Multi-Agent Justification Logic: Communication and Evidence Elimination

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Abstract

This paper presents a logic combining *Dynamic Epistemic Logic*, a framework for reasoning about multi-agent communication, with a new multi-agent version of *Justification Logic*, a framework for reasoning about evidence and justification. This novel combination incorporates a new kind of *multi-agent evidence elimination* that cleanly meshes with the multi-agent communications from Dynamic Epistemic Logic, resulting in a system for reasoning about multi-agent communication and evidence elimination for groups of interacting rational agents.

1 Introductory Example

Consider the following email exchange among friends planning a party.

x_1 { To: Bob, Charlie   From: Anne
If the cheese store is still open, then I'll bring the cheese tonight.

x_2 { To: Anne, Bob   From: Charlie   Re: x_1
OK, Anne. If you bring the cheese, then I'll bring the crackers.

x_3 { To: Bob, Charlie   From: Anne
I just checked. The cheese store is still open!

x_4 { To: Anne, Bob   From: Charlie   Re: x_3
It's great the store's still open! I'll go get the crackers.

After this sequence of messages, Bob and Charlie are each able to conclude that Anne is bringing the cheese. After all, each of Bob and Charlie has message x_1 as evidence that "Anne will bring the cheese if the store is open" along with message x_3 as evidence that the store is indeed open. So by combining their evidence x_1 and x_3, they each have evidence that Anne will bring the cheese.

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Furthermore, since Charlie replied to each of Anne's messages x_1 and x_3 with his own messages x_2 and x_4, which indicates that Charlie has received and understood each of Anne's messages, Bob not only has his combined evidence x_1 and x_3 that Anne will bring the cheese but can also see on the basis of Charlie's messages x_2 and x_4 that Charlie too has combined evidence x_1 and x_3 to conclude that Anne will bring the cheese.

But now we add a twist to the story with the following private message from Anne to Bob.

$$x_5 \left\{ \begin{array}{l} \text{To: Bob} \quad \text{From: Anne} \\ \text{Bob, I messed up! The store is closed! Charlie is going to flip!} \end{array} \right.$$

Clearly, this message has the effect of causing Bob to set aside (or *eliminate*) his evidence x_3 relevant to the assertion that the store is open. But since x_5 was sent privately to Bob,[1] Bob has no reason to believe that Charlie will doubt the evidence x_3 that the store is open. Therefore, while message x_5 causes Bob to eliminate his evidence x_3 relevant to the assertion that the store is open—and so to abandon his combined evidence x_1 and x_3 that Anne is bringing the cheese—Bob will nevertheless maintain his belief based on the combination of evidence x_2 and x_4 that Charlie still believes based on the combination of evidence x_1 and x_3 that Anne will bring the cheese.

2 Introduction

As indicated by our email example, this paper concerns reasoning about multi-agent communication and evidence elimination. Our work here is part of a larger project aimed at joining two areas of study: *Justification Logic*,[2] a family of logics for reasoning about evidence and justification, and *Dynamic Epistemic Logic*,[3] a family of logics for reasoning about multi-agent communication and belief. Before we discuss our rationale for seeking a combination of these theories, let us first say a few words about each of them in isolation.

Justification Logic has been promoted as a logic for reasoning about evidence and justification [2, 12]. The idea here is to remedy a deficiency found in the standard use of modal logic for reasoning about the justifications agents have for the beliefs that they possess. To illustrate this deficiency, consider a valid modal formula of the form $B_i\varphi \to B_i\psi$, where B_i is a modal operator and each of φ and ψ are formulas. This formula, read, "if agent i believes φ, then agent i believes ψ," stipulates a certain connection between agent i believing one thing, ψ, and agent i believing another thing, φ. But we see that this formula does not provide a *reason* as to why agent i's belief in one thing follows whenever he or she has belief in another. The formula merely asserts that such conditional belief holds without providing any explanation as to why this is the case.

Justification Logic aims to better explicate this situation by first introducing structured syntactic objects called *terms* and then allowing us to form new formulas of the form $t{:}_i\,\varphi$

[1] For simplicity, we assume that neither CC ("carbon copy") nor BCC ("blind carbon-copy") is possible. The recipients of a message are therefore those individuals mentioned in the "To" field.

[2] See: [1, 2, 3, 11, 12, 15, 20].

[3] See: [4, 5, 7, 21, 24, 27, 28].

whenever t is a term, i is an agent, and φ is a formula we have already formed. The idea is to identify the structure of the term t with an abstract description of a derivation within a given theory of Justification Logic in a way that satisfies the *Internalization Theorem* [1]: if p is a proof of a theorem χ in the logic, then there is a systematic way to construct a term u_p whose structure reflects the structure of p such that $u_p :_i \chi$ is also a theorem of the logic. This provides us with a *proof-based notion of evidence*, in the sense that the appearance of a term t in a theorem $t :_i \varphi$ encodes a description as to why it is (according to the theory) that agent i believes that φ holds. This leads us to read the formula $t :_i \varphi$ as "t is agent i's evidence that φ is true."

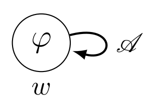
Returning to the deficiency of modal logic, Justification Logic enables us to replace the statement $B_i\varphi \to B_i\psi$ of conditional (modal) belief with a more nuanced statement of conditional *evidence-based belief*: for a term s_t built up from t, the formula $t :_i \varphi \to s_t :_i \psi$ says, "if t is agent i's evidence for φ, then s_t is agent i's evidence for ψ." Since s_t is built up from t, we see that this tells us that in case agent i has justification t for his or her belief in φ, then agent i may justify his or her belief in ψ by inserting the argument t into the appropriate places in the argument s_t, thereby yielding an argument supporting ψ. In this way, Justification Logics provide an in-language notion of *proof-based evidence*. To distinguish this notion of evidence from other notions of evidence, it may be helpful to think of proof-based evidence in the following intuitive way: to have *proof-based evidence* for φ is to have deductive argumentation supporting φ.

One immediate limitation for the proof-based notion of evidence found in many Justification Logics is that it is *static*: in these theories, either agent i has an argument supporting φ, in which case he or she has and will always have evidence for φ, or else agent i does not have an argument supporting φ, in which case he or she does not and will never have evidence for φ. Such Justification Logics do not allow agents to *learn* new evidence, to *forget* old evidence, or to *have a change of mind* about the reliability of existing evidence. This is clearly at odds with everyday experience where, for example, agent i may believe ψ on the basis of an argument s_t only to find out later that the evidence t on which s_t is crucially dependent is completely unreliable, thereby forcing agent i to throw out both t and s_t and hence the belief in ψ on the basis of s_t. What is missing from basic Justification Logics is a certain *evidence dynamics*, whereby certain basic pieces of evidence can change their status from "good" to "bad" (or the other way around).

A further criticism of the proof-based notion of evidence is that it neglects the intuitive *social* role that evidence plays as a vehicle for *persuasion*: if agent i has an argument supporting φ but agent j does not, then agent i ought to be able to tell agent j of i's argument supporting φ, which might have the effect of convincing agent j to believe φ. Here what is missing is a means of describing how evidence dynamics can be brought about as a result of *communication*. But note that the way in which this communication takes place must also be taken into account: i's argument might be told to j in public, which then might also effect the beliefs of other agents who hear of i's argument; alternatively, i's argument may be told to j in private, which would probably not effect the others' beliefs (though they might develop suspicions as to what i and j are talking about if they see i and j depart for a closed-door meeting).

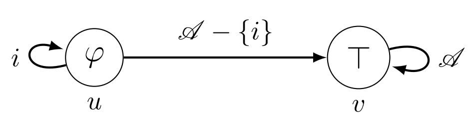
One framework that elegantly describes a wide range of multi-agent communications—whether public or private, with or without deception, having or not having suspicion, and so on—is the framework of *Dynamic Epistemic Logic*. In Dynamic Epistemic Logic, a communication is modeled using what we call an *update frame*, which is a finite Kripke frame[4] whose worlds have been labeled by formulas. The idea, developed by Baltag, Moss, and Solecki (BMS) [4, 5] based on work on *public announcements* by Plaza [17, 18] and by Gerbrandy and Groeneveld [13, 14], is that each world w in an update frame represents a possible communication of the formula χ_w that labels w. The structure of the arrows in the update frame describes the agents' conditional uncertainties as to which communication is actually taking place: in case there is an i-arrow from world w to world w', then agent i will think it possible that the formula $\chi_{w'}$ is communicated if it is the case that the formula χ_w is actually communicated. This allows us to use update frames to represent a wide variety of communications.

As an example, for a nonempty finite set $\mathscr{A}$ of agents, the update frame



represents the *public announcement of* φ. If $\chi_w = \varphi$ is actually communicated, then each agent $i \in \mathscr{A}$ thinks that the only formula that can possibly be communicated is the formula $\chi_w = \varphi$ itself.[5] Thus to have such a communication occur is to make it the case that every agent will simultaneously come to know φ, which is intuitively what happens when a public announcement of φ occurs. Of course, the update frame above for the public announcement of φ also says something about *higher-order knowledge:* since there is a unique i-arrow from w to w for an agent $i \in \mathscr{A}$ and since there is also a unique j-arrow from w to w for another agent $j \in \mathscr{A}$, j knows that i knows that φ is to be communicated. After all, at the unique world w that j considers possible, agent i knows that χ_w is the formula that is communicated. It is not too hard to see that the update frame above describes a situation in which it is *common knowledge* that φ is communicated; that is, each agent knows that φ is communicated, each agent knows that each agent knows that φ is communicated, each agent knows that each agent knows that each agent knows that φ is communicated, and so on for any finite number of occurrences of the phrase "each agent knows that" appearing before the phrase "φ is communicated."

As another example, the update frame



[4]A *finite Kripke frame* is a finite directed graph whose nodes are called "worlds" and whose edges are labeled by agents' names. We will call the edges of a finite Kripke frame "arrows" and we will use the phrase "X-arrow" to refer to an arrow that is labeled by X.

[5]We adopt the following drawing convention for update frames: for a (possibly empty) set S of agents, if a drawing of an update frame contains an S-arrow from a world w to a world w', then what is meant is that for each agent $a \in S$, there is an a-arrow from w to w'.

represents the *private announcement of* φ *to agent* i, since in case $\chi_u = \varphi$ is the formula that is actually communicated, the unique i-arrow from u to u says that i believes that the only formula that can possibly be communicated is the formula $\chi_u = \varphi$, whereas the unique j-arrow from u to v for each $j \in \mathscr{A} - \{i\}$ says that any other agent j believes that the formula $\chi_v = \top$ (the propositional constant for truth, which conveys no new information) was communicated.[6] Thus if φ is actually communicated, then the information about φ is sent to agent i while all the other agents gain no new information. This is intuitively what it is to have a private announcement of φ to agent i.

Using symbols to denote update frames, Dynamic Epistemic Logic allows us to form new formulas of the form $[U,u]\varphi$, where U is a symbol for an update frame, u is a symbol naming a world in the update frame U (here u indicates that the formula χ_u is what is actually communicated), and φ is a formula that has already been formed. The formula $[U,u]\varphi$ then says, "after the update (U,u), where the formula χ_u is actually communicated, we have that φ is true." (Note that we use the words "update" and "communication" interchangeably because very complicated update frames are better thought of as generalized "informational updates" due to the fact that there is not always an intuitive, everyday communicative type corresponding to each complex update frame; see the discussion in [19].) Since the formula φ occurring in $[U,u]\varphi$ can be a formula describing various agents' beliefs, we see that Dynamic Epistemic Logic can be used to describe how communications affect the beliefs of agents; for example, $[U,u]B_i\psi$ says that agent i believes ψ after the occurrence of the update (U,u).

While a communication can affect agent belief, the language of Dynamic Epistemic Logic—itself an extension of the language of modal logic—does not provide us with the means to describe *reasons* for such changes in agent belief. To illustrate, let p be a propositional letter and then consider a valid formula of the form $p \to [U,u]B_ip$ ("if p is true, then, after update (U,u), agent i believes p"). This formula stipulates a connection between the occurrence of the update (U,u) and agent i coming to believe the atom p. In essence, the content of the update (U,u) brings about this change in agent i's belief, though the reason why this particular change in belief is brought about is left unstated—the formula says only that this communication brings about i's change in belief *for some reason*. (This is similar to the case of modal logic itself, where the formula $B_i\varphi \to B_i\psi$ does not say *why* it is that i's belief in ψ follows whenever he or she believes φ.) Indeed, despite the fact that an explanation based on the structure of U justifies this change in i's belief, this explanation is not expressible within the language itself. So for example, if U is the public announcement of p, then the explanation of why (U,u) causes i to believe p would go something like "since a public announcement of a fact causes all agents to come to learn that fact, agent i will come to believe the fact p after it is publicly announced"; however, such an explanation cannot be formulated within the language. Furthermore, if we suppose that another agent j believes

[6]More precisely: the $\mathscr{A}$-arrow from v to v makes it so that in case $\chi_u = \varphi$ is the formula that is actually communicated, each of the non-i agents mistakenly believes that it is common belief that $\top$ is communicated (and so no new information is conveyed). So while i learns the new information φ—this due to the i-arrow from u to u—the other agents have the mistaken common belief that no new information has been communicated.

that update (U, u) will cause agent i to come to believe the fact p,

$$B_j(p \to [U, u]B_i p) \ ,$$

then, just as in modal logic, there is no way for the language to describe agent j's justification for this belief. In short, the language is not only lacking the capacity to express the agents' reasons for *holding* their beliefs (as is the case with modal logic) but it is also unable to express the agents' reasons for *changing* their beliefs.

To summarize what we have said thus far, we have argued that each of Justification Logic and Dynamic Epistemic Logic would be better suited for modeling, describing, and studying rational agency if it had additional features. In the case of Justification Logic, we have argued for the ability to handle *evidence dynamics*, taking into account the *social* role that evidence plays as a vehicle for persuasion in various kinds of communication, be it public or private, with or without deception, having or not having suspicion, and so on. In the case of Dynamic Epistemic Logic, we have argued for an *in-language account of evidence* so that agents can provide justifications not only as to why they hold certain beliefs but also as to why they would change these beliefs. We thus think it natural to look for a combined framework—something that might eventually be called *Dynamic Justification Logic*—that brings together the complementary strengths from each of Dynamic Epistemic Logic and Justification Logic so as to address each area's respective weakness.

In this paper, we begin the project of Dynamic Justification Logic by proposing a theory called JLCE that combines multi-agent communication and belief from Dynamic Epistemic Logic with a new multi-agent Justification Logic for reasoning about evidence and justification. JLCE, the *theory of Justification Logic with Communication and Elimination*, allows us to reason not only about multi-agent belief dynamics arising as the result of Dynamic Epistemic Logic-style communications—hereafter called *(multi-agent) communications*—but also about a kind of multi-agent evidence dynamics we call *evidence elimination*, whereby a piece of evidence t relevant to a particular claim is to be set aside as part of a multi-agent communication whose content effectively undermines the basic pieces of evidence that make up t. Our email example indicates one of the many kinds of multi-agent evidence eliminations that JLCE can reason about: with Bob's basic piece of evidence x_3 undermined, his more complex evidence obtained by combining the rotten piece of evidence x_3 with another piece of evidence x_1 is also undermined.

After defining the language, axiomatics, and semantics of JLCE, we will show how this theory can be used to formalize the evidence elimination from our email example. Along the way, we will prove a number of results about JLCE.

3 Syntax

Our language, called UL (for "update language"), is used to reason about multi-agent communication and evidence elimination for a finite nonempty set $\mathscr{A}$ of rational agents. The atoms of UL are given as follows.

Definition 3.1. We define the sets

$$\begin{aligned}\mathscr{P} &:= \{p_k \mid k \in \mathbb{N}\} \text{ of } \textit{propositional letters},\\ \mathscr{C} &:= \{c_k \mid k \in \mathbb{N}\} \text{ of } \textit{constants}, \text{ and}\\ \mathscr{V} &:= \{x_k \mid k \in \mathbb{N}\} \text{ of } \textit{variables}.\end{aligned}$$

To define the full language of UL, we introduce the notion of *update frame*. Update frames are an extension of the "action models" found in the Dynamic Epistemic Logic literature [7, 21, 28].

Definition 3.2. Given a nonempty set F of formulas, to say that U is an *F-update frame* means that U is a tuple $(W^U, R^U, \mathsf{p}^U, \mathsf{v}^U)$ whose components satisfy the following.

- W^U is a nonempty finite set of *worlds* (in U).
- $R^U : \mathscr{A} \to \wp(W^U \times W^U)$ assigns a transitive binary relation R^U_i to each agent $i \in \mathscr{A}$.[7]
- $\mathsf{p}^U : W^U \to F$ labels worlds with formulas.
- $\mathsf{v}^U : W^U \to \wp(\mathscr{V} \times \mathscr{A} \times F)$ labels worlds with finite sets of variable-agent-formula triples (x, i, φ).[8]

An F-update frame U and a world $u \in W^U$ form a *pointed F-update frame* with *point u*. For a set F' of formulas, we write $\mathscr{U}(F')$ to denote the set of pointed F'-update frames.

Pointed update frames are used to describe communications along with eliminations of relevant evidence. Informally speaking, to say that t *is relevant for agent i to φ* means that i considers evidence t to be probative for φ—that is, i believes that t generally tends to demonstrate or prove φ—though i need not be persuaded to believe φ on the basis of evidence t. So to have i take evidence t as relevant to φ is just to say that i considers t to be admissible as evidence for φ, regardless of whether this admissibility also comes with belief. We will write "$t \gg_i \varphi$" to mean that t is relevant for i to φ. Note that agent i may consider many different pieces of evidence to be relevant to a single assertion φ. The statement $t \gg_i \varphi$ merely asserts that t is among the possibly many pieces of evidence that are relevant to φ.

When agent i has a piece of evidence t relevant to φ and agent i also believes that φ is true, then we will say that t is *agent i's evidence for believing* φ and we will write "$t :_i \varphi$". So we have two notions of evidence: the weaker notion $t \gg_i \varphi$ of relevant evidence and the stronger notion $t :_i \varphi$ of relevant evidence plus belief.

Pointed update frames (U, u) will be used to describe an *update*, which is a multi-agent communication. A pointed update frame (U, u) describes the communication of the formula $\mathsf{p}^U(u)$ within a collection $\{\mathsf{p}^U(v) \mid v \in W^U\}$ of formulas that might be communicated. So while the update (U, u) actually communicates the formula $\mathsf{p}^U(u)$, the agents have uncertainty as to which communication actually occurs according to the structure of the Kripke

[7] Our reasons for taking R^U_i transitive are for convenience only. We shall discuss this point later in Remark 3.13.

