Definition. We introduced three power series with an infinite radius of convergence:

\[
\begin{align*}
\exp(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n; \\
\sin(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}; \\
\cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.
\end{align*}
\]

Let us define two more power series, analogous to those of \(\sin\) and \(\cos\):

\[
\begin{align*}
\sinh(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}; \\
\cosh(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}.
\end{align*}
\]

Exercise 34. Prove that the power series \(\sinh\) and \(\cosh\) have infinite radii of convergence. Show that all the resulting functions \(\sin: \mathbb{C} \to \mathbb{C}\), \(\cos: \mathbb{C} \to \mathbb{C}\), \(\sinh: \mathbb{C} \to \mathbb{C}\), and \(\cosh: \mathbb{C} \to \mathbb{C}\) are continuous.

Exercise 35. Verify the following identities for any real numbers \(x\) and \(y\).

\[
\begin{align*}
\exp(ix) &= \cos(x) + i \sin(x); \\
\exp(x) &= \cosh(x) + \sinh(x); \\
cosh(x + iy) &= \cosh(x) \cos(y) + i \sinh(x) \sin(y); \\
\sinh(x + iy) &= \sinh(x) \cos(y) + i \cosh(x) \sin(y).
\end{align*}
\]

Definition. Consider the power series

\[
\arctan(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}.
\]

Exercise 36. Show that the power series \(\arctan\) has radius of convergence 1. Show that \(\arctan(-1)\) and \(\arctan(1)\) each converge. For which other complex numbers \(z\) of modulus 1 can you say something about the convergence of the series \(\arctan(z)\)? Define \(\pi := 4 \arctan(1)\), and show that \(\arctan\) defines a homeomorphism \([-1, 1] \to [-\pi/4, \pi/4]\) whose inverse is the map given by \(\tan(x) := \sin(x)/\cos(x)\).

Exercise 37. Show that the composite \(\text{gd}(x) := \arctan(\sinh(x))\) defines a continuous bijection

\[
\text{gd}: \left[ \log(\sqrt{2} - 1), \log(\sqrt{2} + 1) \right] \to \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right].
\]

This function is called the Gudermannian.

Exercise 38. Prove that for any real number \(x\) with \(\log(\sqrt{2} - 1) < x < \log(\sqrt{2} + 1)\), one has

\[
\cos(\text{gd}(x)) \cosh(x) = 1 \quad \text{and} \quad \sin(\text{gd}(x)) \cosh(x) = \sinh(x).
\]
Definition. Let us now define two interesting rational sequences, which will serve as coefficients for some interesting power series.

The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are defined recursively in the following manner. Set $B_0 := 1$, and for any integer $m \geq 1$, define $B_m$ as the unique rational number so that

$$\sum_{k=0}^{m} \binom{m+1}{k} B_k = 0.$$ 

Similarly, the (even) Euler numbers $E_0, E_2, E_4, \ldots$ are defined recursively in the following manner. Set $E_0 := 1$, and for any integer $m \geq 1$, define $E_{2m}$ as the unique rational number so that

$$\sum_{k=0}^{m} \binom{2m}{2k} E_{2k} = 0.$$ 

Exercise 39. Show that the Euler numbers $E_{2m}$ are all integers. Show that not all Bernoulli numbers are integers, but for any integer $m \geq 1$, the Bernoulli number $B_{2m+1} = 0$.

Exercise* 40. Show that

$$\lim_{n \to \infty} \sin \left( \sqrt{\frac{1}{8}} \frac{B_{2n}}{B_{2n+2}} \right) = \lim_{n \to \infty} \cos \left( \sqrt{\frac{1}{8}} \frac{B_{2n}}{B_{2n+2}} \right) = \frac{\sqrt{2}}{2}.$$ 

Conclude that the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{B_{n} x^{n}}{n!}$$

is $2\pi$, where $\pi$ is as defined above.

Exercise* 41. Compute the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{E_{2n}}{(2n+1)!} x^{2n+1},$$

and show that it converges to the Gudermannian $\text{gd}(x)$ when $\log(\sqrt{2} - 1) < x < \log(\sqrt{2} + 1)$.

**OPTIONAL EXERCISES**

Definition. Suppose $k$ an integer. Then the $k$-th polylogarithm is the series

$$\text{Li}_k(z) := \sum_{n=1}^{\infty} \frac{1}{n^k} z^n.$$ 

Exercise. Compute the radius of convergence of $\text{Li}_k$, and note that it is at least $1$.

Exercise. Show that if $k$ is a negative integer, then $(z - 1)^{-1-k} \text{Li}_k(z)$ is a polynomial of degree $-k$ with integral coefficients. What are the roots of this polynomial? What can you say about its coefficients?

Definition. If $k$ is a positive integer, then the generalized harmonic numbers $H_n(k)$ are the partial sums of the generalized harmonic series:

$$H_n(k) := \sum_{j=1}^{n} \frac{1}{j^k}.$$ 

Exercise. Show that for any integer $n \geq 2$, the generalized harmonic number $H_n(k)$ is not an integer.

Exercise. Compute the radius of convergence of the power series

$$M_k(z) := \sum_{n=1}^{\infty} H_n(k) z^n.$$ 

Show, moreover, that within this radius of convergence, one has

$$(z - 1)M_k(z) = \text{Li}_k(z).$$