PROBLEM SET V

DUE FRIDAY, 24 MARCH

Exercise 24. Show that no continuously differentiable function \( \mathbb{R}^m \to \mathbb{R}^n \) with \( m > n \) can be injective.

Exercise 25. Consider the function \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) given by the formula
\[
F(\rho, \theta, \varphi) := (\rho \sin \theta \cos \varphi, \rho \sin \theta \sin \varphi, \rho \cos \varphi).
\]
Show that \( F \) is continuously differentiable, and find open sets \( U, V \subset \mathbb{R}^3 \) such that \( F \) defines a homeomorphism \( U \to V \) with a differentiable inverse.

Exercise \* 26. Fix real numbers \( \lambda > 0 \) and \( y_0 \), and consider the curves \( \gamma : \mathbb{R} \to \mathbb{R}^3 \) and \( \eta : \mathbb{R} \to \mathbb{R}^3 \) given by \( \gamma(t) := (1, t, 2 \arctan(\exp(\lambda t))) \) and \( \eta(t) := (1, y_0, t) \). Using \( F \) from the previous problem, at what angle does the curve \( F \circ \gamma \) intersect the curve \( F \circ \eta \)?

Exercise 27. Suppose \( G : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \) a function, continuously differentiable in an open neighborhood of a point \( (x, y) \in \mathbb{R}^m \times \mathbb{R}^n \). Suppose, in addition, that \( G(x, y) = 0 \), and the \( n \times n \) matrix
\[
\left( \frac{\partial G_i}{\partial y_j} \right)
\]
is invertible. Then show that there exist: an open neighborhood \( U \) of \( x \) in \( \mathbb{R}^m \), an open neighborhood of \( y \) in \( \mathbb{R}^n \), and a differentiable function \( s : U \to V \) with the property that for any \( u \in U \), the point \( s(u) \in V \) is the unique point with the property that \( G(u, s(u)) = 0 \).

This result permits us to do a certain amount of geometry on subsets of Euclidean space defined by systems of equations.

Exercise 28. Consider the cylinder
\[
K = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - y = 0\}.
\]
Let \( C \) be the intersection \( K \cap S^2(1) \) with the 2-sphere of radius 1. At any point \( (x, y, z) \in C \) except for \( (0, 1, 0) \), write an equation for the line tangent to \( C \). What goes wrong at \( (0, 1, 0) \)? If we take \( K \cap S^2(r) \) for \( r \neq 1 \), do we still have such a “bad” point? Explain.