What, and for what is
Higher geometric quantization*

Urs Schreiber†

July 13, 2014

Abstract

The process of full non-perturbative quantization is so profoundly fundamental for modern physics, and also for modern mathematics, as the mysteries involved are proverbial. Formulating a general theory of quantization that would lift Kostant-Souriau-Bott-Kirillov geometric quantization from mechanics to covariant local Lagrangian (higher-)gauge field theory ought to be one of the big open problems in mathematics. We list arguments that any such lift is a higher analog of geometric quantization, with "higher" in the sense of geometric homotopy theory referring to homotopy n-types for higher n. Here symplectic phase spaces/polarized varieties are refined to higher moduli stacks of fields equipped with pre-quantum ∞-line bundles constituted by sheaves of spectra. Quantization itself is given by indices in twisted generalized cohomology.

These are expository notes prepared for a pair of talks given at

Workshop on Higher Gauge Theory and Higher Quantization
Heriot-Watt University, Edinburgh – June 26-27 2014
www.christiansaemann.de/higherworkshop2014

as well as for a talk given at

Symmetries and correspondences in number theory, geometry, algebra, physics: intra-disciplinary trends
Oxford – June 26-27 2014
www.maths.nottingham.ac.uk/personal/ibf/files/sc3.html

as well as for a talk to be given at

Operator and geometric analysis on quantum theory,
Trento, September 15-19 2014
www.science.unitn.it/ moretti/convegno/convegno.html

*ncatlab.org/schreiber/show/What+and+what+for+is+Higher+geometric+quantization
†ncatlab.org/nlab/show/Urs+Schreiber

Contents

$$
\begin{tikzcd}
 & \begin{matrix}\mathbf{Fields}_{\mathrm{traj}} \\ 2.1.4\end{matrix} \arrow[dl, "(-)_{\mathrm{in}}"'] \arrow[dr, "(-)|_{\mathrm{out}}"] \arrow[dd, phantom, "\overset{\exp\left(\frac{i}{\hbar}S\right)}{\underset{2.1.1,2.1.4}{\Swarrow}}"] & \\
\begin{matrix}\mathbf{Fields}_{\mathrm{in}} \\ 2.1.3,2.2.2\end{matrix} \arrow[dr, "2.2.1{,}2.3.1"] \arrow[ddr, "\mathbf{L}_{\mathrm{in}}"'] & & \begin{matrix}\mathbf{Fields}_{\mathrm{out}} \\ 2.1.3,2.2.2\end{matrix} \arrow[dl, "2.2.1{,}2.3.1"'] \arrow[ddl, "\mathbf{L}_{\mathrm{out}}"] \\
 & \begin{matrix}\mathbf{Phases} \\ 2.1.2\end{matrix} \arrow[d, "\substack{\mathrm{superposition} \\ 2.3.2}"] & \\
 & \begin{matrix}\mathbf{B}\mathrm{GL}_1(E) \\ 2.3.2\end{matrix} &
\end{tikzcd}
$$

$$
E^{\bullet+\mathbf{L}_{\mathrm{in}}}(\mathbf{Fields}_{\mathrm{in}}) \xrightarrow[\int\limits_{\phi\in\mathbf{Fields}_{\mathrm{traj}}} \exp\left(\frac{i}{\hbar}S(\phi)\right)d\mu]{2.3.3} E^{\bullet+\mathbf{L}_{\mathrm{out}}}(\mathbf{Fields}_{\mathrm{out}})
$$

1 Preliminaries

Before entering discussion of higher geometric quantization in 2, here some lead-in words on the problem of quantization as such, on traditional geometric quantization, on the modern picture of quantum field theory, and finally on higher geometry.

1.1 Quantization, an open problem

Quantization is some process that takes differential geometric input to linear algebraic output. For reviews of mathematical aspects of fundamental quantum field theory and for further pointers see for instance [Deligne-Morgan 99, Sati-Schreiber 11, Moore 14].

Put bluntly, higher geometric quantization is meant to eventually complete the mathematical formalization of quantization as such. To appreciate this, it may serve to briefly recall the glaring open problem here, waiting to be solved.

That the process of quantization is central to modern fundamental physics is a tautology. But beyond toy examples, the majority of quantization procedures considered in the literature are in *perturbation theory*, both in in the coupling constants as well as in Planck's constant $\hbar$ (see remark 1.2 below a tad more on this). While the results of perturbation theory are impressive, perturbation theory by definition considers only the infinitesimal neighbourhood of the "classical" locus in a large and rich realm, and for a fundamental understanding of that realm this is hardly sufficient.

classical locus	infinitesimal neighbourhood	full quantum realm
$\hbar = 0$	formal power series in $\hbar$	analytic functions in $\hbar$
classical field theory	deformation quantization/ Feynman diagram loop expansion/ perturbation theory	geometric quantization/ non-perturbative quantum field theory

An instructive class of examples that illustrate the need for genuine non-perturbative quantization are Chern-Simons-type field theories in dimension $4k+3$ together with their holographically related self-dual higher gauge theories in dimension $4k+2$. The rich subtleties involved in the non-perturbative geometric quantization of these systems were understood and explained mainly in [Witten 96, Witten 99] and turned into mathematical theorems in [Hopkins-Singer 02, MS-P-S 11, Hopkins-Quick 12]. These quantum theories capture (we will keep returning to this, e.g. in 1.3 and 2.2.4 below) at least some sectors of the core archetypes of examples of quantum field theories of interest, such as notably (for $k = 1$) 4d Yang-Mills theory descending from the 6d theory ([Witten 04, Witten 09]), but also (for $k = 0$) the 2d sigma-model (the fundamental string) and, last not least for $k = 2$, the 10d target space type II string theory itself.

k	$(4k+3)$-dimensional Chern-Simons theory	form θ-characteristic of prequantum line bundles $\longrightarrow$	$(4k+2)$d self-dual higher gauge theory
0	3d Chern-Simons theory		chiral WZW fields in 2d
1	7d Chern-Simons theory		self-dual 2-form in 6d
2	11d Chern-Simons theory		RR-fields in 10d

Here the θ-characteristic step involves dividing the Chern class $c_1(L)$ of the prequantum line bundle L by 2, which is non-trivial. Since $c_1(L) \propto \hbar^{-1}$ this is also intrinsically non-perturbative. It goes along with maximizing Planck's constant by doubling it, an issue that is invisible in perturbation theory.

Moreover, the dominant theme of theoretical high energy physics in the 21st century so far (dominant in terms of numbers and citations of articles, for whatever that is worth) is the conjecture [Witten 98a] of "$\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$-duality" (reviewed e.g. in [Nastase 07]), saying that lifting this state of affairs from the Chern-Simons-type sectors to their supergravity completions (see [Witten 98b]) produces the full quantization, in particular of the $6d$ $(2,0)$-theory with its compactification to 4d Yang-Mills theory. Whichever precise mathematical form AdS/CFT duality will eventually take, it will crucially involve a mathematical formulation of non-perturbative quantization of field theory.

Remarkably, non-perturbative quantization is also known to be the theme underlying deep topics in modern mathematics. Examples include (not even to mention quantum groups, quantum cohomology etc.):

- geometric representation theory, where for instance Kirillov's orbit method constructs unitary irreps of compact Lie groups by quantizing Wilson loops (a perspective indicated in [Witten 89], more details are in [Fiorenza-Sati-Schreiber 13a]);
- knot theory, where the (Jones polynomial) knot invariants are identified with the Wilson line observables in 3d Chern-Simons theory [Witten 89];
- Gromov-Witten theory, where the String path integral quantization exists rigorously via integration against virtual fundamental classes of moduli stacks of algebraic curves;
- elliptic cohomology, where the Ochanine and the Witten elliptic genus and the string-orientation of tmf [Ando-Hopkins-Strickland 01, Ando-Hopkins-Rezk 10] are understood as the partition function of the quantum string (type II and heterotic, respectively) [Witten 87].
- string topology operations, which find their natural interpretation as a TQFT obtained by a homological pull-push path integral quantization [Cohen-Godin] and [Lurie 09] 4.2.16, 4.2.17;
- geometric Langlands duality – we come to this below in 1.3;
- also Donaldson theory, mirror symmetry, etc...

That various of these aspects of quantization are appreciated in mathematics is witnessed by various Fields medals: those awarded to Borcherds (vertex operator algebra), Kontsevich (deformation quantization and Gromov-Witten theory), Witten (super-QFT indices) and also Perelman (renormalization group flow of 2d sigma model in gravity-dilaton background).

It would therefore seem reasonable to expect that the problem

"What is quantization?"

were regarded as one of the outstanding open problems of modern mathematics; much like the widely accepted big problems "What are motives?" (now pretty much solved) or "What is absolute geometry over $\mathbb{F}_1$?" (still in the making). This question is all the more pressing as the *codomain* of quantization of local topological field theories has full mathematical incarnation since [Lurie 09] (this we come back to at the end in 2.3.4). But the sheer profoundness of quantization may at times make it hard to see the forest for the trees. Some aspects of the general question do receive due attention, for instance the sub-question

"What is quantization of 4d Yang-Mills theory
in the full non-perturbative sense exhibiting confinement and the mass gap?"

which was effectfully declared and is now widely appreciated as one of the "Millenium Problems" of mathematics [Jaffe-Witten-Douglas].

The full question is really part of one grand mathematical problem posed already in the previous millenium – namely Hilbert's 6th problem [Hilbert 1900][1].

While the idea of higher geometric quantization will not change the fact that this is a hard question, we argue now that if there is any general mathematical theory of quantization at all which will eventually provide systematic answers to non-perturbative questions, then higher geometric quantization is a central part of the picture.[2]

[1] Discussion in this vein, similar to the present note but with more of a perspective from the foundations of mathematics, we gave in [Schreiber 13b, Schreiber 13c]. The theme of Hilbert's 6th was also picked up again in [Moore 14]

[2] To clarify again: It has become almost commonplace that quant*ized* local field theory is incarnated in higher category theory, but consideration of quant*ization* in higher geometry is rare at the moment. A visionary precursor is [Freed 92].

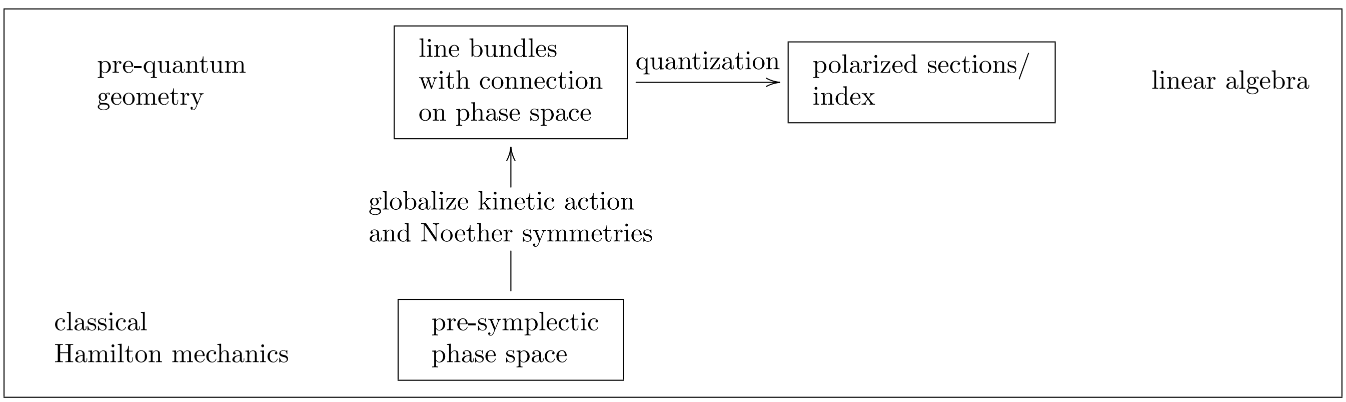
1.2 Traditional Geometric quantization

In the comparatively simple special case of 1-dimensional field theory, hence for mechanics (see e.g. [Arnold 89]), a fairly comprehensive mathematical formulation of quantization does exist: this is *geometric quantization*, going back to Kostant-Souriau, developed further by Bott and many others. A classical survey is [Bates-Weinstein 97], a review including more modern developments is in [Bongers 14, Nuiten 13].

Mathematically, Kostant-Souriau-Bott geometric quantization is essentially the process that is known

- in **differential geometry** as lifting a symplectic 2-form to a polarized line bundle with connection and forming the polarized sections;
- in **complex-analytic geometry** as lifting a Kähler form to a holomorphic line bundle and forming its holomorphic sections:
- in **algebraic geometry** as polarizing a variety and then forming its θ-characteristic;
- in **operator algebra** as choosing a Spin^c-structure and then forming the index of the Spin^c-Dirac operator;
- in **stable homotopy theory** as choosing a KU-orientation and forming the push-forward of the prequantum line bundle in KU-cohomology.

Or rather, geometric quantization is the G-equivariant version of these constructions, producing a space of quantum states on which the given group G of quantum observables acts (notably the Hamiltonian energy observable for $G = \mathbb{R}$). Applied to coadjoint orbits of a (compact) Lie group G then geometric quantization produces unitary irreducible representations of G and as such is famous as "Kirillov's orbit method" in geometric representation theory



In conclusion, geometric quantization is indeed intrinsically geometric and any variant or refinement of geometry may be expected to have an impact on geometric quantization.

Remark 1.1. The upward step from 2-forms to line bundles may be understood as giving global meaning to the kinetic action and to (Noether-)symmetries; we discuss this below in 2.1.1. This point is glossed over in much standard physics discussion when restricting attention to a local picture in perturbation theory.

Remark 1.2. Geometric quantization has a sibling known as *algebraic deformation quantization*, introduced as a concept in [BFFLS 78] with existence results due to Fedosov and Kontsevich, see [Cattaneo-Felder 00], which we come to below in 2.3.2. A good mathematical discussion of deformation quantization in the context of perturbative quantum field theory is in [Dütsch-Fredenhagen 00]. Deformation quantization tends to be more widely familiar than geometric quantization. But as the name indicates, deformation quantization concerns only the infinitesimal approximation to the full process of quantization, hence the perturbation theory in Planck's constant $\hbar$, and thus excludes many quantum phenomena. While this is most useful as far as it goes, and in fact the basis of most activity in field theory, here we will not further consider this.

1.3 The web of quantum field theories

One of the most subtle and fruitful applications of the technology of geometric quantization has been to the quantization of $(4k+3)$-dimensional field theories of Chern-Simons-type and their $(4k+2)$-dimensional self-dual higher gauge theory duals; which in turn are part of a web of interrelations of various quantum field theories that surround the class of Einstein-Yang-Mills-Dirac gravity-gauge-matter theories on which phenomenological fundamental physics is based [Witten 95, Witten 96, Witten 98a, Witten 98b, Witten 04, Witten 11]. One part of this web is supposed to look roughly as follows (see also example 2.45 below):

$$
\begin{tikzcd}
\text{11d supergravity} \arrow[d, "\text{KK-compactification}"'] & \\
\text{7d supergravity} \arrow[d, "\mathrm{AdS}_7/\mathrm{CFT}_6\text{ holography}"] & \text{7d Chern-Simons theory} \arrow[l, hook'] \arrow[d, "\substack{\text{Chern-Simons/WZW-type}\\ \text{holography}}"'] \\
\text{6d (2,0) theory on M5 brane} \arrow[d, "\substack{\text{KK-compactification/}\\ \text{AGT correspondence}}"'] & \text{self-dual 2-form theory} \arrow[l, hook'] \\
\text{4d super Yang-Mills theory} \arrow[d, "\text{KK-compactification}"'] & \\
\text{Kapustin-Witten theory} &
\end{tikzcd}
$$

To indicate just some aspects of this web that will be recurring below:

- the very bottom layer – the most reduced version of the full phenomenon – has been suggested by Witten to correspond to the (geometric) Langlands program (see e.g. [Frenkel 09]).

  [Kapustin-Witten 06, Witten 09, Kapustin 10] say: The geometric Langlands correspondence is about higher (co-)dimension quantum operators ("Wilson operators", "'t Hooft operators") acting on higher quantum states (D-brane states) via correspondences of moduli stacks of gauge fields.

  but [Langlands 14] cautions: What this really means and whether it is true remains mathematically unclear.

  Such a state of affairs highlights the urgency of the open problem in 1.1: We need to formulate a general mathematical theory of quantization of higher codimension field theory in which arguments such as in [Kapustin-Witten 06] would become actual mathematical theorems.

  Langlands duality suggests an additional desideratum:
  The theory should be *inter-geometric*. (e.g. Is there a p-adic version?) This we come to below in 2.3.1.

- The second layer involves a more symmetric version of Yang-Mills theory for which many of the central questions in the quantization of Yang-Mills theory have found answers [Seiberg-Witten 94].

- The right part of the third and fourth layer has been largely turned into theorems [Hopkins-Singer 02] using differential generalized cohomology (we come to this below in 2.3.1).

In any case, most parts of the web of quantum field theories remain an open mathematical problem, the name of which seems be "What is non-perturbative string theory?" (see [Moore 14] for a recent reminder). This has been already so since the 1970s and more pronouncedly so in the 1990s, when however the feeling was that some general conceptual insight was missing, as in the saying due to [Amati 7x] that "string theory is part of 21st-century physics that fell by chance into the 20th century".

But the 21st century has arrived meanwhile, and it has brought with it a new foundations of mathematics in higher geometry. In view of this the above web of interrelations deserves a new look. This we turn to now in 1.4.

1.4 Higher geometry

In the 1980s the grand figure of 20th century mathematics, Alexander Grothendieck, circulated a text [Grothendieck 83] that called for the pursuit of a theory of *stacks*. Starting with [Brown 73] and via work including [Jardine 87], Simpson, [Toën-Vezzosi 02], and Rezk, culminating in the monograph [Lurie 06], the theory of stacks was successfully pursued, and is now known as *higher topos theory*. As all topos theory it is Janus-faced. From one perspective, it is about *higher geometry* (also: "derived geometry"), and this is what we are concerned with here. (From the other perspective it is about *higher logic*, but this we will not consider here.)

