Exercise 41. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^{1/3}$. Note that $f$ is continuous, and show that $f$ is differentiable at all points $x \in \mathbb{R}$ such that $x \neq 0$. Show that $f$ is not differentiable at 0 according Df. 4.4 in Apostol. Nevertheless, make precise the idea that there is a tangent line to $f$ at 0, and its slope is infinite. Does your idea apply to the function $g(x) := \sqrt{|x|}$ as well?

Definition. Suppose $U \subset \mathbb{R}$ an open set, and suppose $k$ a natural number. Recall that a function $f : U \to \mathbb{R}$ is said to be of class $C^k$ if for every natural number $n \leq k$, the $n$-fold derivative $f^{(n)}$ exists and is continuous on $U$. We denote by $C^k(U)$ the set of such functions.

A function $f : U \to \mathbb{R}$ is said to be smooth or infinitely differentiable or of class $C^\infty$ if it is of class $C^k$ for every natural number $k$. We denote by $C^\infty(U)$ the set of such functions.

Exercise 42. Suppose $n$ a positive integer. Consider the function $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) := |x|^n$.

What is the largest natural number $k$ such that $f_n \in C^k(\mathbb{R})$?

Definition. We will now define, for any natural number $n$, a polynomial $p_n$. We do this recursively. First, set $p_0(x) := 1$ and $p_1(x) := x$.

Now for any $n \geq 1$, set $p_{n+1}(x) := 2x p_n(x) - p_{n-1}(x)$.

We will regard these polynomials as functions $\mathbb{R} \to \mathbb{R}$.

Exercise 43. Let’s establish some basic properties of the polynomials $p_n$.

(a) Compute $p_n$ for $0 \leq n \leq 6$ explicitly.

(b) Formulate a conjectural formula for the coefficient of the highest power of $p_n$. Prove your conjecture.

(c) Show that the constant term $p_n(0)$ is given by the formula

$$p_n(0) = \begin{cases} (-1)^k & \text{if } n = 2k; \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

(d) Show that for any natural number $n$, one has $p_n(1) = 1$ and $p_n(-1) = (-1)^n$.

(e) Show that for any integers $m, n \geq 0$ and any $x \in \mathbb{R}$, one has $p_m(p_n(x)) = p_{mn}(x)$.

Exercise 44. Show that for any integer $n \geq 0$ and for any $x \in \mathbb{R}$, one has

$$p_n'(x) = 2n \sum_{i=1}^{n/2} p_{2i-1}(x)$$

if $n$ is even and

$$p_n'(x) = n + 2n \sum_{i=1}^{[n/2]} P_{2i}(x)$$

if $n$ is odd.
Definition. Let us say that a function $\phi: \mathbb{R} \to \mathbb{R}$ is of C-type if for any integer $n \geq 0$ and any $x \in \mathbb{R}$, one has $\phi(nx) = p_n(\phi(x))$.

Exercise 45. Suppose $\phi, \psi: \mathbb{R} \to \mathbb{R}$ are functions such that for any $x, y \in \mathbb{R}$, one has

$$\phi(x + y) = \phi(x)\phi(y) - \psi(x)\psi(y), \quad \psi(x + y) = \phi(x)\psi(y) + \psi(x)\phi(y),$$

and $\phi(x)^2 + \psi(x)^2 = 1$. Show that $\phi$ is of C-type.

Exercise 46. Suppose $\phi: \mathbb{R} \to \mathbb{R}$ a function of C-type. Show that for any integer $n \geq 1$ and any $x \in \mathbb{R}$, one has

$$\phi'((n+1)x) - \phi'((n-1)x) = 2\phi(nx)\phi'(x).$$