Bisimulation and public announcements in logics of evidence-based knowledge

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Abstract. This paper introduces a notion of bisimulation for Artemov's logics of evidence-based knowledge. Bisimulation allows us to study the effect of dynamic epistemic operations on language expressivity. It is shown that public announcements, a basic dynamic epistemic operation, add expressivity to the language of evidenced-based knowledge. It is also shown that public announcements are definable in the language of evidence-based knowledge augmented with an evidence admissibility relation.

1 Introduction

Plato defined knowledge as *justified true belief.* Following the ideas in (Hintikka 1962), modal logics have been used as a formal means of modeling the informal notion of knowledge. If the modal is K and φ is a formula, then the formula $K\varphi$ is accordingly read, "φ is known." While theories in this language can make various knowledge assertions such as $K\varphi \supset K\psi$, the language has no means of expressing a reason as to why one assertion follows from another, contrary to the first component of Plato's three-part definition. $K\varphi$ is thus an assertion of *implicit* knowledge because φ is known for some unspecified reason.

Explicit modal logics extend the language of classical propositional logic by introducing formula-labeling terms t, allowing formation of the formula $t{:}\varphi$. In these systems, the structure of t in a theorem $t{:}\varphi$ corresponds to a particular derivation of this theorem, so modeling knowledge using explicit modal logics naturally incorporates a notion of justification. We may thus assign to $t{:}\varphi$ the reading "φ is known for reason t." These logics—called *justification logics*—may thus be viewed as logics of *evidence-based* knowlege.

So far, justification logics have only been studied in a static setting. In this paper, we study the effect on language expressivity of public announcements, a basic dynamic epistemic operation. Defining a notion of bisimulation appropriate for justification logics, we show that public announcements add expressivity to the language of evidence-based knowledge. We also show that the addition of an evidence admissibility relation to the basic language of evidence-based knowlege makes public announcements definable within the extended language.

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2 Justification logics

2.1 LP: Artemov's basic logic of evidence-based knowledge

LP, Artemov's Logic of Proofs (Artemov 2001), is the basic logic of evidence-based knowledge. The language of LP extends that of propositional logic by introducing a countable collection of *variables* $x_1, x_2, x_3, \dots$, a countable collection of *constants* $c_1, c_2, c_3, \dots$, the colon for forming assertions of evidence-based knowledge, the binary function symbols $+$ and $\cdot$, and the unary function symbol $!$. *Terms* are built up from variables and constants using the function symbols. The rules of LP formula formation are those of propositional logic in addition to the following: if t is a term and φ is an LP formula, then $t{:}\varphi$ is also an LP formula. The intended reading of $t{:}\varphi$ is "t is sufficient evidence for φ." The Hilbert-style theory of LP consists of the following axiom and rule schemas:

- *Classical propositional logic*

  C. A finite collection of axiom schemas for classical propositional logic

  RC. Modus ponens: infer ψ from $\varphi \supset \psi$ and φ

- *Evidence management*

  LP1. $t{:}(\varphi \supset \psi) \supset (s{:}\varphi \supset (t \cdot s){:}\psi)$

  LP2. $t{:}\varphi \supset\, !t{:}(t{:}\varphi)$

  LP3. $t{:}\varphi \vee s{:}\varphi \supset (t + s){:}\varphi$

  LP4. $t{:}\varphi \supset \varphi$

  RLP. Constant necessitation: infer $c{:}A$ from LP axiom A and constant c

For present purposes, we use Fitting's Kripke-style semantics of (Fitting 2005), which is based on Mkrtychev's minimal semantics of (Mkrtychev 1997). Specifically, let an S4 Kripke model (G, R_e, V) be given.[1] A function $\mathcal{E}$ that assigns to each world Γ and term t a set $\mathcal{E}(\Gamma, t)$ of LP formulas is called an *evidence function* if it satisfies each of the following properties:

- *Evidence Closure*
  - *Application.* If $\varphi \supset \psi \in \mathcal{E}(\Gamma, t)$ and $\varphi \in \mathcal{E}(\Gamma, s)$, then $\psi \in \mathcal{E}(\Gamma, t \cdot s)$.
  - *Proof Checker.* If $\varphi \in \mathcal{E}(\Gamma, t)$, then $t{:}\varphi \in \mathcal{E}(\Gamma, !t)$.
  - *Sum.* $\mathcal{E}(\Gamma, t) \cup \mathcal{E}(\Gamma, s) \subseteq \mathcal{E}(\Gamma, t + s)$.
  - *Constant Specification.* $A \in \mathcal{E}(\Gamma, c)$ for each LP axiom A and constant c.

