j=0}^{m} a_j e^j \int_0^j e^{-x}g(x) \, dx \right| \leq \left( \sum_{j=0}^{m} |a_j| \right) e^m \frac{(m^m + 2)^{p-1}}{(p-1)!} < 1, \]
a contradiction.