[8] Finiteness ensures that update frames can in principle be written out (using an appropriate syntactic encoding) and that a forthcoming theory based on update frames (the U-calculus, Definition 3.5) is decidable.

frame (W^U, R^U) underlying U and the way in which p^U labels worlds in U by formulas. We explained this in greater detail above in the introduction, where we gave examples of two update frames, one for the public announcement of φ and one for the private announcement of φ to agent i.

Our contribution in this paper is to add a notion of *multi-agent evidence elimination* to the above-described BMS-picture. We thus view each world in an update frame U as constituting a *possible communication and elimination:* for an update (U, u), a variable $x \in \mathscr{V}$, an agent $i \in \mathscr{A}$, and an assertion φ, if $(x, i, \varphi) \in \mathsf{v}^U(u)$, then the communication and elimination possibility u simultaneously communicates $\mathsf{p}^U(u)$ and eliminates agent i's evidence x relevant to φ, which has the effect of falsifying $x \gg_i \varphi$. But an elimination of a variable x relevant to an assertion can have consequences for other pieces of evidence s_x that are built up from x. We saw this in our email example: Anne's final email x_5 eliminated Bob's basic piece of evidence x_3 relevant to the store being open, and this elimination itself triggered an elimination of Bob's combined evidence x_1 and x_3 that Anne is bringing the cheese.

Definition 3.3. $\mathscr{G}_0$ is the following grammar.

$$\begin{array}{rcl} t & ::= & c \mid x \mid t \cdot_\varphi t \mid t + t \mid !t \\ \varphi & ::= & \bot \mid p \mid \varphi \to \varphi \mid B_i\varphi \mid t \gg_i \varphi \\ & & c \in \mathscr{C},\, x \in \mathscr{V},\, p \in \mathscr{P},\, i \in \mathscr{A} \end{array}$$

Let $\mathscr{T}_0$ be the set of expressions built using t as a start symbol in grammar $\mathscr{G}_0$, and let $\mathscr{F}_0$ be the set of expressions built using φ as a start symbol in grammar $\mathscr{G}_0$. Then, whenever the pair $(\mathscr{T}_k, \mathscr{F}_k)$ is defined, we define grammar $\mathscr{G}_{k+1}$ as follows.

$$\begin{array}{rcl} t & ::= & s \mid t \cdot_\varphi t \mid t + t \mid !t \\ \varphi & ::= & \psi \mid \varphi \to \varphi \mid B_i\varphi \mid t \gg_i \varphi \mid [U, u]\varphi \\ & & s \in \mathscr{T}_k,\, \psi \in \mathscr{F}_k,\, i \in \mathscr{A},\, (U, u) \in \mathscr{U}(\mathscr{F}_k) \end{array}$$

Let $\mathscr{T}_{k+1}$ be the set of expressions built using t as a start symbol in grammar $\mathscr{G}_{k+1}$, and let $\mathscr{F}_{k+1}$ be the set of expressions built using φ as a start symbol in grammar $\mathscr{G}_{k+1}$. Finally, we define the set $\mathscr{T} := \bigcup_{k \in \mathbb{N}} \mathscr{T}_k$ of UL-*terms* (also called *terms*) and the set $\mathscr{F} := \bigcup_{k \in \mathbb{N}} \mathscr{F}_k$ of UL-*formulas* (also called *formulas*). We adopt the usual abbreviations for other logical connectives. The *update language*, written UL, is the pair $(\mathscr{T}, \mathscr{F})$ consisting of the set $\mathscr{T}$ of terms and the set $\mathscr{F}$ of formulas. We define the set $\mathscr{U} := \mathscr{U}(\mathscr{F})$ of *pointed* UL-*update frames* (also called *pointed update frames*).

Definition 3.4. $t :_i \varphi$ abbreviates $(t \gg_i \varphi) \wedge B_i\varphi$.

The constants and variables together constitute the *atomic terms*. The symbol B_i is called an i-*belief modal* (or simply a *belief modal*), the symbol $\gg_i$ is called an i-*arrow* (or simply an *arrow*), the symbol $:_i$ is called an i-*colon* (or simply a *colon*), and the symbol

Rules

$$\frac{(x_k, i, \varphi) \in \mathsf{v}^U(u)}{U, u \vdash (x_k, i, \varphi)}\ \text{(EV)}$$

$$\frac{U, u \vdash (t, i, \varphi \to \psi)}{U, u \vdash (t \cdot_\varphi s, i, \psi)}\ \text{(EAL)} \qquad \frac{U, u \vdash (s, i, \varphi)}{U, u \vdash (t \cdot_\varphi s, i, \psi)}\ \text{(EAR)}$$

$$\frac{U, u \vdash (t, i, \varphi) \qquad U, u \vdash (s, i, \varphi)}{U, u \vdash (t + s, i, \varphi)}\ \text{(ES)} \qquad \frac{U, u \vdash (t, i, \varphi)}{U, u \vdash (!t, i, t :_i \varphi)}\ \text{(EC)}$$

$$\frac{U, u' \vdash (x_k, i, \varphi) \qquad u R_i^U u'}{U, u \vdash (x_k, i, \varphi)}\ \text{(EM)}$$

Figure 1: The U-calculus

$[U, u]$ is called an *update modal.* There are a number of interesting sub-languages of UL; in particular, the *basic fragment* is the pair $(\mathscr{T}_b, \mathscr{F}_b)$ obtained from $(\mathscr{T}_0, \mathscr{F}_0)$ by restricting formulas containing belief modals or arrows to the form $t :_i \varphi$. We assign the following informal readings to the key formulas in UL.

- $B_i\varphi$ is read, "agent i believes φ."
- $t \gg_i \varphi$ is read, "t is agent i's evidence relevant to φ."
- $t :_i \varphi$ is read, "t is agent i's evidence for believing φ."
- $[U, u]\varphi$ is read, "after update (U, u), formula φ is true."

Each term describes a way of obtaining derived evidence from given evidence through the use of logical principles; let us discuss this now. The constants $c \in \mathscr{C}$ play a special role as evidence relevant to the axioms of our theory, and the variables $x \in \mathscr{V}$ are used as contingent relevant evidence that may be directly affected by an evidence elimination. The operation $\cdot_\varphi$ takes evidence t relevant to an implication $\varphi \to \psi$ and evidence s relevant to the antecedent φ and produces the evidence $t \cdot_\varphi s$ relevant to the consequent ψ, in accord with the rule of Modus Ponens. The operation $+$ is the union of pieces of relevant evidence: $t + s$ is relevant evidence for anything for which one or both of t and s is relevant evidence. The operation ! ("bang") is an evidence checker: in case t is relevant evidence for φ, then $!t$ ("bang t") checks that t is evidence for φ.

Definition 3.5. Given an update frame U, the *U-calculus* is defined in Figure 1.

The U-calculus will be used to describe the effect evidence eliminations have on evidence relevance in the following way: the update (U, u) eliminates agent i's evidence t relevant to φ—meaning that $t \gg_i \varphi$ will be false after update (U, u)—if and only if $U, u \vdash (t, i, \varphi)$ is derivable in the U-calculus.

Remark 3.6 (Variable-Driven Elimination)**.** We observe that our formulation of the U-calculus in Figure 1 restricts evidence elimination in such a way that whenever update (U, u) eliminates agent i's evidence t relevant to φ—thereby falsifying $t \gg_i \varphi$—this elimination can be traced back to an elimination of one or more variables $x \in \mathscr{V}$ as relevant evidence for agent i. This restriction, which we call *variable-driven elimination*, guarantees that whenever a formula of the form $s \gg_i \psi$ is derivable in our forthcoming theory JLCE (Definition 3.12), it will not be possible for an update (U', u') to eliminate s as agent i's evidence relevant to ψ. Variable-driven elimination therefore maintains a property of *elimination consistency:* to say that $s \gg_i \psi$ is derivable in our forthcoming theory JLCE means that s is *always* agent i's evidence relevant to ψ, and hence no elimination can eliminate s as agent i's evidence relevant to ψ. While the author conjectures that it is possible to develop an alternative to the U-calculus that provides for elimination consistency under a condition weaker than that of variable-driven elimination, allowing eliminations to be traced back to a term $t \in \mathscr{T}$ that need not be a variable, various syntactic consistency checks would need to be imposed on the structure of update-frame–eliminable terms in order to maintain elimination consistency. In the interest of simplicity in a first paper on evidence elimination in Justification Logic, the author has gone the route of variable-driven elimination. This may seem problematic: in everyday life, one often discovers a problem with his or her reasoning only after it has gone on for some time and then suddenly faces a contradiction. This suggests that one should eliminate the more complex piece of evidence s directly and then see what consequences this has not only for the various pieces of complex evidence that make use of s but also those pieces of simpler evidence that themselves make up s. Put another way, what is suggested is a general theory of *evidence-based Belief Revision*, in which an agent reasons in a step-by-step manner according to his or her evidence, adjusting this evidence on-the-fly whenever contradictions are encountered or new information comes in. While this is an extremely interesting and promising line of work, the scope of this study is too large for the purposes of a single paper.

The goal of this paper is to provide a basic account of the way in which complex pieces of evidence are affected by changes in basic pieces of evidence. So evidence elimination is not at all a general account of evidence-based Belief Revision but instead an account of part of an adversarial argumentative process that plays an important role in the overall evidence-based Belief Revision picture. In particular, this process is one in which an agent's reasoning is successfully undermined by an attack on the premises underlying that reasoning. To illustrate, think of the friends in our email example as employees of a catering company managed by Bob and assume that an angry client Daisy sues Bob in court, alleging that the employees of the company ruined her party by providing crackers without any cheese, and this was due to the manager Bob's negligence. Bob might show emails x_1 through x_4 to the court to argue that he was not negligent. But when Daisy manages to obtain a copy of email x_5 and shows this to the court, thereby eliminating Bob's evidence that he did not know of any problem his employees might have in delivering both the crackers and the cheese, the court will presumably side with Daisy. So we see how evidence elimination is about this fundamental adversarial process of undermining an agent's reasoning by attacking the premises. But while this process is very important to a general theory of evidence-based Belief Revision, it is merely one way in which evidence changes in light of new information within a generalized framework of evidence-based Belief Revision. Other changes that directly affect more complex pieces of evidence need to be studied as well, and future work in Dynamic Justification Logic will aim toward establishing a general account of evidence-based Belief Revision that does just this. In the meantime, the work in the present paper shall stick to the study of variable-based evidence elimination.

Given (U, u) and (t, i, φ), either $U, u \vdash (t, i, \varphi)$ or $U, u \nvdash (t, i, \varphi)$, and determining which of these is the case is a decidable question. Further, the U-calculus satisfies the following anti-monotonicity property.

Lemma 3.7 (Anti-Monotonicity)**.** $U, u' \vdash (t, i, \psi)$ and $u R_i^U u'$ together imply that $U, u \vdash (t, i, \psi)$.

Proof. By a straightforward induction on the length of the derivation of $U, u' \vdash (t, i, \psi)$. □

To state the axiomatics, we define a notion of *composition* for update frames that allows us to take pointed update frames (U, u) and (U', u') and build a pointed update frame $\big(U \circ U', (u, u')\big)$ whose execution has the same effect as the sequential execution of update (U, u) followed by update (U', u').

Notation 3.8 (Functions on Pairs)**.** We shall often write "$f(a, b)$" instead of "$f((a, b))$" when we wish to denote the value of function f on the pair (a, b); our intended meaning will be clear from context.

Definition 3.9. To define the *composition* $U \circ U'$, we use standard definitions from Dynamic Epistemic Logic [7, 21, 28] for the first three components: set $W^{U \circ U'} := W^U \times W^{U'}$, allow $(u, u') R_i^{U \circ U'} (v, v')$ if and only if $u R_i^U v$ and $u' R_i^{U'} v'$, and set $\mathsf{p}^{U \circ U'}(u, u') := \neg[U, u]\neg \mathsf{p}^{U'}(u')$. To define the final component $\mathsf{v}^{U \circ U'}$, we set $\mathsf{v}^{U \circ U'}(u, u') := \mathsf{v}^U(u) \cup \mathsf{v}^{U'}(u')$.

Theorem 3.10 (Composition Correctness)**.** If U and U' are update frames, then so is $U \circ U'$.

Proof. It suffices to verify that $R_i^{U \circ U'}$ is transitive for each $i \in \mathscr{A}$, but this follows by the definition of $R_i^{U \circ U'}$ and the fact that each of R_i^U and $R_i^{U'}$ is transitive. □

The definition of composition ensures that $\big(U \circ U', (u, u')\big)$ eliminates agent i's evidence t relevant to the assertion that φ if and only if at least one of (U, u) or (U', u') eliminates i's evidence t relevant to the assertion that φ.

Lemma 3.11 (Composition)**.** Given update frames U and U', the following statements are equivalent.

- $U \circ U', (u, u') \vdash (t, i, \varphi)$.
- $U, u \vdash (t, i, \varphi)$ or $U', u' \vdash (t, i, \varphi)$.

Belief/Evidence Schemes

CL. Classical propositional tautologies

B1. $B_i(\varphi \to \psi) \to (B_i\varphi \to B_i\psi)$

B2. $B_i\varphi \to B_iB_i\varphi$

E1. $t \gg_i (\varphi \to \psi) \to ((s \gg_i \varphi) \to (t \cdot_\varphi s) \gg_i \psi)$

E2. $(t \gg_i \varphi) \to (t + s) \gg_i \varphi$

$(s \gg_i \varphi) \to (t + s) \gg_i \varphi$

E3. $(t \gg_i \varphi) \to\, !t \gg_i (t :_i \varphi)$

E4. $(t \gg_i \varphi) \to B_i(t \gg_i \varphi)$

Elimination/Update Schemes

U1. $[U, u]q \leftrightarrow (\mathsf{p}^U(u) \to q)$ if $q \in \mathscr{P} \cup \{\bot\}$

U2. $[U, u](\varphi \to \psi) \leftrightarrow ([U, u]\varphi \to [U, u]\psi)$

U3. $[U, u]B_i\varphi \leftrightarrow (\mathsf{p}^U(u) \to \bigwedge_{uR_i^U v} B_i[U, v]\varphi)$

U4. $[U, u](t \gg_i \varphi) \leftrightarrow \neg\mathsf{p}^U(u)$ if $U, u \vdash (t, i, \varphi)$

$[U, u](t \gg_i \varphi) \leftrightarrow (\mathsf{p}^U(u) \to t \gg_i \varphi)$ if $U, u \nvdash (t, i, \varphi)$

U5. $[U, u][U', u']\varphi \leftrightarrow [U \circ U', (u, u')]\varphi$

Rule: $\dfrac{\varphi}{c_k :_i \varphi}$ (CN)

Figure 2: The theory AX

Proof. By induction on the construction of t. The base case $t \in \mathscr{C}$ follows immediately and the base case $t \in \mathscr{V}$ follows because $\mathsf{v}^{U \circ U'}(u, u') = \mathsf{v}^U(u) \cup \mathsf{v}^{U'}(u')$. The induction cases $t = s_1 \cdot_\varphi s_2$, $t = s_1 + s_2$, and $t = !s$ follow by the induction hypothesis and respective use of rules EAL and EAR, ES, and EC (Figure 1). □

Finally, we state in two parts the axioms and rules of our *theory of Justification Logic with Communication and Elimination*, written JLCE. The first part is the axiomatic theory AX (Figure 2), and the second part is our theory JLCE itself (Figure 3).

Definition 3.12. The theory AX is defined in Figure 2. Regarding Axiom U3, we stipulate that a conjunction $\bigwedge_{uR_i^U v} \chi_v$ ranging over the set $S := \{v \in W^U \mid uR_i^U v\}$ is to be identified with a fixed tautology denoted by "$\top$" whenever $S = \emptyset$. The theory JLCE is defined in Figure 3. The theory JLCE^* consists of Rules AX and MP from Figure 3.

We will see later that the theories JLCE and JLCE^* derive the same theorems. This result will be useful: while our primary interest is in the full theory JLCE, it is sometimes easier to prove a given result using the simpler theory JLCE^*.

Rules

$$\frac{\mathsf{AX} \vdash \varphi}{\varphi}\ (\text{AX}) \qquad \frac{\varphi \to \psi \quad \varphi}{\psi}\ (\text{MP})$$

$$\frac{\varphi}{B_i\varphi}\ (\text{BN}) \qquad \frac{\varphi}{[U,u]\varphi}\ (\text{UN})$$

Figure 3: The theory JLCE

Remark 3.13. We observe two points. First, in the statement of Axiom U4 (Figure 2), we have offloaded to the U-calculus the work of determining the particular form the axiom should take for a given triple (t, i, φ). It is possible to eliminate this reliance on an external theory by including new formulas in the language that allow us to embed the U-calculus into the overall theory. See [22] for details on how this is done in a theory of simple evidence elimination for Justification Logic. The reason we have offloaded the work in this way in this paper is to save space and to simplify the presentation of the axiomatics.

Second, we have assumed that agent belief is governed by the modal theory K4 and hence that agent belief is introspective (meaning it satisfies Axiom B2, which says that if agent i believes something, then she believes she believes it). While other choices are possible in Justification Logic [15, 20], they would introduce complications here that would distract us from the focus of this paper: multi-agent communication and evidence elimination.

Notation 3.14 (Provable Equivalence $\varphi \Leftrightarrow \varphi'$ and $\varphi \Leftrightarrow^* \varphi'$)**.** We write $\varphi \Leftrightarrow \varphi'$ to mean that $\vdash \varphi \leftrightarrow \varphi'$, and we write $\varphi \Leftrightarrow^* \varphi'$ to mean that $\vdash^* \varphi \leftrightarrow \varphi'$. Given $n \in \mathbb{N}$, we write

$$\varphi_0 \Leftrightarrow \varphi_1 \Leftrightarrow \varphi_2 \Leftrightarrow \cdots \Leftrightarrow \varphi_n \tag{1}$$

to mean that we have $\varphi_i \Leftrightarrow \varphi_{i+1}$ for each $i \in \mathbb{N}$ with $i < n$. Similarly, given $n \in \mathbb{N}$, we write (1) with each "$\Leftrightarrow$" replaced by "$\Leftrightarrow^*$" to mean that we have $\varphi_i \Leftrightarrow^* \varphi_{i+1}$ for each $i \in \mathbb{N}$ with $i < n$.