Higher geometry may be understood as the pairing of the traditional theories of geometry and homotopy theory.

$$\boxed{\text{higher geometry} = \text{geometry} + \text{homotopy theory}}$$

Homotopy theory is the theory of symmetry and of symmetries-of-symmetries and deals with what technically are called *homotopy n-types*, or *n-groupoids* (in the most general sense), where a 0-groupoid is just a set, a pointed connected 1-groupoid is a group of symmetries, and where, inductively, an $(n+1)$-groupoid is like an n-groupoid with one further stage of symmetries-of-symmetries allowed. The term "higher" in "higher topos theory" and in "higher geometry" refers to this parameter n, also known as the *Postnikov stage*, being allowed to be greater than 0.

Hence where traditional geometry deals with geometric spaces that consist of a set of points, hence with geometric 0-groupoids, so higher geometry deals with geometric n-groupoids. Examples include topological groupoids and Lie groupoids (in particular orbifolds), which are the 1-groupoid generalization of topological groups and Lie groups.

In general, given any kind of basic geometric spaces (such as topological manifolds, smooth manifolds, supermanifolds, complex-analytic manifolds, varieties, schemes, dg-manifolds, dg-schemes etc.) then an n-groupoid which is equipped with the kind of geometry consistently probeable by these kinds of spaces is called an *n-stack* "on the site" of these test spaces. Here 0-stacks are just sheaves and chain complexes of sheaves (as in a derived category) are special abelian stacks.

$$\boxed{\text{abelian stacks} = \text{objects of derived categories}}$$

Stacks that are not just probeable by test geometries but are suitably locally equivalent to test geometries are called *geometric stacks*. In gauge physics these geometric stacks are familar in their *infinitesimal* (Lie theoretic) approximation, where they are incarnated as BV-BRST complexes [Henneaux-Teitelboim 92]: the nth-order ghost-of-ghost fields of the BRST complex are the cotangents to the space of n-fold symmetries in a higher stack, the BRST diffferential is a linearized approximation to the simplicial identities which hold the cells of a higher stack together. From this perspective higher geometry may be thought of as all about globalizing (Lie integrating) BV-BRST problems, taking all global consistency constraints (anomaly cancellation) properly into account.

$$\boxed{\text{geometric stacks} = \text{globalized/Lie integrated BV-BRST complexes}}$$

We sketch the definition and meaning of stacks a tad more below in 2.1.3 and 2.2.1, where we highlight how the concept of stack is the natural formalization what in physics are the gauge principle and the locality principle.

$$\boxed{\text{stack principle} = \text{gauge principle} + \text{locality principle}}$$

More exposition of and introduction to higher geometry from the point of view of relevance here is in [Fiorenza-Sati-Schreiber 13a, Schreiber 13c] and in section 1 of [Schreiber 13a].

2 Aspects of higher geometric quantum theory

We go now through a list of topics in quantum field theory that each, as we will indicate, calls for a higher geometric analogue of one aspect or other of traditional Konstant-Souriau-Bott-Kirillov geometric quantization (as recalled in 1.2 above).

2.1

- 2.1.1 – Kinetic action functionals;
- 2.1.2 – Local covariant field theory;
- 2.1.3 – Field bundles of gauge fields;
- 2.1.4 – Prequantized Lagrangian correspondences

2.2

- 2.2.1 – Local Chern-Simons Lagrangians;
- 2.2.2 – Higher gauge fields and higher-order ghosts;
- 2.2.3 – Kaluza-Klein reduction and Transgression;
- 2.2.4 – Polarization and Self-dual higher gauge fields;
- 2.2.5 – Boundary conditions and Brane intersection laws;

2.3

- 2.3.1 – Higher background fields and Differential cohomology;
- 2.3.2 – Cohomological quantization and Brane charges;
- 2.3.3 – The path integral and Secondary integral transform;
- 2.3.4 – Quantization of local topological defect field theory.

The topics are ordered by homotopy-theoretic degree. Part I. is meant for readers with no background in any homotopy theory or category theory and in fact is supposed to gently introduce some relevant concepts. Part II also tries to be self-contained but a background in homological algebra and elements of simplicial homotopy theory would help. Part III invokes deeper concepts of stable homotopy theory which we try to give a rough idea of, but details of which are beyond the scope of a brief survey, the reader in need of more detail is referred to pointers given in [Schreiber 13a, Schreiber 14].

2.1 One

We consider here some basic aspects of mechanics eventually motivating and producing a formulation in a context of stacks of groupoids.

2.1.1 Kinetic action functionals

The role of prequantum line bundles in traditional geometric quantization carries in it the seed of all of the considerations that we will be concerned with here. Therefore to get started it serves to first highlight what mechanism it is that makes prequantum line bundles appear. The key point is that prequantum line bundles serve to *globalize* structure, passing beyond formal neighbourhoods.

1. **Globalizing kinetic action functionals.**

   Given a phase space X with symplectic form $\omega \in \Omega^2_{\mathrm{cl}}(X)$, by the Poincaré lemma there is a good cover $\{U_i \hookrightarrow X\}_i$ and smooth 1-forms $\theta_i \in \Omega^1(U_i)$ such that $\mathbf{d}\theta_i = \omega_{|U_i}$. Physically such a 1-form is (up to a factor of 2) a choice of *kinetic energy density* called a *kinetic Lagrangian* L_{kin}:

   $$\theta_i = 2L_{\mathrm{kin},i}\,.$$

   By Darboux's theorem each θ_i has the form

   $$\theta_i = -p_a \wedge \mathbf{d}q^a$$

   for Darboux coordinates $\{q^a, p_q\}$ on $U_i \simeq \mathbb{R}^{2n}$, which is the familiar expression for the phase space kinetic Lagrangian.

   Each θ_i induces a exponentiated kinetic action for trajectories that stay within U_i:

   $$\exp(\tfrac{i}{\hbar}S_{\mathrm{kin}})_i : [S^1, U_i] \longrightarrow U(1)$$

   by

   $$\gamma \mapsto \exp(\tfrac{i}{\hbar}\int_{S^1}\gamma^*\theta_i)\,.$$

   However, in order for these functionals to globalize to produce a well-defined global action functional

   $$\exp(\tfrac{i}{\hbar}S_{\mathrm{kin}}) : [S^1, X] \longrightarrow U(1)$$

   more data is needed. By a classical argument (which is essentially that of Dirac's charge quantization argument) one finds that the extra data needed is precisely that of smooth functions

   $$g_{ij} : U_i \cap U_j \longrightarrow \mathbb{R}$$

   which serve as gauge transformations between the local kinetic terms

   $$\theta_j - \theta_i = \mathbf{d}g_{ij} \quad \text{on } U_i \cap U_j\,.$$

   and which satisfy the consistency condition

   $$g_{ij} + g_{jk} = g_{ik} \quad \text{on } U_i \cap U_j \cap U_K\,.$$

   The data of a cover $\{U_i \to X\}$ and forms $\{\theta_i, g_{ij}\}$ satisfying the above conditions is called a *complex line bundle with conection* on X. Its curvature is the globally defined differential 2-form ω given by $\omega|_{U_i} = \theta_i$. Conversely one says that the line bundle lifts the curvature, and if the curvature is a (pre-symplectic)form that it is a *prequantum line bundle* for ω.

2. **Globalizing Noether symmetries.** The pre-symplectic forms ω that drop out of variational calculus of local Lagrangians defined on jet bundles of fields bundles on the "covariant phase space" Y are always globally exact

$$\omega = \mathbf{d}\theta$$

, Therefore superficially the above discussion might seem to be superfluous. However, the pre-symplectic forms obtained this way in general have symmetries (vector in their kernel) and the actual phase space is the *reduced phase space* $X = Y//G$, where G is the group of symmetries.

Therefore to proceed with quantization one needs to "descend" the trivial line bundle on which *theta* is a connection along the quotient map

$$\begin{tikzcd} Y \arrow[d] & . \\ Y//G & = X \end{tikzcd}$$

This may be thought of as analogous to the descent along the open cover that we considered above

$$\begin{tikzcd} \sqcup_i U_i \arrow[d] \\ X \end{tikzcd}$$

Accordingly, the data and conditions needed for that to work is again of the above kind, namely

- for each $g \in G$ an equivalence

$$\eta_g : \theta \xrightarrow{\simeq} \rho(g)^*\theta$$

between θ and the pullback of θ along the action of g, hence a smooth function $\eta_g \in C^\infty(X, \mathbb{R}/\Gamma)$ with

$$\rho(g)^*\theta - \theta = \mathbf{d}\eta_g$$

such that

(a) the assignment $g \mapsto \eta_g$ is smooth;

(b) for all pairs $(g_1, g_2) \in G \times G$ there is an equality

$$\eta_{g_2}\eta_{g_1} = \eta_{g_2 g_1} .$$

Equipped with this equivariance data that globally defined pre-symplecitic potential θ on Y will be equivalent to a generically non-trivial prequantum line bundle on $X = Y//G$.

From this kind of discussion it is fairly immediate to say what an *n-line bundle with n-connection* on some X is, namely:

- a choice of cover $\{U_i \to X\}$;
- a collection of n-forms $\theta_i \in \Omega^n(U_i)$;
- a collection of $(n-1)$-forms $B_{ij} \in \Omega^{n-1}(U_i \cap U_j)$ such that on double overlaps they relate $\theta_j - \theta_i = \mathbf{d}g_{ij}$
- and so on.

Such line n-bundles with n-connection globalize n-form potentials such as to induce a well-defined global action functional on n-dimensional trajectories. This is what we turn to now in 2.1.2.

2.1.2 Local covariant field theory

A traditional approach to considering d-dimensional field theory is to regard it as encoding time/paramater evolution between $(d-1)$-dimensional spatial hypersurfaces. Quantization in this perspective is essentially what goes by the name *canonical quantization* (where "canonical" is meant with the connotation of "standard"). As much as this perspective is traditional, its problems are notorious:

- **wild phase spaces** – Where for 1-dimensional field theory/mechanics one typically has finite dimensional symplectic/polarized phase spaces (being spaces of fields on the point), for $(d > 1)$-dimensional field theory one finds at best infinite-dimensional versions of such spaces (spaces of fields on the $(d-1)$-dimensional hyperslices) on which many desired operations familiar from finite dimensional geometry are not well defined. Regularization and renormalization techniques serve to deal with this problem in perturbation theory, but not for the non-perturbative theory.

- **non-covariance under diffeomorphisms** – The choice of spatial hyperslices breaks the natural invariance of the field theory.

Both of these problems may be understood as aspects of one single problem of canonical quantization:

- **non-locality** – Canonical quantization breaks the *locality* of the field theory by evaluating it globally on $(d-1)$-dimensional hypersurfaces Σ and disregarding the fact that the assignments of the theory to Σ should arise by integrating up local data.

At the classical level the solution to this problem is essentially as old as the problem itself, even if not as widely known.

Fact 2.1. *For every local Lagrangian field theory of dimension n with fields that are sections of a field bundle, then the pre-symplectic form on the critical locus is the transgression of a canonical $(n+1)$-form on the jet bundle of the field bundle, the "symplectic current" ω.*

Part of this statement is the old De Donder-Weil formulation of variational calculus (reviewed e.g. in section 2 of [Hélein 02]). Another part is the theory of covariant phase spaces that was being (re-)discovered by [Zuckerman 87] and others. The relation is discussed around remark 3.4 and example 3.5 of [Schreiber 13c].

n-dimensional local field theory	$\xrightarrow{\text{transgression}}$	canonical time evolution
pre-symplectic current ω_{n+1}		pre-symplectic form $\omega_2(\phi) = \int_\Sigma \phi^* \omega_{n+1}$
Hamilton-De Donder-Weyl equation $(\iota_{v_n} \cdots (\iota_{v_1}))\omega_{n+1} = \mathbf{d}\mathcal{H}$		Hamilton equation $\iota_v \omega_2 = \mathbf{d}H$

Higher pre-quantization This clearly suggests that the symplectic current should be regarded as a higher degree analog of the symplectic form and that a higher degree analog of the classical theory of symplectic geometry is called for. The development of this idea proceeded in three stages:

1. **Multisymplectic geometry.** One traditional proposal for such a theory is known as *mutlisymplectic geometry* (see e.g. [Román-Roy 09] for a review). Authors working in this context notice that the evident generalization of the definition of the Poisson bracket associated with an $(n+1)$-form does not in general satisfy the Jacobi identity, hence does not define a Lie algebra. To nevertheless force a Lie algebra structure on it some authors propose to quotient out the (fairly large) subspace of observables in which the failure of the Jacobi identity takes values.

2. n-**Plectic geometry.** However, a "coherent" failure of the Jacobi identity is precisely the hallmark of higher (homotopy theoretic) Lie theory. In [Rogers 10] it was observed that the failure of the

multisymplectic Jacobi identity is indeed coherent in a canonical way, hence that there is naturally not a Lie algebra, but a *Lie n-algebra* (n-truncated L_∞-algebra)

$$\mathfrak{pois}(X,\omega) \in L_\infty\mathrm{Alg}$$

associated with a closed $(n+1)$-form. To distinguish this homotopy theoretic perspective from the classical multisymplectic geometry proposal the term *n-plectic geometry* was coined, see [Rogers 11].

3. **Prequantum n-bundles**

   The relation between pre-symplectic forms and pre-quantum line bundles in traditional geometric quantization has the a generalization to higher degree closed forms and higher line bundles incarnated as cocycles in ordinary differential cohomology. For readers with a background in sheaf cohomology we review this below in 2.1.2. Here for the moment we just state the impact for pre-quantization of local field theory.

   In [Fiorenza-Rogers-Schreiber 13] it was shown that this n-plectic Poisson Lie n-algebra of a manifold with closed $(n+1)$-form as proposed in [Rogers 10] is indeed equivalent to that of the infinitesimal symmetries of any prequantum line n-bundle, in higher analogy of what is famously the case in traditional geometric quantization (for $n = 1$). Schematically the situation is hence the following.

$$\begin{tikzcd}
\boxed{\begin{array}{l}\text{prequantum}\\ n\text{-line bundle}\\ \text{with } n\text{-connection}\end{array}} \arrow[r, "\text{symmetries}"] \arrow[d, "\text{curvature}"'] & \boxed{\begin{array}{l}\text{quantomorphism}\\ n\text{-group}\end{array}} \arrow[d, "\text{Lie differentiation}"] \\
\boxed{\begin{array}{l}\text{closed}\\ (n+1)\text{-form}\end{array}} \arrow[r, "\text{induces}"] & \boxed{\text{Poisson Lie } n\text{-algebra}}
\end{tikzcd}$$

Fact 2.2. *In analogy to how a Hamilonian and Hamilton's equations on a symplectic manifold (X,ω_2) are equivalently a Lie algebra homomorphism $\mathbb{R} \longrightarrow \mathfrak{pois}(X,\omega_2)$ so the Hamilton-De Donder-Weyl equation on an n-plectic manifold (X,ω_{n+1}) are equivalently given by an L_∞-algebra homomorphism $\mathbb{R}^n \longrightarrow \mathfrak{pois}(X,\omega_{n+1})$.* (This is prop. 3.28 in [Schreiber 13c].)

In conclusion, the schematic picture of 1.2 now becomes this:

$$\begin{tikzcd}
\begin{array}{l}\text{pre-quantum}\\ n\text{-geometry}\end{array} & \boxed{\begin{array}{l}n\text{-line bundles}\\ \text{with } n\text{-connection}\\ \text{on field jet bundle}\end{array}} \arrow[r, "\text{quantization}"] & \boxed{???} \\
\begin{array}{l}\text{classical}\\ \text{Hamilton-De Donder-Weyl}\\ \text{mechanics}\end{array} & \boxed{\begin{array}{l}\text{pre-}n\text{-plectic}\\ \text{field jet bundle}\end{array}} \arrow[u, "\text{globalize Noether symmetries}"] &
\end{tikzcd}$$

While the n-plectic picture thus provides a sensible higher analog of prequantum geometry including higher Poisson brackets and higher Hamiltonian flows, it does not yet answer what the actual higher quantization step should be (the higher analog of passing to spaces of polarized sections). This we turn to below in 2.3.2.

Ordinary differential cohomology For readers with background in traditional sheaf cohomology we end this section here by recalling some basics of the definition of the equivalence classes of line n-bundles with n-connection referred to above, in terms of sheaf cohomology with coefficients in the Deligne complex. Readers without such background might first go to section 2.2.1 which offers a little more background.