[1] G is a nonempty set of elements that are referred to as *worlds*, R_e is a reflexive and transitive binary relation on G, and V assigns to each world Γ a set $V(\Gamma)$ of propositional letters that are the propositional letters taken to be true at Γ.

- *Evidence Monotonicity.* If $\varphi \in \mathcal{E}(\Gamma, t)$ and $\Gamma R_e \Delta$, then $\varphi \in \mathcal{E}(\Delta, t)$.

Informally, $\mathcal{E}(\Gamma, t)$ is understood as the set of formulas for which t is admissible as evidence at world Γ.[2] A *Fitting model* is then a tuple $M = (G, R_e, \mathcal{E}, V)$, where $\mathcal{E}$ is an evidence function. For a world Γ of a model M, we will write $M, \Gamma \models \varphi$ to mean that the formula φ is true at Γ in M. The negation will be written $M, \Gamma \not\models \varphi$. Truth at a world is defined by induction on the construction of φ, where the propositional cases are given as usual for Kripke models. For the LP case, $M, \Gamma \models t{:}\varphi$ holds exactly when we have that $\varphi \in \mathcal{E}(\Gamma, t)$ and that $M, \Delta \models \varphi$ whenever $\Gamma R_e \Delta$.

2.2 Adding implicit knowledge

To incorporate implicit knowledge in the language of evidence-based knowledge, we wish to extend the language of LP by introducing modals K_i for each $i = 1, 2, \ldots, n$. We call this extended language the *language of evidence-based knowledge* or, more briefly, the *EBK language*. Fitting models for the EBK language are obtained from the Fitting models defined above by adding a reflexive relation R_i corresponding to each modal K_i. Thus a general Fitting model is a tuple $M = (G, \{R_i\}_{i=1}^n, R_e, \mathcal{E}, V)$. In single-agent logics, where $n = 1$, the subscript on both the modal and the relation will be dropped.

There are a number of ways to connect evidence-based and implicit knowledge, though work on this has just begun. The connection studied thus far is given by the axiom schema $t{:}\varphi \supset K_i\varphi$ for $i = 1, 2, \ldots, n$ (Artemov 2004; Artemov and Nogina 2005). This connection schema may be read, "An agent knows those things that have a reason." The class of Fitting models satisfying this connection principle is the class of Fitting models that have $R_i \subseteq R_e$ for each R_i.

Restricting further the relation R_i, we may obtain models in which the modal K_i behaves as in any of the epistemic logics T, $\mathsf{S4}$, or $\mathsf{S5}$. A justification logic corresponding to these models may then be defined by stipulating the LP axiom and rule schemas, the connection schema, and the corresponding modal logic axiom and rule schemas for the modal K_i. We thus have a family of justification logics—each satisfying the connection schema—named according to how the modality K_i behaves.[3] For example, $\mathsf{S4}_n\mathsf{LP}$ is the system in which each of n agents has an $\mathsf{S4}$ modality. Similarly, we have $\mathsf{T}_n\mathsf{LP}$, $\mathsf{S5}_n\mathsf{LP}$, and various mixed logics in which agents' reasoning powers differ (for example, $\mathsf{S4S5LP}$ is the two-agent logic in which K_1 is an $\mathsf{S4}$ modal and K_2 is an $\mathsf{S5}$ modal).

[2]To say that t is *admissible (as evidence)* for φ means that t is possible evidence for φ. Possible evidence is not the same as actual evidence. If t is *possible* evidence for φ, then t may be taken into account when considering the truth of φ. However, it need not be the case that t is itself sufficient to guarantee the truth of φ, something we require of *actual* evidence. Thus the viewpoint of this paper is that (actual) evidence is a rather strong notion because it is *conclusive*. This is not to say that weaker notions of evidence are not of interest—it's just that weaker notions have yet to be addressed because LP originated from proof-theoretic considerations (and a proof is quite a strong notion of evidence).

[3]See (Artemov 2004; Artemov and Nogina 2005; Fitting 2004) for detailed studies of this wide-ranging family of logics.

3 Bisimulation for justification logics

We now define a notion of bisimulation for the EBK language. This allows us to study expressivity issues related to dynamic epistemic operations, something we take up in the next section of the paper.

Definition 3.1. Given the model $M = (G, \{R_i\}_{i=1}^n, R_e, \mathcal{E}, V)$, a world $\Gamma \in G$, and a formula φ in the EBK language, to say that φ is *knowable* at Γ means that $M, \Delta \models \varphi$ whenever $\Gamma R_e \Delta$.