4 The Internalization Theorem

Artemov [1] first identified a key result of single-agent Justification Logics (where $|\mathscr{A}| = 1$) called the property of *internalization:* if $\mathscr{A} = \{a\}$, then for each theorem φ, there is a term t such that $t :_a \varphi$ is also a theorem. Internalization specifies the way in which a Justification Logic allows us to describe proofs of the theory using terms, which bolsters our reading of terms as pieces of evidence for the formulas that they label. In this section, we will prove the internalization property for JLCE. To do this, we will show first that JLCE* has internalization (this is the forthcoming Lemma 4.2) and then prove that JLCE* and JLCE derive the same theorems (this is the forthcoming Theorem 4.13). The proofs of these results require quite a bit of work, so we shall proceed incrementally. We begin with a lemma that

1.	$t :_i (\varphi \to \psi) \to t \gg_i (\varphi \to \psi)$	CL
2.	$s :_i \varphi \to s \gg_i \varphi$	CL
3.	$t \gg_i (\varphi \to \psi) \to (s \gg_i \varphi \to (t \cdot_\varphi s) \gg_i \psi)$	E1
4.	$t :_i (\varphi \to \psi) \to (s :_i \varphi \to (t \cdot_\varphi s) \gg_i \psi)$	PR 1–3
5.	$t :_i (\varphi \to \psi) \to B_i(\varphi \to \psi)$	CL
6.	$s :_i \varphi \to B_i\varphi$	CL
7.	$B_i(\varphi \to \psi) \to (B_i\varphi \to B_i\psi)$	B1
8.	$t :_i (\varphi \to \psi) \to (s :_i \varphi \to B_i\psi)$	PR 5–7
9.	$(t \cdot_\varphi s) :_i \psi \leftrightarrow (t \cdot_\varphi s) \gg_i \psi \wedge B_i\psi$	CL
10.	$t :_i (\varphi \to \psi) \to (s :_i \varphi \to (t \cdot_\varphi s) :_i \psi)$	PR 4, 8, 9

Figure 4: Proof that $\vdash^* t :_i (\varphi \to \psi) \to \big((s :_i \varphi) \to (t \cdot_\varphi s) :_i \psi\big)$

1.	$t :_i \varphi \to t \gg_i \varphi$	CL
2.	$t \gg_i \varphi \to (t + s) \gg_i \varphi$	E2
3.	$t :_i \varphi \to B_i\varphi$	CL
4.	$(t + s) :_i \varphi \leftrightarrow (t + s) \gg_i \varphi \wedge B_i\varphi$	CL
5.	$t :_i \varphi \to (t + s) :_i \varphi$	PR 1–4
6.	$s :_i \varphi \to s \gg_i \varphi$	CL
7.	$s \gg_i \varphi \to (t + s) \gg_i \varphi$	E2
8.	$s :_i \varphi \to B_i\varphi$	CL
9.	$s :_i \varphi \to (t + s) :_i \varphi$	PR 4, 6–8
10.	$\big((t :_i \varphi) \vee (s :_i \varphi)\big) \to (t + s) :_i \varphi$	PR 5, 9

Figure 5: Proof that $\vdash^* \big((t :_i \varphi) \vee (s :_i \varphi)\big) \to (t + s) :_i \varphi$

shows that in JLCE^*, evidence—and not just relevant evidence—is closed under the term-forming operations $\cdot_\varphi$ and $+$.

Lemma 4.1. We have each of the following.

1. $\vdash^* t :_i (\varphi \to \psi) \to \big((s :_i \varphi) \to (t \cdot_\varphi s) :_i \psi\big)$.
2. $\vdash^* \big((t :_i \varphi) \vee (s :_i \varphi)\big) \to (t + s) :_i \varphi$.

Proof. Using PR to denote use of classical propositional reasoning, the proofs appear in Figures 4 and 5. □

Lemma 4.2 (JLCE^* Internalization; [1, 20])**.** If $\vdash^* \varphi$ and $i \in \mathscr{A}$, then there there is a term $t \in \mathscr{T}$ such that $\vdash^* t :_i \varphi$.

Proof. This argument is a straightforward adaptation of the standard proof of Artemov's original internalization result [1]. Choosing $i \in \mathscr{A}$, the proof is by induction on the length of JLCE^* derivations. If the last step of the proof was Rule AX, then set $t := c_0$ and observe that $\vdash^* c_0 :_i \varphi$ by Rule AX. If the last step of the proof was Rule MP from premises $\psi \to \varphi$

1.	$t :_i \varphi \to B_i\varphi$	CL
2.	$B_i\varphi \to B_iB_i\varphi$	B2
3.	$t :_i \varphi \to t \gg_i \varphi$	CL
4.	$(t \gg_i \varphi) \to B_i(t \gg_i \varphi)$	E4
5.	$t :_i \varphi \to B_i(t \gg_i \varphi) \wedge B_iB_i\varphi$	PR 1–4
6.	$B_i(t :_i \varphi) \leftrightarrow B_i(t \gg_i \varphi) \wedge B_iB_i\varphi$	MR
7.	$t :_i \varphi \to B_i(t :_i \varphi)$	PR 5, 6
8.	$t \gg_i \varphi \to !t \gg_i (t :_i \varphi)$	E3
9.	$t :_i \varphi \to !t \gg_i (t :_i \varphi)$	PR 3, 8
10.	$!t :_i (t :_i \varphi) \leftrightarrow !t \gg_i (t :_i \varphi) \wedge B_i(t :_i \varphi)$	CL
11.	$t :_i \varphi \to !t :_i (t :_i \varphi)$	PR 7, 9, 10

Figure 6: Proof that $\vdash^* t :_i \varphi \to !t :_i (t :_i \varphi)$

and ψ, then it follows by the induction hypothesis that there is a term $s_1 \in \mathscr{T}$ and a term $s_2 \in \mathscr{T}$ such that $\vdash^* s_1 :_i (\psi \to \varphi)$ and $\vdash^* s_2 :_i \psi$. So set $t := s_1 \cdot_\psi s_2$ and observe that $\vdash^* (s_1 \cdot_\psi s_2) :_i \varphi$ by Lemma 4.1 and classical propositional reasoning. □

While JLCE^* does not explicitly include Rule BN, the following lemma shows that this rule is admissible. Since JLCE is just JLCE^* plus the rules BN and UN, this lemma gets us one step closer to showing that the theories JLCE^* and JLCE derive the same theorems.

Lemma 4.3 (Belief Admissibility)**.** If $\vdash^* \varphi$ and $i \in \mathscr{A}$, then $\vdash^* B_i\varphi$.

Proof. Suppose $\vdash^* \varphi$ and $i \in \mathscr{A}$. Then for some term $t \in \mathscr{T}$, we have $\vdash^* t :_i \varphi$ by Lemma 4.2. Since $t :_i \varphi$ abbreviates $(t \gg_i \varphi) \wedge B_i\varphi$, it follows by classical propositional reasoning that $\vdash^* B_i\varphi$. □

Once we know that Rule BN is admissible, we can show that evidence (and not just relevant evidence) is closed under the term-forming operation !.

Lemma 4.4. We have $\vdash^* t :_i \varphi \ \to\ !t :_i (t :_i \varphi)$.

Proof. Using PR to denote the use of classical propositional reasoning and MR to denote the use of modal reasoning (using Lemma 4.3 in place of Rule BN), the proof appears in Figure 6. □

Since Rule BN is admissible in JLCE^*, to show that the theories JLCE^* and JLCE derive the same theorems, it suffices for us to show that Rule UN is also admissible in JLCE^*. Before we begin the proof of this result, we need the following auxiliary lemmas. The first concerns preconditions of a composition and the second concerns associativity of composition.

Lemma 4.5. $\mathsf{p}^{U \circ U'}(u, u') \Leftrightarrow^* [U, u]\mathsf{p}^{U'}(u') \wedge \mathsf{p}^U(u)$.

Proof. Using PR to denote use of classical propositional reasoning:

$$\begin{array}{lll} & \mathsf{p}^{U\circ U'}(u,u') & \\ \Leftrightarrow^* & \neg[U,u]\neg\mathsf{p}^{U'}(u') & \text{CL, Def. 3.9} \\ \Leftrightarrow^* & \neg([U,u]\mathsf{p}^{U'}(u') \to [U,u]\bot) & \text{U2, PR} \\ \Leftrightarrow^* & \neg([U,u]\mathsf{p}^{U'}(u') \to (\mathsf{p}^{U}(u) \to \bot)) & \text{U1, PR} \\ \Leftrightarrow^* & [U,u]\mathsf{p}^{U'}(u') \wedge \mathsf{p}^{U}(u) & \text{PR} \end{array}$$

□

Lemma 4.6. $[(\bar{U}\circ U)\circ U',((\bar{u},u),u')]\varphi \Leftrightarrow^* [\bar{U}\circ(U\circ U'),(\bar{u},(u,u'))]\varphi$.

Proof. By induction on the construction of φ.[9] In what follows, update frames will be named using the symbol U perhaps with additional marks such as primes or bars. Worlds in update frames will be named using the symbol u with corresponding marks. This allows us to write $[U']$ to denote the update modal $[U',u']$, to write $\mathsf{p}^{\bar{U}}$ to denote the formula $\mathsf{p}^{\bar{U}}(\bar{u})$, to write $U \nvdash (t,i,\psi)$ to denote the expression $U,u \nvdash (t,i,\psi)$, to write $[U\circ U']$ to denote the update modal $[U\circ U',(u,u')]$, and so on. Proceeding with the proof, many cases require us to prove that

$$\mathsf{p}^{(\bar{U}\circ U)\circ U'} \Leftrightarrow^* \mathsf{p}^{\bar{U}\circ(U\circ U')} \quad . \tag{2}$$

Using PR to denote the use of classical propositional reasoning, we provide the text of the proof for this assertion in Figure 7 with the understanding that a copy of this text is to be construed as occurring at the beginning of the proofs of those cases where the result (2) is needed. We now provide the proofs of the each case. In what follows, we use IH to denote the use of the induction hypothesis.

- Base (or induction) case $q \in \mathscr{P}\cup\{\bot\}$ is proved using U1, (2) and PR, and U1.
- Induction case $\psi\to\chi$ with is proved using U2, IH and PR, and U2.
- Induction case $B_i\psi$ is proved in Figure 8; MR denotes the use of modal reasoning (with Lemma 4.3 used in place of Rule BN).
- For induction case $t \gg_i \psi$, it follows by Lemma 3.11 that we have the "left" statement $(\bar{U}\circ U)\circ U' \vdash (t,i,\psi)$ if and only if $\bar{U}\vdash(t,i,\psi)$ or $U\vdash(t,i,\psi)$ or $U'\vdash(t,i,\psi)$. But the latter disjunctive trio is equivalent by Lemma 3.11 to the "right" statement $\bar{U}\circ(U\circ U')\vdash(t,i,\psi)$. The result is proved using U4 (with the left statement), (2) and PR, and U4 (with the right statement).
- Induction case $[\hat{U}]q$ with $q\in\mathscr{P}\cup\{\bot\}$ is proved in Figure 9.
- Induction case $[\hat{U}](\psi\to\chi)$ is proved using U5, U2, U5 and PR, IH and PR, U5 and PR, U2, and U5.
- Induction case $[\hat{U}]B_i\psi$ is proved in Figure 10.

[9]Note that this is actually an induction on $k\in\mathbb{N}$ with a sub-induction on the construction of $\varphi\in\mathscr{F}_k$. However, since the sub-inductive arguments are essentially independent of k, we shall only present the sub-inductive arguments themselves. It will be apparent from context whether an appeal to the "induction hypothesis" concerns the "inner" sub-induction hypothesis (the statement that the result holds for certain formulas in $\mathscr{F}_k$ that have already been constructed) or the "outer" induction hypothesis (the statement that the result holds for formulas in $\bigcup_{i=0}^{k-1}\mathscr{F}_i$, all of which have already been constructed).

$$
\begin{array}{lll}
& \mathsf{p}^{(\bar{U}\circ U)\circ U'} & \\
\Leftrightarrow^* & [\bar{U}\circ U]\mathsf{p}^{U'} \wedge [\bar{U}]\mathsf{p}^{U} \wedge \mathsf{p}^{\bar{U}} & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & [\bar{U}\circ U]\mathsf{p}^{U'} \wedge [\bar{U}]\mathsf{p}^{U} \wedge \mathsf{p}^{\bar{U}} \wedge \mathsf{p}^{\bar{U}} & \text{PR} \\
\Leftrightarrow^* & \neg[\bar{U}\circ U]\neg\mathsf{p}^{U'} \wedge \mathsf{p}^{\bar{U}} & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & \neg[\bar{U}][U]\neg\mathsf{p}^{U'} \wedge \mathsf{p}^{\bar{U}} & \text{U5, PR} \\
\Leftrightarrow^* & ([\bar{U}][U]\neg\mathsf{p}^{U'} \to \bot) \wedge \mathsf{p}^{\bar{U}} & \text{PR} \\
\Leftrightarrow^* & ([\bar{U}][U]\neg\mathsf{p}^{U'} \to [\bar{U}]\bot) \wedge \mathsf{p}^{\bar{U}} & \text{U1, PR} \\
\Leftrightarrow^* & [\bar{U}]\neg[U]\neg\mathsf{p}^{U'} \wedge \mathsf{p}^{\bar{U}} & \text{U2, PR} \\
\Leftrightarrow^* & [\bar{U}]\mathsf{p}^{U\circ U'} \wedge \mathsf{p}^{\bar{U}} & \text{CL} \\
\Leftrightarrow^* & \mathsf{p}^{\bar{U}\circ(U\circ U')} & \text{Lem. 4.5}
\end{array}
$$

Figure 7: Proof that $\mathsf{p}^{(\bar{U}\circ U)\circ U'} \Leftrightarrow^* \mathsf{p}^{\bar{U}\circ(U\circ U')}$, proof of Lemma 4.6

$$
\begin{array}{lll}
& [(\bar{U}\circ U)\circ U']B_i\psi & \\
\Leftrightarrow^* & \mathsf{p}^{(\bar{U}\circ U)\circ U'} \to \bigwedge_{u_\ell R_i^{(\bar{U}\circ U)\circ U'} v_\ell} B_i[(\bar{U}\circ U)\circ U', v_\ell]\psi & \text{U3} \\
\Leftrightarrow^* & \mathsf{p}^{\bar{U}\circ(U\circ U')} \to \bigwedge_{u_\ell R_i^{(\bar{U}\circ U)\circ U'} v_\ell} B_i[(\bar{U}\circ U)\circ U', v_\ell]\psi & \text{(2), PR} \\
\Leftrightarrow^* & \mathsf{p}^{\bar{U}\circ(U\circ U')} \to \bigwedge_{u_\ell R_i^{(\bar{U}\circ U)\circ U'} v_\ell} B_i[\bar{U}\circ(U\circ U'), v_r]\psi & \text{IH, MR} \\
\Leftrightarrow^* & \mathsf{p}^{\bar{U}\circ(U\circ U')} \to \bigwedge_{u_r R_i^{\bar{U}\circ(U\circ U')} v_r} B_i[\bar{U}\circ(U\circ U'), v_r]\psi & \text{def'n of } \circ \\
\Leftrightarrow^* & [\bar{U}\circ(U\circ U')]B_i\psi & \text{U3}
\end{array}
$$

$$
\begin{array}{lllllll}
u_\ell & := & ((\bar{u},u),u') & \qquad & v_\ell & := & ((\bar{v},v),v') \\
u_r & := & (\bar{u},(u,u')) & \qquad & v_r & := & (\bar{v},(v,v'))
\end{array}
$$

Figure 8: Proof of induction case $B_i\psi$, proof of Lemma 4.6

$$
\begin{array}{lll}
& [(\bar{U}\circ U)\circ U'][\hat{U}]q & \\
\Leftrightarrow^* & [((\bar{U}\circ U)\circ U')\circ\hat{U}]q & \text{U5} \\
\Leftrightarrow^* & \mathsf{p}^{((\bar{U}\circ U)\circ U')\circ\hat{U}} \to q & \text{U1} \\
\Leftrightarrow^* & \left(\mathsf{p}^{(\bar{U}\circ U)\circ U'} \wedge [(\bar{U}\circ U)\circ U']\mathsf{p}^{\hat{U}}\right) \to q & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & \left(\mathsf{p}^{\bar{U}\circ(U\circ U')} \wedge [\bar{U}\circ(U\circ U')]\mathsf{p}^{\hat{U}}\right) \to q & \text{(2), IH, PR} \\
\Leftrightarrow^* & \mathsf{p}^{(\bar{U}\circ(U\circ U'))\circ\hat{U}} \to q & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & [(\bar{U}\circ(U\circ U'))\circ\hat{U}]q & \text{U1} \\
\Leftrightarrow^* & [\bar{U}\circ(U\circ U')][\hat{U}]q & \text{U5}
\end{array}
$$