For X a manifold and $A \in \mathrm{Ab}$ an abelian group, then the ordinary cohomology groups $H^n(X, A)$ are invariants of the underlying homotopy type of X, in particular they are *homotopy invariant* in that the canonical maps

$$H^n(X, A) \xrightarrow{\simeq} H^n(X \times \mathbb{R}, A)$$

are equivalences. In order to bring the actual geometry of manifolds into the picture, consider

Definition 2.3. Let S denote the *site* (Grothendieck topology) of

$$S = \left\{ \begin{array}{ll} & \{\text{smooth manifolds}\} \\ \text{or} & \{\text{complex analytic manifolds}\} \\ \text{or} & \{\text{super-manifolds}\} \\ \text{or} & \{\text{formal manifolds }\} \\ \text{or} & \{\text{formal super-manifolds}\} \\ & \\ \text{or} & \begin{array}{l} \text{any locally étale-contractible site with terminal object } * \\ \text{such that } \mathrm{Hom}(*, -) \text{ preserves split hypercovers} \end{array} \end{array} \right.$$

Then for $\mathbf{A} \in \mathrm{Ab}(\mathrm{Sh}(S))$ a sheaf of abelian groups, there are the *abelian sheaf cohomology groups* $H^n(X, \mathbf{A})$ which need not be homotopy invariant. Only when $\mathbf{A} := \mathrm{LConst}(A)$ is locally constant then its sheaf cohomology reproduces ordinary cohomology:

$$H^n(X, \mathrm{LConst} A) \simeq H^n(X, A)\,.$$

More generally for $A_\bullet \in \mathrm{Ch}_\bullet(\mathrm{Sh}(S))$ a chain complex of abelian sheaves, then there is the abelian sheaf hypercohomology

$$H^n(X, A_\bullet) := H^0(X, \mathbf{B}^n A_\bullet) \simeq \mathbb{R}\mathrm{Hom}(\mathbb{Z}[X], A_\bullet[-n])\,.$$

Example. Write

$$\mathbb{G}_a = \left\{ \begin{array}{ll} (\underline{\mathbb{R}}, +) & \text{for } S = \{\text{smooth manifolds}\} \\ (\underline{\mathbb{C}}, +) & \text{for } S = \{\text{complex analytic manifolds}\} \end{array} \right. .$$

for the sheaf of (smooth, or holomorphic, etc.) functions and write

$$\flat \mathbb{G}_a := \mathrm{LConst}(\mathbb{G}_a(*))$$

for the sheaf of locally constant functions. Write

$$\Omega^\bullet \in \mathrm{Ch}^{-\bullet}(\mathrm{Sh}(S)) = \mathrm{Ch}_\bullet(\mathrm{Sh}(S))$$

for the (smooth, or holomorphic, etc.) de Rham complex. The Poincaré lemma says that

$$H^n(X, \Omega^\bullet) \simeq H^n(X, \flat \mathbb{G}_a)$$

In fact the local quasi-isomorphism

$$\flat \mathbb{G}_a \xrightarrow{\simeq} \Omega^\bullet$$

exhibits (just) a resolution, but this particular resolution serves to naturally induce the Hodge filtration:

$$\begin{tikzcd} {} \arrow[r, dotted] & F^{p+1}\Omega^\bullet \arrow[r] \arrow[dr, dashed] & F^p\Omega^\bullet \arrow[r] \arrow[d, dashed] & F^{p-1}\Omega^\bullet \arrow[r, dotted] \arrow[dl, dashed] & {} \\ & & \Omega^\bullet & & \end{tikzcd}$$

given by the degree-filtration of the de Rham complex:

$$F^p \Omega^\bullet := \Omega^{\bullet \geq p}\,.$$

Using this we find "genuinely geometric" cohomology groups of differential forms, such as

$$H^n(X, \Omega^{\bullet \geq n}) \simeq \Omega^n_{\mathrm{cl}}(X)\,.$$

Specifically in the complex-analytic case the above filtration reproduces the traditional Hodge filtration

$$H^k(X, \Omega^{\bullet \geq p}) \simeq \underset{k-q \geq p}{\oplus} H^{k-q,q}(X)$$

and thus the group of Hodge cocycles is given by the following fiber product of sheaf hypercohomology groups:

$$\mathrm{Hdg}^p(X) := H^{p,p}(X)_{\mathrm{integral}} \simeq H^{2p}(X, \mathbb{Z}) \underset{H^{2p}(X, \flat \mathbb{G}_a)}{\times} H^{2p}(X, \Omega^{\bullet \geq p})\,.$$

(see e.g. [Esnault-Viehweg 88] for review of this and some of the following)

Definition 2.4. Pulling back the above Hodge filtration along the exponential sequence

$$\mathbb{Z} \overset{\mathrm{ch}_{\mathbb{Z}}}{\hookrightarrow} \mathbb{G}_a \xrightarrow{\exp\left(\frac{i}{\hbar}(-)\right)} \mathbb{G}_m$$

we obtain a tower of homotopy pullbacks in $\mathrm{Ch}_\bullet(\mathrm{Sh}(S))$:

$$\begin{tikzcd}
(\mathbf{B}^n \mathbb{G}_m)_{\mathrm{conn}} \arrow[r] \arrow[d, dashed] & \mathbf{B}^{n+1}\Omega^{\bullet \geq n+1} \arrow[d, dashed] \\
F^3(\mathbf{B}^n \mathbb{G}_m)_{\mathrm{conn}} \arrow[r] \arrow[d] & \mathbf{B}^{n+1}\Omega^{\bullet \geq 3} \arrow[d] \\
F^2(\mathbf{B}^n \mathbb{G}_m)_{\mathrm{conn}} \arrow[r] \arrow[d] & \mathbf{B}^{n+1}\Omega^{\bullet \geq 2} \arrow[d] \\
\mathbf{B}^n \mathbb{G}_m \arrow[r, "\theta"] \arrow[d] & \mathbf{B}^{n+1}\Omega^{\bullet \geq 1} \arrow[d] \\
\mathbf{B}^{n+1}\mathbb{Z} \arrow[r, "\mathrm{ch}"] & \mathbf{B}^{n+1}\Omega^{\bullet} \simeq \mathbf{B}^{n+1}\flat\mathbb{G}_a
\end{tikzcd}$$

Proposition 2.5. *These homotopy fiber products are given by the Deligne complexes:*

$$F^p(\mathbf{B}^n \mathbb{G}_m)_{\mathrm{conn}} \simeq \left(\mathbb{Z} \hookrightarrow \Omega^0 \xrightarrow{d_{\mathrm{dR}}} \Omega^1 \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^{p-1} \longrightarrow 0 \longrightarrow \cdots \right)$$

(with $\mathbb{Z}$ in degree $n+1$).

[Fiorenza-Schreiber-Stasheff 10]

Examples.

1. $\mathbf{B}\mathbb{G}_m$ modulates line bundles;
2. $(\mathbf{B}\mathbb{G}_m)_{\mathrm{conn}}$ modulates line bundles with connection;
3. $(\mathbf{B}^2\mathbb{G}_m)_{\mathrm{conn}}$ modulates bundle gerbes with connection and curving;
4. $F^2(\mathbf{B}^2\mathbb{G}_m)_{\mathrm{conn}}$ modulates bundle gerbes with connection but without curving,
   the symmetries of these are given by Courant algebroids;
5. generally $(\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}}$ modulates line n-bundles with n-connection;
   the symmetries of these are higher Kostant-Souriau quantomorphism group extensions;
6. $H^\bullet(-, (\mathbf{B}^\bullet\mathbb{G}_m)_{\mathrm{conn}})$ is called *ordinary differential cohomology*;
   the deformation theory of ordinary differential cohomology (before analytification, in positive characteristic) is given by Artin-Mazur formal groups;
7. on a complex manifold X the fiber
   $$J^p(X) \longrightarrow H^0(X, F^p(\mathbf{B}^{2p}\mathbb{G}_m)_{\mathrm{conn}}) \longrightarrow \mathrm{Hdg}^p(X)$$
   is (the abelian group underlying) the pth *higher Jacobian* ("intermediate Jacobian") of X.

[Fiorenza-Rogers-Schreiber 13]

2.1.3 Field bundles of gauge fields

We discuss here how the "field bundle" for a non-perturbative gauge field does not actually exist in ordinary geometry but exists in the higher geometry of stacks, and we use this perspective to motivate and introduce the concept of stacks in the first place.

Traditionally, texts in mathematical physics state that a local Lagrangian field theory on a spacetime Σ is defined by a fiber bundle $V \to \Sigma$ – called the *field bundle*, as above in 2.1.2 – together with a horizontal differential d-form on the jet bundle. Then the fields of the theory are the sections of the field bundle. For instance in the simplest case of a complex scalar field, then the field bundle is simply the trivial bundle $\mathbb{C} \times \Sigma \to \Sigma$; or a spinor field is a section of a spinor bundle $S \to \Sigma$.

However, for what is arguably the most important example of a physical field, namely a gauge field, the concept of field bundle does not in fact work beyond perturbation theory. We discuss now why that is and then discuss how the way to fix this is precisely to pass from traditional differential geometry to "1-higher" differential geometry, namely to stacks. Here the field bundle for non-perturbative gauge fields does exist as a stacky bundle or 2-bundle.

For G a Lie group, the gauge group, then a G-gauge field on Σ is equivalently a G-principal connection ∇ on Σ. Underlying a G-principal connection is a G-principal bundle. In terms of physics the class of this bundle is known as the *magnetic monopole charge* (for $G = U(1)$) or *instanton number* (for $G = SU(n)$ in $d = 4$).

Example 2.6 (Dirac monopoles and instantons)**.** Consider space 3-space with a point removed, – $\mathbb{R}^3 - \{0\}$ – which for the purpose of topological effects we may think of as just the 2-sphere $\Sigma = S^2$. To describe this locally we may cover by two coordinate patches $D_\pm \simeq \mathbb{R}^2$, one being the sphere without the north pole

$$D_+ := S^2 - \{(0,0,1)\}$$

and the other one, $D-$, being an ϵ-neighbourhood of that northpole.

An electromagnetic field on D_+ is given by a vector potential $A_\pm \in \Omega^1(D_\pm)$. We might think of this as being restrictions of a section of the cotangent bundle of S^2. However, then the gluing condition would be the equality $A|_{D_+ \cap D_-} = A|_{D_+ \cap D_-}$ which would say that the gauge field is given by a globally defined differential form $A \in \Omega^1(S^2)$. In that case the Faraday tensor is globally excat, $F = \mathbf{d}A$, and hence by Stokes' theorem the magnetic flux of the electromagnetic field through the spehere would necessarily vanish:

$$\text{no local gauge transformations} \quad \Rightarrow \quad \Phi_{\mathrm{mag}} := \int_{S^2} F = 0$$

.

This is indeed one possible electromagnetic field configuration, but clearly not the most general one. Instead, Dirac's famous charge chantization argument in modern language says precisely that when comparing A_+ with A_- on $D_+ \cap D_-$ then we need to allow for a possibly non-trivial gauge transformation given by a function $g : D_+ \cap D_- \to U(1)$ and such that this is a gauge transformation

$$A_+ \xrightarrow[\simeq]{g} A_- \ \text{ on } D_+ \cap D_-$$

hence such that $A_+ - A_- = g^{-1}\mathbf{d}g$ on $D_+ \cap D-$.

(Here $U_0 \cap U_1$ is an ϵ-thickening of the circle and such gauge transformations are effectively equivalent to functions $S^1 \to U(1)$ which are classified by their winding number. Mathematically this is the first Chern-class of the $U(1)$-principal bundle underlying the gauge field, physically this is the magnetic charge hidden at the point 0 which had been removed from spacetime.)

$$\text{with local gauge transformations} \quad \Rightarrow \quad \Phi_{\mathrm{mag}} := \int_{S^2} F \in \mathbb{Z}\,.$$

If in this example one replaces the gauge group $U(1)$ with SU(2) and replaces the 2-sphere with the 4-sphere (the one-point compactification of Minkowski spacetime) then the verbatim discussion yields SU(2) instanton fields, where now the integer class (that used to be magnetic charge in the previous case) is now the *instanton number* (see e.g. [Schaefer-Shuryak 96]).

Remark 2.7 ("only gauge equivalence classes are relevant")**.** This standard example proves wrong the naive version of the statement that "in physics only gauge equivalence classes of fields are relevant". The gauge equivalence class of the local vector potential $A_\pm$ is its image $[A_\pm]$ in the quotient space $\Omega^1(D_\pm)/\mathrm{im}(d_{\mathrm{dR}})$. Requiring these gauge equivalence classes to coincide on $D_+ \cap D_-$ would make the integral quantization of magnetic charge and instanton number disappear, in contradiction to reality. On the other hand, once we consider the full global field configuration, *then* dividing out gauge equivalences makes sense, and indeed the charge/instanton number is precisely the invariant of these global gauge equivalence classes.

From this example it is clear that there is no ordinary kind of field bundle such that its sections would be general gauge fields. Only if we are given a *fixed* principal bundle $P \to \Sigma$, hence a fixed monopole/instanton sector, then a principal connection on it is equivalently a section of its Atiyah bundle $TP/G \to T\Sigma$. Hence in perturbation theory about a given charge/instanton sector then gauge fields are sections of a field bundle. But in full non-perturbative quantum gauge theory where the charge/instanton sector is part of the field content this is not possible.

field	field bundle
complex scalar	trivial complex line bundle
spinor	spinor bundle
gauge field in fixed instanton sector	Atiyah bundle
	higher field bundle
monopole/instanton	trivial G-gerbe/ trivial $\mathbf{B}G$-fiber 2-bundle
non-perturbative gauge field	trivial $\mathbf{B}G_{\mathrm{conn}}$-fiber 2-bundle

Remark 2.8. One might speculate that for phenomenology fixing a single instanton sector is sufficient, that maybe"the universe" indeed sits in one charge/instanton sector and its full quantum description may be obtained in perturbing abput that topological sector. However, that is not the case: the physical vacuum of QCD in the standard model of particle physics is a quantum superposition of all possible instanton sectors [Schaefer-Shuryak 96]. The relevant phases in this superposition are governed by the θ-angle of QCD, an observable quantity. This means that the non-perturbative phenomena of gauge fields are not a negligible subtlety, but control the very nature of the vacuum that we inhabit.

We now finally come to the concept of *stacks* as the direct formalization of these kinds of phenomena. The mathematical concept of *stack* may be thought of as precisely the concept that appears when combining the *gauge principle* with the *locality principle* of field theory

$$\boxed{\text{gauge principle} + \text{locality principle} = \text{theory of stacks}}$$

We indicate the central idea, for an exposition along these lines see [Fiorenza-Sati-Schreiber 13a]).

To start with, notice that by the above it makes sense to call the space $\mathbb{C}$ of complex numbers the *universal moduli space* of complex scalar fields: because given a spacetime Σ then the collection of scalar fields on Σ is *represented* by $\mathbb{C}$ in that there is a natural equivalence

$$\{\text{complex scalar fields on } \Sigma\} \simeq \{\text{maps } \Sigma \longrightarrow \mathbb{C}\}\,.$$

This being *natural* means that both sides of the equivalence canonically pull back along smooth maps $\Sigma' \longrightarrow \Sigma$ and that they do so compatibly.

Trivial as this may be, notice that the analogous statement for gauge fields has no chance to work with a moduli *space*. Because the gauge principle says that G-gauge fields on Σ form not a set, but a groupoid, and a manifold such as $\mathbb{C}$ does not represent a groupoid.

Stacks. The concept of smooth pre-stack is simply the minimal mathematical structure that makes gauge fields be representable in the above form. This may be thought of as a straightforwardly operational definition, closely akin to what the physics demands.

The smooth pre-stack $\mathbf{B}G_{\mathrm{conn}}$ is *defined* to be something such that for each smooth manifold Σ there is a *groupoid* whose objects are smooth maps $\Sigma \to \mathbf{B}G_{\mathrm{conn}}$ and whose morphisms are smooth homotopies between such maps, and such this groupoid is naturally equivalent to that of G-gauge fields in arbitrary monopole/instanton sectors (G-principal connections) on Σ and gauge transformations between them:

$$\left\{ \begin{array}{l} \text{gauge fields on } \Sigma \\ \text{with gauge transformations between them} \end{array} \right\} \simeq \left\{ \begin{array}{ll} \text{maps and} \\ \text{homotopies} \end{array} \begin{tikzcd} \Sigma \arrow[r, bend left, "\nabla_1"{name=U}] \arrow[r, bend right, "\nabla_2"'{name=D}] & \mathbf{B}G_{\mathrm{conn}} \arrow[from=U, to=D, phantom, "\Downarrow\simeq"] \end{tikzcd} \right\}$$

This equivalence being a *natural* equivalence means that it respects pullback (e.g. restriction) of gauge fields along smooth maps $\Sigma_1 \to \Sigma_2$ of (parts of) spacetime manifolds.

Techncially this means that a pre-stack of this form is a functor

$$(\mathbf{B}G_{\mathrm{conn}}) : \mathrm{SmthMfd}^{\mathrm{op}} \longrightarrow \mathrm{Grpd}$$

hence an assignment to each smooth manifold Σ of a groupoid $\mathbf{B}G_{\mathrm{conn}}(\Sigma)$ of gauge fields on Σ and to each smooth function $f : \Sigma_1 \to \Sigma_2$ of a pullback (restriction) map $f^* : \mathbf{B}G_{\mathrm{conn}}(\Sigma_2) \to \mathbf{B}G_{\mathrm{conn}}(\Sigma_1)$ which respects composition of functions and identity functions.

Notice that also every ordinary smooth manifold X gives rise to a prestack this way, by the assignment $\Sigma \mapsto C^\infty(\Sigma, X)$, where on the right we regard the set of smooth functions as a groupoid with only identity morphisms. This way of looking at a smooth manifold as a stack is very much in the spirit of modern physics: it says that we determine what a space (spacetime) X is like by considering all possible ways that stuff may propagate through it.

This construction gives a map

$$\begin{tikzcd} \mathrm{SmthMfd} \arrow[r, hook] & \mathrm{SmthPreStack} \\ X \arrow[r, mapsto] & C^\infty(-, X) \end{tikzcd}$$

from the collection (the category) of all smooth manifold to that of pre-stacks. The idea of 1-higher geometry is now simply to pass along this inclusion from the world of smooth manifolds into the more general, more flexible, and more physical world of pre-stacks.

That this is a consistent step to do is the statement of the *Yoneda lemma*. The Yoneda lemma says first of all that this inclusion is *fully faithful*. This is meant in the technical sense of category theory, but it means verbatim what it says: the inclusion faithfully respects all the properties of the collection of smooth manifolds. Hence passing from smooth manifolds to pre-stacks is indeed a genuine generalization, a passage to a larger world where none of the original structure is lost, but only new structzure (namely gauge transformations) is added.

But the Yoneda lemma says more: it also says that if we regard a smooth manifold X as a pre-stack as above and then consider maps of pre-stacks from that into $\mathbf{B}G_{\mathrm{conn}}$, then the groupoid of such maps is equivalently $\mathbf{B}G_{\mathrm{conn}}(X)$. Hence: if we regard ordinary spaces as generalized spaces then they still have the same maps between them and their maps into genuinely generalized spaces (pre-stacks) are precisely what these were meant to be.