Definition 3.2. Given models

$$M_1 = (G_1, \{R_i\}_{i=1}^n, R_e, \mathcal{E}_1, V_1) \text{ and } M_2 = (G_2, \{S_i\}_{i=1}^n, S_e, \mathcal{E}_2, V_2),$$

a nonempty binary relation $B \subseteq \mathcal{P}(G_1 \times G_2)$ is a *bisimulation* between M_1 and M_2 if each of the following conditions hold.

- The *frame bisimulation* conditions:

  For each relation R of M_1 and S of M_2 both sharing the same subscript:

  1. If $\Gamma_1 R \Delta_1$ and $\Gamma_1 B \Gamma_2$, then there is a $\Delta_2 \in G_2$ such that $\Gamma_2 S \Delta_2$ and $\Delta_1 B \Delta_2$;
  2. If $\Gamma_2 S \Delta_2$ and $\Gamma_1 B \Gamma_2$, then there is a $\Delta_1 \in G_1$ such that $\Gamma_1 R \Delta_1$ and $\Delta_1 B \Delta_2$.

- *Agreement of propositional valuation:* if $\Gamma_1 B \Gamma_2$, then $V_1(\Gamma_1) = V_2(\Gamma_2)$.

- *Agreement of evidence for knowable formulas:* if $\Gamma_1 B \Gamma_2$ and φ is knowable at Γ_1 or at Γ_2, then $\varphi \in \mathcal{E}_1(\Gamma_1, t)$ iff $\varphi \in \mathcal{E}_2(\Gamma_2, t)$ for each term t.

Two models are said to be *bisimilar* if there exists a bisimulation between them. World Γ of model M and world Δ of model N are said to be *bisimilar* if there is a bisimulation B between M and N satisfying $\Gamma B \Delta$. For such a Γ of M and a Δ of N, we write $(M, \Gamma) \simeq_B (N, \Delta)$, though the subscript B may be omitted when doing so ought not cause confusion.

Remark 3.3. By restricting to the case $n = 0$ in Definition 3.2, we obtain bisimulation for formulas in the the language of LP itself.

What's new about Definition 3.2 is the condition for evidence agreement on knowable formulas. While we could have defined bisimulation so that there is evidence agreement for *all* formulas, this turns out to be too strong of a requirement because it obscures the expressivity results we are able to obtain with this weaker notion of bisimulation. Regardless, our notion of bisimulation is correct, as the following proposition shows.

Proposition 3.4. Let M_1 and M_2 be as in Definition 3.2. If $(M_1, \Gamma_1) \simeq_B (M_2, \Gamma_2)$ and φ is any formula in the EBK language, then $M_1, \Gamma_1 \models \varphi$ iff $M_2, \Gamma_2 \models \varphi$.

Proof. By induction on the construction of φ. All cases are routine except the LP inductive case. We check this remaining case in detail, as follows. $M_1, \Gamma_1 \models t{:}\varphi$ means φ is knowable at Γ_1 and $\varphi \in \mathcal{E}_1(\Gamma_1, t)$. By the induction hypothesis, $\Gamma_1 B \Gamma_2$ implies φ is knowable at Γ_2 and, by the definition of bisimulation, we have $\varphi \in \mathcal{E}_2(\Gamma_2, t)$. Hence $M_2, \Gamma_2 \models t{:}\varphi$. Interchanging the models M_1 and M_2 in this argument gives the converse. □

We now give two examples of bisimilar models, both of which will be important for later results in the paper. The reader may wish to skip over the examples until they are later referenced.

Example 3.5. Let p be a propositional letter, x be a variable, $G = \{\Gamma, \Delta\}$, R_e be the smallest reflexive relation satisfying $\Gamma R_e \Delta$, $V(\Gamma) = \{p\}$, and $V(\Delta) = \emptyset$. We will define models

$$M_1 = (G, R_e, \mathcal{E}_1, V) \text{ and } M_2 = (G, R_e, \mathcal{E}_2, V)$$

so that M_1 and M_2 are bisimilar, $p \in \mathcal{E}_1(\Gamma, x)$, and $p \notin \mathcal{E}_2(\Gamma, x)$. We first specify $\mathcal{E}_1$ and $\mathcal{E}_2$ and then show that M_1 and M_2 are bisimilar.

