ANALYSIS I: PROBLEM SET # 7
DUE FRIDAY, 22 APRIL

Notation. For any topological space $X$, denote by $B(X)$ the set of bounded, continuous functions $f : X \to \mathbb{R}$. On this set, we have the uniform metric given by

$$d(f, g) := \sup_{x \in X} |g(x) - f(x)|.$$  

Recall that we have shown that this is a complete metric space, and in the topology induced by this metric, a sequence $(f_n)_{n \geq 0}$ of elements of $B(X)$ converges to an element $f \in B(X)$ if and only if $(f_n)_{n \geq 0}$ converges uniformly to $f$.

Exercise 52. Is every function $f \in B(\mathbb{R}^n)$ uniformly continuous?

Definition. Suppose $X$ a topological space, and suppose $E \subset X$ a subspace. Then the support of a function $f : E \to \mathbb{R}$ is the closure in $X$ of the subset

$$\{ x \in E \mid f(x) \neq 0 \} \subset X.$$

Notation. Suppose $X$ a topological space, and suppose $E \subset X$ a subspace. We denote by supp$(f)$ the support of a function $f : E \to \mathbb{R}$.

Suppose $E_1, E_2 \subset X$ two subspaces, and suppose $f_1 : E_1 \to \mathbb{R}$ and $f_2 : E_2 \to \mathbb{R}$ two functions. Then we write $f_1 \leq f_2$ (or $f_2 \geq f_1$) if and only if supp$(f_1) \subset$ supp$(f_2)$, and for any point $x \in$ supp$(f_2)$, one has $f_1(x) \leq f_2(x)$.

Denote by $C^0_0(X)$ the set of continuous functions $f : X \to \mathbb{R}$ whose support is compact.

Exercise 53. Show that $C^0_0(\mathbb{R}^n)$ is a subspace of $B(\mathbb{R}^n)$, and describe its closure.

Exercise 54. Suppose $S : C^0_0(\mathbb{R}) \to \mathbb{R}$ a function with the following three properties.

1. The function $S$ is linear; that is, for any $f, g \in C^0_0(\mathbb{R})$ and any $\gamma \in \mathbb{R}$, one has

$$S(f + g) = S(f) + S(g) \quad \text{and} \quad S(\gamma f) = \gamma S(f).$$

2. The function $S$ is positive definite; that is, for any $f \in C^0_0(\mathbb{R})$, if $f \geq 0$, then $S(f) \geq 0$, and if both $f \geq 0$ and $S(f) = 0$, then $f = 0$.

3. The function $S$ is translation invariant; that is, for any $f \in C^0_0(\mathbb{R})$ and any $t \in \mathbb{R}$, one has $S(f) = S(tf)$, where $tf \in C^0_0(\mathbb{R})$ is defined by

$$tf(x) = f(t + x).$$

Prove that there exists a positive real quantity $\alpha$ such that for any $f \in C^0_0(\mathbb{R})$, one has

$$S(f) = \alpha \int_{a}^{b} f,$$

where $[a, b]$ is a closed interval containing supp$(f)$.

Notation. A consequence of the previous exercise is that there is precisely one linear, positive definite, translation invariant function $I : C^0_0(\mathbb{R}) \to \mathbb{R}$ with the property that $I(b) = 1$, where

$$b(x) = \begin{cases} 
0 & \text{if } x \notin [-1, 1]; \\
1 + x & \text{if } x \in [-1, 0]; \\
1 - x & \text{if } x \in [0, 1].
\end{cases}$$

$$I(f) = \alpha \int_{a}^{b} f,$$
Exercise 55. Show that, for any Riemann integrable function $g : [a, b] \to \mathbb{R}$ such that $g(x) \geq 0$ for every $x \in [a, b]$ one has
\[
\int_a^b g = \operatorname{sup}\{I(f) \mid f \in C_0^\infty(\mathbb{R}), f \leq g\}.
\]

Exercise 56. Is $I$ continuous?

Exercise 57. Suppose $f, g \in C_0^\infty(\mathbb{R})$. Show that, for any $u \in \mathbb{R}$, the function $c_{f, g, u} : \mathbb{R} \to \mathbb{R}$ defined by
\[
c_{f, g, u}(v) := f(v)g(u - v)
\]
is an element of $C_0^\infty(\mathbb{R})$. Now the convolution of $f$ and $g$ is defined as
\[
(f \ast g)(u) := I(c_{f, g, u}).
\]
Show that $f \ast g \in C_0^\infty(\mathbb{R})$ as well. Deduce that the convolution product defines a map
\[
C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}).
\]

Exercise 58. Show that the convolution is associative and commutative; that is, for any $f, g, h \in C_0^\infty(\mathbb{R})$,
\[
f \ast (g \ast h) = (f \ast g) \ast h \quad \text{and} \quad f \ast g = g \ast f.
\]

Exercise 59. Prove that for any $f, g \in C_0^\infty(\mathbb{R})$, one has
\[
I(f \ast g) = I(f)I(g).
\]

Notation. For any nonnegative integer $k$, set
\[
C_0^k(\mathbb{R}) := C^k(\mathbb{R}) \cap C_0^\infty(\mathbb{R}).
\]
Similarly, set
\[
C_0^\infty(\mathbb{R}) := C^\infty(\mathbb{R}) \cap C_0^\infty(\mathbb{R}).
\]

Exercise 60. Show that if $f \in C_0^1(\mathbb{R})$ and $g \in C_0^2(\mathbb{R})$, then $f \ast g \in C_0^1(\mathbb{R})$, and
\[
(f \ast g)' = f' \ast g.
\]
Deduce that the convolution product restricts to a map
\[
C_0^k(\mathbb{R}) \times C_0^l(\mathbb{R}) \to C_0^{k+l}(\mathbb{R}).
\]

Exercise 61. Use Exercise 51 to construct a function $\varphi \in C_0^\infty(\mathbb{R})$ such that the following conditions hold.

1. One has $\varphi \geq 0$.
2. One has $I(\varphi) = 1$.
3. If $\varphi_n$ is defined by
   \[
   \varphi_n(x) := n\varphi(nx),
   \]
   then $\operatorname{supp} \varphi_n \subset B(0, 1/n)$.

For any $f \in C_0^\infty(\mathbb{R})$, show that $\varphi_n \ast f \to f$ uniformly as $n \to \infty$. Deduce that, for any real number $m$, the subspace of $C_0^\infty(\mathbb{R})$ consisting of those $f \in C_0^\infty(\mathbb{R})$ such that $I(f) = m$ is dense in the subspace of $C_0^\infty(\mathbb{R})$ consisting of those $f \in C_0^\infty(\mathbb{R})$ such that $I(f) = m$, and thus $C_0^\infty(\mathbb{R})$ is dense in $C_0^\infty(\mathbb{R})$. 