Fact 2.9 (Yoneda lemma)**.** *For $\mathbf{A}$ a stack on smooth manifolds and for Σ a smooth manifold regarded as a stack, then the groupoid $\mathbf{A}(\Sigma)$ assigned by $\mathbf{A}$ to Σ is indeed naturally equivalent to the groupoid of maps (of stacks) from Σ into $\mathbf{A}$:*

$$\mathbf{A}(\Sigma) \simeq \left\{ \begin{tikzcd} \Sigma \arrow[r, bend left, "\nabla_1"{name=U}] \arrow[r, bend right, "\nabla_2"'{name=D}] & \mathbf{A} \arrow[from=U, to=D, phantom, "\Downarrow\simeq"] \end{tikzcd} \right\}$$

$\boxed{\text{Yoneda lemma: regarding pre-stacks as generalized smooth manifolds is consistent.}}$

On the other hand, while consistent, not all pre-stacks are of interest. As motivated above, we want to focus on those pre-stacks of fields which satisfy the locality principle in that the groupoid of fields that they assign to any manifold X is equivalent to the collection of local fields which are glued by gauge transformations on intersections. Mathematically, this locality principle is called the *descent property*. A pre-stack that satisfies this locality/descent property is called a *stack*. While standard, this is not a very suggestive terminology, and the reader is encouraged to stick to thinking "stack = space of gauge fields that satisfy the locality princple" and realize that beneath the mathematical terminology, the concept of stacks is most basic and natural to gauge theory.

principle	physical meaning	mathematical incarnation
gauge principle	identify field configurations by gauge equivalences	smooth pre-stack has groupoid of maps into it from any manifold
locality principle	global field configurations are equivalent to local field configurations on local coordinate charts identified (by gauge transformations!) on intersections	stack is pre-stack whose global assignments are equivalent to local assignments identified on intersections

In the existing physics literature discussion of the stack of gauge fields appears for instance in section 6.1 of [Witten 08].

To sum up, the above inclusion of smooth manifolds into prestacks factors through genuine stacks as follows

$$\begin{array}{ccccc} \mathrm{SmthMfd} & \hookrightarrow & \mathrm{SmthStack} & \underset{\text{stackification}}{\overset{\longleftarrow}{\hookrightarrow}} & \mathrm{SmthPreStack}\,. \\ \boxed{\text{geometry}} & & \boxed{\text{locality}} & & \boxed{\text{gauge}} \end{array}$$

Here we have indicated that there is a construction, called *stackification*, that completes any pre-stack to a stack. Conceptually the way this works is most obvious: given a pre-stack (of gauge fields, possibly not local) the global fields that the corresponding stackification assigns to a manifold are those which are by definition those that locally of the given kind and glued together by gauge transformation on intersections. This construction is most useful for computations, because it allows to induce genuine stacks that are physically relevant, by pre-stacks which may be easier to write to paper.

Example 2.10. Every Lie groupoid $[\ \mathcal{G}_1 \rightrightarrows \mathcal{G}_0\ ]$ defines a prestack by

$$X \mapsto [\ C^\infty(X, \mathcal{G}_1) \rightrightarrows C^\infty(X, \mathcal{G}_0)\ ]\,.$$

The stacks which arise from stackificartion of Lie groupoids are precisely the *geometric stacks* in the smooth context, usually called *differentiable stacks*.

For instance the stackification of $[\ G \rightrightarrows *\ ]$ is $\mathbf{B}G$.

Example 2.11. Every orbifold is naturally regarded as a stack. In the spirit of thinking of all stacks as "moduli spaces of local gauge fields" one is to think of as orbifold as being the moduli space of fields of a gauged σ-model: the maps into the ordifold from some worldvolume Σ are the fields, the homotopies between them are the gauge transformations that induce what are known as the *twisted sectors* of the orbifold σ-model.

For the purpose of gauge theory it is important that smooth stacks also include examples which are in a sense far from the cases of example 2.10.

Example 2.12. Write Ω^n_{cl} for the pre-stack that assigns to a smooth manifold X the set $\Omega^n(X)$ of smooth differential forms on X, regarded as a groupoid with only identity gauge transformations. (For $n = 2$ it is useful to think of this as the stack of *Faraday tensors*, i.e. electromagnetic field strenghts, which are indeed invariant under gauge transformation.) This pre-stack is in fact a stack.

Example 2.13. Similarly, for G a Lie group, then the assignment $X \mapsto \Omega^1(X, \mathfrak{g})$ of the set of Lie algebra valued differential forms is a pre-stack which is a stack. Consider the prestack

$$\Omega^1(-, \mathfrak{g})//G : X \mapsto \Omega^1(X, \mathfrak{g})//C^\infty(X, G)$$

which sends each manifold X to the groupoid of $\mathfrak{g}$-valued differential 1-forms on X with gauge transformations (by G-valued smooth functions on X) between them. This is the pre-stack of G-gauge fields in trivial instanton sectors. Its stackification is the stack $\mathbf{B}G_{\mathrm{conn}}$ of all gauge field configurations discussed above.

Finally, to come back to the problem of missing field bundles for non-perturbative gauge fields in differential geometry: by the above we see that once we pass from differential geometry to 1-higher differential geometry where smooth manifolds are generalized to stacks, then then non-perturbative field bundle for G-gauge theory on some Σ is simpy

$$\begin{array}{c} \Sigma \times \mathbf{B}G_{\mathrm{conn}} \;\; . \\ \downarrow \\ \Sigma \end{array}$$

If we forget the connection itself and just consider the underlying bundle, which in terms of physics means to consider just the underlying *instanton sector*, then the corresponding field 2-bundle is

$$\begin{array}{c} \Sigma \times \mathbf{B}G \\ \downarrow \\ \Sigma\,. \end{array}$$

This is an example of a (Giraud, non-abelian) *G-gerbe*. In fact as far as G-gerbes go this is the *trivial* G-gerbe.

Remark 2.14. While smooth stacks, simple and pertinent to gauge theory as they are, have not yet made it into the standard physics monographs, it is notworthy that their infinitesimal approximation is well known: the infinitesimal version (Lie differentiation) of the moduli stack of G gauge fields on Σ is the action Lie algebroid whose Chevalley-Eilenberg algebra is nothing but the (off-shell) **BRST complex** as known from [Henneaux-Teitelboim 92].

2.1.4 Prequantized Lagrangian correspondences

Modern developments in quantum field theory briefly touched on in 1.3 indicate a deep role of transfer through correspondences in the formulation of field theory and its quantization. Here we survey [Schreiber 13c] how a version of this is secretly at the heart of classical Hamilton-Jacobi-Lagrange mechanics.

The suggestion that (geometric) quantization is fundamentally about transfer through correspondences goes back to [Weinstein 71]. There it was suggested that the geometric quantization of symplectic phase spaces should extend to transfer through *Lagrangian correspondences* between two symplectic phase spaces (X_i, ω_i):

$$\begin{tikzcd} & Y \arrow[d, hook, "\iota"] & \\ & X_1 \times X_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & & X_2 \end{tikzcd} \qquad (p_1^*\omega_1 - p_2^*\omega_2)|_Y = 0\,.$$

Such correspondences subsume plain symplectomorphisms, but also the more general transformations that physics textbooks know as *canonical transformations* [Weinstein 83].

Example 2.15. A function $f : X_1 \to X_2$ between symplectic manifolds (X_i, ω_i) is a symplectomorphism, in that $f^*\omega_2 = \omega_1$, precisely if the graph $\mathrm{graph}(f) \stackrel{(p_1,p_2)}{\longrightarrow} X_1 \to X_2$ of f consitutes a Lagrangian correspondence.

Example 2.16. A Lagrangian correspondence out of the point is equivalently a Lagrangian submanifold.

The idea – sometimes known as *Weinstein's dictionary* – is that where a symplectic phase space induces under geometric quantization a Hilbert space of quantum states, so a Lagrangian correspondence should induce a linear map between these Hilbert spaces, hence a *quantum operator*.

Weinstein's dictionary

classical mechanics	symplectic manifold	Lagrangian correspondence
quantum mechanics	Hilbert space	quantum operator

From 1.3 we know that more recent developments suggest that a higher geometric theory of quantum operators is called for, hence a higher geometric version of such a dictionary.

To this end, we observe that Lagrangian corresondence have a natural and suggestive reformulation in sheaf theory. As before, we regard the category SmthMfd of smooth manifolds with smooth functions between them as a site with its usual Grothendieck topology of open covers. By the Yoneda embedding

$$\mathrm{SmthMfd} \hookrightarrow \mathrm{Sh}(\mathrm{SmthMfd})$$

smooth manifolds themselves are faithfully embedded into the category of sheaves on smooth manifolds, and we may regard the latter hence as a generalization of smooth manifolds, which it makes sense to think of as "generalized smooth spaces".

Example 2.17. A sheaf $X \in \mathrm{Sh}(\mathrm{SmthMfd})$ which is 'supported on points" in that there is a set $X_s \in \mathrm{Set}$ such that the value of X on a given manifold $U \in \mathrm{SmthMfd}$ is naturally a subset of the set of functions of sets $U_s \to X_s$, is equivalently a *diffeological space*. This contains in paticular infinite-dimensional Fréchet manifolds, such as mapping spaces of smooth manifolds out of a compact manifold, which are crucial for applications in physics.

But Sh(SmthMfd) contains also objects which are not supported on points at all, and these are very useful to include in a category of smooth spaces (for convenience we repeat example 2.12 here):

Example 2.18. For each $n \in \mathbb{N}$ with $n \geq 1$, then there is the familiar sheaf of smooth differential forms

$$\Omega^n \in \mathrm{Sh}(\mathrm{SmthMfd})\,.$$

This only has a single underlying point, and yet is a “large”smooth space. In fact the Yoneda lemma says that it is the classifying smooth space for differential n-forms, in that morphisms of sheaves $X \to \Omega^n$ are naturally equivalent to differential forms $\omega \in \Omega^n(X)$. Moreover, under this equivalence the pullback of differential forms along a smooth function corresponds simply to the composition of maps:

$$
\begin{tikzcd}
X_2 \arrow[rr, "f"] \arrow[dr, "f^*\omega_1"'] & & X_1 \arrow[dl, "\omega_1"] \\
& \Omega^n &
\end{tikzcd}
$$

The analogous statements hold for the sub-sheaf $\Omega^n_{\mathrm{cl}} \hookrightarrow \Omega^n$ of closed differential forms.

Using this one finds the following reformulation of correspondences as above[3]

Proposition 2.19. *Correspondences of pre-symplectic manifolds* (X_i, ω_2) *of the above form are equivalently correspondences of smooth spaces* over Ω^2_{cl}*:*

$$
\left\{
\begin{tikzcd}
& Y \arrow[d, "\iota"] & \\
& X_1 \times X_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X_1 & & X_2
\end{tikzcd}
\quad (p_1^*\omega_1 - p_2^*\omega_2)|_Y = 0
\right\}
\simeq
\left\{
\begin{tikzcd}
& Y \arrow[dl, "p_1\iota"'] \arrow[dr, "p_2\iota"] & \\
X_1 \arrow[dr, "\omega_1"'] \arrow[rr, phantom, "="] & & X_2 \arrow[dl, "\omega_2"] \\
& \Omega^2_{\mathrm{cl}} &
\end{tikzcd}
\right\}
$$

Several authors have made proposals to make geometric quantization a functor on some kind of category of Lagrangian correspondence, but it is maybe fair to say that the results remain somewhat inconclusive as far as the general idea of quantization is concerned. On the other hand, from 1.2 we notice that it is not so much symplectic phase spaces themselves that are quantized, but pre-quantized spaces. This means that one ought to pass to some kind of pre-quantized Lagrangian correspondences first.

To see what these should be, observe that pre-quantization has the following nice formulation in terms of stacks.

Definition 2.20. Write

$$
\mathbf{H} := L_{\mathrm{le}}\mathrm{Grpd}(\mathrm{Sh}(\mathrm{SmthMfd})) := \left(\begin{array}{l} \text{the homotopy theory obtained} \\ \text{from sheaves of groupoids} \\ \text{by universally turning stalkwise equivalences} \\ \text{into homotopy equivalences} \end{array} \right)
$$

This is the homotopy theory of smooth stacks (as we introduced in more elementary terms above in 2.1.3).

[3]For the definition of correspondences between phase spaces to make sense as a bare definition one does not need that the ω_i are symplectic, they may be just pre-symplectic, and one does not need that S is really Lagrangian in $(X_1 \times X_2, p_1^*\omega_1 - p_2^*\omega_2)$, it is sufficient that it be isotropic. We will consider “isotropic correspondences” in this more general sense here. The symplectic/Lagrangian condition may always be imposed if desired.

Example 2.21. Write $\mathbf{B}U(1) \in \mathbf{H}$ for the sheaf of groupoids which assigns to any manifold Σ the groupoid with a single object and with the group of smooth $U(1)$-valued function on Σ as automorphisms of that object. Then in $\mathbf{H}$ we have

$$\left\{ \begin{tikzcd} \Sigma \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathbf{B}U(1) \end{tikzcd} \right\} \simeq \left\{ \begin{array}{l} U(1)\text{-principal bundles on } \Sigma \\ \text{and isomorphisms between these} \end{array} \right\}$$

In particular the automorphisms of the trivial bundle 0 form the group of $U(1)$-valued functions

$$\left\{ \begin{tikzcd} \Sigma \arrow[r, bend left, "0"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{B}U(1) \end{tikzcd} \right\} \simeq C^\infty(\Sigma, U(1)) \,.$$

Example 2.22. We write

$$\mathbf{B}U(1)_{\mathrm{conn}} := \Omega^1 // \underline{U}(1) \in \mathbf{H}$$

for the stack presented by the quotient of the sheaf of 1-forms by the sheaf of $U(1)$-valued functions, acting by the evident gauge transformations.

$$\left\{ \begin{tikzcd} \Sigma \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathbf{B}U(1)_{\mathrm{conn}} \end{tikzcd} \right\} \simeq \left\{ \begin{array}{l} U(1)\text{-principal bundles on } \Sigma \\ \text{with } U(1)\text{-principal connections} \\ \text{and isomorphisms between these} \end{array} \right\}$$

The map that forgets the differential forms gives a morphism

$$\mathbf{B}U(1)_{\mathrm{conn}} \longrightarrow \mathbf{B}U(1) \quad \in \mathbf{H} \,.$$

On the other hand, the de Rham differential $\mathbf{d} : \Omega^1 \to \Omega^2_{\mathrm{cl}}$ descents to this quotient and provides a universal curvature map

$$F_{(-)} : \mathbf{B}U(1)_{\mathrm{conn}} \longrightarrow \Omega^2_{\mathrm{cl}} \quad \in \mathbf{H}$$

One finds that a pre-quantization of a pre-symplectic manifold (X, ω), hence a choice of $U(1)$-principal bundle whose curvature 2-form in ω, is equivalently a lift through this map of stacks:

Fact 2.23.

$$\{\textit{pre-quantizations of } (X,\omega)\} \;\simeq\; \left\{ \begin{tikzcd} & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[d, "F_{(-)}"] \\ X \arrow[ur, dashed, "\nabla"] \arrow[r, "\omega"'] & \Omega^2_{\mathrm{cl}} \end{tikzcd} \right\} .$$

In view of all this it is compelling to set [Schreiber 13c]:

Definition 2.24. A *prequantized Lagrangian correspondence* lifting a given Lagrangian correspondence as above as a diagram in $\mathbf{H}$ of the form

$$
\begin{tikzcd}
 & Y \arrow[dl, "p_1\iota"'] \arrow[dr, "p_2\iota"] & \\
X_1 \arrow[dr, "\omega_1"'] \arrow[rr, phantom, "="] & & X_2 \arrow[dl, "\omega_2"] \\
 & \Omega^2_{\mathrm{cl}} &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
 & Y \arrow[dl] \arrow[dr] & \\
X_1 \arrow[dr, "\nabla_1"'] \arrow[ddr, bend right, "\omega_1"'] \arrow[rr, phantom, "\Swarrow \simeq"] & & X_2 \arrow[dl, "\nabla_2"] \arrow[ddl, bend left, "\omega_2"] \\
 & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[d, "F_{(-)}"] & \\
 & \Omega^2_{\mathrm{cl}} &
\end{tikzcd}
$$

One finds (section 2.10 and 2.11 of [Schreiber 13c]) that such prequantized Lagrangian correspondences neatly subsume within them all the central concepts of classical Hamilton-Lagrange-Jacobi mechanics (as discussed e.g. in [Arnold 89]).

Proposition 2.25. *1-parameter flows of prequantized Lagrangian correspondences as above on a prequantized phase space (X, ∇) are equivalent to choices $H \in C^\infty(X)$ of Hamiltonian functions on X, where the flow of H sends parameter (time) $t \in \mathbb{R}$ to the correspondence*

1. *whose underlying diffeomorphism is the Hamiltonian flow* $\exp(t\{H, -\})$*;*
2. *whose homotopy is in components the Hamilton-Jacobi action functional* $\exp(\frac{i}{\hbar} S_t)$*, which*
3. *is the exponentiated integral of the Lagrangian L of H obtained by Legendre transform.*

$$
\begin{tikzcd}
 & \mathrm{graph}\left(\exp\left(t\{H,-\}\right)\right) \arrow[dl] \arrow[dr] & \\
X \arrow[dr, "\nabla"'] \arrow[rr, phantom, "\Swarrow \exp\left(\frac{i}{\hbar} S_t\right) = \exp\left(\frac{i}{\hbar}\int_0^t L\, dt\right)"] & & X \arrow[dl, "\nabla"] \\
 & \mathbf{B}U(1)_{\mathrm{conn}} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{space of trajectories} \arrow[dl, "\text{initial values}"'] \arrow[dr, "\text{Hamiltonian evolution}"] & \\
\text{incoming configurations} \arrow[dr, "\text{prequantum bundle}"'] \arrow[rr, phantom, "\Swarrow \text{action functional}"] & & \text{outgoing configurations} \arrow[dl, "\text{prequantum bundle}"] \\
 & \text{2-group of phases} &
\end{tikzcd}
.
$$

More generally:

Proposition 2.26 ([Fiorenza-Rogers-Schreiber 13])**.** *The concretified automorphisms of ∇ over $\mathbf{B}U(1)_{\mathrm{conn}}$ is the quantomorphism group of (X, ω). Its Lie algebra is the Poisson Lie algebra of (X, ω).*

This formulation of classical Hamilton-Lagrange-Jacobi mechanics via correspondences over the moduli stack $\mathbf{B}U(1)_{\mathrm{conn}}$ in prop. 2.25 is in itself just a reformulation of entirely classical theory. One may find it pleasant, and it may have the advantage of lending itself more to full formalization (see [Schreiber 13c]), but the real impact of this formulation is that in contrast to its classical incarnation it is immediately clear how it generalizes to a mathematical theory of prequantization of higher dimensional Hamilton-Lagrange-Jacobi mechanics, hence of local De Donder-Weyl field theory as in 2.1.2:

essentially all that one needs to do is to pass in prop. 2.25 from the moduli $\mathbf{B}U(1)_{\mathrm{conn}}$ of line bundles with connection to moduli $\mathbf{B}^n U(1)_{\mathrm{conn}}$ of line n-bundles with n-connection as in 2.4 and interpret the resulting diagrams in a homotopy theory of geometric higher groupoids. This we turn to in 2.2 below.