Let $\mathcal{E}_1$ be the (unique) evidence function with the smallest graph that also satisfies $p \in \mathcal{E}_1(\Gamma, x)$. It then follows that $\varphi \notin \mathcal{E}_1(\Gamma, x)$ for any formula $\varphi \neq p$. We also have that $\mathcal{E}_1(\Delta, t) = \mathcal{E}_1(\Gamma, t)$ for all terms t.

For $w \in G$ and t any term, $\mathcal{E}_2$ is defined as follows:

$$\mathcal{E}_2(w, t) = \begin{cases} \mathcal{E}_1(w, t) & \text{if } t \neq x, \\ \emptyset & \text{if } t = x. \end{cases}$$

It is not difficult to show that $\mathcal{E}_2$ is also an evidence function; most of the evidence function properties follow immediately or else from the fact that $\mathcal{E}_1$ is itself an evidence function. It's also clear that $p \notin \mathcal{E}_2(\Gamma, x)$.

M_1 and M_2 are clearly frame bisimulations and agree on their propositional valuations. What remains is to show that they also satisfy the condition on the evidence functions $\mathcal{E}_1$ and $\mathcal{E}_2$. So suppose that φ is knowable at Γ in M_1. Certainly it cannot be the case that φ is p, for p is not knowable at Γ in M_1. Thus we have $\varphi \notin \mathcal{E}_1(\Gamma, x)$ because $\varphi \neq p$, and we also have $\varphi \notin \mathcal{E}_2(\Gamma, x)$ because $\mathcal{E}_2(\Gamma, x) = \emptyset$. So, in the case $t = x$, the evidence functions agree on the knowable formula φ. In case $t \neq x$, then we have $\mathcal{E}_1(\Gamma, t) = \mathcal{E}_2(\Gamma, t)$ by definition, so the evidence functions clearly agree on the knowable formula φ. The case where φ is knowable at Γ in M_2 is shown in the same way. A similar argument also applies at the world Δ. Hence M_1 and M_2 are bisimilar.

Example 3.6. Let $G = \{\Gamma\}$, $R_e = \{(\Gamma, \Gamma)\}$, and $V(\Gamma) = \emptyset$. Let x be a variable. Then there are models

$$M_1 = (G, R_e, \mathcal{E}_1, V) \text{ and } M_2 = (G, R_e, \mathcal{E}_2, V)$$

such that M_1 and M_2 are bisimilar, $\bot \in \mathcal{E}_1(\Gamma, x)$, and $\bot \notin \mathcal{E}_2(\Gamma, x)$. The construction of the evidence functions $\mathcal{E}_1$ and $\mathcal{E}_2$ is analogous to that given in Example 3.5, as is the verification of the bisimulation condition on the evidence functions.

4 Public announcements and expressivity

A public announcement of the formula φ operates on an epistemic model by deleting all those worlds in which φ does not hold (Plaza 1989). Public announcements appear as labeled modalities: $[\varphi]\psi$ means that ψ holds after the public announcement of φ. If L is an epistemic language, then the language L *with public announcements* is the extension of L obtained by adding brackets (for formation of public announcement formulas) and admitting an additional rule of formula formation for public announcement formulas: if φ and ψ are formulas, then so is $[\varphi]\psi$. We now define the truth of a public announcement formula at a world of a model.

Definition 4.1. If Γ is a world of the model $M = (G, \{R_i\}_{i=1}^n, R_e, \mathcal{E}, V)$, then $M, \Gamma \models [\varphi]\psi$ means that either $M, \Gamma \not\models \varphi$ or that $M|\varphi, \Gamma \models \psi$, where $M|\varphi$ is the submodel of M obtained by deleting all those worlds of M in which φ does not hold. That is,

$$M|\varphi := (G^\varphi, \{R_i^\varphi\}_{i=1}^n, R_e^\varphi, \mathcal{E}^\varphi, V^\varphi)$$

where

- $G^\varphi := \{\Gamma \in G \mid M, \Gamma \models \varphi\}$
- $R_i^\varphi := R_i \cap (G^\varphi \times G^\varphi)$ for $i = 1, 2, \ldots, n$
- $R_e^\varphi := R_e \cap (G^\varphi \times G^\varphi)$
- $\mathcal{E}^\varphi(\Delta, t) := \mathcal{E}(\Delta, t)$ for $\Delta \in G^\varphi$ and t a term
- $V^\varphi(\Gamma) := V(\Gamma)$ for $\Gamma \in G^\varphi$

This definition also works for the public announcement of φ in a multi-agent Kripke model M—a model that does not contain the evidence function $\mathcal{E}$—by omitting the mention of evidence functions.

