ANALYSIS I: PROBLEM SET # 3

DUE FRIDAY, 25 FEBRUARY

Exercise 19. Suppose \( s \in \mathbb{R} \). Show that the function \( r_s : (0, \infty) \to (0, \infty) \) given by \( r_s(x) = x^s \) is continuous, where \((0, \infty)\) is given the subspace topology.

Exercise 20. For any real number \( s \geq 1 \), one can define a metric \( d_s \) on \( \mathbb{R}^n \) by the formula

\[
d_s(x, y) = \left( \sum_{i=1}^{n} (y_i - x_i)^s \right)^{1/s}.
\]

Thus \( d_2 \) is the standard metric on \( \mathbb{R}^n \). Furthermore, if one lets \( s \to \infty \), then one obtains another metric

\[
d_\infty(x, y) = \sup_{i=1}^{n} |y_i - x_i|.
\]

Show that for any two quantities \( s, t \in [1, \infty) \cup \{\infty\} \), the metrics \( d_s \) and \( d_t \) are equivalent.

Exercise 21. We mentioned that the inclusion of a subspace is a continuous map. In fact, the subspace topology is rigged precisely to make this happen. Show that if \( X \) is a space, and \( A \subset X \) is a subset, then the subspace topology on \( A \) is the coarsest topology on \( A \) such that the inclusion map \( i : A \to X \) is continuous.

Exercise 22. Classify all connected compact subsets of \( \mathbb{R} \). Justify your claims.

Example. Here’s an example that merits some contemplation. Consider the function \( f(x) = 1/x \). This can regarded as a map

\[
f: \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}.
\]

Since \( f \) is its own inverse, \( f \) here is a homeomorphism. (In general, a homeomorphism from a space to itself is called an automorphism.) Of course we have removed the point \( 0 \in \mathbb{R} \), because in elementary school we were told that \( 1/0 \) is “undefined.” But let’s try to define it anyhow.

We note that, as \( x \) approaches \( 0 \) from the right, \( 1/x \) increases without bound; as \( x \) approaches \( 0 \) from the left, \( 1/x \) decreases without bound. If we wanted to add a point that would play the role of \( 1/0 \), then this leads us to the following idea: consider the set \( \mathbb{R} \cup \{\infty\} \), where \( \infty \) here is just a formal symbol that we’ve tacked on. Let’s topologize this set in the following way: we will contemplate the set

\[
\{(a, b) \mid a, b \in \mathbb{R}\} \cup \{(-\infty, c) \cup \{\infty\} \cup (d, \infty) \mid c, d \in \mathbb{R}\} \subset \mathcal{P}(\mathbb{R} \cup \{\infty\}),
\]

and we will contemplate the topology generated by this set. That is, we will look at all unions of all finite intersections of open intervals in \( \mathbb{R} \) along with sets

\[
(-\infty, c) \cup \{\infty\} \cup (d, \infty)
\]

that “wrap around” our added point \( \infty \).

Now we think of our map \( f: \mathbb{R} - \{0\} \to \mathbb{R} - \{0\} \), and we note that we can extend it to a map \( F: \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\} \), where we set \( F(x) := 1/x \) for any \( x \in \mathbb{R} - \{0\} \), we set \( F(0) := \infty \), and we set \( F(\infty) := 0 \). With the topology we’ve given \( \mathbb{R} \cup \{\infty\} \), this is continuous!

The space \( \mathbb{R} \cup \{\infty\} \) we’ve constructed here is called a compactification of \( \mathbb{R} \). (We’ll explain more about that later.)

The subspace \( \mathbb{R} \subset \mathbb{R} \cup \{\infty\} \) is open, and \( \infty \) is contained in the closure of any ray \((-\infty, c)\) or \((d, \infty)\).

Exercise 23. A little bit of thought should convince you that the space \( \mathbb{R} \cup \{\infty\} \) we’ve described in the previous example is in fact a circle. Let’s find a homeomorphism that makes this explicit. Translate our circle \( S^1 \) so that we are looking at the unit circle around the point \((0, 1)\) in \( \mathbb{R}^2 \):

\[
S^1(1) := \{ \theta \in \mathbb{R}^2 \mid ||\theta - (0, 1)|| = 1 \}.
\]
Now let us define a map \( g : S^1(1) - \{(0,2)\} \to \mathbb{R} \) in the following way. For any point \( \theta \in S^1(1) - \{(0,2)\} \), let \( \ell_\theta \) be the unique line passing through \((0,2)\) and \( \theta \) in \( \mathbb{R}^2 \). Now define \( g(\theta) \) to be the \( x \)-coordinate of the point of intersection between \( \ell_\theta \) and the \( x \)-axis \( \{(x,y) \in \mathbb{R}^2 \mid y = 0\} \).