2.2 Infinity

Above in 2.1 we considered "1-higher" differential geometry of stacks, which assign groupoids of fields and their gauge transformations to manifolds (or to varieties etc.). This is just the first stage in general higher geometry, which deals with higher stacks that assign higher groupoids of higher gauge fields with gauge transformations between these, gauge-of-gauge-transformations between those and so on. Aspects of this we turn to now.

2.2.1 Local Chern-Simons Lagrangians

It is compelling to combine the aspects of the previous two sections 2.1.2 and 2.1.3 and ask for higher prequantum bundles defined not just on manifolds (say on jet bundles) but on moduli stacks of fields. Doing this leads to fully localized and gauge equivariant versions of Chern-Simons-type Lagrangians. This is what we discuss now. To do so we need beyond the 1-stacks already discussed also "abelian stacks" given by chain complexes of sheaves, as well as the unification of both in general ∞-stacks, which are the substrate of fully higher geometry, and so we introduce this first.

Higher stacks. A good bit of theoretical physics is controled by one or another equation of the simple form

$$d^2 = 0\,,$$

namely by the existence of a differential operator d which squares to 0. In the context of BRST cohomology and of topological twists of supersymmetric field theory, the differential here is the quantum operator of a conserved "supercharge", and the richness of the physical structures encoded by just $d^2 = 0$ has variously been the cause of some amazement, an echo of which is the common term "master equation" (e.g. [Henneaux-Teitelboim 92]).

But in the grand scheme of things, this "master equation" is but the abelian shadow of a system of equations which governs homotopy theory and goes by the more mundane term "simplicial identities". The key idea here is familiar from basic topology: given a topological space X, there is its *singular simplicial complex* $\mathbb{Z}[\mathrm{Sing}(X)]$, which is the system of abelian groups of singular k-chains in X, for all natural numbers k. Given a singular k-chain c, there is for each $0 \le i \le k$ a $(k-1)$-chain $\partial_i c$, obtained by restricting the formal linear combination of k-simplices which constitute c to their faces opposite the ith vertex. This map ∂_i taking $(k+1)$-chains to k-chains is hence called the *ith singular face map*. It satisfies the *simplicial identities*

$$\partial_i \partial_{i+j+1} = \partial_{i+j}\partial_i \qquad i, j \in \mathbb{N}$$

(which simply express that removing an element from a linear order means to shift all its succesors one place down). A system of abelian group such as $\mathbb{Z}[\mathrm{Sing}(X)]$ which is equipped with a system of face operators of this form is called a *semi-simplicial abelian group*. Typically one considers also compatible reverse co-face maps that regard k-simplices as degenerate $(k+1)$-simplices in all $(k+1)$ different ways, and then speaks of a *simplicial abelian group*.

A key fact is that a (semi-)simplicial abelian group (A, ∂) induces a chain complex of abelian groups by forming the alternating sum of all the face maps in a given degree:

$$d := \sum_{i=0}^{k} (-1)^i \partial_i\,.$$

The simplicial identities then *imply* the master equation:

$$\begin{aligned} d^2 &= \sum_{j=0}^{n}\sum_{i=0}^{n-1} (-1)^{i+j}\partial_i\partial_j \\ &= 0 \end{aligned}.$$

A fundamental theorem known as the Dold-Kan correspondence says that this construction establishes an equivalence between non-negatively graded chain complexes and simplicial abelian groups, thereby embedding homological algebra into homotopy theory.

While simplicial abelian groups with their simplicial identities look a tad more involved than chain complexes with their "master equation" $d^2 = 0$, the big advantage of them is that they generalize to richer, non-abelian structures: since the simplicial identities do not require lineratity to make sense, one may consider systems of plain sets (instead of abelian groups) equipped with simplicial maps satisfying the simplicial identities. Such simplicial sets consist of sets of abstract k-simplices for all k, related by the face and co-face maps.

But the simplicial sets which underly simplicial abelian groups turn out have a special property: given any collection of k-simplices that look like the boundary of an $k+1$-simplex except that one k-faces is missing (a "$(k+1)$-horn"), then there exists an actual $(k+1)$-simplex with such boundary.

$$\begin{tikzcd} & {} \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ {} \arrow[ur] \arrow[rr, dashed] & {} & {} \end{tikzcd}$$

This property encodes a homotopy-theoretic composition operation of k-simplicies: whenever $k+1$ of them meet appropriately, then there is a single k-simplex that plays the role of their composite up to a homotopy exhibited by a $(k+1)$-simplex. Moreover, with this concept of composition every k-simplex has an inverse, up to higher simplices.

Therefore simplicial sets satisfying this property – called the Kan property – behave like higher dimensional versions of groupoids: they have k-dimensional transformation for every k which may be composed and which all behave as k-fold symmetries. Indeed, every ordinary groupoid gives rise to such a Kan complex via its "nerve", which is the simplicial set whose k-simplices are the k-fold sequences of composable morphisms in the original groupoid. In this fashion Kan simplicial sets are a joint generalization of groupoids and of non-negatuvely graded chain complexes[4], which combine the possible non-abelianness of groupoids with the higher grading of chain complexes:

$$\begin{tikzcd} \boxed{\text{groupoids}} \arrow[dr, "\text{nerve}"'] & & \boxed{\text{chain complexes}} \arrow[dl, "\text{Dold-Kan}"] \\ & \boxed{\text{Kan simplicial sets}} & \end{tikzcd} .$$

By construction, there is an evident concept of homotopies between maps between Kan simplicial sets. The resulting homotopy theory is the homotopy theory of ∞-*groupoids*

$$\infty\mathrm{Grpd} := (\text{homotopy theory of Kan simplicial sets}) .$$

From this it is clear that equipping chain complexes and Kan simplicial sets with geometric structure in direct analogy to discussion of stacks above in 2.1.3 yields a concept of higher stacks:

$$\begin{tikzcd} \boxed{\text{sheaves of groupoids}} \arrow[dr] & & \boxed{\text{chain complexes of sheaves}} \arrow[dl] \\ & \boxed{\text{Kan simplicial sheaves}} & \end{tikzcd} .$$

The homotopy theory of ∞-*stacks* is that obtained from the category of sheaves of Kan simplicial sets by univerally turning all maps between them which *locally* are homotopy equivalences of Kan simplicial sets into actual homotopy equivalences.

[4]If one includes also negatively graded chain complexes then the analogous story leads to spectra, this we turn to below in 2.3

Hence in the evident generalization of def. 2.20:

Definition 2.27. Write

$$\mathbf{H} := L_{\mathrm{le}}\,\mathrm{sSet}(\mathrm{Sh}(\mathrm{SmthMfd})) := \left(\begin{array}{l} \text{the homotopy theory obtained} \\ \text{from sheaves of (Kan-)simplicial sets} \\ \text{by universally turning local homotopy equivalences} \\ \text{into homotopy equivalences} \end{array} \right)$$

Remark 2.28. This $\mathbf{H}$ satisfies some abstract properties ("cohesion") from which it follows on general grounds that it is a good context for higher geometrical structures appearing in physics. This we turn to below in 2.3.1.

The higher stacks with values in chain complexes (under the Dold-Kan map) are equivalently chain complexes of sheaves (for instance of quasicoherent sheaves) that are already familiar in the theoretical physics literature, at least in the string theory literature.

$$\mathrm{DK} : \text{chain complexes in non-negative degree} \xrightarrow{\simeq} \text{simplicial abelian groups} \xrightarrow{\text{forget grp. structure}} \text{Kan simplicial sets}\,.$$

This gives a map of homotopy theories

$$\mathrm{DK} : (\text{chain complexes of sheaves}) \longrightarrow \mathbf{H}\,.$$

Example 2.29. For every n write

$$\mathbf{B}^n U(1) := \mathrm{DK}\,(\underline{U}(1)[-n])$$

for the ∞-stack given by the image under the Dold-Kan correspondence of the $U(1)$-valued smooth functions regarded as chain complex concentrated in degree n. Similarly write

$$\mathbf{B}^n U(1)_{\mathrm{conn}} := L\left(\mathbb{Z} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^n\right)$$

Using this we may now express fully local Chern-Simons Lagrangians.

Local Chern-Simons functionals.

Example 2.30 ([Fiorenza-Schreiber-Stasheff 10])**.** For $\mathfrak{g}$ a semisimple Lie algebra with simply connected Lie group denoted G and with Killing form invariant polynomial denoted $\langle -,- \rangle$, then there is a differential Lie integration of the the canonical Lie algebra cocycle

$$\langle -, [-,-] \rangle : \mathfrak{g} \longrightarrow \mathbb{R}[2]$$

to a morphism of higher moduli stacks which differentially refines the second Chern class c_2:

$$\begin{tikzcd} \mathbf{B}G_{\mathrm{conn}} \arrow[r, "\mathbf{L}_{\mathrm{CS}_3}"] \arrow[d] & \mathbf{B}^3 U(1)_{\mathrm{conn}} \arrow[d] \\ BG \arrow[r, "c_2"'] & K(\mathbb{Z},4) \end{tikzcd}.$$

This $\mathbf{L}_{\mathrm{CS}_3}$ is the fully local Lagrangian of Chern-Simons theory. Applied to a globally defined $\mathfrak{g}$-valued differential form

$$A : \Sigma \longrightarrow \Omega^1(-,\mathfrak{g})$$

it produces the Chern-Simons 3-form

$$\mathrm{CS}(A) : \Sigma \xrightarrow{A} \Omega^1(-,\mathfrak{g}) \longrightarrow \mathbf{B}G_{\mathrm{conn}}\,.$$

Transgressed to maps out of a closed oriented Σ is produces the exponentiated action functional of Chern-Simons theory.

Example 2.31 ([Fiorenza-Sati-Schreiber 12c]). The cup product on Deligne cohomology refines to a morphism of higher stacks

$$\mathbf{B}^{2k+1}U(1)_{\mathrm{conn}} \xrightarrow{(-)\cup(-)} \mathbf{B}^{4k+3}U(1)_{\mathrm{conn}}$$

These are the fully local Lagrangians of $4k+3$-dimensional abelian Chern-Simons theory.

Example 2.32 ([Fiorenza-Rogers-Schreiber 11]). For (X,π) a Poisson manifold, its Poisson Lie algebroid $\mathfrak{P}(X,\pi)$ naturally carries a Lie algebroid 2-cocycle exhibited by π:

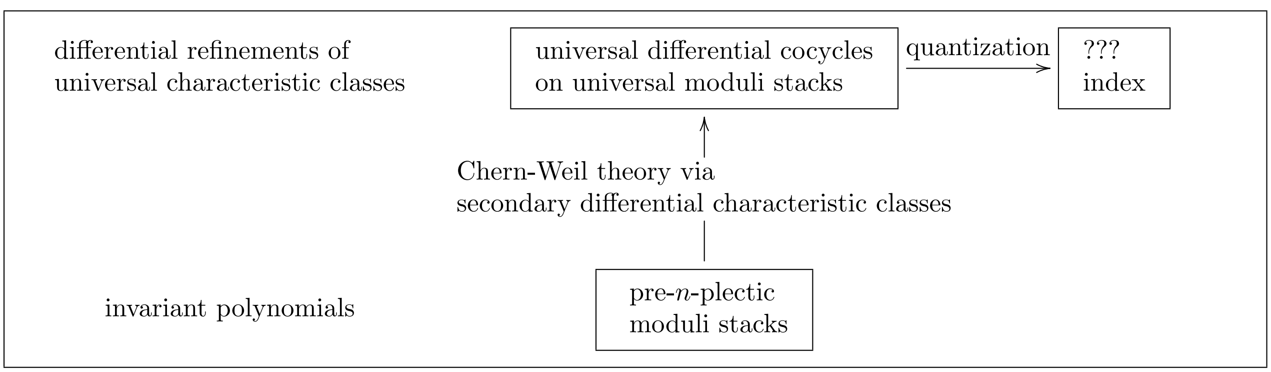
$$\mathfrak{P}(X,\pi) \longrightarrow \mathbb{R}[-2]$$

which is transgressive and which is integral precisely if (X,π) is integral in the standard sense. By the process of [Fiorenza-Schreiber-Stasheff 10] this Lie-integrates to the local Lagrangian of a 2d Poisson-Chern-Simons theory of the form

$$\begin{tikzcd}
\mathrm{SymplGrp}_{\mathrm{conn}} \arrow[r, "\mathbf{L}_{\mathrm{CS}_2}"] \arrow[d, "\text{forget connection data}"] & \mathbf{B}^2U(1)_{\mathrm{conn}} \arrow[d] \\
\mathrm{SymplGrpd} \arrow[r] & \mathbf{B}(\mathbf{B}U(1)_{\mathrm{conn}})
\end{tikzcd}$$

whose domain is a differential refinement of the stack represented by the symplectic groupoid SymplGrpd of (X,π), such that on local differential form data (hence in perturbation theory, ignoring instanton sectors) this Lagrangian is that of the Poisson σ-model. On instanton sectors this descends to a line 2-bundle with 1-form connection which is equivalent to what is traditionally known as the pre-quantization of the symplectic groupoid [Bongers 14].

We discuss below in 2.3.2 that the boundary field theory of this non-perturbative 2d Poisson-Chern-Simons theory is the traditional quantum mechanics on X.



2.2.2 Higher gauge fields and Higher-order ghosts

Above we saw ordinary non-abelian G-gauge fields modulated by $\mathbf{B}G_{\mathrm{conn}}$ as well as abelian n-form fields modulated by $\mathbf{B}^nU(1)_{\mathrm{conn}}$. In general these combine to higher non-abelian gauge groups.

Example 2.33 ([Fiorenza-Schreiber-Stasheff 10, Schreiber 13a]). The homotopy fiber of the local Chern-Simons Lagrangian of example 2.30 is the moduli 2-stack of String 2-connection fields.

$$\begin{tikzcd}
\mathbf{B}\mathrm{String}_{\mathrm{conn}} \arrow[r] \arrow[d] & \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \arrow[r] \arrow[d] & \mathbf{B}^3U(1)_{\mathrm{conn}} \arrow[d] \\
BO\langle 8\rangle \arrow[r] & B\mathrm{Spin} \arrow[r] & K(\mathbb{Z},4)
\end{tikzcd}$$

The relevance of this to physics is the following.

Example 2.34 ([Fiorenza-Sati-Schreiber 09, Fiorenza-Sati-Schreiber 12b])**.** The flux quantization condition [Witten 96] on the C-field of 7-dimensional Chern-Simons theory, in the mathematical formulation of [Hopkins-Singer 02], when *localized* (hence lifted from gauge equivalence classes to moduli stacks) says that the C-field is modulated by the homotopy fiber product of higher stacks $\mathbf{B}\mathrm{String}^{2a}_{\mathrm{conn}}$ in

$$\begin{tikzcd}
\mathbf{B}\mathrm{String}^{2a}_{\mathrm{conn}} \arrow[r] \arrow[d] & \mathbf{B}^3 U(1)_{\mathrm{conn}} \arrow[d] \\
\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 \arrow[r, "\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}"] & \mathbf{B}^3 U(1)
\end{tikzcd} ,$$

where $\frac{1}{2}\mathbf{p}_1$ is the local Spin-Chern-Simons Lagrangian of example 2.30. Here Spin and E_8 are ordinary non-abelian gauge Lie groups, and $\mathbf{B}^2 U(1)$ is the higher abelian gauge group for abelian 3-form fields, but their homotopy fiber product is a non-abelian higher group called String^{2a}, a higher analog of the group Spin^c. Upon restriction to the Hořava-Witten orientifold plane the moduli of String^{2a}-2-connections turn into the moduli of Green-Schwarz anomaly free background fields for the heterotic string.

