Power law rheology by distributed yielding times of spring-like elements

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A system of parallel springs can respond to an applied stress ramp with a fractional power-law creep $\Delta x(t) \propto t^{\beta}$, when individual springs yield with a suitable distribution $p(\tau)$ of life times and when immediately after a yield event the spring is replaced in a strain-free connfiguration. Such a system is equivalent to parallel Maxwell elements with distributed visco-elastic relaxation times.

Review of the Maxwell Element

A Maxwell element (ME) is a spring of stiffness $k = F/\Delta x$ connected in series with a dashpot of friction constant $\gamma = F/v$. In the Laplace domain with frequency variable $s = i\omega$, the applied force $F(s)$ and the corresponding length change $\Delta x(s)$ of the ME are related by $F(s) = g(s)\Delta x(s)$, with the frequency dependent stiffness

$$g(s) = k\frac{s\tau}{1+s\tau} \quad (1)$$

An important quantity of the ME is the visco-elastic relaxation time $\tau = \gamma/k$. Using the function $g(s)$, it is possible to compute the linear mechanical response of the ME to an arbitrary experimental protocol. The most common protocols are as follows:

(1) The **stress step experiment**: applying a force step $F(t) = F_0\theta(t) \leftrightarrow F(s) = F_0/s$ and measuring the deformation response. Here one obtains in the Laplace domain $\Delta x(s) = F(s)/g(s) = \frac{F_0}{k}\left(\frac{1/\tau}{s^2}+\frac{1}{s}\right)$. Transforming back to the time domain yields $\Delta x(t \geq 0) = \frac{F_0}{k}\left(1+\frac{t}{\tau}\right)$. The length of the ME increases abruptly and then continues to grow linearly with time.

(2) The **stress ramp experiment**: applying a force ramp $F(t) = rt \leftrightarrow F(s) = r/s^2$ and measuring the deformation response. Here one obtains in the Laplace domain $\Delta x(s) = F(s)/g(s) = \frac{r\tau^2}{k}\left(\frac{1}{(s\tau)^3}+\frac{1}{(s\tau)^2}\right)$. Transforming back to the time domain yields $\Delta x(t \geq 0) = \frac{r}{k}\left(t+\frac{t^2}{2\tau}\right)$. The length of the ME increases with time like a polynomial of second order. The linear term comes from the elastic spring, the quadratic term from the viscous dashpot.

(3) The **strain step experiment**: applying a deformation stap $\Delta x(t) = \Delta x_0\theta(t) \leftrightarrow \Delta x(s) = \Delta x_0/s$ and measuring the force response. Here one obtains in the Laplace domain $F(s) = g(s)\Delta x(s) = k\Delta x_0\frac{\tau}{1+(s\tau)}$. Transforming back to the time domain yields $\Delta F(t \geq 0) = k\Delta x_0 e^{-t/\tau}$. The force of the ME decays exponentially with time.

Systems of springs with finite lifetime

From the theoretical point of view, the strain step experiment is the easiest. It is also useful to reinterprete the ME as a parallel system of conventional springs, however with finite average lifetime τ. At time $t = 0$, all springs are still alive and in a streched state, due to the applied strain step. After this, more and more individual springs yield (Poisson process, as in radioactive decay) and thereafter do not contribute to the total elastic force any longer. This stochastic exponential decay leads effectively to the same force response as in a ME. Generally, the deterministic viscosity, or friction, of a system is just the macroscopic effect of many small stochastic yielding events of purely elastic elements on the microscopic level.

Pareto distributed lifetimes

This re-interpretation leads naturally to extensions of the model. For example, we can consider now a system in which the average lifetime τ is not the same for all springs, but distributed throughout the spring ensemble according to some $p(\tau)$. If we perform a strain step experiment with such an inhomogeneous ensemble, the force response is a weighted average of the individual springs:

$$F(t)_{\text{strain step}} = k\Delta x_0 \int_0^\infty p(\tau)\, e^{-t/\tau}\, d\tau \quad (2)$$

As a concrete example, we consider a **Pareto distribution** $p(\tau) \propto \tau^{-\alpha-1}$ of average lifetimes **with parameter** α. In this case, we can use the integral

$$\int_0^\infty \frac{e^{-t/\tau}}{\tau^{1+\alpha}}\, d\tau = \frac{\Gamma(\alpha)}{t^\alpha}, \quad (3)$$

which is valid for α-parameters larger then zero. It follows that $F(t)_{\text{strain step},\alpha} \propto t^{-\alpha}$. In order to test this result numerically, we draw 1000 random lifetimes τ_i from a Pareto distribution and compute the resulting weighted response function by summing over the corresponding exponentials e^{-t/τ_i}. The result is shown in Fig.1.

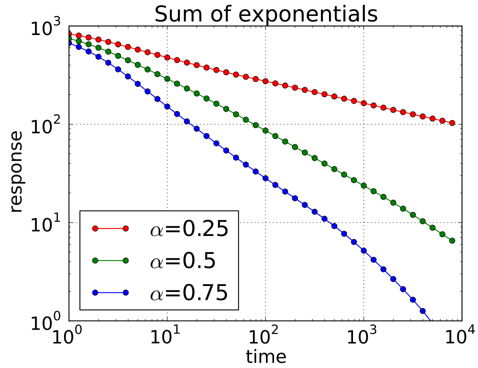

Fig.1: *Numerically computed force response of a system of 1000 parallel springs to a strain step, when individual springs yield with an average lifetime τ that in turn is distributed according to $p(\tau) \propto \tau^{-\alpha-1}$. At least within a certain time window, the response approximates the powerlaw $t^{-\alpha}$.*

Therefore, with the parameter α we can adjust any fractional powerlaw rheology for the inhomogeneous system, ranging from purely elastic behaviour at $\alpha = 0$ to purely viscous behaviour at $\alpha = 1$.

Effective system response function

The above result means in the Laplace domain that the effective frequency dependent stiffness of the inhomogeneous system is

$$g(s,\alpha) \propto s^\alpha. \quad (4)$$

Using this effective stiffness, we can now also calculate how the inhomogeneous system would respond to a stress ramp experiment: $\Delta x(s) = F(s)/g(s,\alpha) \propto (r/s^2)/s^\alpha \propto s^{-2-\alpha}$. In the time domain, that means

$$\Delta x(t)_{\text{stress ramp},\alpha} \propto t^{1+\alpha}. \quad (5)$$