Figure 9: Proof of induction case $[\hat{U}]q$, proof of Lemma 4.6

$$
\begin{array}{lll}
 & [(\bar{U}\circ U)\circ U'][\hat{U}]B_i\psi & \\
\Leftrightarrow^* & [((\bar{U}\circ U)\circ U')\circ \hat{U}]B_i\psi & \text{U5} \\
\Leftrightarrow^* & \mathsf{p}^{((\bar{U}\circ U)\circ U')\circ\hat{U}} \to & \\
 & \quad \bigwedge_{\hat{u}_\ell R^{((\bar{U}\circ U)\circ U')\circ\hat{U}}\hat{v}_\ell} B_i[((\bar{U}\circ U)\circ U')\circ\hat{U},\hat{v}_\ell]\psi & \text{U3} \\
\Leftrightarrow^* & [(\bar{U}\circ U)\circ U']\mathsf{p}^{\hat{U}} \wedge \mathsf{p}^{(\bar{U}\circ U)\circ U'} \to & \\
 & \quad \bigwedge_{\hat{u}_\ell R^{((\bar{U}\circ U)\circ U')\circ\hat{U}}\hat{v}_\ell} B_i[((\bar{U}\circ U)\circ U')\circ\hat{U},\hat{v}_\ell]\psi & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & [(\bar{U}\circ U)\circ U']\mathsf{p}^{\hat{U}} \wedge \mathsf{p}^{(\bar{U}\circ U)\circ U'} \to & \\
 & \quad \bigwedge_{\hat{u}_\ell R^{((\bar{U}\circ U)\circ U')\circ\hat{U}}\hat{v}_\ell} B_i[(\bar{U}\circ U)\circ U',v_\ell][\hat{U}]\psi & \text{U5, MR} \\
\Leftrightarrow^* & [\bar{U}\circ(U\circ U')]\mathsf{p}^{\hat{U}} \wedge \mathsf{p}^{\bar{U}(\circ U\circ U')} \to & \\
 & \quad \bigwedge_{\hat{u}_\ell R^{((\bar{U}\circ U)\circ U')\circ\hat{U}}\hat{v}_\ell} B_i[(\bar{U}\circ U)\circ U',v_\ell][\hat{U}]\psi & \text{IH, (2), PR} \\
\Leftrightarrow^* & [\bar{U}\circ(U\circ U')]\mathsf{p}^{\hat{U}} \wedge \mathsf{p}^{\bar{U}(\circ U\circ U')} \to & \\
 & \quad \bigwedge_{\hat{u}_\ell R^{((\bar{U}\circ U)\circ U')\circ\hat{U}}\hat{v}_\ell} B_i[\bar{U}\circ(U\circ U'),v_r][\hat{U}]\psi & \text{IH, MR} \\
\Leftrightarrow^* & [\bar{U}\circ(U\circ U')]\mathsf{p}^{\hat{U}} \wedge \mathsf{p}^{\bar{U}(\circ U\circ U')} \to & \\
 & \quad \bigwedge_{\hat{u}_\ell R^{((\bar{U}\circ U)\circ U')\circ\hat{U}}\hat{v}_\ell} B_i[(\bar{U}\circ(U\circ U'))\circ\hat{U},\hat{v}_r]\psi & \text{U5, MR} \\
\Leftrightarrow^* & \mathsf{p}^{(\bar{U}\circ(U\circ U'))\circ\hat{U}} \to & \\
 & \quad \bigwedge_{\hat{u}_\ell R^{((\bar{U}\circ U)\circ U')\circ\hat{U}}\hat{v}_\ell} B_i[(\bar{U}\circ(U\circ U'))\circ\hat{U},\hat{v}_r]\psi & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & \mathsf{p}^{(\bar{U}\circ(U\circ U'))\circ\hat{U}} \to & \\
 & \quad \bigwedge_{\hat{u}_r R^{(\bar{U}\circ(U\circ U'))\circ\hat{U}}\hat{v}_r} B_i[(\bar{U}\circ(U\circ U'))\circ\hat{U},\hat{v}_r]\psi & \text{def'n of } \circ \\
\Leftrightarrow^* & [(\bar{U}\circ(U\circ U'))\circ\hat{U}]B_i\psi & \text{U3} \\
\Leftrightarrow^* & [\bar{U}\circ(U\circ U')][\hat{U}]B_i\psi & \text{U5}
\end{array}
$$

$$
\begin{array}{lllllll}
\hat{u}_\ell & := & (((\bar{u},u),u'),\hat{u}) & \qquad & \hat{v}_\ell & := & (((\bar{v},v),v'),\hat{v}) \\
\hat{u}_r & := & ((\bar{u},(u,u')),\hat{u}) & & v_\ell & := & ((\bar{v},v),v') \\
 & & & & v_r & := & (\bar{v},(v,v')) \\
 & & & & \hat{v}_r & := & ((\bar{v},(v,v')),\hat{v})
\end{array}
$$

Figure 10: Proof of induction case $[\hat{U}]B_i\psi$, proof of Lemma 4.6

$$
\begin{array}{lll}
& [(\bar{U} \circ U) \circ U'][\hat{U}]t \gg_i \psi & \\
\Leftrightarrow^* & [((\bar{U} \circ U) \circ U') \circ \hat{U}]t \gg_i \psi & \text{U5} \\
\Leftrightarrow^* & \mathsf{p}^{((\bar{U} \circ U) \circ U') \circ \hat{U}} \to X & \text{U4} \\
\Leftrightarrow^* & [(\bar{U} \circ U) \circ U']\mathsf{p}^{\hat{U}} \wedge \mathsf{p}^{(\bar{U} \circ U) \circ U'} \to X & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & [\bar{U} \circ (U \circ U')]\mathsf{p}^{\hat{U}} \wedge \mathsf{p}^{\bar{U} \circ (U \circ U')} \to X & \text{IH, (2), PR} \\
\Leftrightarrow^* & \mathsf{p}^{(\bar{U} \circ (U \circ U')) \circ \hat{U}} \to X & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & [(\bar{U} \circ (U \circ U')) \circ \hat{U}]t \gg_i \psi & \text{U4} \\
\Leftrightarrow^* & [\bar{U} \circ (U \circ U')][\hat{U}]t \gg_i \psi & \text{U5} \\
& X = \bot \quad \text{or} \quad X = t \gg_i \psi &
\end{array}
$$

Figure 11: Proof of induction case $[\hat{U}]t \gg_i \psi$, proof of Lemma 4.6

$$
\begin{array}{lll}
& [(\bar{U} \circ U) \circ U'][U_1][U_2]\psi & \\
\Leftrightarrow^* & [(((\bar{U} \circ U) \circ U') \circ U_1) \circ U_2]\psi & \text{U5} \\
\Leftrightarrow^* & [((\bar{U} \circ U) \circ U') \circ (U_1 \circ U_2)]\psi & \text{IH} \\
\Leftrightarrow^* & [(\bar{U} \circ U) \circ U'][U_1 \circ U_2]\psi & \text{U5} \\
\Leftrightarrow^* & [\bar{U} \circ (U \circ U')][U_1 \circ U_2]\psi & \text{IH} \\
\Leftrightarrow^* & [(\bar{U} \circ (U \circ U')) \circ (U_1 \circ U_2)]\psi & \text{U5} \\
\Leftrightarrow^* & [((\bar{U} \circ (U \circ U')) \circ U_1) \circ U_2]\psi & \text{IH} \\
\Leftrightarrow^* & [\bar{U} \circ (U \circ U')][U_1][U_2]\psi & \text{U5}
\end{array}
$$

Figure 12: Proof of induction case $[U_1][U_2]\psi$, proof of Lemma 4.6

- For induction case $[\hat{U}](t \gg_i \psi)$, we observe that, similar to induction case $t \gg_i \psi$ above, we have $((\bar{U} \circ U) \circ U') \circ \hat{U} \vdash (t, i, \psi)$ if and only if $(\bar{U} \circ (U \circ U')) \circ \hat{U} \vdash (t, i, \psi)$ by Lemma 3.11. We make use of this biconditional in our proof in Figure 11 (by assuming that U4 can be used in the second provable equivalence if and only if U4 can be used in the second-to-last provable equivalence).
- Induction case $[U_1][U_2]\psi$ is proved in Figure 12. □

In one part of our proof that Rule UN is JLCE^*-admissible, we will show that Rule UN is AX-admissible. A key case of this proof requires an important lemma about "preconditions" (the forthcoming Lemma 4.7) along with an interesting lemma about substitution (the forthcoming Lemma 4.10).

Lemma 4.7 (Precondition)**.** For each $\varphi \in \mathscr{F}$, we have $[U, u]\varphi \Leftrightarrow^* \mathsf{p}^U(u) \to [U, u]\varphi$.

Proof. By induction on the construction of φ, with PR denoting use of classical propositional reasoning and IH denoting the use of the induction hypothesis.

- Base case $q \in \mathscr{P} \cup \{\bot\}$. See Figure 13.

$$
\begin{array}{lll}
[U,u]q & \Leftrightarrow^* \quad \mathsf{p}^U(u) \to q & \text{U1} \\
& \Leftrightarrow^* \quad \mathsf{p}^U(u) \to (\mathsf{p}^U(u) \to q) & \text{PR} \\
& \Leftrightarrow^* \quad \mathsf{p}^U(u) \to [U,u]q & \text{U1}
\end{array}
$$

Figure 13: Base case $q \in \mathscr{P} \cup \{\bot\}$, Lemma 4.7

$$
\begin{array}{lll}
& [U,u](\psi \to \chi) & \\
\Leftrightarrow^* & [U,u]\psi \to [U,u]\chi & \text{U2} \\
\Leftrightarrow^* & (\mathsf{p}^U(u) \to [U,u]\psi) \to (\mathsf{p}^U(u) \to [U,u]\chi) & \text{IH, PR} \\
\Leftrightarrow^* & \mathsf{p}^U(u) \to ([U,u]\psi \to [U,u]\chi) & \text{PR} \\
\Leftrightarrow^* & \mathsf{p}^U(u) \to [U,u](\psi \to \chi) & \text{U2, PR}
\end{array}
$$

Figure 14: Induction case $\psi \to \chi$, Lemma 4.7

- Induction case $\psi \to \chi$. See Figure 14.
- Induction case $B_i\psi$. See Figure 15.
- Induction case $t \gg_i \psi$ is similar to case $B_i\psi$, though U4 is used in place of U3. Note that each time U4 is used, the particular form that U4 takes depends on the truth of the single assertion $U, u \vdash (t, i, \psi)$.
- Induction case $[U', u']\psi$. See Figure 16. □

Definition 4.8 ($L(\varphi)$, $S(\varphi)$, Substitutions $\psi\sigma$ and $\psi\sigma^{U,u}$)**.** Let $\varphi \in \mathscr{F}$. $L(\varphi)$ is the set of propositional letters occurring in φ (note that a propositional letter p_k occurring in φ may occur within an update modal $[U,u]$ that itself occurs in φ). $S(\varphi)$ is the set of functions $\sigma : L(\varphi) \to \mathscr{F}$ that map the natural-number subscript k of propositional letter p_k occurring in φ to formula $\sigma(k) \in \mathscr{F}$. To define substitution of formulas for propositional letters: given $\sigma \in S(\varphi)$ and $\psi \in \mathscr{F}$, we let $\psi\sigma$ denote the formula obtained from ψ by simultaneously replacing for each $k \in L(\varphi)$ all occurrences of the propositional letter p_k in ψ by the formula $\sigma(k)$. Note that for p_k to be replaced in ψ by a substitution $\sigma \in S(\varphi)$, the propositional letter p_k must occur both in ψ and in φ. Given $(U,u) \in \mathscr{U}$, $\varphi \in \mathscr{F}$, and $\sigma \in S(\varphi)$, we define the substitution $\sigma^{U,u} : L(\varphi) \to \mathscr{F}$ by setting $\sigma^{U,u}(k) := [U,u]\sigma(k)$ for each $k \in L(\varphi)$.

Definition 4.9 (Propositional Logic PL)**.** PL is the set of formulas in the language of proposi-

$$
\begin{array}{lll}
& [U,u]B_i\psi & \\
\Leftrightarrow^* & \mathsf{p}^U(u) \to \bigwedge_{uR_i^U v} B_i[U,v]\psi & \text{U3} \\
\Leftrightarrow^* & \mathsf{p}^U(u) \to (\mathsf{p}^U(u) \to \bigwedge_{uR_i^U v} B_i[U,v]\psi) & \text{PR} \\
\Leftrightarrow^* & \mathsf{p}^U(u) \to [U,u]B_i\psi & \text{U3, PR}
\end{array}
$$

Figure 15: Induction case $B_i\psi$, Lemma 4.7

$$\begin{array}{lll}
& [U,u][U',u']\psi & \\
\Leftrightarrow^* & [U \circ U', (u,u')]\psi & \text{U5} \\
\Leftrightarrow^* & \mathsf{p}^{U\circ U'}(u,u') \to [U \circ U', (u,u')]\psi & \text{IH} \\
\Leftrightarrow^* & \mathsf{p}^{U}(u) \wedge [U,u]\mathsf{p}^{U'}(u') \to [U \circ U', (u,u')]\psi & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & \mathsf{p}^{U}(u) \to \big(\mathsf{p}^{U}(u) \wedge [U,u]\mathsf{p}^{U'}(u') \to [U \circ U', (u,u')]\psi\big) & \text{PR} \\
\Leftrightarrow^* & \mathsf{p}^{U}(u) \to \big(\mathsf{p}^{U\circ U'}(u,u') \to [U \circ U', (u,u')]\psi\big) & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & \mathsf{p}^{U}(u) \to [U \circ U', (u,u')]\psi & \text{IH, PR} \\
\Leftrightarrow^* & \mathsf{p}^{U}(u) \to [U,u][U',u']\psi & \text{U5, PR}
\end{array}$$

Figure 16: Induction case $[U',u']\psi$, Lemma 4.7

$$\begin{array}{lll}
& [U,u]\big((\psi_1 \to \psi_2)\sigma\big) & \\
\Leftrightarrow^* & [U,u](\psi_1\sigma \to \psi_2\sigma) & \text{def'n of substitution} \\
\Leftrightarrow^* & [U,u](\psi_1\sigma) \to [U,u](\psi_2\sigma) & \text{U2} \\
\Leftrightarrow^* & \big(\mathsf{p}^{U}(u) \to (\psi_1\sigma^{U,u})\big) \to \big(\mathsf{p}^{U}(u) \to (\psi_2\sigma^{U,u})\big) & \text{IH, PR} \\
\Leftrightarrow^* & \mathsf{p}^{U}(u) \to (\psi_1\sigma^{U,u} \to \psi_2\sigma^{U,u}) & \text{PR} \\
\Leftrightarrow^* & \mathsf{p}^{U}(u) \to \big((\psi_1 \to \psi_2)\sigma^{U,u}\big) & \text{def'n of substitution}
\end{array}$$

Figure 17: Induction case $\psi_1 \to \psi_2$, Lemma 4.10

tional logic having atoms $q \in \mathscr{P} \cup \{\bot\}$ and the binary Boolean connective $\to$ for implication. Use of logical connectives not in this language is to be understood as an abbreviation of the expression that uses connectives that do appear in this language. We observe that PL is a sublanguage of UL.

Lemma 4.10 (Substitution)**.** For each $\psi \in \mathsf{PL}$, each $\sigma \in S(\psi)$, and each $(U,u) \in \mathscr{U}$, we have $[U,u](\psi\sigma) \Leftrightarrow^* \mathsf{p}^U(u) \to (\psi\sigma^{U,u})$.

Proof. By induction on the construction of $\psi \in \mathsf{PL}$, with IH denoting use of the induction hypothesis and PR denoting use of classical propositional reasoning.

- Base case: $\psi = p \in \mathscr{P}$. We have $[U,u](p\sigma) \Leftrightarrow^* \mathsf{p}^U(u) \to [U,u](p\sigma)$ by Lemma 4.7, and we have $[U,u](p\sigma) = p\sigma^{U,u}$ by the definition of $\sigma^{U,u}$.
- Base case: $\psi = \bot$. We have $\bot\sigma = \bot$ by the definition of substitution, and we have $[U,u]\bot \Leftrightarrow^* \mathsf{p}^U(u) \to \bot$ by Axiom U1.
- Induction case: $\psi = \psi_1 \to \psi_2$. See Figure 17. □

We are now ready to prove the JLCE^*-admissibility of Rule UN. We begin by showing that this rule is AX-admissible.

Lemma 4.11 (AX Update Admissibility)**.** $\mathsf{AX} \vdash \varphi$ implies $\vdash^* [\bar{U},\bar{u}]\varphi$.

1. $\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}([\bar{U},\bar{v}]\varphi \to [\bar{U},\bar{v}]\psi) \;\to$
   $(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}[\bar{U},\bar{v}]\varphi \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}[\bar{U},\bar{v}]\psi)$ PR
2. $\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}[\bar{U},\bar{v}](\varphi \to \psi) \;\to$
   $(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}[\bar{U},\bar{v}]\varphi \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}[\bar{U},\bar{v}]\psi)$ U2, PR 1
3. $\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}](\varphi \to \psi) \;\to$
   $(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]\varphi \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]\psi)$ MR 2
4. $(\mathsf{p}^{\bar{U}}(\bar{u}) \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}](\varphi \to \psi)) \;\to$
   $((\mathsf{p}^{\bar{U}}(\bar{u}) \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]\varphi) \;\to$
   $(\mathsf{p}^{\bar{U}}(\bar{u}) \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]\psi))$ PR 3
5. $[\bar{U},\bar{u}]B_i(\varphi \to \psi) \to ([\bar{U},\bar{u}]B_i\varphi \to [\bar{U},\bar{u}]B_i\psi)$ U3, MR 4
6. $[\bar{U},\bar{u}](B_i(\varphi \to \psi) \to (B_i\varphi \to B_i\psi))$ U2, PR 5

Figure 18: Base case B1, proof of Lemma 4.11

Proof. By induction on the length of AX derivations, using PR to denote the use of classical propositional reasoning and MR to denote the use of modal reasoning (using Lemma 4.3 in place of rule BN).