Lemma 4.2 (Correctness)**.** Let φ be a formula in the EBK language with public announcements. If Γ is a world of a Fitting model M and $M, \Gamma \models \varphi$, then $M|\varphi$ is a Fitting model.

Proof. A straightforward verification. □

Suppose that T is an epistemic theory that is sound and complete with respect to a fixed class of Kripke models. To say that *public announcements are definable within* T means that for every formula φ in the language of T with public announcements, there is a formula ψ in the language of T without public announcements such that the biconditional

$\varphi \equiv \psi$ is valid. Plaza was the first to show that public announcements are definable within $\mathsf{S5}_n$ (Plaza 1989), and this result extends naturally to the epistemic logics T_n and $\mathsf{S4}_n$.[4]

The next theorem exhibits a formula in the language of LP with public announcements that is equivalent to no formula in the language of LP without public announcements. Therefore, public announcements are not definable within LP.

Theorem 4.3. The language of LP with public announcements is strictly more expressive than the language of LP.

Proof. In the models M_1 and M_2 of Example 3.5, we have that $(M_1, \Gamma_1) \simeq (M_2, \Gamma_2)$, from which it follows by Proposition 3.4 that no LP formula distinguishes Γ_1 and Γ_2. Since $[p]x\!:\!p$ holds at Γ_1 and not at Γ_2, we have that $[p]x\!:\!p$ is equivalent to no LP formula. □

Since our justification logics extend LP, this theorem extends naturally to justification logics in the EBK language. Therefore, public announcements are not definable in any justification logic. This is the statement of following corollary.

Corollary 4.4. The EBK language with public announcements is strictly more expressive than the EBK language (without public announcements).

Proof. Let M_1' be the trivial extension of the model M_1 of Theorem 4.3; that is, $R_i = \{(\Gamma, \Gamma)\}$ for $i = 1, 2, \ldots, n$. Define M_2' similarly. The same phenomenon then occurs with the formula $[p]x\!:\!p$. □

As Evan Goris observed, the formula $[p]x\!:\!p$ has an interesting interpretation that is summarized by the following proposition.

Proposition 4.5. Let Γ be a world of a model $M = (G, R_e, \mathcal{E}, V)$. Then

$$M, \Gamma \models [p]x\!:\!p \text{ iff } \Big(M, \Gamma \models p \text{ implies } p \in \mathcal{E}(\Gamma, x)\Big).$$

Proof. Suppose that both $M, \Gamma \models [p]x\!:\!p$ and $M, \Gamma \models p$. It then follows from the definition of truth of a public announcement formula at a world that $M|p, \Gamma \models x\!:\!p$. This implies $p \in \mathcal{E}^p(\Gamma, x)$ by the definition of truth of a formula of the form $x\!:\!\varphi$. But then $p \in \mathcal{E}(\Gamma, x)$ because $\mathcal{E}^p(\Gamma, x) = \mathcal{E}(\Gamma, x)$ by definition.

[4] The result breaks down in multi-agent logics with common knowledge (Baltag, Moss, and Solecki 1999; Baltag, Moss, and Solecki 2005), though Kooi and van Benthem show in (Kooi and van Benthem 2004) that public announcements are again definable if the language of epistemic logic with common knowledge is extended by introducing a notion of *relativized* common knowledge (see also their paper (van Benthem, van Eijck, and Kooi 2005) with van Eijck). Whereas the (unary) common knowledge modality C behaves as reachability—$M, \Gamma \models C\varphi$ iff φ holds in each world reachable from Γ via a path in the reflexive transitive closure of $\bigcup_{i=1}^n R_i$, written $(\bigcup_{i=1}^n R_i)^*$—relativized common knowledge is a binary modality C^r that behaves as restricted reachability—$C^r(\varphi, \psi)$ holds iff ψ holds in each world reachable from Γ via a path path in $(\bigcup_{i=1}^n R_i)^*$ whose worlds all satisfy φ. Then $C\varphi \equiv C^r(\top, \varphi)$ is valid, and it can be shown that public announcements are definable in the extended language containing C^r.

Conversely, suppose $M, \Gamma \models p$ implies $p \in \mathcal{E}(\Gamma, x)$. In the case $M, \Gamma \not\models p$, we have $M, \Gamma \models [p]x\!:\!p$ trivially, so assume $M, \Gamma \models p$ and thus that $p \in \mathcal{E}(\Gamma, x)$. To see that $M|p, \Gamma \models x\!:\!p$, it remains to be shown that p is knowable at Γ in $M|p$. But this follows immediately from the fact that p holds at every world of $M|p$ by definition. The result follows. □

So the language of LP with public announcements can describe evidence admissibility for some true formulas, though evidence admissibility is usually a strictly semantic notion. This perhaps provides the reader with some intuition as to why public announcements add expressivity to the EBK language.