To appreciate how these higher groups work, the key fact is that the operation forming loop space objects $\Omega X := * \underset{X}{\times} *$ of pointed ∞-stacks constitutes an equivalence between ∞-group objects in ∞-stacks and pointed connected ∞-stacks, in generalization of the classical statement for plain homotopy types:

Fact 2.35 (Quillen, May, ... [Lurie 1x])**.** *:*

$$\{\mathit{smooth}\ \infty\text{-}\mathit{groups}\} \underset{\mathbf{B}}{\overset{\Omega}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \{\mathit{pointed\ connected\ smooth}\ \infty\text{-}\mathit{stacks}\}$$

Theorem 2.36 ([Nikolaus-Schreiber-Stevenson 12])**.** *For G an ∞-group then $\mathbf{B}G$ is the moduli ∞-stack of G-principal ∞-bundles, hence of G-instanton sectors.*

Example 2.37 ([Fiorenza-Sati-Schreiber 12a])**.** homotopy fiber sequence

$$\mathbf{B}\mathrm{String}_{\mathrm{conn}} \longrightarrow \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

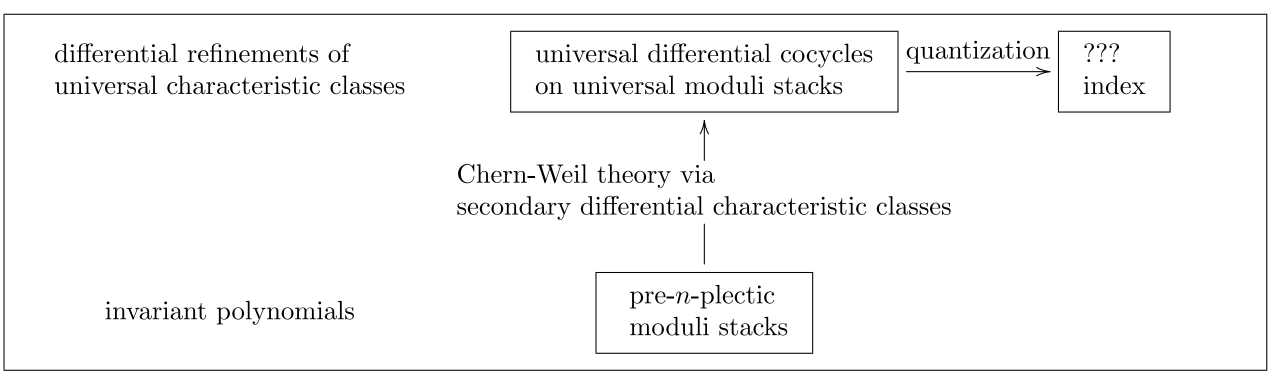
7d Chern-Simons

$$\mathbf{L}_{\mathrm{CS}_7} : \mathbf{B}\mathrm{String}_{\mathrm{conn}} \longrightarrow \mathbf{B}^7 U(1)_{\mathrm{conn}}$$

Example 2.38 ([Fiorenza-Sati-Schreiber 12c])**.**

$$\mathbf{L}_{\mathrm{CS}_{4k+3}} : \mathbf{B}^{2k+1} U(1)_{\mathrm{conn}} \overset{(-)\cup(-)}{\longrightarrow} \mathbf{B}^{4k+3} U(1)_{\mathrm{conn}}$$

In conclusion, the picture of geometric quantization here looks like this:



2.2.3 Kaluza-Klein reduction and Transgression

Central to the definition of local action functionals in physics is the operation of integration of differential forms, of course, and particularly of integration over fibers: For Σ_k a closed oriented smooth manifold of dimension k, and for $n+1 \geq k$, then for every coordinate chart U integration is a map of the form

$$\int_{\Sigma \times U/U} (-) : \; \Omega^{n+1}_{\mathrm{cl}}(U \times \Sigma_k) \longrightarrow \Omega^{n+1-k}(U) \; .$$

Moreover, this operation is *natural in* U. In the language of sheaves and stacks, this is succinctly captured by stating that fiber integration over Σ is a morphism of stacks

$$\int_{\Sigma} \; : \; [\Sigma, \Omega^{n+1}_{\mathrm{cl}}] \longrightarrow \Omega^{n+1-k}_{\mathrm{cl}} \, .$$

If here the closed differential forms are thought of as curvature forms, then ... generalize to n-connections.

$$\exp\left(\tfrac{i}{\hbar}\int_{\Sigma}(-)\right) \; : \; [\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}] \longrightarrow \mathbf{B}^{n-k} U(1)_{\mathrm{conn}}$$

$$\begin{tikzcd}
{[\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}]} \arrow[r, "\exp\left(\frac{i}{\hbar}\int_\Sigma(-)\right)"] \arrow[d, "{[\Sigma_k, F_{(-)}]}"] & \mathbf{B}^{n-k} U(1)_{\mathrm{conn}} \arrow[d, "F_{(-)}"] \\
{[\Sigma_k, \Omega^{n+1}_{\mathrm{cl}}]} \arrow[r, "\int_\Sigma(-)"] & \Omega^{n+1}_{\mathrm{cl}}
\end{tikzcd}$$

For $\mathbf{Fields}_n$ the moduli stack of fields of an n-dimensional theory, then for Σ_k a k-dimensional manifold of dimension $k < n$ we may regard the mapping stack

$$\mathbf{Fields}_{n-k} := [\Sigma_k, \mathbf{Fields}_n]$$

as a moduli stack of fields for an $(n-k)$-dimensional theory. This then is such that a field configuration over an spacetime Y_{n-k} of dimension $n-k$

$$\phi : Y_{n-k} \longrightarrow \mathbf{Fields}_{n-k}$$

is equivalently, by the defining universal property of mapping stacks, a field configuration of the n-dimensional theory on the product

$$Y_{n-k} \times \Sigma_k \longrightarrow \mathbf{Fields}_n \, .$$

in the context of S-duality and geometric Langlands duality: section 6 of [Witten 08]

2.2.4 Polarization and Self-dual higher gauge theory

Geometric quantization of Chern-Simons-type field theories in dimension $4k+3$ serves in a mathematically precise way to express a relationship to quantization of $4k+2$-dimensional self-dual higher gauge fields which is of the type that physicists have come to call *holography*. This means that the quantum states of the higher dimensional theory in the form of wave functions on the space of fields are identified with the generating functions for the quantum correlators (n-point functions) of the lower dimensional theory as functions on the space of sources.

prequantum 1-line bundle by transgression

$$[\Sigma_{4k+2}, \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}}] \xrightarrow{[\Sigma,\mathbf{L}]} [\Sigma_{4k+2}, \mathbf{B}^{4k+3}U(1)_{\mathrm{conn}}] \xrightarrow{\int} \mathbf{B}U(1)_{\mathrm{conn}}$$

higher differential intersection pairing. The corresponding curvature symplectic form is the actual intersection pairing on $(2k+1)$-forms.

polarized phase space is Griffiths higher intermediate Jacobian (...)

$$J^{k+1}(X, \hat{E}) \to \hat{H}^{2k+2}(X, \mathbb{Z} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^k \to 0 \to \cdots) \longrightarrow \mathrm{Hdg}(X, \hat{E})\,.$$

For $k=0$ this self-dual gauge theory is the WZW string (the fundamental 1-brane), which has been the focus of much attention in the past, for $k=1$ it is the self-dual gauge theory on the 5-brane. which has become the focus of much attention these days, as it relates to 4d Yang-Mills

Definition 2.39. A Hodge filtration on $\hat{E} \in \mathrm{Spectra}(\mathrm{Sh}(S))$ is is a filtration $F^\bullet \flat_{\mathrm{dR}} \hat{E}$ such that

1. each stage has the same image under Π;
2. each stage is in the kernel of $\flat$.

Proposition 2.40. *The induced sequence of homotopy fiber products*

$$F^p \hat{E} := \Pi \hat{E} \underset{\Pi \flat_{\mathrm{dR}} \hat{E}}{\times} F^p \flat_{\mathrm{dR}} \hat{E}$$

exhibits $(\Pi \dashv \flat)$*-fractures as in theorem 2.50*

$$\begin{tikzcd}[column sep=small] & F^p\flat_{\mathrm{dR}}\hat{E} \arrow[dr] & & & & \flat_{\mathrm{dR}}(F^p\hat{E}) \arrow[dr] & \\ F^p\hat{E} \arrow[ur] \arrow[dr] & & \Pi\flat_{\mathrm{dR}}\hat{E} & \simeq & F^p\hat{E} \arrow[ur] \arrow[dr] & & \Pi\flat_{\mathrm{dR}}(F^p\hat{E}) \\ & \Pi\hat{E} \arrow[ur] & & & & \Pi F^p(\hat{E}) \arrow[ur] & \end{tikzcd}$$

Definition 2.41. Denote the Moore-Postnikov tower of the $\sharp$-unit by:

$$\begin{tikzcd} & \vdots \arrow[d, dotted, no head] & \\ & \sharp_3 X \arrow[d] \arrow[ddr] & \\ & \sharp_2 X \arrow[d] \arrow[dr] & \\ X \arrow[uur] \arrow[ur] \arrow[r] & \sharp_1 X \arrow[r, hook] & \sharp X \end{tikzcd}$$

Hence $\sharp_n X$ is the "n-image" of the $\sharp$-unit on X.

Definition 2.42. Given a $\hat{E}$-Hodge filtration, def. 2.39, and given any $X \in \mathrm{sSet}(S)$ with k the lowest number such that $H(X, F^{k+1}\flat_{\mathrm{dR}}\hat{E}) \simeq 0$, then the *differential moduli* of $\hat{E}$ on X is the iterated homotopy fiber product

$$\hat{E}(X) := \sharp_1[X, F^k \hat{E}] \underset{\sharp_1[X, F^{k-1}\hat{E}]}{\times} \sharp_2[X, F^{k-1}\hat{E}] \underset{\sharp_2[X, F^{k-2}\hat{E}]}{\times} \sharp_3[X, F^{k-2}\hat{E}] \underset{\sharp_3[X, F^{k-3}\hat{E}]}{\times} \cdots\,.$$

Proposition 2.43. *The underlying homotopy type of $\hat{E}(X)$ is that of $\hat{E}$-cocyles on X in kth Hodge filtration stage:*

$$\flat(\hat{E}(X)) \simeq \flat[X, F^k\hat{E}] \simeq \mathbf{H}(X, F^k\hat{E})\,.$$

Proof. Use that cohesion implies $\flat \circ \sharp_n \simeq \flat$ for all n, and that $\flat$ preserves homotopy limits. ◻

Proposition 2.44. *For $\hat{E} = (\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}}$ this reproduces the Artin-Mazur moduli:*

$$(\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}}(X) \;:\; U \mapsto \{U\text{-parameterized Deligne cocycles on } X\ \}$$

[Fiorenza-Rogers-Schreiber 13]

Example. *The homotopy fiber of*

$$\hat{E}(X) \xrightarrow{\simeq} (\Pi\hat{E})(X) \underset{(\Pi\flat_{\mathrm{dR}}\hat{E})(X)}{\times} (\flat_{\mathrm{dR}}\hat{E})(X) \longrightarrow \tau_0(\Pi\hat{E})(X) \underset{\tau_0(\Pi\flat_{\mathrm{dR}}\hat{E})(X)}{\times} \tau_0(\flat_{\mathrm{dR}}\hat{E})(X)\,,$$

is a stack whose 0-truncation is is the higher Jacobian $J^{k+1}(X)$ with its Griffiths complex structure.

2.2.5 Boundary conditions and Brane intersection laws

By Weinstein's dictionary in 2.1.4, a Lagrangian correspondence from the point to a given phase space X determines a vector $\Psi \in \mathcal{H}$ in the Hilbert space associated with X, hence a quantum state. In the context of quantum operators, this should be thought of as an "initial condition" for further quantum propagation, a boundary condition

$$
\begin{tikzcd}
 & Y_{\mathrm{in}} \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & & {} \arrow[dl, dotted] \\
\ast \arrow[dr] \arrow[rr, phantom, "\swarrow"] & & X \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\swarrow"] & & X \arrow[dl] \arrow[dr, dotted] & \\
 & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[rr, equal] & & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[rr, equal, dotted] & & {} \\
 & \boxed{\text{boundary}} & & \boxed{\text{propagation}} & & \\
\mathbb{C} \arrow[rr, "\Psi"] & & \mathcal{H} \arrow[rr] & & \mathcal{H} \arrow[r, dotted, no head] & {}
\end{tikzcd}
$$

This has an evident generalization to local higher gauge theory as in 2.2.2: given a moduli stack $\mathbf{Fields}_{\mathrm{bulk}}$ of bulk fields equipped with a local Lagrangian $\mathbf{L} : \mathbf{Fields}_{\mathrm{bulk}} \longrightarrow \mathbf{B}^n U(1)_{\mathrm{conn}}$, then a *codimension-1 boundary condition* on this is a diagram of the form

$$
\begin{tikzcd}
 & \mathbf{Fields}_{\mathrm{bdr}} \arrow[dl] \arrow[dr] & \\
\ast \arrow[dr, "0"'] \arrow[rr, phantom, "\overset{\simeq}{\underset{\phi_{\mathrm{bdr}}}{\swarrow}}"] & & \mathbf{Fields}_{\mathrm{bulk}} \arrow[dl, "\mathbf{L}"] \\
 & \mathbf{B}^n U(1)_{\mathrm{conn}} &
\end{tikzcd},
$$

Below in 2.3.4 we indicate how to "derive from first principles" this definition[5], but here and in the following we are content with its plausibility and with various examples and applications.

By the general laws of homotopy theory, there is always a *universal* boundary condition for a given local Lagrangian $\mathbf{L}$, namely its homotopy fiber (formed in the homotopy theory of higher stacks). In σ-model quantum field theories this universal boundary condition has a useful and interesting interpretation: it determines "brane intersection laws":

Example 2.45 (brane intersection laws [Fiorenza-Sati-Schreiber 13b])**.** For σ-model field theory describing the propagation of a p-brane in some target space, then the moduli space of fields $\mathbf{Fields}_{\mathrm{bulk}}$ is just that target space. For a gauged σ-model it is a suitable differential gerbe over that target space. The Lagrangian $\mathbf{L} : \mathbf{Fields}_{\mathrm{blk}} \to \mathbf{B}^{p+1}U(1)_{\mathrm{conn}}$ in this case encodes the globally defined WZW term of the brane, in that (as in 2.2.3) given a closed orientied brane worldvolume $\Sigma_{p+1} \to \mathbf{Fields}_{\mathrm{bulk}}$, then the WZW-part of the action functional is

$$
\exp(\tfrac{i}{\hbar} S_{\mathrm{WZW}}) : \; [\Sigma_{p+1}, \mathbf{Fields}_{\mathrm{bulk}}] \longrightarrow [\Sigma_{p+1}, \mathbf{B}^{p+1}U(1)_{\mathrm{conn}}] \xrightarrow{\exp(\frac{i}{\hbar}\int_{\Sigma_{p+1}(-)})} U(1) \; .
$$

[5] Which was originally highlighted to us by Domenico Fiorenza

But if the brane worldvolume has a boundary $\partial\Sigma$, hence if we consider an "open brane" (e.g. an open string for $p = 1$), then a field configuration of the sigma-model is not just any map $\Sigma_{p+1} \to \mathbf{Fields}_{\mathrm{bulk}}$ but one that satisfies certain boundary conditions. In particular there may be a background brane $Q \to \mathbf{Fields}_{\mathrm{bulk}}$ and the Dirichlet-type boundary condition that the p-brane ends on that background brane. This means that a boundary field configuration is a diagram of the form

$$\begin{tikzcd}
\partial\Sigma \arrow[r, "\phi_{\mathrm{bdr}}"] \arrow[d, hook] & Q \arrow[d] \\
\Sigma \arrow[r, "\phi_{\mathrm{bulk}}"'] & \mathbf{Fields}_{\mathrm{bulk}}
\end{tikzcd}$$

If here $p = 0$ and we are considering a charged particle, then the correct boundary condition is a "quark" state, being a trivialization of the background gauge bundle. (...) In view of this it is plausible that we should demand that on Q the local Lagrangian should trivialize

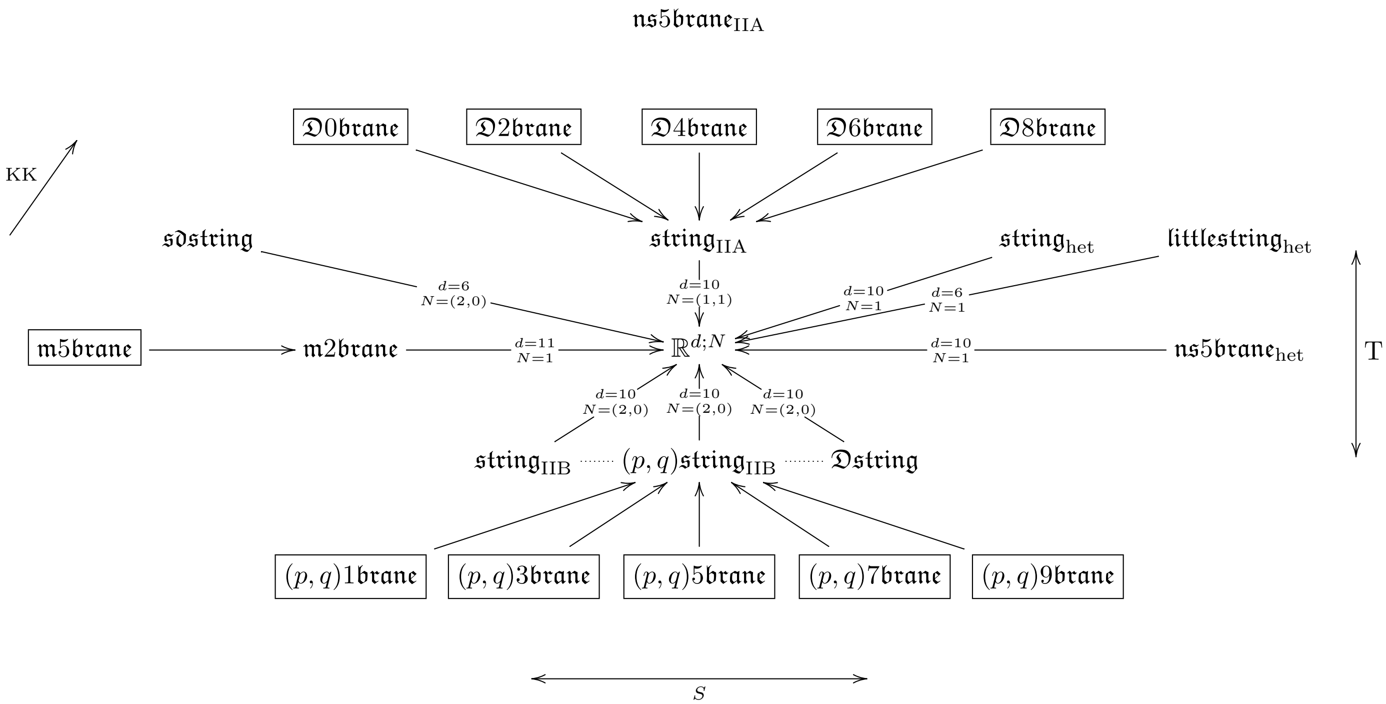
$$\begin{tikzcd}
\partial\Sigma \arrow[r, "\phi_{\mathrm{bdr}}"] \arrow[d, hook] & Q \arrow[d] \arrow[r] & * \arrow[d] \arrow[dl, Rightarrow] \\
\Sigma \arrow[r, "\phi_{\mathrm{bulk}}"'] & \mathbf{Fields}_{\mathrm{bulk}} \arrow[r, "\mathbf{L}"'] & \mathbf{B}^{p+1}U(1)_{\mathrm{conn}}
\end{tikzcd}$$

Write here $\mathbf{Fields}^{\mathrm{univ}}_{\mathrm{bdr}}$ for the homotopy fiber of $\mathbf{L}$. By the universal property of the homotopy fiber, the above diagram factors essentially uniquely as

$$\begin{tikzcd}
\partial\Sigma \arrow[r, "\phi_{\mathrm{bdr}}"] \arrow[d, hook] & Q \arrow[d] \arrow[r, "\Phi_{\mathrm{bulk}}"] & \mathbf{Fields}^{\mathrm{univ}}_{\mathrm{bdr}} \arrow[d] \arrow[r] & * \arrow[d] \arrow[dl, Rightarrow] \\
\Sigma \arrow[r, "\phi_{\mathrm{bulk}}"'] & \mathbf{Fields}_{\mathrm{bulk}} \arrow[r, equal] & \mathbf{Fields}_{\mathrm{bulk}} \arrow[r, "\mathbf{L}"'] & \mathbf{B}^{p+1}U(1)_{\mathrm{conn}}
\end{tikzcd}$$

But this means that the background brane Q on which the "fundamental" brane Σ ends has itself bulk fields given by $\Phi_{\mathrm{bulk}} : Q \to \mathbf{Fields}^{\mathrm{univ}}_{\mathrm{bdr}}$. If now that space is itself equipped with a Lagrangian $\mathbf{L}^Q$, then we Q itself obtains the structure of a sigma-model field theory and becomes a brane in its own right. The fact that the brane Σ is allowed to end on the brane Q is thus all endcoded in the fact that the bulk fields of Q are a higher extension of the bulk fields of Σ, classified by the local Lagrangian of Σ.