- Base case: Axiom CL. Given an instance $\varphi \in \mathscr{F}$ of CL, there exists a classical propositional tautology $\psi \in \mathsf{PL}$ and a substitution $\sigma \in S(\psi)$ such that $\varphi = \psi\sigma$. Choose an arbitrary $(\bar{U},\bar{u}) \in \mathscr{U}$. Since ψ is a tautology, it follows that $\vdash^* \psi\sigma^{\bar{U},\bar{u}}$ by Axiom CL and hence that $\vdash^* \mathsf{p}^{\bar{U}}(\bar{u}) \to (\psi\sigma^{\bar{U},\bar{u}})$ by PR. Applying Lemma 4.10, it follows that $\vdash^* [\bar{U},\bar{u}](\psi\sigma)$. Since $\varphi = \psi\sigma$, we have shown that $\vdash^* [\bar{U},\bar{u}]\varphi$, as desired.
- Base case: Axiom B1. See Figure 18.
- Base case: Axiom B2. See Figure 19.
- Base case: Axiom E1. For each $t \in \mathscr{T}$, $i \in \mathscr{A}$, $\psi \in \mathscr{F}$, and $(U,u) \in \mathscr{U}$, define the formula $(t,i,\psi)^{U,u}$ by setting

$$(t,i,\psi)^{U,u} := \begin{cases} \neg\mathsf{p}^U(u) & \text{if } U,u \vdash (t,i,\psi), \\ \mathsf{p}^U(u) \to t \gg_i \psi & \text{if } U,u \nvdash (t,i,\psi). \end{cases}$$

  We let $(t,i,\psi)_1^{U,u}$ mean that $(t,i,\psi)^{U,u} = \neg\mathsf{p}^U(u)$, and $(t,i,\psi)_0^{U,u}$ denote the negation of $(t,i,\psi)_1^{U,u}$. By inspection of the $\bar{U}$-calculus (Figure 1), it is clear that we have

$$\big((t,i,\varphi \to \psi)_1^{\bar{U},\bar{u}} \text{ or } (s,i,\varphi)_1^{\bar{U},\bar{u}}\big) \text{ iff } (t \cdot_\varphi s,i,\psi)_1^{\bar{U},\bar{u}} \;. \tag{3}$$

  Further, we have the following chain of provable equivalences.

$$[\bar{U},\bar{u}](t \gg_i (\varphi \to \psi) \to (s \gg_i \varphi \to t \cdot_\varphi s \gg_i \psi))$$
$$\Leftrightarrow^* \; [\bar{U},\bar{u}](t \gg_i (\varphi \to \psi)) \to ([\bar{U},\bar{u}](s \gg_i \varphi) \to [\bar{U},\bar{u}](t \cdot_\varphi s \gg_i \psi)) \quad \text{U2, PR}$$
$$\Leftrightarrow^* \; (t,i,\varphi \to \psi)^{\bar{U},\bar{u}} \to ((s,i,\varphi)^{\bar{U},\bar{u}} \to (t \cdot_\varphi s,i,\psi)^{\bar{U},\bar{u}}) \quad \text{U4, PR}$$

1.	$B_i(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}[\bar{U},\bar{v}]\varphi) \to B_iB_i(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}[\bar{U},\bar{v}]\varphi)$	B2
2.	$B_i(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}[\bar{U},\bar{v}]\varphi) \to B_i(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]\varphi)$	MR 1
3.	$(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}[\bar{U},\bar{v}]\varphi) \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}(\bigwedge_{\bar{v}R_i^{\bar{U}}\bar{w}}[\bar{U},\bar{w}]\varphi)$	$R_i^{\bar{U}}$ trans.
4.	$(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]\varphi) \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}(\bigwedge_{\bar{v}R_i^{\bar{U}}\bar{w}} B_i[\bar{U},\bar{w}]\varphi)$	MR 3
5.	$(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]\varphi) \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}(\mathsf{p}^{\bar{U}}(\bar{v}) \to \bigwedge_{\bar{v}R_i^{\bar{U}}\bar{w}} B_i[\bar{U},\bar{w}]\varphi)$	PR 4
6.	$B_i(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]\varphi) \to$	
	$\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i(\mathsf{p}^{\bar{U}}(\bar{v}) \to \bigwedge_{\bar{v}R_i^{\bar{U}}\bar{w}} B_i[\bar{U},\bar{w}]\varphi)$	MR 5
7.	$B_i(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}[\bar{U},\bar{v}]\varphi) \to$	
	$\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i(\mathsf{p}^{\bar{U}}(\bar{v}) \to \bigwedge_{\bar{v}R_i^{\bar{U}}\bar{w}} B_i[\bar{U},\bar{w}]\varphi)$	PR 2, 6
8.	$B_i(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}[\bar{U},\bar{v}]\varphi) \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]B_i\varphi$	U3, MR 7
9.	$(\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]\varphi) \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]B_i\varphi$	MR 8
10.	$(\mathsf{p}^{\bar{U}}(\bar{u}) \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]\varphi) \to$	
	$(\mathsf{p}^{\bar{U}}(\bar{u}) \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]B_i\varphi)$	PR 9
11.	$[\bar{U},\bar{u}]B_i\varphi \to [\bar{U},\bar{u}]B_iB_i\varphi$	U3, PR 10
12.	$[\bar{U},\bar{u}](B_i\varphi \to B_iB_i\varphi)$	U2, PR 11

Figure 19: Base case B2, proof of Lemma 4.11

If $(t\cdot_\varphi s, i, \psi)_1^{\bar U,\bar u}$, then it follows by (3) that $(t, i, \varphi\to\psi)_1^{\bar U,\bar u}$ or $(s, i, \varphi)_1^{\bar U,\bar u}$ and hence that the last formula in the above chain of provable equivalences is an instance of Axiom CL. If $(t\cdot_\varphi s, i, \psi)_0^{\bar U,\bar u}$, then it follows by (3) that $(t, i, \varphi\to\psi)_0^{\bar U,\bar u}$ and $(s, i, \varphi)_0^{\bar U,\bar u}$; the last formula in the above chain is therefore JLCE^*-provably equivalent (by PR) to

$$\mathsf{p}^{\bar U}(\bar u) \to (t \gg_i (\varphi\to\psi) \to ((s\gg_i \varphi) \to (t\cdot_\varphi s \gg_i \psi))) \ ,$$

which is itself JLCE^*-provable by PR from Axiom E1.

- Base case: Axiom E2. By inspection of the U-calculus (Figure 1), it is clear that we have

$$\big((t, i, \varphi)_1^{\bar U,\bar u} \text{ and } (s, i, \varphi)_1^{\bar U,\bar u}\big) \text{ iff } (t+s, i, \varphi)_1^{\bar U,\bar u} \ . \tag{4}$$

Further, we have the following chain of provable equivalences.

$$\begin{array}{lll} & [\bar U,\bar u](t\gg_i \varphi \to t+s\gg_i\varphi) & \\ \Leftrightarrow^* & [\bar U,\bar u](t\gg_i\varphi) \to [\bar U,\bar u](t+s\gg_i\varphi) & \text{U2} \\ \Leftrightarrow^* & (t,i,\varphi)^{\bar U,\bar u} \to (t+s,i,\varphi)^{\bar U,\bar u} & \text{U4, PR} \end{array}$$

If $(t+s, i, \varphi)_1^{\bar U,\bar u}$, then it follows by (4) that $(t, i, \varphi)_1^{\bar U,\bar u}$ and hence that the last formula in the above chain of provable equivalences is an instance of Axiom CL. If $(t+s, i, \varphi)_0^{\bar U,\bar u}$, then it follows by (4) that $(t, i, \varphi)_0^{\bar U,\bar u}$ and hence that the last formula in the above chain is JLCE^*-provably equivalent to

$$\mathsf{p}^{\bar U}(\bar u) \to ((t\gg_i\varphi) \to (t+s\gg_i\varphi)) \ ,$$

which is itself JLCE^*-provable by PR from Axiom E2. The argument for the other form of Axiom E2, $(s\gg_i\varphi)\to(t+s\gg_i\varphi)$, is similar.

- Base case: Axiom E3. The argument is similar to the argument for Axiom E2, though Axiom E3 is used at the end in place of Axiom E2.
- Base case: Axiom E4. Consider the following chain of provable equivalences.

$$\begin{array}{lll} & [\bar U,\bar u](t\gg_i\varphi \to B_i(t\gg_i\varphi)) & \\ \Leftrightarrow^* & [\bar U,\bar u](t\gg_i\varphi) \to [\bar U,\bar u]B_i(t\gg_i\varphi) & \text{U2} \\ \Leftrightarrow^* & (t,i,\varphi)^{\bar U,\bar u} \to [\bar U,\bar u]B_i(t\gg_i\varphi) & \text{U4, PR} \\ \Leftrightarrow^* & (t,i,\varphi)^{\bar U,\bar u} \to (\mathsf{p}^{\bar U}(\bar u) \to \bigwedge_{\bar u R_i^{\bar U}\bar v} B_i[\bar U,\bar v](t\gg_i\varphi)) & \text{U3, PR} \\ \Leftrightarrow^* & (t,i,\varphi)^{\bar U,\bar u} \to (\mathsf{p}^{\bar U}(\bar u) \to \bigwedge_{\bar u R_i^{\bar U}\bar v} B_i(t,i,\varphi)^{\bar U,\bar v}) & \text{U4, MR} \end{array}$$

If $(t, i, \varphi)_1^{\bar U,\bar u}$, then the last line of the above chain of provable equivalences is an instance of Axiom CL. So let us assume that $(t, i, \varphi)_0^{\bar U,\bar u}$ and hence that $(t, i, \varphi)_0^{\bar U,\bar u}$ by Lemma 3.7. The following is then a JLCE^*-derivation of the last line of the above chain.

$$\begin{array}{lll} 1. & (t\gg_i\varphi) \to B_i(t\gg_i\varphi) & \text{E4} \\ 2. & (t\gg_i\varphi) \to \bigwedge_{\bar u R_i^{\bar U}\bar v} B_i(\mathsf{p}^{\bar U}(\bar v) \to t\gg_i\varphi) & \text{MR 1} \\ 3. & (\mathsf{p}^{\bar U}(\bar u) \to t\gg_i\varphi) \ \to & \\ & \quad (\mathsf{p}^{\bar U}(\bar u) \to \bigwedge_{\bar u R_i^{\bar U}\bar v} B_i(\mathsf{p}^{\bar U}(\bar v) \to t\gg_i\varphi)) & \text{PR 2} \end{array}$$

$$\begin{array}{lll} & [\bar{U},\bar{u}][U,u]q & \\ \Leftrightarrow^* & [\bar{U}\circ U,(\bar{u},u)]q & \text{U5} \\ \Leftrightarrow^* & \mathsf{p}^{\bar{U}\circ U}(\bar{u},u)\to q & \text{U1} \\ \Leftrightarrow^* & (\mathsf{p}^{\bar{U}}(\bar{u})\wedge[\bar{U},\bar{u}]\mathsf{p}^{U}(u))\to q & \text{Lem. 4.5, PR} \\ \Leftrightarrow^* & [\bar{U},\bar{u}]\mathsf{p}^{U}(u)\to(\mathsf{p}^{\bar{U}}(\bar{u})\to q) & \text{PR} \\ \Leftrightarrow^* & [\bar{U},\bar{u}]\mathsf{p}^{U}(u)\to[\bar{U},\bar{u}]q & \text{U1, PR} \\ \Leftrightarrow^* & [\bar{U},\bar{u}](\mathsf{p}^{U}(u)\to q) & \text{U2} \end{array}$$

Figure 20: Base case U1, proof of Lemma 4.11

$$\begin{array}{lll} & [\bar{U},\bar{u}][U,u](\varphi\to\psi) & \\ \Leftrightarrow^* & [\bar{U}\circ U,(\bar{u},u)](\varphi\to\psi) & \text{U5} \\ \Leftrightarrow^* & [\bar{U}\circ U,(\bar{u},u)]\varphi\to[\bar{U}\circ U,(\bar{u},u)]\psi & \text{U2} \\ \Leftrightarrow^* & [\bar{U},\bar{u}][U,u]\varphi\to[\bar{U},\bar{u}][U,u]\psi & \text{U5, PR} \\ \Leftrightarrow^* & [\bar{U},\bar{u}]([U,u]\varphi\to[U,u]\psi) & \text{U2} \end{array}$$

Figure 21: Base case U2, proof of Lemma 4.11

- Base case: Axiom U1. By Figure 20, Axiom U2, and PR.
- Base case: Axiom U2. By Figure 21, Axiom U2, and PR.
- Base case: Axiom U3. By Figure 22, Axiom U2, and PR.
- Base case: Axiom U4. We consider three sub-cases.
  1. First sub-case: $U,u\vdash(t,i,\varphi)$. Axiom U4 then has the form $[U,u](t\gg_i\varphi)\leftrightarrow\neg\mathsf{p}^U(u)$. The result follows by Figure 23, Axiom U2, and PR.
  2. Second sub-case: $U,u\nvdash(t,i,\varphi)$ and $\bar{U},\bar{u}\nvdash(t,i,\varphi)$. Axiom U4 then has the form $[U,u](t\gg_i\varphi)\leftrightarrow(\mathsf{p}^U(u)\to t\gg_i\varphi)$. The result follows by Figure 24, Axiom U2, and PR.
  3. Third sub-case: $U,u\nvdash(t,i,\varphi)$ and $\bar{U},\bar{u}\vdash(t,i,\varphi)$. Axiom U4 then has the form $[U,u](t\gg_i\varphi)\leftrightarrow(\mathsf{p}^U(u)\to t\gg_i\varphi)$. The result follows by Figure 25, Axiom U2, and PR.
- Base case: Axiom U5. By Figure 26, Axiom U2, and PR.
- Induction case: Rule CN. Suppose $c_k:_i\varphi$ was derived by Rule CN from AX-theorem φ. Applying the definition of $c_k:_i\varphi$, it follows that $\vdash^*\varphi$, $\vdash^* c_k\gg_i\varphi$, and $\vdash^* B_i\varphi$. By the induction hypothesis, we have $\vdash^*[\bar{U},\bar{v}]\varphi$ for arbitrary $\bar{v}\in W^{\bar{U}}$ and hence that $\vdash^*\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}}B_i[\bar{U},\bar{v}]\varphi$ by MR. But then it follows by PR that the first line of the chain of equivalences in Figure 27 is JLCE^*-derivable. The result follows by PR. □

Lemma 4.12 (Update Admissibility)**.** $\vdash^*\varphi$ implies $\vdash^*[\bar{U},\bar{u}]\varphi$.

$$
\begin{array}{lll}
 & [\bar{U},\bar{u}][U,u]B_i\varphi & \\
\Leftrightarrow^* & [\bar{U}\circ U,(\bar{u},u)]B_i\varphi & \text{U5} \\
\Leftrightarrow^* & \mathsf{p}^{\bar{U}\circ U}(\bar{u},u) \to \bigwedge_{(\bar{u},u)R_i^{\bar{U}\circ U}(\bar{v},v)} B_i[\bar{U}\circ U,(\bar{v},v)]\varphi & \text{U3} \\
\Leftrightarrow^* & \mathsf{p}^{\bar{U}\circ U}(\bar{u},u) \to \bigwedge_{(\bar{u},u)R_i^{\bar{U}\circ U}(\bar{v},v)} B_i[\bar{U},\bar{v}][U,v]\varphi & \text{U5, MR} \\
\Leftrightarrow^* & \mathsf{p}^{\bar{U}\circ U}(\bar{u},u) \to \bigwedge_{uR_i^U v}\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}][U,v]\varphi & \text{def'n of }\circ\text{, PR} \\
\Leftrightarrow^* & \mathsf{p}^{\bar{U}}(\bar{u}) \wedge [\bar{U},\bar{u}]\mathsf{p}^U(u) \ \to & \\
 & \qquad \bigwedge_{uR_i^U v}\bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}][U,v]\varphi & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}]\mathsf{p}^U(u) \to & \\
 & \qquad \bigwedge_{uR_i^U v}(\mathsf{p}^{\bar{U}}(\bar{u}) \ \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}][U,v]\varphi) & \text{PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}]\mathsf{p}^U(u) \to \bigwedge_{uR_i^U v}[\bar{U},\bar{u}]B_i[U,v]\varphi & \text{U3, PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}]\mathsf{p}^U(u) \to [\bar{U},\bar{u}]\bigwedge_{uR_i^U v} B_i[U,v]\varphi & \text{U5, PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}](\mathsf{p}^U(u) \to \bigwedge_{uR_i^U v} B_i[U,v]\varphi) & \text{U5, PR}
\end{array}
$$

Figure 22: Base case U3, proof of Lemma 4.11

$$
\begin{array}{lll}
 & [\bar{U},\bar{u}][U,u]t \gg_i \varphi & \\
\Leftrightarrow^* & [\bar{U}\circ U,(\bar{u},u)]t \gg_i \varphi & \text{U5} \\
\Leftrightarrow^* & \neg\mathsf{p}^{\bar{U}\circ U}(\bar{u},u) & \text{U4, +} \\
\Leftrightarrow^* & \neg(\mathsf{p}^{\bar{U}}(\bar{u}) \wedge [\bar{U},\bar{u}]\mathsf{p}^U(u)) & \text{Lem 4.5, PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}]\mathsf{p}^U(u) \to \neg\mathsf{p}^{\bar{U}}(\bar{u}) & \text{PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}]\mathsf{p}^U(u) \to [\bar{U},\bar{u}]\bot & \text{U1, PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}]\neg\mathsf{p}^U(u) & \text{U2}
\end{array}
$$

$$
\begin{array}{rll}
\text{Assumption of sub-case:} & U,u \vdash (t,i,\varphi) & (*) \\
\text{By (*) and Lem. 3.11:} & \bar{U}\circ U,(\bar{u},u) \vdash (t,i,\varphi) & (+)
\end{array}
$$

Figure 23: Base case U4 (first sub-case), proof of Lemma 4.11

$$
\begin{array}{lll}
& [\bar{U},\bar{u}][U,u]t \gg_i \varphi & \\
\Leftrightarrow^* & [\bar{U}\circ U,(\bar{u},u)]t \gg_i \varphi & \text{U5} \\
\Leftrightarrow^* & \mathsf{p}^{\bar{U}\circ U}(\bar{u},u) \to t \gg_i \varphi & \text{U4, ++} \\
\Leftrightarrow^* & (\mathsf{p}^{\bar{U}}(\bar{u}) \wedge [\bar{U},\bar{u}]\mathsf{p}^{U}(u)) \to t \gg_i \varphi & \text{Lem. 4.5, PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}]\mathsf{p}^{U}(u) \to (\mathsf{p}^{\bar{U}}(\bar{u}) \to t \gg_i \varphi) & \text{PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}]\mathsf{p}^{U}(u) \to [\bar{U},\bar{u}]t \gg_i \varphi & \text{U4, **, PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}](\mathsf{p}^{U}(u) \to t \gg_i \varphi) & \text{U2}
\end{array}
$$

Assumption of sub-case: $U,u \nvdash (t,i,\varphi)$ and $\bar{U},\bar{u} \nvdash (t,i,\varphi)$ (**)
By (**) and Lem. 3.11: $\bar{U}\circ U,(\bar{u},u) \nvdash (t,i,\varphi)$ (++)

Figure 24: Base case U4 (second sub-case), proof of Lemma 4.11

$$
\begin{array}{lll}
& [\bar{U},\bar{u}][U,u]t \gg_i \varphi & \\
\Leftrightarrow^* & [\bar{U}\circ U,(\bar{u},u)]t \gg_i \varphi & \text{U5} \\
\Leftrightarrow^* & \neg\mathsf{p}^{\bar{U}\circ U}(\bar{u},u) & \text{U4, +++} \\
\Leftrightarrow^* & \neg(\mathsf{p}^{\bar{U}}(\bar{u}) \wedge [\bar{U},\bar{u}]\mathsf{p}^{U}(u)) & \text{Lem 4.5, PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}]\mathsf{p}^{U}(u) \to \neg\mathsf{p}^{\bar{U}}(\bar{u}) & \text{PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}]\mathsf{p}^{U}(u) \to [\bar{U},\bar{u}]t \gg_i \varphi & \text{U4, ***, PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}](\mathsf{p}^{U}(u) \to t \gg_i \varphi) & \text{U2}
\end{array}
$$

Assumption of sub-case: $U,u \nvdash (t,i,\varphi)$ and $\bar{U},\bar{u} \vdash (t,i,\varphi)$ (***)
By (***) and Lem. 3.11: $\bar{U}\circ U,(\bar{u},u) \vdash (t,i,\varphi)$ (+++)

Figure 25: Base case U4 (third sub-case), proof of Lemma 4.11

$$
\begin{array}{lll}
& [\bar{U},\bar{u}][U,u][U',u']\varphi & \\
\Leftrightarrow^* & [\bar{U}\circ U,(\bar{u},u)][U',u']\varphi & \text{U5} \\
\Leftrightarrow^* & [(\bar{U}\circ U)\circ U',((\bar{u},u),u')]\varphi & \text{U5} \\
\Leftrightarrow^* & [\bar{U}\circ (U\circ U'),(\bar{u},(u,u'))]\varphi & \text{Lem. 4.6} \\
\Leftrightarrow^* & [\bar{U},\bar{u}][U\circ U',(u,u')]\varphi & \text{U5}
\end{array}
$$

Figure 26: Base case U5, proof of Lemma 4.11

$$
\begin{array}{lll}
 & (\mathsf{p}^U \to c_k \gg_i \varphi) \wedge (\mathsf{p}^{\bar{U}}(\bar{u}) \to \bigwedge_{\bar{u}R_i^{\bar{U}}\bar{v}} B_i[\bar{U},\bar{v}]\varphi) & \\
\Leftrightarrow^* & (\mathsf{p}^U \to c_k \gg_i \varphi) \wedge [\bar{U},\bar{u}]B_i\varphi & \text{U3, PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}](c_k \gg_i \varphi) \wedge [\bar{U},\bar{u}]B_i\varphi & \text{U4, PR} \\
\Leftrightarrow^* & [\bar{U},\bar{u}]c_k :_i \varphi & \text{U2, PR}
\end{array}
$$

Figure 27: Induction case CN, proof of Lemma 4.11

Proof. By induction on the length of JLCE^* derivations. In the base and induction cases where Rule AX was used, the result follows by Lemma 4.11. In the induction case where Rule MP was used with premises $\varphi \to \psi$ and φ, it follows by the induction hypothesis that $\vdash^* [\bar{U},\bar{u}](\varphi \to \psi)$ and $\vdash^* [\bar{U},\bar{u}]\varphi$. Applying Axiom U2 and classical propositional reasoning, it follows that $\vdash^* [\bar{U},\bar{u}]\varphi \to [\bar{U},\bar{u}]\psi$. Therefore, by Rule MP, we conclude that $\vdash^* [\bar{U},\bar{u}]\psi$. □

Theorem 4.13 (Admissibility)**.** $\vdash \varphi$ if and only if $\vdash^* \varphi$.

