Definition. Define functions arccos: \((-1, 1) \rightarrow \mathbb{R}\) and arcsin: \((-1, 1) \rightarrow \mathbb{R}\) by the formulas

\[
arccos(x) := \int_x^1 \frac{dt}{\sqrt{1 - t^2}} \quad \text{and} \quad \arcsin(x) := \int_0^x \frac{dt}{\sqrt{1 - t^2}}.
\]

Exercise 54. Show that arccos and arcsin are \(C^\infty\) bijections \((-1, 1) \rightarrow (0, \pi)\) and \((-1, 1) \rightarrow (-\pi/2, \pi/2)\) (respectively). Prove that cos defines an inverse to arccos, and sin defines an inverse to arcsin.

Exercise 55. Show that one may extend arccos and arcsin to continuous functions \([-1, 1] \rightarrow \mathbb{R}\) by setting

\[
arccos(-1) = \pi, \quad \arccos(1) = 0, \quad \arcsin(-1) = -\pi/2, \quad \text{and} \quad \arcsin(1) = \pi/2.
\]

Reflection: can these functions be regarded as differentiable at \(-1\) and \(1\)?

Exercise 56. Prove the following identity:

\[
\prod_{k=1}^{n-1} \sin(k \pi/n) = 2^{1-n} n.
\]

Definition. Define the hyperbolic cosine \(\cosh: \mathbb{R} \rightarrow \mathbb{R}\) and the hyperbolic sine \(\sinh: \mathbb{R} \rightarrow \mathbb{R}\) by means of the following formulas.

\[
\cosh(x) := \frac{1}{2} (\exp(x) + \exp(-x)) \quad \text{and} \quad \sinh(x) := \frac{1}{2} (\exp(x) - \exp(-x)).
\]

Exercise 57. Prove the following properties of cosh and sinh.

(a) Check that \(\cosh(x) = \cosh(-x)\) and that \(\sinh(x) = -\sinh(-x)\).

(b) Show that cosh and sinh are each of class \(C^\infty\). Write formulas describing \(\cosh''\) and \(\sinh''\).

(c) Show that for any \(x \in \mathbb{R}\), one has \(\cosh(x)^2 - \sinh(x)^2 = 1\).

(d) Show that cosh restricts to a bijection \([0, +\infty) \rightarrow [1, +\infty)\), and show that sinh: \(\mathbb{R} \rightarrow \mathbb{R}\) is a bijection.

(e) Show that the function \(\text{arccosh}: [1, +\infty) \rightarrow [0, +\infty)\) given by

\[
\text{arccosh}(x) := \log(x + \sqrt{x^2 - 1})
\]

is an inverse to cosh, and show that the function \(\text{arcsinh}: \mathbb{R} \rightarrow \mathbb{R}\) given by

\[
\text{arcsinh}(x) := \log(x + \sqrt{x^2 + 1})
\]

is an inverse to sinh. (In particular, you must show that these formulas make sense on the purported domains.)

For which integers \(k \geq 0\) are \(\text{arccosh}\) and \(\text{arcsinh}\) of class \(C^k\)?

Exercise 58. Since \(\exp: \mathbb{R} \rightarrow \mathbb{R}\) can be extended to a function \(\exp: \mathbb{C} \rightarrow \mathbb{C}\), the functions cosh and sinh can each also be extended to functions \(\mathbb{C} \rightarrow \mathbb{C}\). With this extended definition, prove that for any \(x \in \mathbb{R}\),

\[
\cosh(ix) = \cos(x) \quad \text{and} \quad \sinh(ix) = i \sin(x).
\]

Reflection: how do the formulas for \(\cosh''\) and \(\sinh''\) you found above relate to formulas for \(\cos''\) and \(\sin''\)?

Exercise 59. Show that the composite \(\text{gd}(x) := \arctan(\sinh(x))\) defines a \(C^\infty\) bijection

\[
\text{gd}: \mathbb{R} \rightarrow (-\pi/2, \pi/2).
\]

This function is called the Gudermannian.
Exercise 60. Prove that for any real number \( x \), one has 
\[
\cos(gd(x)) \cosh(x) = 1 \quad \text{and} \quad \sin(gd(x)) \cosh(x) = \sinh(x).
\]

Exercise* 61. Prove that \( \cosh \) is a function of \( C \)-type. Use all the facts you have collected about these to show that 
\[
p_n(x) = \begin{cases} (-1)^n \cosh(n \arccosh(-x)) & x \in (-\infty, -1] \\
\cosh(n \arccosh(x)) & x \in (1, +\infty). 
\end{cases}
\]
Reflection: would you have immediately recognized that the formula above described a polynomial?

Exercise* 62. (a) Show that the function \( f : \mathbb{R} \to \mathbb{R} \) given by 
\[
f(x) := \begin{cases} \exp(-1/x^2) & \text{if } x > 0; \\
0 & \text{if } x \leq 0.
\end{cases}
\]
is a \( C^\infty \) function with the property that \( f(x) \) vanishes for negative \( x \), is positive and monotonic for positive \( x \), and its image is \([0,1)\).

(b) Similarly, show that the function \( g : \mathbb{R} \to \mathbb{R} \) given by 
\[
g(x) := \begin{cases} \exp \left(1/(x^2 - 1)\right) & \text{if } |x| < 1; \\
0 & \text{if } |x| \geq 1.
\end{cases}
\]
is a \( C^\infty \) function with the property that it vanishes outside the interval \((-1,1)\) and is positive inside \((-1,1)\).

(c) Using the ideas from the previous two parts, construct, for any closed interval \( I \) and any open interval \( U \) containing \( I \), a function \( \rho_{I,U} \in C^\infty(\mathbb{R}) \) with all the following properties: \(^1\)

1. For any \( x \in \mathbb{R} \), one has \( 0 \leq \rho_{I,U}(x) \leq 1 \).
2. For any \( x \in \mathbb{R} - U \), one has \( \rho_{I,U}(x) = 0 \).
3. For any \( x \in I \), one has \( \rho_{I,U}(x) = 1 \).

Exercise** 63. (a) Show that the function 
\[
b(x) := \begin{cases} x/(\exp(x) - 1) & \text{if } x \neq 0; \\
1 & \text{if } x = 0
\end{cases}
\]
is of class \( C^\infty \).

(b) For any integer \( m \geq 0 \), denote by \( b_m \) the value \( b^{(m)}(0) \) of the \( m \)-th derivative of \( b \) at \( 0 \). Show that \( b_0 = 1 \), and for any integer \( m \geq 1 \), show that \( b_m \) is the unique rational number so that 
\[
\sum_{k=0}^{m} \frac{(m+1)!}{(m+1-k)!} b_k = 0.
\]

(c) Use this characterization to show that for any integer \( m \geq 1 \), one has \( b_{2m+1} = 0 \) and \( b_{2m} = (-1)^{m-1}|b_{2m}| \).

(d) Suppose now that \( p \) is any polynomial, and suppose that \( N \geq 0 \) an integer such that for any integer \( k > N \) and any \( x \in \mathbb{R} \), one has \( p^{(k)}(x) = 0 \). Show that 
\[
\int_0^1 p(x) \, dx = p(0) - \sum_{k=1}^{N} b_k \left( p^{(k-1)}(1) - p^{(k-1)}(0) \right).
\]
Reflection: you have shown that integrals of polynomials can be computed by the values of their derivatives at the endpoints! How can this be?

\(^1\)For an additional challenge, construct a function \( \rho_{I,U} \in C^\infty(\mathbb{R}) \) with these properties, where \( I \) is any compact subset of \( \mathbb{R} \) and \( U \) is an open set containing \( I \).