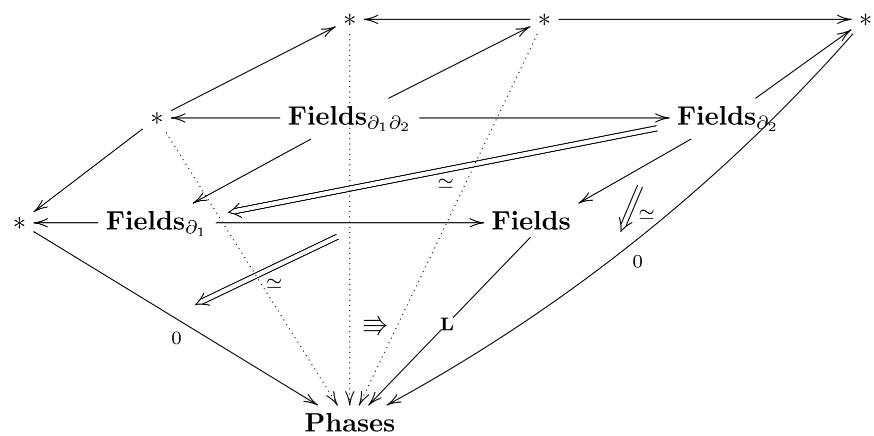
In particular all the Green-Schwarz super p-branes of string/M-theory are generalized WZW models of this type. By the above homotopy-theoretic analysis of their local Lagrangians, one finds the following bouquet of super p-branes and their intersection laws [Fiorenza-Sati-Schreiber 13b]:



Notice that this diagram, which expresses a mathematical theorem in strong homotopy Lie theory, captures a fair bit of the general structure of the web of field theories in 1.3. For instance the sequence extending horizontally to the left expresses the existence of the M5-brane worldvolume which carries the 6-dimensional theory (whose KK-reduction to 4d is supposed to be super-Yang-Mills theory whose S-duality captures geometric Langlands duality) and the existence of the arrow $\mathfrak{m5brane} \longrightarrow \mathfrak{m2brane}$ here exhibits the fact that the M2-brane may end on the M5-brane, which in turn is the fact that there are strings on the M5-brane (the boundaries of the M2) that are charged under a self-dual 2-connection form field, the very hallmark of the 6d theory.

Indeed, the above bouquet diagram is of the shape of the infamous "M-theory amoeba" which physicists draw (e.g. fig. 1 in [Witten 98]) as a map of the web of string and field theories, much the way the ancient mariners had drawn maps of the world. The point here being that it is possible to make modest steps towards turning modern fundamental string physics lore into mathematics, and that it crucially involves higher geometry.

In view of local field theory in higher codimension a key aspect here is that all these constructions iterate in a hierarchical fashion. For instance where the diagrams at the beginning of this section express the pre-quantum operation exhibiting a codimension-1 boundary, then a codimension-2 corner is accordingly a boundary-of-boundaries expressed by a 3-morphism (a homotopy-of-homotopies) in a diagram as follows[6]



Such higher codimension "corner operators" appear in the web of field theories 1.3 when an $(n+1)$-dimensional twisting field theory is related first to an n-dimensional topological field theory and then further to a $(n-1)$-dimensional, a fact highligted maybe first in [Sati 11]. The formalization as indicated here is spelled out in section 3.9.14 of [Schreiber 13a]. We come back to the quantization of corner field theory below in 2.3.3.

[6]Thanks to Hisham Sati for help with preparation of this diagram.

2.3 $\pm$Infinity

After all the discussion of higher pre-quantum geometry above, we now come to higher geometric quantization proper. Naturally, the linearization involved in quantization turns out to go along with passing from homotopy types to *linear* homotopy types, namely spectra, and hence from higher geometry in terms of higher stacks, i.e. sheaves of higher groupoids, to linear geometry in terms of sheaves of spectra.

higher pre-quantum geometry	higher quantum geometry
non-linear	linear
higher stacks = sheaves of higher groupoids	sheaves of spectra

2.3.1 Higher background fields and Differential cohomology

Whatever string theory is otherwise, it is the substrate of the web of quantum field theories in 1.3. To the extent that this web of relations is pertinent to understanding quantum (Einstein-)Yang-Mills-Dirac theory at a deep level, so are string vacua and their non-perturbative effects. In particular, to the extent that 4d Yang-Mills is at a deep level to be understood in terms of the compactification of the 6d theory on the 5-brane [Witten 04], then a sizable chunk of perturbative and non-perturbative string theory is involved (e.g. [Witten 11]) in questions that are fundamentally field theoretic, irrespective of what becomes of grand unification and quantum gravity.

That string backgrounds are inherently objects in higher geometry (in higher category theory) has been clear since the Kalb-Ramond B-field – the stringy version of the electromagnetic field – was globally understood as a 2-connection.

Proceeding from there into deeper stringy territory, it is for instance hopeless to speak about the global (non-perturbative) nature of the M-theory C-field (the one whose holographic dual is the self-dual 2-form on the 5-brane) without at least mentioning the word "groupoid" [DFM 03] and a proper description arguably takes place only in moduli 3-stacks on smooth manifolds [Fiorenza-Sati-Schreiber 12b]. The (Hořava-Witten-)boundary restriction of these produces the moduli stacks of Green-Schwarz anomaly free globally defined background fields of the heterotic string, whose description in [Fiorenza-Sati-Schreiber 09] was the source for many of the developments discussed here.

But the story ranges much deeper still. Ever since the words "D-brane" and "K-theory" appeared in the same sentence ([Freed-Hopkins 00] is not the first but maybe the first mathematical reference), it was essentially clear that a full description of string backgrounds requires *stable* homotopy theory of spectra representing generalized (Eilenberg-Steenrod-type) cohomology – and that differential geometric refinements of these to *differential generalized cohomology* are necessary, as highlighted in [Freed 00], to discuss subtle "anomaly-cancellation mechanisms", i.e. lifts of obstructions to quantization. This line of thought has been much refined since then [Distler-Freed-Moore 11] and has shown the clear need to formulate string backgrounds in twisted differential generalized cohomology.

However, an actual mathematical theory of twisted differential generalized cohomology had been missing, apart from the plausible conviction that the constructions in [Hopkins-Singer 02] ought to be examples. Attention to this open problem of producing a sensible axiomatics for differential generalized cohomology was drawn in [Simons-Sullivan 07], where the following was noticed:

On smooth manifolds, the functor $H(-, (\mathbf{B}^n U(1))_{\mathrm{conn}})$ that sends a smooth manifold to its ordinary differential cohomology of degree $(n+1)$ (recall def. 2.4), hence to the group of equivalence classes of n-connecion fields on the manifold, is uniquely characterized by sitting in a hexagon of presheaves of abelian groups

$$
\begin{tikzcd}
& H(-,\Omega^n)/\mathrm{im}(d_{\mathrm{dR}}) \arrow[rr, "d_{\mathrm{dR}}"] \arrow[dr] & & H(-,\Omega^{n+1}_{\mathrm{cl}}) \arrow[dr] & \\
H(-,\mathbf{B}^n\flat\mathbb{R}) \arrow[ur] \arrow[dr] & & H(-,(\mathbf{B}^nU(1))_{\mathrm{conn}}) \arrow[ur, "\theta"] \arrow[dr] & & H(-,\mathbf{B}^{n+1}\flat\mathbb{G}_a) \\
& H(-,\mathbf{B}^n\flat U(1)) \arrow[ur] \arrow[rr, "\beta"'] & & H(-,\mathbf{B}^{n+1}\mathbb{Z}) \arrow[ur, "\mathrm{ch}"] &
\end{tikzcd}
$$

where the diagonals and the two boundaries are exact sequences. The meaning of the hexagon is this:

$$
\begin{tikzcd}
& \boxed{\begin{array}{l}\text{connect.}\\ \text{on trivial}\\ \text{bundles}\end{array}} \arrow[rr, "\text{de Rham differential}"] \arrow[dr, "\text{regard as}"] & & \boxed{\begin{array}{l}\text{curvature}\\ \text{forms}\end{array}} \arrow[dr, "\text{de Rham theorem}"] & \\
\boxed{\begin{array}{l}\text{closed}\\ \text{differ.}\\ \text{forms}\end{array}} \arrow[ur, "\text{regard as}"] \arrow[dr, "\text{regard as}"'] & & \boxed{\begin{array}{l}\text{connections}\\ \text{on geometric}\\ \text{bundles}\end{array}} \arrow[ur, "\text{curvature}"] \arrow[dr, "\text{topol. class}"'] & & \boxed{\begin{array}{l}\text{ration.}\\ \text{bundles}\end{array}} \\
& \boxed{\begin{array}{l}\text{flat}\\ \text{connect.}\end{array}} \arrow[ur, "\text{regard as}"'] \arrow[rr, "\begin{array}{l}\text{Bockstein homom./}\\ \text{comparison map}\end{array}"] & & \boxed{\begin{array}{l}\text{shape of}\\ \text{bundles}\end{array}} \arrow[ur, "\text{Chern character}"'] &
\end{tikzcd}
$$

The outer parts of the hexagon all involve plain ordinary cohomology and all have a classical generalization from ordinary to generalized Eilenberg-Steenrod-type cohomology (we will write ES-type cohomology for short, e.g. K-theory, elliptic cohomology, ... cobordism cohomology). By Brown's representability theorem, generalized ES-type cohomology is what is represented by spectra $E \in \mathrm{Spectra}$ instead of just chain complexes.

In view of this, [Simons-Sullivan 07] were led to evident question:
Question: *Does such a hexagon also characterize differential generalized cohomology?*
or: *What* **is** *differential generalized cohomology, axiomatically?*

Before answering this, it is worthwhile to notice that there is more "generalization" to cohomology than just ES-type generalization:

Generalizations of ordinary cohomology	**needed for**
ES-type abelian generalized cohomology	type II superstring RR-fields in twisted KR-theory, higher geometric quantization
non-abelian cohomology	Chern-Simons theory, Wess-Zumino-Witten theory, modular functors, equivariant elliptic cohomology
geometry other than plain smooth	Kähler geometric quantization, supersymmetric field theory, Artin-Mazur deformation theory
twisted cohomology	quantum anomaly cancellation, covariant quantization of higher gauge fields

For modern applications to field theory one really needs all these aspects. We indicate now how.

Recall the homotopy theory $\mathbf{H}$ of higher stacks from def. 2.27. For reasons discussed above, we are interested in an "inter-geometric" description of moduli stacks of higher gauge fields in physics that makes sense in smooth geometry, in supergeometry, in (complex-)analytic geometry, etc.. Therefore we now consider the homotopy theory of stacks over sites S of test spaces possibly different from that of smooth manifolds.

$$\mathbf{H} := L_{\mathrm{le}}\,\mathrm{sSet}(\mathrm{Sh}(S)) := \left(\begin{array}{l} \text{the homotopy theory obtained} \\ \text{from sheaves of (Kan-)simplicial sets} \\ \text{by universally turning local homotopy equivalences} \\ \text{into homotopy equivalences} \end{array} \right)$$

The following observation turns out to hold in it the key for a general inter-geometric concept of higher moduli stacks of higher gauge fields.

Proposition 2.46 ([Schreiber 13a])**.** *For S a site of spaces as in def. 2.3, then the derived global section functor $\Gamma : \mathbf{H} \to \infty\mathrm{Grpd}$ is "cohesive" in that it extends to a quadruple of derived adjoints*

$$\begin{tikzcd} \mathbf{H} \arrow[r, shift left=9, "\times" near start] \arrow[r, "\Gamma" description] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=4.5, "\mathrm{LConst}" description] \arrow[l, hook', shift left=4.5] \end{tikzcd}$$

with the bottom right adjoint homotopy fully faithful and the top left adjoint preserving products.

The point of this is that such an adjoint triple induces the following concept formation.

Definition 2.47. Write $\boxed{(\Pi \dashv \flat \dashv \sharp) \;:\; \mathrm{sSh}(S) \to \mathrm{sSh}(S)}$ for induced adjoint triple of derived endofunctors (e.g. $\flat = \mathrm{LConst} \circ \Gamma$).

For G in $\mathrm{Grp}(\mathrm{sSet}(S))$, write $G \xrightarrow{\theta_G} \flat_{\mathrm{dR}} G \longrightarrow \flat \mathbf{B}G \longrightarrow \mathbf{B}G$ for the homotopy fiber sequence of the $\flat$-counit on the delooping.

The following says that key aspects of gauge theory are captured by this.

Proposition 2.48 ([Schreiber 13a])**.**

for G a Lie group with Lie algebra $\mathfrak{g}$	*for $G = \mathbf{B}^n \mathbb{G}_m$*
$\Pi(\mathbf{B}G) \simeq BG$ *and* $\flat(\mathbf{B}G) \simeq K(G,1)$	$\Pi(\mathbf{B}^n\mathbb{G}_m) \simeq K(\mathbb{Z}, n+1)$
$\flat_{\mathrm{dR}} G = \{$*sheaf of flat $\mathfrak{g}$-valued diff. forms*$\}$	$\flat_{\mathrm{dR}} \mathbf{B}^n \mathbb{G}_m \simeq \mathbf{B}^{n+1}\Omega^{\bullet \geq 1}$
θ_G *is the Maurer-Cartan form*	$\theta_{\mathbf{B}^n\mathbb{G}_m}$ *is the Chern character from def. 2.4*
$[X, \mathbf{B}G]$ *is the moduli stack of G-principal bundles*	$[X, \mathbf{B}^n\mathbb{G}_m]$ *is the higher Picard stack*
$\sharp_1[X, \flat\mathbf{B}G] \underset{\sharp_1[X,\mathbf{B}G]}{\times} [X, \mathbf{B}G]$ *is moduli stack of flat connections*	*(details below in **??**)*

Remark 2.49. Hence cohesion is an axiomatics in particular for moduli stacks of connections. Being general abstract, it characterizes those statements about such moduli stacks which hold irrespective of the specific choice of geometry (e.g. smooth, complex-analytic, supergeometric etc). This reminds one of the story of Langlands duality, which supposedly exhibits a parallel between properties of such moduli in smooth/complex-analytic geometry on the one hand, and in arithmetic geometry on the other.

Theorem 2.50 ([Bunke-Nikolaus-Völkl 13])**.** *For $\hat{E}$ a spectrum object in any cohesive homotopy theory as in theorem 2.46, then the canonical hexagon*

$$\begin{tikzcd} & \Pi_{\mathrm{dR}}\hat{E} \arrow[rr, "\mathbf{d}"] \arrow[dr] & & \flat_{\mathrm{dR}}\hat{E} \arrow[dr] & \\ \flat\Pi_{\mathrm{dR}}\hat{E} \arrow[ur] \arrow[dr] & & \hat{E} \arrow[ur, "\theta_{\hat{E}}"] \arrow[dr] & & \Pi\flat_{\mathrm{dR}}\hat{E} \\ & \flat\hat{E} \arrow[ur] \arrow[rr] & & \Pi\hat{E} \arrow[ur, "\mathrm{ch}_E := \Pi\theta_{\hat{E}}"] & \end{tikzcd}$$

formed from homotopy-exact diagonals consists of homotopy fiber sequences.
Moreover, both squares are homotopy Cartesian and hence the outer hexagon uniquely determines $\hat{E}$.

And by prop. 2.48: For $\hat{E} \simeq (\mathbf{B}^n U(1))_{\mathrm{conn}}$, this reproduces on cohomology groups the abive hexagon for ordinary differential cohomology.

In conclusion this says that just as plain Eilenberg-Steenrod-type generalized cohomology is that which is represented by plain stable homotopy types (spectra), so differential generalized cohomology is that which is represented by cohesive sheaves of such spectra, hence by spectra in higher cohesive geometry.

Moreover, twisted such differential generalized cohomology is what is represented by parameterized such sheaves of spectra, namely by bundles of spectra over higher stacks. This is phenomenon that turns out to be central for the formulation of quantization in 2.3.3 below.

2.3.2 Cohomological boundary quantization and Brane charges

We indicate here how traditional geometric quantization (as in 1.2) is equivalently the cohomological boundary quantization of the 2d Poisson-Chern-Simons theory of example 2.32, as discussed in [Nuiten 13]. This provides a perspective on geometric quantization which has evident generalization to higher geometry and to at least key aspects of a prescription for higher geometric quantization of higher pre-quantum geometries that we turn to below in 2.3.3 and 2.3.4.