5 Adding assertions of evidence admissibility

We have seen that adding public announcements to the EBK language adds expressivity, and hence public announcements are not definable within any of our justification logics. In the present section, we address this matter by defining the theory E^J, a conservative extension of LP in which public announcements are definable. We will then describe how nautral extensions of E^J do the job for arbitrary justification logics.

5.1 E^J: a basic logic with public announcement definability

It is now our task to provide a conservative extension of LP in which public announcements are definable. Proposition 4.5 suggests that it might be sufficient to expand the language to include explict assertions of evidence admissibility. This is the route we shall take.

If t is a term and φ is a formula, then we introduce the new formula $t \gg \varphi$ whose intended reading is "t is admissible (as evidence) for φ." $t \gg \varphi$ is true in world Γ of model $M = (G, R_e, \mathcal{E}, V)$ exactly when $\varphi \in \mathcal{E}(\Gamma, t)$.

For purposes of technical simplicity, we will also add an $\mathsf{S4}$ modal J to the language. This makes it straightforward both to express $t\!:\!\varphi$ in terms of evidence admissibility and also to capture Evidence Monotonicity.

All together, the language of E^J is obtained from that of LP by adding the symbols $\gg$ and J. The E^J rules of formula formation are those of LP in addition to the following: if t is a term and φ is an E^J formula, then both $t \gg \varphi$ and $J\varphi$ are also E^J formulas. The Hilbert-style theory of E^J consists of the following axiom and rule schemas:

- *Classical propositional logic*

  A. Finite number of axiom schemas for classical propositional logic

  RA. Modus ponens: infer ψ from φ and $\varphi \supset \psi$

- $\mathsf{S4}$ *knowledge for J*

  J1. $J(\varphi \supset \psi) \supset (J\varphi \supset J\psi)$

J2. $J\varphi \supset \varphi$

J3. $J\varphi \supset JJ\varphi$

RJ. J necessitation: infer $J\varphi$ from φ

- *Evidence admissibility*

E1. $(t \gg (\varphi \supset \psi)) \supset ((s \gg \varphi) \supset ((t \cdot s) \gg \psi))$

E2. $(t \gg \varphi) \supset (!t \gg t{:}\varphi)$

E3. $(t \gg \varphi) \vee (s \gg \varphi) \supset ((t + s) \gg \varphi)$

E4. $(t \gg \varphi) \supset J(t \gg \varphi)$

RE. Infer $c \gg A$ from axiom A and constant c

- *Connection principle*

C. $t{:}\varphi \equiv J\varphi \wedge (t \gg \varphi)$

Proposition 5.1. E^J is a conservative extension of LP and of $\mathsf{S4}$.

Proof. E^J is clearly sound for LP Fitting models. E^J is also complete for these models, as we show in a moment via a canonical model construction. This then gives the desired result. So we proceed with the construction. Note that by sets we mean sets of E^J formulas.

A set is *consistent* if for no finite subset is $\perp$ provable. Any consistent set may be extended to a maximal consistent set as usual. For convenience, if Γ is a set, let $\Gamma^{\#} := \{\varphi \mid J\varphi \in \Gamma\}$. Now define the *canonical model* $M = (G, R_e, \mathcal{E}, V)$ in the usual way: G is the set of all maximal consistent sets, we have $\Gamma R_e \Delta$ if and only if $\Gamma^{\#} \subseteq \Delta$, we set $\mathcal{E}(\Gamma, t) := \{\varphi \mid t \gg \varphi \in \Gamma\}$, and we set $V(\Gamma) := \{p \mid p \in \Gamma\}$.

To verify that M is a Fitting model, two items must be checked. That (G, R_e, V) is an $\mathsf{S4}$ Kripke model is straightforward. That $\mathcal{E}$ is an evidence function follows from the definitions of $\mathcal{E}$ and R_e in M and the axiom schemas E1, E2, E3, and E4.

We then verify a property of M called the *Truth Lemma*: $\varphi \in \Gamma$ if and only if $M, \Gamma \models \varphi$. This then immediately yields completeness: if φ is not provable, then $\{\neg\varphi\}$ is consistent and may be thus be extended to a world Γ of M. Applying the Truth Lemma, we have that $M, \Gamma \not\models \varphi$, as desired. So what remains is the proof of the Truth Lemma.