(23.1) Write an explicit formula for \( g \).

(23.2) Show that \( g \) is a homeomorphism.

(23.3) Show that \( g \) can be extended to a homeomorphism

\[ G : S^1(1) \to \mathbb{R} \cup \{\infty\} \]

by setting \( G(\theta) := g(\theta) \) for any \( \theta \in S^1(1) - \{(0,2)\} \) and \( G((0,2)) := \infty \).

(23.4) In the previous example, we constructed a homeomorphism \( F : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\} \) (which is not the identity). Using the homeomorphism \( G \), we can rewrite this as a homeomorphism

\[ G^{-1} \circ F \circ G : S^1(1) \to S^1(1). \]

What homeomorphism is this? How might you draw its graph?

**Exercise 24.** The space \( \mathbb{R} \cup \{\infty\} \) is not the only compactification of \( \mathbb{R} \); we were led to consider it by our example. If we thought of another example, we might be led to a different compactification. Think, for example, of the function \( b(x) = x^2 \), which we regard as a map

\[ b : \mathbb{R} \to \mathbb{R}. \]

(24.1) Show that \( b \) is a homeomorphism. (Be careful if you decide to study \( b^{-1} \)!) 

(24.2) Show that \( b(x) \) increases without bound as \( x \) does, and likewise \( b(x) \) decreases without bound as \( x \) does.

(24.3) Describe a space \( \mathbb{R} \cup \{-\infty, \infty\} \) (where \( -\infty \) and \( \infty \) are each just formal symbols) with the following properties.

1. \( \mathbb{R} \) is an open subspace of \( \mathbb{R} \cup \{-\infty, \infty\} \). 
2. The closure of a ray \((-\infty, a) \subset \mathbb{R} \) in \( \mathbb{R} \cup \{-\infty, \infty\} \) is the set

\[ [-\infty, a] := \{-\infty\} \cup (-\infty, a], \]

and the closure of a ray \((b, \infty) \subset \mathbb{R} \) in \( \mathbb{R} \cup \{-\infty, \infty\} \) is the set

\[ [b, \infty] := [b, \infty) \cup \{\infty\}. \]

(24.4) Describe an extension of \( b \) to a homeomorphism

\[ H : \mathbb{R} \cup \{-\infty, \infty\} \to \mathbb{R} \cup \{-\infty, \infty\}. \]

(24.5) Show that the space \( \mathbb{R} \cup \{-\infty, \infty\} \) is homeomorphic to the interval \([-1, 1]\).

(24.6) Show that the map

\[ p : \mathbb{R} \cup \{-\infty, \infty\} \to \mathbb{R} \cup \{\infty\} \]

(with the topologies defined here and above) is continuous. Show, moreover, that for any topological space \( X \), and any continuous map \( f : \mathbb{R} \cup \{-\infty, \infty\} \to X \) such that \( f(-\infty) = f(\infty) \), there exists a unique map \( f' : \mathbb{R} \cup \{\infty\} \to X \) such that \( f = f' \circ p \).

**Exercise 25.** Suppose \((X,d)\) a metric space, and suppose \((x_n)_{n \in \mathbb{N}}\) a sequence therein. Show that if \((x_n)_{n \in \mathbb{N}}\) converges to a point \(x \in X\), then \(x\) is the unique limit point of \((x_n)_{n \in \mathbb{N}}\). Give an example with \(X = \mathbb{R}\) to show that the converse is false.

**Exercise 26.** Suppose we are given a finite collection \(\{X_i\}\) of compact spaces, \(i = 1, 2, \ldots, n\). Show that the product space \(\prod_{i=1}^n X_i\) is compact. (This is the easy case of Tychonoff’s Theorem, to which we shall return.)

**Exercise 27.** Suppose \(K\) a compact space, and suppose \((X,d)\) a metric space. Then show that the set \(\mathcal{C}(K,X)\) of continuous maps \(K \to X\) can be given a metric in the following manner:

\[ d(f, g) := \sup \{d(g(x), f(x)) \mid x \in X\}. \]

Does this definition give you a metric if you don’t assume that \(K\) is compact?

**Definition.** Suppose \(X\) and \(Y\) metric spaces. Then a map \(f : X \to Y\) is uniformly continuous if for any \(\varepsilon > 0\), there exists a quantity \(\delta > 0\) such that for any points \(x_0, x_1 \in X\), if \(d(x_0, x_1) < \delta\), then \(d(f(x_0), f(x_1)) < \varepsilon\).

**Exercise 28.** Suppose \(X\) and \(Y\) metric spaces. Show that a uniformly continuous map \(X \to Y\) is continuous, and show that the converse holds if \(X\) is compact.