Proof. The left-to-right direction is shown by induction on the length of JLCE derivation and makes use of Belief Admissibility and Update Admissibility (Lemmas 4.3 and 4.12). The right-to-left direction is immediate: a JLCE^* derivation is a JLCE derivation. □

Theorem 4.14 (JLCE Internalization)**.** If $\vdash \varphi$ and $i \in \mathscr{A}$, then there is a term $t \in \mathscr{T}$ such that $\vdash t :_i \varphi$.

Proof. By JLCE^* Internalization (Lemma 4.2) and Theorem 4.13. □

5 Depth and Reduction

One of the central results of many Dynamic Epistemic Logics is the *Reduction Theorem*, which says that every formula φ containing an update modal $[U,u]$ can be "reduced" to a provably equivalent formula $\varphi^\dagger$ that does not contain any update modals. The Reduction Theorem typically plays an important role in the proof of completeness, and this is so for the theories JLCE and JLCE^* as well. However, in these theories, Reduction has the following variant form: each formula φ can be "reduced" to a provably equivalent formula $\varphi^\dagger$ whose update modals occur *only within the scope of a term*, by which we mean that update modals appear only in subformulas ψ that themselves occur within a subformula having the form $t \gg_i \psi$. As we will see later, this version of Reduction is sufficient for proving completeness for JLCE and JLCE^*.

Since we saw in Theorem 4.13 that JLCE and JLCE^* derive the same theorems, it suffices for us to prove Reduction for JLCE^*. As in Dynamic Epistemic Logic [28], we proceed by defining a notion of "depth" for the language; this notion is sometimes called "complexity" by other authors.

Definition 5.1 (Depth)**.** The equations in Figure 28 define a function $d : \mathscr{F} \cup \mathscr{T} \to \mathbb{N}$ that maps each $o \in \mathscr{F} \cup \mathscr{T}$ to a natural number $d(o)$ called the *depth of o*. For each $n \in \mathbb{N}^+$, we

$$
\begin{array}{lcl}
d(q) & := & 1, \text{ for } q \in \mathscr{P} \cup \{\bot\} \\
d(\varphi \to \psi) & := & 1 + \max\{d(\varphi), d(\psi)\} \\
d(B_i\varphi) & := & 1 + d(\varphi) \\
d(t \gg_i \varphi) & := & 2 + \max\{d(t), d(\varphi)\} \\
d([U,u]\varphi) & := & (4 + d(U))^{4+d(\varphi)} \cdot d(\varphi) \\
d(U) & := & |W^U| + \max_{u \in W^U} d(\mathsf{p}^U(u)) \\
d(c_k) & := & 1 \\
d(x_k) & := & 1 \\
d(t \cdot_\varphi s) & := & 1 + \max\{d(t), d(s), d(\varphi)\} \\
d(t + s) & := & 1 + \max\{d(t), d(s)\} \\
d(!t) & := & 1 + d(t)
\end{array}
$$

Note: This definition is adapted from [28].

Figure 28: Definition of a function $d : \mathscr{F} \cup \mathscr{T} \to \mathbb{N}$

define the sets

$$
\begin{array}{lcl}
\mathscr{F}^{(n)} & := & \{\varphi \in \mathscr{F} \mid d(\varphi) \leq n\} \\
\mathscr{T}^{(n)} & := & \{t \in \mathscr{T} \mid t \text{ occurs in some } \varphi \in \mathscr{F}^{(n)}\} \\
\mathscr{U}^{(n)} & := & \mathscr{U}(\mathscr{T}^{(n)}, \mathscr{F}^{(n)})
\end{array}
$$

and define the language $\mathsf{UL}^{(n)} := (\mathscr{T}^{(n)}, \mathscr{F}^{(n)})$.

With the definition of depth in hand, we now define how to "reduce" a given formula to a JLCE^*-equivalent formula whose update modals occur only within the scope of a term.

Theorem 5.2 (Reduction)**.** The schematic equations in Figure 29 define a function $\dagger : \mathscr{F} \to \mathscr{F}$ that maps each formula φ to a formula $\varphi^\dagger$ such that $\varphi \Leftrightarrow^* \varphi^\dagger$. Further, the schematic equations in Figure 29 are *depth-respecting:* for each $\mathscr{F}$-instance of a schematic equation in Figure 29, the function $\dagger$ is applied on the left-hand side to a formula whose depth is strictly greater than that of any formula on the right-hand side to which $\dagger$ is applied.

Proof. To show that equations in Figure 29 define a function that takes each formula φ to a formula $\varphi^\dagger$, we argue by induction on $n \in \mathbb{N}^+$ that the equations in Figure 29 define a function $\dagger_n : \mathscr{F}^{(n)} \to \mathscr{F}^{(n)}$. The equations defining $\dagger_n$ are obtained from those in Figure 29 as follows: the $\dagger$ on the left-hand side of an equation is to be replaced by $\dagger_n$, each $\dagger$ on the right-hand side of an equation is to be replaced by $\dagger_{n-1}$, and any equation that then contains $\dagger_0$ on its right-hand side is to be omitted. It is easy to see that $\dagger_1$ is well-defined, and it is similarly easy to argue that $\dagger_{n+1}$ is well-defined if $\dagger_n$ is well-defined. The latter argument requires us to prove that our equations in Figure 29 are depth-respecting, which is a dull exercise in unravelling definitions and reasoning with inequalities. To get a flavor for

$$
\begin{array}{lcl}
q^\dagger & = & q \text{ if } q \in \mathscr{P} \cup \{\bot\} \\
(\varphi \to \psi)^\dagger & = & \varphi^\dagger \to \psi^\dagger \\
(B_i\varphi)^\dagger & = & B_i(\varphi^\dagger) \\
(t \gg_i \varphi)^\dagger & = & t \gg_i \varphi \\
([U,u]q)^\dagger & = & (\mathsf{p}^U(u))^\dagger \to q \text{ if } q \in \mathscr{P} \cup \{\bot\} \\
([U,u](\varphi \to \psi))^\dagger & = & ([U,u]\varphi)^\dagger \to ([U,u]\psi)^\dagger \\
([U,u]B_i\varphi)^\dagger & = & (\mathsf{p}^U(u))^\dagger \to \bigwedge_{uR_i^U v} B_i([U,v]\varphi)^\dagger \\
([U,u](t \gg_i \varphi))^\dagger & = & \begin{cases} \neg(\mathsf{p}^U(u))^\dagger & \text{if } U,u \vdash (t,i,\varphi) \\ (\mathsf{p}^U(u))^\dagger \to t \gg_i \varphi & \text{if } U,u \nvdash (t,i,\varphi) \end{cases} \\
([U,u][U',u']\varphi)^\dagger & = & ([U \circ U',(u,u')]\varphi)^\dagger
\end{array}
$$

Figure 29: Definition of a function $\dagger : \mathscr{F} \to \mathscr{F}$, Theorem 5.2

$$
\begin{array}{cl}
 & d\big([U \circ U',(u,u')]\varphi\big) \\
= & (4 + d(U \circ U'))^{4+d(\varphi)} \cdot d(\varphi) \\
= & \big(4 + |W^U| \cdot |W^{U'}| + \max_{(v,v') \in W^{U \circ U'}} \{d(\neg[U,v]\neg\mathsf{p}^{U'}(v'))\}\big)^{4+d(\varphi)} \cdot d(\varphi) \\
= & \big(5 + |W^U| \cdot |W^{U'}| + \\
 & \quad \max_{v' \in W^{U'}}\{(4+d(U))^{5+d(\mathsf{p}^{U'}(v'))} \cdot (1 + d(\mathsf{p}^{U'}(v')))\}\big)^{4+d(\varphi)} \cdot d(\varphi) \\
\leq & \big(5 + |W^U| \cdot |W^{U'}| + (4+d(U))^{4+d(U')} \cdot d(U')\big)^{4+d(\varphi)} \cdot d(\varphi) \\
< & \big(2 \cdot (4+d(U))^{4+d(U')} \cdot d(U')\big)^{4+d(\varphi)} \cdot d(\varphi) \\
< & \big((4+d(U))^{5+d(U')} \cdot d(U')\big)^{4+d(\varphi)} \cdot d(\varphi) \\
< & (4+d(U))^{(5+d(U'))\cdot(4+d(\varphi))} \cdot (4+d(U'))^{4+d(\varphi)} \cdot d(\varphi) \\
= & (4+d(U))^{20+5\cdot d(\varphi)+4\cdot d(U')+d(U')\cdot d(\varphi)} \cdot (4+d(U'))^{4+d(\varphi)} \cdot d(\varphi) \\
< & (4+d(U))^{4+(4+d(U'))^5 \cdot d(\varphi)} \cdot (4+d(U'))^{4+d(\varphi)} \cdot d(\varphi) \\
\leq & (4+d(U))^{4+(4+d(U'))^{4+d(\varphi)} \cdot d(\varphi)} \cdot (4+d(U'))^{4+d(\varphi)} \cdot d(\varphi) \\
= & d\big([U,u][U',u']\varphi\big)
\end{array}
$$

Figure 30: Proof that $d\big([U \circ U',(u,u')]\varphi\big) < d\big([U,u][U',u']\varphi\big)$.

these arguments, see Figure 30. Arguments for some of the other inequalities can be found by adapting Figures 4.2–4.4 and 4.6 from [20]; arguments for the remaining inequalities are not too difficult to obtain. We therefore have by induction on $n \in \mathbb{N}^+$ that $\dagger_n$ is well-defined for each $n \in \mathbb{N}^+$. Next, we define a function $\dagger$ mapping formulas to formulas by setting $\varphi^\dagger := \varphi^{\dagger_{d(\varphi)}}$ and then argue that $\dagger$ is the unique function satisfying the equations in Figure 29.[10]

All that remains is for us to argue by induction on formula depth that $\varphi \Leftrightarrow^* \varphi^\dagger$. Almost all of the cases are straightforward adaptations of the standard arguments in Dynamic Epistemic Logic [28], with the exception of the following cases that we handle in detail.

- Case $B_i\psi$.

  Since $\dagger$ is depth-respecting, we have $d(\psi) < d(B_i\psi)$ and hence that $\psi \Leftrightarrow^* \psi^\dagger$ by the induction hypothesis. Thus $B_i\psi \Leftrightarrow^* B_i\psi^\dagger$ by modal reasoning (using Lemma 4.3 instead of Rule BN). Since $B_i\psi^\dagger = (B_i\psi)^\dagger$, it follows that $B_i\psi \Leftrightarrow^* (B_i\psi)^\dagger$.

- Case $[U,u]B_i\psi$.

  Let $S := \{v \in W^U \mid uR_i^U v\}$. If $S = \emptyset$, then a conjunction $\bigwedge_{uR_i^U v} \chi_v$ ranging over S is equal to $\top$. It therefore follows by Axiom U3 and classical propositional reasoning that $[U,u]B_i\psi \Leftrightarrow^* \top$, and it follows by the definition of $\dagger$ and classical propositional reasoning that $([U,u]B_i\psi)^\dagger \Leftrightarrow^* \top$. We therefore have that $[U,u]B_i\psi \Leftrightarrow^* ([U,u]B_i\psi)^\dagger$, as desired.

  So suppose $S \neq \emptyset$ and hence that there is a $v \in S$. Since $\dagger$ is depth-respecting, we have $d(\mathsf{p}^U(u)) < d([U,u]B_i\psi)$ and $d([U,v]\psi) < d([U,u]B_i\psi)$ and hence that $\mathsf{p}^U(u) \Leftrightarrow^* (\mathsf{p}^U(u))^\dagger$ and $[U,v]\psi \Leftrightarrow^* ([U,v]\psi)^\dagger$ by the induction hypothesis. Applying modal reasoning (using Lemma 4.3 instead of Rule BN), we have $\mathsf{p}^U(u) \Leftrightarrow^* (\mathsf{p}^U(u))^\dagger$ and $B_i[U,v]\psi \Leftrightarrow^* B_i([U,v]\psi)^\dagger$. Since $v \in S$ was chosen arbitrarily, it follows by Axiom U3, the definition of $\dagger$, and classical propositional reasoning that $[U,u]B_i\psi \Leftrightarrow^* ([U,u]B_i\psi)^\dagger$, as desired.

- Case $[U,u](t \gg_i \psi)$.

  Since $\dagger$ is depth-respecting, we have $d(\mathsf{p}^U(u)) < d([U,u](t \gg_i \psi))$ and hence that $\mathsf{p}^U(u) \Leftrightarrow^* (\mathsf{p}^U(u))^\dagger$ by the induction hypothesis. Applying classical propositional reasoning and Axiom U4, it follows that $[U,u](t \gg_i \psi) \Leftrightarrow^* \big([U,u](t \gg_i \psi)\big)^\dagger$. □

We have therefore shown that every formula φ can be reduced to a JLCE^*-equivalent formula $\varphi^\dagger$.

6 Semantics

The semantics of UL is our adaptation of a Kripke-style semantics for Justification Logic due to Fitting [11] and Mkrtychev [16].

[10] In a bit more detail, making frequent use of the fact that the equations in Figure 29 are depth-respecting, we proceed in the following way. First, we argue by induction on $n \in \mathbb{N}^+$ that for each $\varphi \in \mathscr{F}^{(n)}$ and each $k \in \mathbb{N}$, we have $\varphi^{\dagger_{d(\varphi)}} = \varphi^{\dagger_{d(\varphi)+k}}$. Using this, we argue that the function $\dagger$ defined by $\varphi^\dagger := \varphi^{\dagger_{d(\varphi)}}$ satisfies the equations in Figure 29. Finally, we argue by induction on $n \in \mathbb{N}^+$ that if $r : \mathscr{F} \to \mathscr{F}$ is another function satisfying the equations in Figure 29, then $\varphi^r = \varphi^\dagger$ for each $\varphi \in \mathscr{F}^{(n)}$.

Definition 6.1. To say that M is a *Fitting model* means that M is a tuple (W^M, R^M, V^M, A^M) whose components satisfy the following.

- W^M is a nonempty set whose members are called *worlds* (in M).
- $R^U : \mathscr{A} \to \wp(W^U \times W^U)$ assigns a transitive binary relation $R_i^U \subseteq W^U \times W^U$ on W^U to each agent $i \in \mathscr{A}$.
- $V^U : \mathscr{P} \to \wp(W^U)$ assigns a set $V^U(p)$ of worlds in M to each propositional letter $p \in \mathscr{P}$.
- $A^U : \mathscr{A} \to (\mathscr{T} \times \mathscr{F} \to \wp(W^U))$ assigns a set $A_i(t, \varphi)$ of worlds in M to each term-agent-formula triple $(t, i, \varphi) \in \mathscr{T} \times \mathscr{A} \times \mathscr{F}$ subject to the following schematic conditions that together make a function of type $\mathscr{A} \to (\mathscr{T} \times \mathscr{F} \to \wp(W^U))$ an *evidence function.*

  Constant Specification: $c \in \mathscr{C}$ and $\mathsf{AX} \vdash \varphi$ imply $A_i^M(c, \varphi) = W^M$.

  Application: $A_i^M(t, \varphi \to \psi) \cap A_i^M(s, \varphi) \subseteq A_i^M(t \cdot_\varphi s, \psi)$.

  Sum: $A_i^M(t, \varphi) \cup A_i^M(s, \varphi) \subseteq A_i^M(t + s, \varphi)$.

  Checker: $A_i^M(t, \varphi) \subseteq A_i^M(!t, t :_i \varphi)$.

  Monotonicity: $\Gamma R_i^M \Delta$ and $\Gamma \in A_i^M(t, \varphi)$ imply $\Delta \in A_i^M(t, \varphi)$.

A *pointed Fitting model* is a pair (M, Γ) consisting of a Fitting model M and a world $\Gamma \in W^U$; Γ is said to be the *point* of (M, Γ).

Definition 6.2 (Truth)**.** Given a pointed Fitting model (M, Γ) and a formula $\varphi \in \mathscr{F}$, we write $M, \Gamma \models \varphi$ to mean that φ is *true at* (M, Γ); the negation of $M, \Gamma \models \varphi$ is written $M, \Gamma \not\models \varphi$. We define when it is that a formula is true at a pointed Fitting model according to the following induction on formula depth.