The proof is by induction on the construction of φ. Most cases are standard, following from the definition of M and the induction hypothesis. We will handle the case $t{:}\varphi$ and leave the rest for the reader.

If $t{:}\varphi \in \Gamma$, then $J\varphi \in \Gamma$ by **C**, and thus $\Gamma R_e \Delta$ implies $\varphi \in \Delta$ and hence $M, \Delta \models \varphi$ by the induction hypothesis. Since $t{:}\varphi \in \Gamma$ also implies $t \gg \varphi \in \Gamma$ by **C**, we have $\varphi \in \mathcal{E}(\Gamma, t)$. Thus $M, \Gamma \models t{:}\varphi$.

If $\neg t{:}\varphi \in \Gamma$, then $\neg J\varphi \in \Gamma$ or $\neg(t \gg \varphi) \in \Gamma$ by **C**. In the latter case, $\varphi \notin \mathcal{E}(\Gamma, t)$, so $M, \Gamma \not\models t{:}\varphi$, as desired. In the case $\neg J\varphi \in \Gamma$, we claim that $\Gamma^{\#} \cup \{\neg\varphi\}$ is consistent. Were it not, then for a finite $\Gamma_1 \subseteq \Gamma^{\#}$, we would have $\mathsf{E}^J \vdash \bigwedge\!\!\!\bigwedge \Gamma_1 \supset \varphi$, and thus $\mathsf{E}^J \vdash J(\bigwedge\!\!\!\bigwedge \Gamma_1) \supset J\varphi$. Since J is an $\mathsf{S4}$ modal and $\Gamma_1 \subseteq \Gamma^{\#}$, it follows that $J(\bigwedge\!\!\!\bigwedge \Gamma_1) \in \Gamma$, and thus $J\varphi \in \Gamma$, contradicting the consistency of Γ. Hence $\Gamma^{\#} \cup \{\neg\varphi\}$ is consistent and thus may be extended to a world Δ of M. We then have $M, \Delta \not\models \varphi$ by the induction hypothesis, and, since $\Gamma R_e \Delta$, we have shown $M, \Delta \not\models t{:}\varphi$, as desired. □

Proposition 5.2. Public announcements are definable within E^J.

Proof. A complete list of reduction schemas for E^J is as follows.

$$\begin{array}{lcl}
[\varphi]p & \equiv & \varphi \supset p \\
[\varphi](\psi \supset \chi) & \equiv & [\varphi]\psi \supset [\varphi]\chi \\
[\varphi]J\psi & \equiv & \varphi \supset J[\varphi]\psi \\
[\varphi][\psi]\chi & \equiv & [\varphi \wedge [\varphi]\psi]\chi \\
[\varphi]t{:}\psi & \equiv & \varphi \supset (J[\varphi]\psi \wedge (t \gg \psi)) \\
[\varphi](t \gg \psi) & \equiv & \varphi \supset (t \gg \psi)
\end{array}$$

Here p is an atom. Each schema is valid for LP Fitting models. □

We saw in Proposition 4.5 that the language of LP with public announcements can describe evidence admissibility for some true formulas. Since E^J can do so for all formulas, whether true or not, it seems as though E^J can say more. This is in fact the case, as the following theorem shows.

Theorem 5.3. The language of E^J is strictly more expressive than that of LP with public announcements.

Proof. We will prove the following fact: no formula φ in the language of LP with public announcements can distinguish the models M_1 and M_2 of Example 3.6. Since we have that $x \gg \bot$ holds at Γ in M_1 but not at Γ in M_2, it then follows that $x \gg \bot$ is equivalent to no formula in the language of LP with public announcements. So what remains is to prove the above-stated fact; we do this by induction on the construction of φ. The base and propositional inductive cases are straightforward, so we handle only the other two cases.

Suppose $M_1, \Gamma \models t{:}\varphi$. This implies $M_1, \Gamma \models \varphi$ and thus $M_2, \Gamma \models \varphi$ by the induction hypothesis. It follows that $\varphi \neq \bot$, and so $\varphi \in \mathcal{E}_1(\Gamma, t)$ if and only if $\varphi \in \mathcal{E}_2(\Gamma, t)$ by the construction of $\mathcal{E}_2$. But since we have that $\varphi \in \mathcal{E}_1(\Gamma, t)$ from our assumption $M_1, \Gamma \models t{:}\varphi$, it follows that $\varphi \in \mathcal{E}_2(\Gamma, t)$. We have then shown that $M_2, \Gamma \models t{:}\varphi$, as desired.