- $M, \Gamma \models p_k$ means that $\Gamma \in V^M(p_k)$.
- $M, \Gamma \not\models \bot$.
- $M, \Gamma \models \varphi_1 \to \varphi_2$ means that $M, \Gamma \not\models \varphi_1$ or $M, \Gamma \models \varphi_2$.
- $M, \Gamma \models B_i\varphi$ means that $M, \Delta \models \varphi$ for each $\Delta \in W^M$ with $\Gamma R_i^M \Delta$.
- $M, \Gamma \models t \gg_i \varphi$ means that $\Gamma \in A_i^M(t, \varphi)$.
- $M, \Gamma \models [U, u]\varphi$ means that either $M, \Gamma \not\models \mathsf{p}^U(u)$ or else both $M, \Gamma \models \mathsf{p}^U(u)$ and $M[U], (\Gamma, u) \models \varphi$, where the components of the tuple $M[U]$ are defined as follows.

$$\begin{aligned} W^{M[U]} &:= \{(\Delta, v) \in W^M \times W^U \mid M, \Delta \models \mathsf{p}^U(v)\} \\ R_i^{M[U]} &:= \{((\Delta, v), (\Delta', v')) \mid \Delta R_i^M \Delta' \ \&\ v R_i^U v'\} \\ A_i^{M[U]}(t, \psi) &:= \{(\Delta, v) \mid \Delta \in A_i^M(t, \psi) \ \&\ U, v \nvdash (t, i, \psi)\} \\ V^{M[U]}(p_k) &:= \{(\Delta, v) \mid \Delta \in V^M(p_k)\} \end{aligned}$$

To say that $\varphi \in \mathscr{F}$ is *valid*, written $\models \varphi$, means that we have $M, \Gamma \models \varphi$ for each pointed Fitting model (M, Γ).

Lemma 6.3 (Update Correctness)**.** Let (M, Γ) be a pointed Fitting model and (U, u) be a pointed update frame. If $M, \Gamma \models \mathsf{p}^U(u)$, then $M[U]$ is a Fitting model.

Proof. $W^{M[U]}$ is nonempty because $M, \Gamma \models \mathsf{p}^U(u)$. $R_i^{M[U]}$ is transitive because each of R_i^M and R_i^U is transitive. To prove that $A^{M[U]}$ is an evidence function, we check each of the defining properties in turn.

- Constant Specification: $c \in \mathscr{C}$ and $\mathsf{AX} \vdash \varphi$ imply $A_i^{M[U]}(c, \varphi) = W^{M[U]}$.

  Suppose $c \in \mathscr{C}$ and $\mathsf{AX} \vdash \varphi$. Choose an arbitrary $(\Delta, v) \in W^{M[U]}$. We have that $\Delta \in A_i^M(c, \varphi)$ because A^M is an evidence function. Further, we have $U, v \nvdash (c, i, \varphi)$ (see Figure 1). Hence $(\Delta, v) \in A_i^{M[U]}(c_k, \varphi)$ by the definition of $A_i^{M[U]}$.

- Application: $A_i^{M[U]}(t, \varphi \to \psi) \cap A_i^{M[U]}(s, \varphi) \subseteq A_i^{M[U]}(t \cdot_\varphi s, \psi)$.

  Suppose $(\Delta, v) \in A_i^{M[U]}(t, \varphi \to \psi) \cap A_i^{M[U]}(s, \varphi)$. It follows by the definition of $A_i^{M[U]}$ that $\Delta \in A_i^M(t, \varphi \to \psi) \cap A_i^M(s, \varphi)$, $U, v \nvdash (t, i, \varphi \to \psi)$, and $U, v \nvdash (s, i, \varphi)$. The first of these items implies $\Delta \in A_i^M(t \cdot_\varphi s)$ because A^M is an evidence function. The second two of the three items together imply that $U, v \nvdash (t \cdot_\varphi s, i, \psi)$ (see Figure 1). But the latter and $\Delta \in A_i^M(t \cdot_\varphi s)$ together imply that $(\Delta, v) \in A_i^{M[U]}(t \cdot_\varphi s, \psi)$ by the definition of $A_i^{M[U]}$.

- Sum: $A_i^{M[U]}(t, \varphi) \cup A_i^{M[U]}(s, \varphi) \subseteq A_i^{M[U]}(t + s, \varphi)$. Similar to the argument for Application.

- Checker: $A_i^{M[U]}(t, \varphi) \subseteq A_i^{M[U]}(!t, t :_i \varphi)$. Similar to the argument for Application.

- Monotonicity: $(\Delta, v) R_i^{M[U]} (\Delta', v')$ and $(\Delta, v) \in A_i^{M[U]}(t, \varphi)$ imply $(\Delta', v') \in A_i^{M[U]}(t, \varphi)$.

  Suppose $(\Delta, v) \in A_i^{M[U]}(t, \varphi)$ and $(\Delta, v) R_i^{M[U]} (\Delta', v')$. It follows that $\Delta \in A_i^M(t, \varphi)$ and $U, v \nvdash (t, i, \varphi)$ by the definition of $A_i^{M[U]}$ and that $\Delta R_i^M \Delta'$ and $v R_i^U v'$ by the definition of $R_i^{M[U]}$. Since A^M is an evidence function, it follows that $\Delta' \in A_i(t, \varphi)$. Since $U, v \nvdash (t, i, \varphi)$ and $v R_i^U v'$, it follows by Lemma 3.7 that $U, v' \nvdash (t, i, \varphi)$. But the latter and $\Delta' \in A_i^M(t, \varphi)$ together imply that $(\Delta', v') \in A_i^{M[U]}(t, \varphi)$ by the definition of $A_i^{M[U]}$. □

In the definition of truth (Definition 6.2), the cases for formulas having the form $[U, u]\varphi$ delegate part of their work to the U-calculus (Figure 1). This may seem strange because we are admitting the U-calculus—a syntactic notion—into our semantics. However, it is our intention for evidence eliminations to respect the intended meanings of the term-forming operations (described earlier in the section on syntax). This ensures that the elimination of one or more parts of a combination t of multiple pieces of evidence shall affect t itself. Therefore, some simple theory describing the logical consequences that the elimination of a simple term has on more complex terms is a necessary part of the semantics. One may also take some comfort in the fact that the U-calculus is a simple, decidable theory.

Definition 6.4 (Isomorphism)**.** To say that f is an *isomorphism* between Fitting models M and M' means that f is a function of type $W^M \to W^{M'}$ satisfying each of the following: f is a bijection, $\Gamma R_i^M \Delta$ if and only if $f(\Gamma) R_i^{M'} f(\Delta)$ for each $(\Gamma, \Delta) \in W^M \times W^M$ and each $i \in \mathscr{A}$, $\Gamma \in V^M(p)$ if and only if $f(\Gamma) \in V^{M'}(p)$ for each $\Gamma \in W^M$ and each $p \in \mathscr{P}$, and $\Gamma \in A_i^M(t, \psi)$ if and only if $f(\Gamma) \in A_i^{M'}(t, \psi)$ for each $\Gamma \in W^M$ and each $(t, i, \psi) \in \mathscr{T} \times \mathscr{A} \times \mathscr{F}$.

Theorem 6.5 (Isomorphism Equivalence)**.** For each isomorphism f between Fitting models M and M', each formula $\varphi \in \mathscr{F}$, and each pointed update frame $(U, u) \in \mathscr{U}$ such that there exists an $\Omega \in W^M$ satisfying $M, \Omega \models \mathsf{p}^U(u)$ and $M', f(\Omega) \models \mathsf{p}^U(u)$, we have each of the following items.

1. $M, \Gamma \models \varphi$ if and only if $M', f(\Gamma) \models \varphi$ for each $\Gamma \in W^M$.
2. $f^U : W^{M[U]} \to W^{M'[U]}$ defined by setting $f^U(\Delta, v) := (f(\Delta), v)$ is an isomorphism between $M[U]$ and $M'[U]$.

Proof. By induction on $n := \max\{d(U), d(\varphi)\} \in \mathbb{N}^+$. The base case $n = 1$ is vacuously true because $d(U) \geq 2$. So we proceed to the induction case. Item 1 is proved by considering the possible syntactic forms that φ might have. First, if φ has one of the forms $q \in \mathscr{P} \cup \{\bot\}$, $\psi \to \chi$, or $B_i\psi$, then the argument is just as in modal logic [8]. So let us consider the remaining forms that $\varphi \in \mathscr{F}$ might have.

- Case: φ has the form $t \gg_i \psi$.

  $M, \Gamma \models t \gg_i \psi$ means that $\Gamma \in A_i^M(t, \psi)$. Since f is an isomorphism between M and M', the latter holds if and only if $f(\Gamma) \in A_i^{M'}(t, \psi)$. But this is what it means to have $M', f(\Gamma) \models t \gg_i \psi$.
- Case: φ has the form $[U', u']\psi$.

  Since $d(\mathsf{p}^{U'}(u')) < d([U', u']\psi)$, it follows by the induction hypothesis that $M, \Gamma \models \mathsf{p}^{U'}(u')$ if and only if $M', f(\Gamma) \models \mathsf{p}^{U'}(u')$. Further, under the assumption that both $M, \Gamma \models \mathsf{p}^{U'}(u')$ and $M', f(\Gamma) \models \mathsf{p}^{U'}(u')$, since $d(U') < d([U', u']\psi)$, it follows by the induction hypothesis that $f^{U'}$ is an isomorphism between $M[U']$ and $M'[U']$ and hence that $M[U'], (\Gamma, u') \models \psi$ if and only if $M'[U'], (f(\Gamma), u') \models \psi$. Applying the definition of truth (Definition 6.2) and the definition of $f^{U'}$, we have proved that $M, \Gamma \models [U', u']\psi$ if and only if $M', f(\Gamma) \models [U', u']\psi$.

This completes the proof of Item 1. To prove Item 2, we prove that f^U is an isomorphism between $M[U]$ and $M'[U]$.

- f^U is a bijection.

  First, we argue that f^U is surjective. Proceeding, $(\Delta', v) \in W^{M'[U]}$ means that $M', \Delta' \models \mathsf{p}^U(v)$. Since f is surjective, there is a $\Delta \in W^M$ such that $f(\Delta) = \Delta'$. Further, since $d(\mathsf{p}^U(v)) < d(U)$, it follows by the induction hypothesis that $M', f(\Delta) \models \mathsf{p}^U(v)$ if and only if $M, \Delta \models \mathsf{p}^U(v)$. But the latter is what it means to have $(\Delta, v) \in W^{M[U]}$. Since $f^U(\Delta, v) = (f(\Delta), v) = (\Delta', v)$, we have shown that f^U is surjective.

  We now argue that f^U is injective. Proceeding, assume that

  $$f^U(\Delta_1, v_1) = f^U(\Delta_2, v_2) \ .$$

  It follows by the definition of f^U that $f(\Delta_1) = f(\Delta_2)$ and $v_1 = v_2$. Since f is injective, it follows that $\Delta_1 = \Delta_2$ and $v_1 = v_2$. So f^U is injective.
- $(\Delta, v)R_i^{M[U]}(\Delta', v')$ if and only if $f^U(\Delta, v)R_i^{M'[U]} f^U(\Delta', v)$.

  $(\Delta, v)R_i^{M[U]}(\Delta', v')$ means that $\Delta R_i^M \Delta'$ and $v R_i^U v'$. Since f is an isomorphism, the latter is equivalent to $f(\Delta)R_i^{M'} f(\Delta')$ and $v R_i^U v'$, which is what it means to have $f^U(\Delta, v)R_i^{M'[U]} f^U(\Delta', v)$.

- $(\Delta, v) \in V^{M[U]}(p_k)$ if and only if $f^U(\Delta, v) \in V^{M'[U]}(p_k)$.

  $(\Delta, v) \in V^{M[U]}(p_k)$ means that $\Delta \in V^M(p_k)$. Since f is an isomorphism, the latter is equivalent to $f(\Delta) \in V^{M'}(p_k)$. But this is what it means to have $f^U(\Delta, v) \in V^{M'[U]}(p_k)$.

- $(\Delta, v) \in A_i^{M[U]}(t, \psi)$ if and only if $f^U(\Delta, v) \in A_i^{M'[U]}(t, \psi)$.

  $(\Delta, v) \in A_i^{M[U]}(t, \psi)$ means that $\Delta \in A_i^M(t, \psi)$ and $U, v \nVdash (t, i, \psi)$. Since f is an isomorphism, the latter conjunction is equivalent to the statement that $f(\Delta) \in A_i^{M'}(t, \psi)$ and $U, v \nVdash (t, i, \psi)$. But this is what it means to have $f^U(\Delta, v) \in A_i^{M'[U]}(t, \psi)$. □

Theorem 6.6 (Soundness). $\vdash \varphi$ implies $\models \varphi$ for each $\varphi \in \mathscr{F}$.

Proof. By Admissibility (Theorem 4.13), it suffices for us to prove that $\vdash^* \varphi$ implies $\models \varphi$. Proceeding, we first prove that $\mathsf{AX} \vdash \chi$ implies $\models \chi$ by induction on the length of AX derivations. In the base case, we must check each of the AX-axioms. Most of the arguments for these axioms are straightforward, making use of standard arguments in Dynamic Epistemic Logic [28] or the fact that the function A^M in a Fitting model M satisfies the defining properties of an evidence function. Perhaps the trickiest Axiom is U5, so we shall handle this axiom in detail.

- Axiom U5 is sound; that is, $\models [U, u][U', u']\varphi \leftrightarrow [U \circ U', (u, u')]\varphi$.

  While much of this argument is standard in Dynamic Epistemic Logic [28], the Justification Logic-specific aspects of AX introduce some complications, so we shall handle this argument in full. Let (M, Γ) be a pointed Fitting model. We may assume without loss of generality that $M, \Gamma \models \neg[U, u]\neg\mathsf{p}^{U'}(u')$ (for the result otherwise follows easily by the definition of truth, Definition 6.2). Proceeding, it suffices by Isomorphism Equivalence (Theorem 6.5) and the definition of truth (Definition 6.2) for us to prove that the function $f : W^{M[U][U']} \to W^{M[U \circ U']}$ defined by setting $f((\Delta, v), v') := (\Delta, (v, v'))$ is an isomorphism between $M[U][U']$ and $M[U \circ U']$.

  - f is a bijection.

    $((\Delta, v), v') \in W^{M[U][U']}$ if and only if $M, \Delta \models \neg[U, v]\neg\mathsf{p}^{U'}(v')$. But the latter is equivalent to the statement that $(\Delta, (v, v')) \in W^{M[U \circ U']}$.

  - $\Omega_1 R_i^{M[U][U']} \Omega_2$ if and only if $f(\Omega_1) R_i^{M[U \circ U']} f(\Omega_2)$.

    We have $((\Delta, v), v') R_i^{M[U][U']} ((\Omega, w), w')$ if and only if $\Delta R_i^M \Omega$, $v R_i^U w$, and $v' R_i^{U'} w'$. But the latter trio is equivalent to the statement that $(\Delta, (v, v')) R_i^{M[U \circ U']} (\Omega, (w, w'))$.

  - $\Omega \in V^{M[U][U']}(p_k)$ if and only if $f(\Omega) \in V^{M[U \circ U']}(p_k)$.

    $((\Delta, v), v') \in V^{M[U][U']}(p_k)$ and $(\Delta, (v, v')) \in V^{M[U \circ U']}(p_k)$ are each equivalent to the statement that $\Delta \in V^M(p_k)$.

  - $\Omega \in A_i^{M[U][U']}(t, \psi)$ if and only if $f(\Omega) \in A_i^{M[U \circ U']}(t, \psi)$.

    $((\Delta, v), v') \in A_i^{M[U][U']}(t, \psi)$ means that $\Delta \in A_i^M(t, \psi)$, $U, v \nVdash (t, i, \psi)$, and $U', v' \nVdash (t, i, \psi)$. Applying Composition (Lemma 3.11), we have $\Delta \in A_i(t, \psi)$ and $(U \circ U'), (v, v') \nVdash (t, i, \psi)$. But this is what it means to have $(\Delta, (v, v')) \in A_i^{M[U \circ U']}(t, \psi)$.

Conclusion: Axiom U5 is valid.

We have argued that each AX-axiom is valid. For the induction case, assume we derived $\mathsf{AX} \vdash c_k :_i \varphi$ by Rule CN from AX-theorem φ. By the induction hypothesis, $\models \varphi$. Letting (M, Γ) be a pointed Fitting model, $\models \varphi$ implies $M, \Delta \models \varphi$ for each $\Delta \in W^M$ satisfying $\Gamma R_i^M \Delta$. Further, since φ is an AX-theorem, it follows that $\Gamma \in A_i^M(c_k, \varphi)$ because A^M is an evidence function. Therefore $M, \Gamma \models c_k :_i \varphi$. Since (M, Γ) was chosen arbitrarily, we have shown that $\models c_k :_i \varphi$. Conclusion: $\mathsf{AX} \vdash \varphi$ implies $\models \varphi$. By the standard argument for the validity of Rule MP [8], it therefore follows that $\vdash^* \varphi$ implies $\models \varphi$. □

Theorem 6.7 (Consistency)**.** JLCE is consistent; that is, $\nvdash \bot$.

Proof. Define the Fitting model M by setting $W^M := \{\Gamma\}$, $R_i^M := \{(\Gamma, \Gamma)\}$ for each $i \in \mathscr{A}$, $V^M(p) := \emptyset$ for each $p \in \mathscr{P}$, and $A_i^M(t, \varphi) = W^M$ for each $(t, i, \varphi) \in \mathscr{T} \times \mathscr{A} \times \mathscr{F}$. (Note that M is indeed a Fitting model: W^M is nonempty, R_i^M is transitive for each $i \in \mathscr{A}$, and A^M is an evidence function.) Since $M, \Gamma \not\models \bot$, it follows by Soundness (Theorem 6.6) that $\nvdash \bot$. □

Theorem 6.8 (Completeness)**.** $\models \varphi$ implies $\vdash \varphi$ for each $\varphi \in \mathscr{F}$.

Proof. Since JLCE^* is a subsystem of JLCE, it suffices for us to argue that $\models \varphi$ implies $\vdash^* \varphi$. Our proof proceeds by a canonical model argument, so let us first make some preliminary definitions. Let S be a set of formulas. If S is finite, then we define

$$\bigwedge S := \begin{cases} \top & \text{if } S = \emptyset, \\ \bigwedge_{\psi \in S} \psi & \text{otherwise.} \end{cases}$$

To say that S is *inconsistent* means that there is a finite subset $S' \subseteq S$ such that $\vdash^* (\bigwedge S') \to \bot$. To say that S is *consistent* means that S is not inconsistent. To say that S is *maximal consistent* means that S is consistent and adding any formula not already present in S will result in a set that is inconsistent. By a Lindenbaum Argument, any consistent set of formulas may be extended to a maximal consistent set of formulas. The *canonical model* is the structure $M^* := (W^*, R^*, V^*, A^*)$ whose components are defined as follows.