Suppose $M_1, \Gamma \not\models t{:}\varphi$. Then we have $M_1, \Gamma \not\models \varphi$ or $\varphi \notin \mathcal{E}_1(\Gamma, t)$. If $M_1, \Gamma \not\models \varphi$, then $M_2, \Gamma \not\models \varphi$ by the induction hypothesis and thus $M_2, \Gamma \not\models t{:}\varphi$, as desired. If $M_1, \Gamma \models \varphi$ and $\varphi \notin \mathcal{E}_1(\Gamma, t)$, then $\varphi \neq \bot$ and so $\varphi \notin \mathcal{E}_2(\Gamma, t)$ by the construction of $\mathcal{E}_2$. Thus $M_2, \Gamma \not\models t{:}\varphi$.

For the inductive case $[\varphi]\psi$, notice that $[\varphi]\psi \equiv \varphi \supset \psi$ is a valid scheme in any one-world model. This inductive case is thus handled by the propositional inductive case. □

5.2 Adding implicit knowledge

To extend LP so as to incorporate implicit knowledge, we added to the language of LP a T, $\mathsf{S4}$, or $\mathsf{S5}$ modal K and to the theory the schema $t{:}\varphi \supset K\varphi$. Similarly, we will now add implicit knowledge to E^J in order to extend arbitrary justification logics to ensure that public announcements are definable. We will show how to do this with a single modal K, which may be T, $\mathsf{S4}$, or $\mathsf{S5}$. This yields the theories TE^J (extending TLP), $\mathsf{S4E}^J$ (extending $\mathsf{S4LP}$), and $\mathsf{S5E}^J$ (extending $\mathsf{S4LP}$). Iterating the process to add additional modals obtains an appropriate extension of the corresponding justification logic. Proceeding, we address the case where we are adding an $\mathsf{S4}$ modal K. The cases where K is T or $\mathsf{S5}$ are handled analogously.

The language of $\mathsf{S4E}^J$ is obtained from that of E^J by adding the new symbol K. The rules of $\mathsf{S4E}^J$ formula formation are those of E^J in addition to the following: if φ is an $\mathsf{S4E}^J$ formula, then $K\varphi$ is also an $\mathsf{S4E}^J$ formula. The Hilbert-style theory of $\mathsf{S4E}^J$ consists of the following axiom and rule schemas:

- *Axiom and rule schemas for* E^J
- $\mathsf{S4}$ *knowledge* for K
- *J-K connection principle*

  C2. $J\varphi \supset K\varphi$

Proposition 5.4. $\mathsf{S4E}^J$ is a conservative extension of $\mathsf{S4LP}$.

Proof. $\mathsf{S4E}^J$ Fitting models are just $\mathsf{S4LP}$ Fitting models. These models have the form $M = (G, R, R_e, \mathcal{E}, V)$, where $R \subseteq R_e$. To interpret E^J formulas, R interprets K and R_e interprets both J and assertions of the form $t\!:\!\varphi$. Since $R \subseteq R_e$, we see that $\mathsf{S4E}^J$ is clearly sound for these models. That $\mathsf{S4E}^J$ is complete for these models follows by extending the E^J canonical model construction in the obvious way to incorporate the modal K and then verifying the additional inductive case for $K\varphi$ in the Truth Lemma. The latter verification makes use of the J-K connection principle (and, in particular, need not make use of the fact that K is $\mathsf{S4}$, and thus we could just as easily have chosen K to be T or $\mathsf{S5}$).

So $\mathsf{S4E}^J$ is sound and complete for $\mathsf{S4LP}$ Fitting models. Thus a $\mathsf{S4E}^J$ theorem φ in the language of $\mathsf{S4LP}$ is an $\mathsf{S4LP}$ validity and is thus also an $\mathsf{S4LP}$ theorem. $\square$

Proposition 5.5. Public announcements are definable within $\mathsf{S4E}^J$.

Proof. Add $[\varphi]K\psi \equiv \varphi \supset K[\varphi]\psi$ to the list of reduction schemas in Proposition 5.2 to obtain a complete list of reduction schemas for $\mathsf{S4E}^J$. These schemas are all valid for $\mathsf{S4E}^J$ Fitting models. $\square$

Theorem 5.6. The language of $\mathsf{S4E}^J$ is strictly more expressive than that of $\mathsf{S4LP}$ with public announcements.

Proof. As in the case for E^J (Theorem 5.3). The extra inductive case is trivial because $K\varphi \equiv \varphi$ in one-world models by the reflexivity of K. $\square$

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