- W^* is the set of all maximal consistent sets of formulas.
- $R^* : \mathscr{A} \to \wp(W^* \times W^*)$ is defined by

$$R_i^* := \{(\Gamma, \Delta) \in W^* \times W^* \mid \forall \varphi \in \mathscr{F} : B_i\varphi \in \Gamma \text{ implies } \varphi \in \Delta\} \ .$$

- $A^* : \mathscr{A} \to (\mathscr{T} \times \mathscr{F} \to \wp(W^*))$ is defined by setting

$$A_i^*(t, \varphi) := \{\Gamma \in W^* \mid (t \gg_i \varphi) \in \Gamma\} \ .$$

- $V^* : \mathscr{P} \to \wp(W^*)$ is defined by setting $V(p) := \{\Gamma \in W^* \mid p \in \Gamma\}$.

To see that M^* is in fact a Fitting model: W^* is nonempty because the set Γ of all formulas true at the pointed Fitting model (M, Γ) from the proof of Theorem 6.7 is maximal consistent; for each $i \in \mathscr{A}$, the relation R_i^* is transitive by the standard argument in modal logic [8]; A^* is an evidence function by Rule AX, classical propositional reasoning, and Axioms E1–E4.

The key property of M^* that we now wish to prove is the *Truth Lemma:* for each $\varphi \in \mathscr{F}$ and each $\Gamma \in W^*$, we have that $\varphi \in \Gamma$ if and only if $M^*, \Gamma \models \varphi$. The proof is by induction on formula depth.

- Cases $q \in \mathscr{P} \cup \{\bot\}$, $\varphi \to \psi$, and $B_i\varphi$ follow by the standard arguments in modal logic [8].
- Case $t \gg_i \varphi$.

  By the definition of A^*, we have $(t \gg_i \varphi) \in \Gamma$ if and only if $\Gamma \in A_i^*(t, \varphi)$. But the latter is what it means to have $M^*, \Gamma \models t \gg_i \varphi$.
- Case $[U, u]\varphi$.

  $[U, u]\varphi \in \Gamma$ is equivalent to $([U, u]\varphi)^\dagger \in \Gamma$ by Reduction (Theorem 5.2) and the maximal consistency of Γ. By the induction hypothesis, we have $([U, u]\varphi)^\dagger \in \Gamma$ if and only if $M^*, \Gamma \models ([U, u]\varphi)^\dagger$. Applying Soundness (Theorem 6.6), Reduction (Theorem 5.2), and the definition of truth (Definition 6.2), it follows $M^*, \Gamma \models ([U, u]\varphi)^\dagger$ is equivalent to $M^*, \Gamma \models [U, u]\varphi$.

This completes the proof of the Truth Lemma. The completeness argument is then easy: suppose $\nvdash^* \chi$. It follows that $\{\neg\chi\}$ is consistent and so may be extended to a maximal consistent set $\Gamma \in W^{M^*}$. Since $\neg\chi \in \Gamma$, it follows by the Truth Lemma that $M^*, \Gamma \models \neg\chi$ and thus that $M^*, \Gamma \not\models \chi$ by the definition of truth (Definition 6.2). Since M^* is a Fitting model, we have shown that $\nvdash^* \chi$ implies $\not\models \chi$. It follows that $\models \chi$ implies $\vdash^* \chi$. □

We conclude this section with a theorem that provides an important connection between JLCE and the U-calculus. The theorem tells us that whenever $\vdash t \gg_i \varphi$, it is not possible to eliminate agent i's evidence t relevant to φ. (See our discussion in Remark 3.6.)

Theorem 6.9 (Non-Elimination)**.** $\vdash t \gg_i \varphi$ implies $U, u \nvdash (t, i, \varphi)$.

Proof. It follows by Soundness (Theorem 6.6) that $\vdash t \gg_i \varphi$ implies $\models t \gg_i \varphi$. Let M be the Fitting model from the proof of Consistency (Theorem 6.7). Suppose toward a contradiction that we have $U, u \vdash (t, i, \varphi)$ for some $(U, u) \in \mathscr{U}$. Since the derivability of $U, u \vdash (t, i, \varphi)$ does not depend on the value of $\mathsf{p}^U(u)$, we could then assume without loss of generality that $\mathsf{p}^U(u) = \top$. Applying the definition of truth (Definition 6.2), we would then have that $\Gamma \notin A_i^{M[U]}(t, \varphi)$ and hence that $M[U], (\Gamma, u) \not\models t \gg_i \varphi$, contradicting the fact that $\models t \gg_i \varphi$. □

7 Formalized Example

We now use JLCE to formalize our email example from the beginning of this paper. Since we are interested only in the evidence and beliefs of Bob ($\mathfrak{B}$) and Charlie ($\mathfrak{C}$), we define our set

$\mathscr{A}$ of agents by $\mathscr{A} := \{\mathfrak{B}, \mathfrak{C}\}$. We let O ("open") abbreviate propositional letter p_0 and C ("cheese") abbreviate propositional letter p_1, and we write $O \to C$ ("open implies cheese") to describe Anne's plan to bring the cheese if the store is open. Email messages x_1 through x_4 provide us with our initial setup X, a conjunction of the formulas

- $x_1 :_{\mathfrak{B}} (O \to C)$ ("x_1 is Bob's evidence that $O \to C$"),
- $x_1 :_{\mathfrak{C}} (O \to C)$ ("x_1 is Charlie's evidence that $O \to C$"),
- $x_2 :_{\mathfrak{B}} (x_1 :_{\mathfrak{C}} (O \to C))$ ("x_2 is Bob's evidence that $x_1 :_{\mathfrak{C}} (O \to C)$"),
- $x_3 :_{\mathfrak{B}} O$ ("x_3 is Bob's evidence that O"),
- $x_3 :_{\mathfrak{C}} O$ ("x_3 is Charlie's evidence that O"), and
- $x_4 :_{\mathfrak{B}} (x_3 :_{\mathfrak{C}} O)$ ("x_4 is Bob's evidence that $x_3 :_{\mathfrak{C}} O$").

It follows by classical propositional reasoning and Lemma 4.1 that $\vdash X \to (x_1 \cdot_O x_3) :_{\mathfrak{B}} C$ and $\vdash X \to (x_1 \cdot_O x_3) :_{\mathfrak{C}} C$; that is, given our setup X, each of Bob and Charlie has evidence $x_1 \cdot_O x_3$ that Anne is bringing the cheese.

Let us now look at Bob's evidence about Charlie's evidence. First, it follows by Lemma 4.1 and Internalization (Theorem 4.14) that there is a term t such that

$$\vdash t :_{\mathfrak{B}} \big(x_1 :_{\mathfrak{C}} (O \to C) \to (x_3 :_{\mathfrak{C}} O \to (x_1 \cdot_O x_3) :_{\mathfrak{C}} C)\big) \ . \tag{5}$$

Defining the term s by setting

$$s := (t \cdot_{(x_1 :_{\mathfrak{C}} (O \to C))} x_2) \cdot_{(x_3 :_{\mathfrak{C}} O)} x_4 \ ,$$

it follows by Lemma 4.1 and classical propositional reasoning that

$$\vdash X \to s :_{\mathfrak{B}} (x_1 \cdot_O x_3) :_{\mathfrak{C}} C \ ; \tag{6}$$

that is, "given setup X, Bob has evidence s that Charlie has evidence $x_1 \cdot_O x_3$ that C." So from the initial situation X given by messages x_1 through x_4, each of Bob and Charlie has evidence $x_1 \cdot_O x_3$ that Anne is bringing the cheese, and Bob has evidence s that Charlie has evidence $x_1 \cdot_O x_3$ that Anne is bringing the cheese.

Now let us examine how Anne's private email x_5 to Bob affects Bob's evidence. Adapting the Dynamic Epistemic Logic definition of the *private announcement to an agent* (described in the introduction; see also: [4, 21, 28]), we define the update frame $\mathsf{PRI}_i^{(k,\varphi)}$, called the *private elimination of agent i's evidence x_k relevant to φ*, as follows.

$$\begin{array}{lcllcl}
W^{\mathsf{PRI}_i^{(k,\varphi)}} & := & \{u, v\} & \mathsf{p}^{\mathsf{PRI}_i^{(k,\varphi)}}(w) & := & \top \text{ for } w \in \{u, v\} \\
R_i^{\mathsf{PRI}_i^{(k,\varphi)}} & := & \{(u,u), (v,v)\} & \mathsf{v}^{\mathsf{PRI}_i^{(k,\varphi)}}(u) & := & \{(x_k, i, \varphi)\} \\
R_j^{\mathsf{PRI}_i^{(k,\varphi)}} & := & \{(u,v), (v,v)\} \text{ if } j \neq i & \mathsf{v}^{\mathsf{PRI}_i^{(k,\varphi)}}(v) & := & \emptyset
\end{array}$$

We picture $\mathsf{PRI}_i^{(k,\varphi)}$ in Figure 31. We will use the update $(\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u)$ to represent the effect of Anne's final message x_5. Using PR to denote the use of classical propositional reasoning, we then have the following.

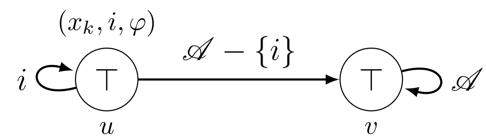


Figure 31: The update frame $\mathsf{PRI}_i^{(k,\varphi)}$

1. $\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u \vdash (x_3, \mathfrak{B}, O)$

   By Axiom EV of the $\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}$-calculus (Figure 1).

2. $\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u \vdash (x_1 \cdot_O x_3, \mathfrak{B}, C)$

   By line 1 and Rule EAR of the $\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}$-calculus (Figure 1).

3. $\vdash [\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u]\big((x_1 \cdot_O x_3) \gg_{\mathfrak{B}} C\big) \to \bot$

   By line 2, Axiom U4 (with $\mathsf{p}^{\mathsf{PRI}_{\mathfrak{B}}^{(3,0)}}(u) = \top$), and PR.

4. $[\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u]\big((x_1 \cdot_O x_3) \gg_{\mathfrak{B}} C\big) \to [\mathsf{PRI}_{\mathfrak{B}}^{(3,0)}, u]\bot$

   By line 3, Axiom U1 (with $\mathsf{p}^{\mathsf{PRI}_{\mathfrak{B}}^{(3,0)}}(u) = \top$), and PR.

5. $\vdash [\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u]\neg\big((x_1 \cdot_O x_3) \gg_{\mathfrak{B}} C\big)$

   By line 4, Axiom U2, and PR.

6. $\vdash [\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u]\neg\big((x_1 \cdot_O x_3) \gg_{\mathfrak{B}} C\big) \to [\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u]\neg\big((x_1 \cdot_O x_3) :_{\mathfrak{B}} C\big)$

   By Def. 3.4, PR, Rule UN, and Axiom U2.

7. $\vdash [\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u]\neg\big((x_1 \cdot_O x_3) :_{\mathfrak{B}} C\big)$

   By lines 5 and 6 and Rule MP.

   This is, after Anne's private message to Bob eliminating Bob's evidence x_3 relevant to O, it is not the case that $x_1 \cdot_O x_3$ is Bob's evidence that C (Anne brings the cheese). And yet we have the following.

8. $\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u \nvdash \big(t, \mathfrak{B}, x_1 :_{\mathfrak{C}} (O \to C) \to (x_3 :_{\mathfrak{C}} O \to (x_1 \cdot_O x_3) :_{\mathfrak{C}} C)\big)$

   By (5), Def. 3.4, PR, and Non-Elimination (Theorem 6.9).

9. $\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u \nvdash (x_2, \mathfrak{B}, x_1 :_{\mathfrak{C}} (O \to C))$

   Inspection of the $\mathsf{PRI}_{\mathfrak{B}}^{(3,0)}$-calculus (Figure 1).

10. $\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u \nvdash (x_4, \mathfrak{B}, x_3 :_{\mathfrak{C}} O)$

    Inspection of the $\mathsf{PRI}_{\mathfrak{B}}^{(3,0)}$-calculus (Figure 1).

11. $\mathsf{PRI}_{\mathfrak{B}}^{(3,O)}, u \nvdash (s, \mathfrak{B}, (x_1 \cdot_O x_3) :_{\mathfrak{C}} C)$

    By lines 8, 9, and 10 and inspection of the $\mathsf{PRI}_{\mathfrak{B}}^{(3,0)}$-calculus.

12. $\vdash X \to s \gg_{\mathfrak{B}} (x_1 \cdot_O x_3) :_{\mathfrak{C}} C$

    By (6), Def. 3.4, and PR.

13. $\vdash X \to [\mathsf{PRI}^{(3,O)}_{\mathfrak{B}}, u] s \gg_{\mathfrak{B}} (x_1 \cdot_O x_3) :_{\mathfrak{C}} C$

    By lines 11 and 12, Axiom U4 (with $\mathsf{p}^{\mathsf{PRI}^{(3,O)}_{\mathfrak{B}}, u}(u) = \top$), and PR.

14. $\vdash X \to B_{\mathfrak{B}}(x_1 \cdot_O x_3) :_{\mathfrak{C}} C$

    By (6), Def. 3.4, and PR.

15. $\vdash X \to [\mathsf{PRI}^{(3,O)}_{\mathfrak{B}}, u] B_{\mathfrak{B}}(x_1 \cdot_O x_3) :_{\mathfrak{C}} C$

    By line 14, Axiom U3 (with $\mathsf{p}^{\mathsf{PRI}^{(3,O)}_{\mathfrak{B}}, u}(u) = \top$ and $R^{\mathsf{PRI}^{(3,O)}_{\mathfrak{B}}}_{\mathfrak{B}}(u) = \{(u, u)\}$), and PR.

16. $\vdash X \to [\mathsf{PRI}^{(3,O)}_{\mathfrak{B}}, u] s :_{\mathfrak{B}} (x_1 \cdot_O x_3) :_{\mathfrak{C}} C$

    By lines 13 and 15, Def. 3.4, Axiom U2, and PR.

In words: given setup X, after Anne's private message to Bob eliminating Bob's evidence x_3 relevant to O, it is (still) the case that Bob has evidence s that Charlie has evidence $x_1 \cdot_O x_3$ that Anne will bring the cheese (line 16), despite the fact that Bob himself does not consider $x_1 \cdot_O x_3$ evidence that Anne will bring the cheese (line 7).

8 Conclusion

The work in this paper is part of a larger project, *Dynamic Justification Logic*, whose aim is to combine the frameworks of Dynamic Epistemic Logic and Justification Logic in order to reason about belief and evidence dynamics arising from multi-agent communications. The role of this paper in the project is to introduce *multi-agent evidence elimination*, one kind of dynamic operation on multi-agent evidence that causes an agent to set aside a piece of evidence relevant to a given assertion and then determine how this affects other pieces of evidence as a result of the ways in which evidence may be logically combined in Justification Logic. In earlier work [20], the author studied a notion of *evidence introduction*, whereby a piece of evidence may be introduced as relevant for a given assertion; however, the particular operation the author defined there lacks the tight integration with the semantics of Dynamic Epistemic Logic used for evidence elimination in this paper, and so the techniques developed here might well be adapted to a notion of *multi-agent evidence introduction* as well.

Yavorskaya [30] studied the first multi-agent languages for Justification Logic. Yavorskaya's multi-agent languages are based on a variation of the basic fragment of UL. The variation may be obtained from the basic fragment of UL by making certain changes to the term-formation grammar in $\mathscr{G}_0$, including, among other changes, the creation of a superscript-labeled copy t^i of each term t for each agent $i \in \mathscr{A}$ and the requirement that a colon-formula $t :_i \varphi$ is well-formed if and only if t is a term s^i labeled by the superscript of agent i. Yavorskaya's work investigates certain "interactions" between the terms of different agents, in the sense of acceptance of principles such as $t^i :_i \varphi \to (!^j_i t^i)^j :_j (t^i :_i \varphi)$ ("agent j can check agent i's evidence") or $t^i :_i \varphi \to (\uparrow_i^j t^i)^j :_j \varphi$ ("agent j trusts agent i's evidence"). Such interactions are essentially static because the Yavorskaya languages lack update models, and hence the interaction principles endorsed by a given model for a Yavorskaya-language remain fixed once and for all. By way of contrast, update modals allow UL to describe transitions between models that bring about changes in the agents' evidence and beliefs, which is an essentially dynamic notion. So it would be natural to investigate "dynamic" versions of Yavorskaya's languages and principles, though this is something we leave for future work. Further, the exact connection between languages with Yavorskaya-style terms (where each agent has his or her own disjoint set of terms) and languages with UL-style terms (where the agents share a single set of terms) is not yet well understood. In particular, a natural question for future research is the following "multi-agent realization" question: given a subtheory T of our theory JLCE such that T is in the basic fragment of UL, is there a theory T' in an appropriate Yavorskaya-style language such that every T-theorem can be converted into (or "realized as") a T'-theorem by adding agent superscripts to terms and, conversely, every T'-theorem can be converted into a T-theorem by omitting all agent superscripts? The author conjectures that this "multi-agent realization" question has an affirmative answer, though investigation of this question is left for future work.

Beyond the operations of evidence elimination and introduction, there are a number of natural directions in which one might proceed, including the introduction of *preferences* and *preference change* over evidence, which would allow us to tie an agent's beliefs (or the relative strength among his or her various beliefs) to his or her preference ordering on evidence. This work would naturally dovetail and complement recent work in Dynamic Epistemic Logic on *Belief Revision* and *preference upgrade* [6, 23, 25]. It might also be interesting to compare the semantic topologic-based approach to evidence recently proposed by van Benthem and Pacuit [26] to the present Justification Logic-style syntactic notion of evidence. Indeed, there notions similar to evidence elimination (and introduction) are studied for a less syntactical point of view.

Finally, our work has natural connections with the leaner public announcement-based Justification Logic system of evidence introduction due to Bucheli, et al. [9]. The latter work is focused on finding the Justification Logic analog of Plaza's modal logic of public announcements. The key in this work is to take the agent's evidence t about the truth of a statement ψ after the hypothetical future announcement of φ as sufficient for the agent having t as evidence for ψ after φ is in fact announced. Thus the key evidence introduction schema (in multi-agent format) is

$$t :_i (\varphi \to [\varphi]\psi) \to [\varphi] t :_i \psi \ .$$

In a follow-up paper by Bucheli, Kuznets, and Studer [10], this approach is varied by having a certain piece of evidence $\Uparrow t$ that is derived from t as support for ψ after the announcement, leading to the evidence introduction schema (again in multi-agent format)

$$t :_i (\varphi \to [\varphi]\psi) \to [\varphi] \Uparrow t :_i \psi \ .$$

The latter is more in the spirit of (a static variation of) the generalized step-by-step inference-based reasoning studied by Velázquez-Quesada [29], which itself suggests a number of interesting new dynamic operations on evidence for future study.

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