The 2d Poisson-Chern-Simons theory constructed in [Fiorenza-Rogers-Schreiber 11] is the non-perturbative (globalized) version of the more familiar Poisson σ-model, in that its fields and Lagrangians are locally that of the Poisson σ-model, but globally it stackifies these fields, as in 2.1.3, to produce non-trivial instanton sectors and to lift the Lagrangian to a secondary characteristic of these (example 2.32)

$$\mathbf{L}_{\mathrm{CS}_2} : \mathrm{SymplGrpd}_{\mathrm{conn}} \longrightarrow \mathbf{B}^2 U(1)_{\mathrm{conn}} .$$

Recall that the construction of *perturbative* algebraic deformation quantization of Poisson manifolds due to Kontsevich is secretly the boundary sector of the perturbative quantization of the perturbative Poisson σ-model [Cattaneo-Felder 00]. What we discuss here is like a non-perturbative lift of that in geometric quantization.

	perturbative	non-perturbative
	algebraic deformation quantization	geometric quantization
boundary of	Poisson σ-model	2d Poisson-Chern-Simons theory
with bulk Lagrangian	Lie algebroid cocycle $\mathfrak{P}(X,\pi) \to \mathbb{R}[2]$	higher stack morphism $\mathrm{SymplGrpd}(X,\pi)_{\mathrm{conn}} \xrightarrow{\mathbf{L}_{\mathrm{CS}_2}} \mathbf{B}^2 U(1)_{\mathrm{conn}}$

By the discussion in 2.2.5 we have that a boundary condition for the 2d Poisson-Chern-Simons theory $\mathbf{L}_{\mathrm{CS}_2}$ is a diagram of the form

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
{*} \arrow[dr] \arrow[rr, phantom, "\Swarrow \xi"] & & \mathrm{SymplGrp}(X,\pi) \arrow[dl, "\mathbf{L}_{\mathrm{CS}_2}"] \\
 & \mathbf{B}(\mathbf{B}U(1)_{\mathrm{conn}}) &
\end{tikzcd} .$$

Indeed, this canonically exists [Bongers 14].

If (X,π) is in fact symplectic then $\mathrm{SymplGrpd} \simeq *$ and hence in this case ξ is equivalently a line bundle with connection – the prequantum line bundle on X.

This situation calls for a higher analog of the familiar linearization map

$$U(1) \longrightarrow \mathrm{GL}_1(\mathbb{C})$$

that controls traditional quantum theory. Here we need a map

$$BU(1) \longrightarrow \mathrm{GL}_1(E)$$

for a suitable "higher ring" E. Higher rings in this sense are E_∞-ring spectra, and the natural canditate example which would induce such a map is the complex K-theory spectrum $E := \mathrm{KU}$.

$$
\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
{*} \arrow[dr] \arrow[ddr] & \arrow[l, phantom, "\Swarrow \xi"] & \mathrm{SymplGrp} \arrow[dl, "\mathbf{L}_{\mathrm{CS}_2}"'] \arrow[ddl, "\mathbf{L}^*_{\mathrm{CS}_2}\rho"] \\
 & \mathbf{B}(\mathbf{B}U(1)_{\mathrm{conn}}) \arrow[d, "\rho"] & \\
 & \mathbf{B}\mathrm{GL}_1(\mathrm{KU}) &
\end{tikzcd}
.
$$

Viewed this way we see that the situation is a higher analog of a prequantized Lagrangian correspondence as in 2.1.4, where however the prequantum line bundle is no longer a $\mathbb{C}$-line bundle, but a KU-line bundle. To quantize this should mean to form sections and interpret the correspondence as defining a linear map between these spaces of sections. The space of sections of an E-line bundle τ is the τ-twisted E-cohomology spectrum. Hence we get an element in $\mathrm{KU}^{\tau+\bullet}(\mathrm{SymplGrpd})$. For X symplectic, this is indeed the quantum Hilbert space [Nuiten 13]. (...)

Example 2.51 (Particle at the boundary of 2d Poisson-Chern-Simons TQFT)**.**

$$
\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
{*} \arrow[dr] & \arrow[l, phantom, "\Swarrow"] & \mathrm{SymGrpd}(X,\pi) \arrow[dl, "\chi"] \\
 & \mathbf{B}^2 U(1) \arrow[d, "\mathbf{B}\rho"] & \\
 & B\mathrm{GL}_1(\mathrm{KU}) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{symplectic manifold} \arrow[dl] \arrow[dr, "\text{atlas}"] & \\
{*} \arrow[dr] & \arrow[l, phantom, "\Swarrow\ \text{prequantum bundle}"] & \text{sympl. grpd = moduli of instanton sector of 2d Poisson-Chern-Simons TQFT} \arrow[dl, "\text{local Lagrangian}"] \\
 & \text{3-group of phases} \arrow[d, "\text{superposition principle}"] & \\
 & \text{moduli for KU-line bundles} &
\end{tikzcd}
$$

$$
\mathrm{KU} \longleftarrow \mathrm{KU}_{\bullet+\chi}(\mathrm{SymGrpd}(X,\pi)) \qquad \text{bundle of Hilbert spaces of quantum states from symplectic leaf-wise geometric quantization}
$$

this has a higher analog, which produces the partition function of the quantum string:

Example 2.52 (Superstring at boundary of 3d Spin-Chern-Simons TQFT).

$$
\begin{tikzcd}
 & X \arrow[d, dashed] \arrow[ddl] \arrow[ddr, "TX"] & \\
 & B\mathrm{String} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\swarrow"] \arrow[ddd, bend right, "J_{\mathrm{String}}"'] & \\
 \ast \arrow[ddr] \arrow[r, phantom, "\overset{\sigma}{\Leftarrow}"] & & B\mathrm{Spin} \arrow[dl, "\frac{1}{2}p_1"] \arrow[ddl, "J_{\mathrm{Spin}}"] \\
 & B^3U(1) \arrow[d, "B\rho"] & \\
 & B\mathrm{GL}_1(\mathrm{tmf}) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{String target spacetime} \arrow[d, dashed] \arrow[ddl] \arrow[ddr] & \\
 & \text{universal boundary for local prequantum Spin-Chern-Simons} \arrow[dl] \arrow[dr] \arrow[ddd, bend right] & \\
 \ast \arrow[ddr] \arrow[r, phantom, "\overset{\text{universal string orientation of tmf}}{\Leftarrow}"] & & \text{moduli for instanton sector of 3d Spin-Chern Simons TQFT} \arrow[dl, "\text{local Lagrangian}"] \\
 & \text{4-group of phases} \arrow[d, "\text{superposition principle}"] & \\
 & \text{moduli for tmf-line bundles} &
\end{tikzcd}
$$

$$
\mathrm{tmf} \longleftarrow X^{TX}
\qquad
\text{integral Witten genus = non-perturbative string partition function}
$$

Example 2.53 (D-Brane Charge and T-Duality).

$$
\begin{tikzcd}
 & E \times_X \tilde{E} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\swarrow"] & \\
 E \arrow[r] \arrow[dr, "B"'] & X & \tilde{E} \arrow[l] \arrow[dl, "\tilde{B}"] \\
 & \mathbf{B}^2U(1) \arrow[d] & \\
 & B\mathrm{GL}_1(\mathrm{KU}) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \arrow[dl] \arrow[dr] \arrow[d, phantom, "\overset{\text{fiberwise Poincaré line bunlde}}{\swarrow}"] & \\
 \text{spacetime being torus bundle} \arrow[d] & & \text{T-dual spacetime} \arrow[d] \\
 \text{B-field} \arrow[r] & \text{3-group of phases} & \text{T-dual B-field} \arrow[l]
\end{tikzcd}
$$

$$
\mathrm{KU}_{\bullet+B}(E) \xleftarrow{\;\simeq\;} \mathrm{KU}_{\bullet-\mathrm{rk}(E)+\tilde{B}}(\tilde{E})
\qquad
\text{T-duality equivalence on D-brane charges in K-theory}
$$

2.3.3 The path integral and Secondary integral transform

The hallmark mystery in the mathematical formulation of quantization of field theory has always been the path integral: on the one hand an essentially undefined heuristics away from toy examples, on the other hand a valuable source of conjectures for profound mathematics.

Holography as in 1.3 and 2.2.4 suggests that what should fundamentally be defined by path integrals are local topological field theories in dimension d, while non-topological field theories in dimension $(d-1)$ will be induced as boundary field theories of these, and they may not and need not have a Lagrangian path integral description themselves. For topological field theories the mysteries of the traditional path integral persist, but, as we indicate now, higher geometry offers a refinement of the concept where integration of functions with values in a ring (of complex numbers) is refined by integration of functions with values in a "higher ring" (an E_∞-ring) – and that does work better: such a higher path integral is given by fiber integration if twisted generalized cohomology for which the required concept of measure is differen than the Lebesgue measures sought in constructive field theory construcitons of path integrals. Moreover, we find that this kind of path integral is a "secondary" version of catgeorified integral transforms as they are familiar from Fourier-Mukai-Hecke transformations, and this makes the n-dimensional topological field theory itself the boundary of an $(d+1)$-dimensional twisting topological field theory

codimension	quantum propagation	
2	categorified integral kernel transform	Fourier-Mukai-Hecke transform
1	path integral	fiber integration in twisted generalized cohomology
0		twisted orientation in generalized cohomology

From the discussion of Lagrangian correspondences in 2.1.4 one sees that what the path integral ought to be is a kind of integral transform acting on functions (sections, "wavefunctions") on a space of physical fields whose integral kernel is the exponentiated action function (functional) on a space of trajectories:

$$
\begin{tikzcd}
 & \boxed{\text{trajectories}} \arrow[dl, "(-)\vert_{\mathrm{in}}"'] \arrow[dr, "(-)\vert_{\mathrm{out}}"] \arrow[dd, "\exp(\frac{i}{\hbar}S(-))"] & \\
\boxed{\text{incoming fields}} & & \boxed{\text{outgoing fields}} \\
 & U(1) &
\end{tikzcd}
$$

If the space of trajectories is or were equipped with a measure $d\mu_{\mathrm{traj}}$, then the path integral is or would be the operation that takes a function ψ_{in} on the space of incoming fields to the function ψ_{out} whose value on an outgoing field configuration ϕ_{out} is given by the integral over the space of trajectories (paths) ending at ϕ_{out} of the action functional weighted by the values of ψ_{in} at trajectory's starting points:

$$\psi_{\mathrm{out}}(\phi_{\mathrm{out}}) := \int\limits_{\substack{\{\phi\in\mathrm{trajectories}\vert \\ \phi\vert_{\mathrm{out}}=\phi_{\mathrm{out}}\}}} \exp(\tfrac{i}{\hbar}S(\phi))\, \psi_{\mathrm{in}}(\phi\vert_{\mathrm{in}})\, d\mu_{\mathrm{traj}}(\phi)\,.$$

Here the function on the space of trajectories $\phi \mapsto \psi_{\mathrm{in}}(\phi\vert_{\mathrm{in}})$ appearing in the integrand is just the pullback of ψ_{in} along the map $(-)\vert_{\mathrm{in}}$. We should write that pullback as $\mathrm{in}^*\psi_{\mathrm{in}}$. On the other hand, the integral itself serves a vaguely dual purpose, in that it sends functions on the space of trajectories down to functions on the space of outgoing fields, and zo highlight this one tends to abbreviate

$$\mathrm{out}_! := \int\limits_{\substack{\{\phi\in\mathrm{trajectories}\vert \\ \phi\vert_{\mathrm{out}}=(-)_{\mathrm{out}}\}}} (-)\, d\mu_{\mathrm{traj}}\,.$$

With this notation the above path integral transform reads more compactly like so:

$$\psi_{\mathrm{out}} := \mathrm{out}_! \left(\exp(\tfrac{i}{\hbar} S) \cdot \mathrm{in}^* \psi_{\mathrm{in}}\right) \ .$$

The famous and essentially only case where this just works verbatim is that where the trajectories are Brownian motions in a finite dimensional manifold, the exponentiated action functional is taken to be real valued ("Wick rotated") and the measure $d\mu_{\mathrm{traj}}$ is the Wiener measure. In this case the above integral transform yields the standard propagator for quantum mechanics, which was essentially the insight that originally led Feynman to introduce the path integral.

Pushing the idea of constructing measures $d\mu_{\mathrm{traj}}$ on spaces of field trajectories as far as possible is the topic of "constructive quantum field theory". There have been impressive results, but typically the fields theories of practical interest are not among them.

On the other hand, if something resists being defined as an integral of complex-number valued functions so stubbornly, then maybe secretly it is not actually an integral of complex-number valued functions. Higher geometry ("derived geometry") offers some hints:

First, a correspondence as above equipped with a $U(1)$-valued function on its space of trajectories is equivalently a plain correspondence *over* $\mathbf{B}U(1)$, with trivial maps from the spaces of fields, and with the homotopy filling the diagram encoding the original $U(1)$-valued function.

$$\left\{ \begin{tikzcd} & \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "(-)|_{\mathrm{out}}"'] \arrow[dr, "(-)|_{\mathrm{in}}"] \arrow[dd, "\exp(\frac{i}{\hbar}S)"] & \\ \mathbf{Fields}_{\mathrm{in}} & & \mathbf{Fields}_{\mathrm{out}} \\ & U(1) & \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} & \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "(-)|_{\mathrm{out}}"'] \arrow[dr, "(-)|_{\mathrm{in}}"] & \\ \mathbf{Fields}_{\mathrm{in}} \arrow[dr, "0"'] \arrow[rr, phantom, "\Swarrow \exp(\frac{i}{\hbar}S)"] & & \mathbf{Fields}_{\mathrm{out}} \arrow[dl, "0"] \\ & \mathbf{B}U(1) & \end{tikzcd} \right\} .$$

This is equivalently the statement that the Lie group $U(1)$ is the loop space object, formed in higher geometry, of the moduli stack $\mathbf{B}U(1)$ of circle-principal bundles, which is the very justification for the notation $\mathbf{B}U(1)$ (the boldface being to indicate that the delooping is taken in higher geometry, and not just in plain homotopy types, which would instead yield just the traditional classifying space $BU(1)$.)

Indeed, the action functional is not actually in general a plain function on the space of trajectories, but is instead a section of a $U(1)$-principal bundle. For instance the gauge-coupling action of an electron on a spacetime X with electromagnetic field $\nabla : X \to \mathbf{B}U(1)_{\mathrm{conn}}$ is

$$\begin{tikzcd} & {[\Sigma_1, X]} \arrow[dl, "(-)|_{\mathrm{in}}"'] \arrow[dr, "(-)|_{\mathrm{out}}"] & \\ X \arrow[dr] \arrow[rr, phantom, "\Swarrow \exp(\frac{i}{\hbar}S_{\mathrm{Lor}})"] & & X \arrow[dl] \\ & \mathbf{B}U(1) & \end{tikzcd} ,$$

where $\Sigma_1 := [0,1] \hookrightarrow \mathbb{R}$ is the interval. This is the same structure as we saw before for prequantized Lagragian correspondences in 2.1.4.

Other famous anomalous action functionals of n-dimensional field theoriey are also of this form, but with $\mathbf{B}U(1)$ generalized to $\mathbf{B}^n U(1)$. The above example in one dimension higher is the action functional of the type II superstring coupled to the Kalb-Ramond B-field (see [Fiorenza-Sati-Schreiber 13a] for a detailed discussion of Freed-Witten-Kapustin anomaly cancellation of the type II superstring in the present fashion).

Second, where traditional geometry is modeled on a commutative ring (of real numbers, of complex numbers), in higher geometry this is allowed to be generalized to a "higher commutative ring", called an E_∞-ring [**?**]. We have already seen that in 2.3.2, where the E_∞-ring KU appeared.

	traditional geometry	higher geometry
	quantum mechanics	n-dimensional QFT
base ring	complex numbers ring $\mathbb{C}$	complex-oriented E_∞-ring E
group of phases	$U(1)$	$\mathbf{B}^n U(1)$
linearization/ superposition principle	$\mathbf{B}U(1) \to \mathbf{B}\mathrm{GL}_1(\mathbb{C})$	$\mathbf{B}^{n+1}U(1) \to \mathbf{B}\mathrm{GL}_1(E)$

It is an old obervation that there is a natural "categorification" of the concept of integral kernel transforms, namely the Fourier-Mukai-type operations. Here instead of functions one considers abelian stacks, namely chain complexes of quasicoherent sheaves, and analogously instead of an action function one considers such a stack τ on the space of trajectories. The Fourier-Mukai-Hecke-type integral transform is then the functor on the categories of such abelian stacks (the "derived categories") given by the same kind of formula as above

$$\psi_{\mathrm{out}} := \mathrm{in}_* \left(\tau \otimes \mathrm{in}^* \psi_{\mathrm{in}}\right) ,$$

where now in^* and in_* are the higher functors ("derived functors") of pullback (inverse image) and push-forward (direct image) of abelian stacks.

Here we have the right adjoint out_* instead of the left adjoint $\mathrm{out}_!$ that we alluded to above. But for certain $E\tau$ (...) they are in fact related by

$$\mathrm{out}_! \simeq \mathrm{out}_*(\tau \otimes (-)) .$$

from the relation to geometric Langlands it is known that these categorified integral transformations appear as quantum operators in codimension 2. This had been highlighted for instance in section 1.2 of [Witten 04].

From the discussion in 2.2.5 we know that the pre-quantum data of a codimension-2 localized field theory (dimensions $(n+1, n, n-1)$) with boundaries and corners looks like this:

corner of cobounding TQFT_{d+1} is boundary of $\mathrm{TQFT}_d^{\mathbf{L}}$ is QFT_{d-1}

boundary of cobounding TQFT_{d+1} is $\mathrm{TQFT}_d^{\mathbf{L}}$

$$\begin{tikzcd}
{*} \arrow[rr, equal] \arrow[d, equal] & & {*} \arrow[rr, equal] & & {*} \arrow[r, dotted, no head] & {} \\
{*} \arrow[d, "\vdash E"'] & \mathbf{Field}^{\partial} \arrow[l] \arrow[r] \arrow[dl, phantom, "\xi"] & \mathbf{Fields}_{\mathrm{out}} \arrow[u] \arrow[d, "\mathbf{L}_{\mathrm{out}}"] & \mathbf{Fields}_{\mathrm{traj}} \arrow[l, "(-)|_{\mathrm{out}}"'] \arrow[r, "(-)|_{\mathrm{in}}"] \arrow[dl, phantom, "\exp(\frac{i}{\hbar}S)"] & \mathbf{Fields}_{\mathrm{in}} \arrow[u] \arrow[d, "\mathbf{L}_{\mathrm{in}}"] & {} \arrow[l, dotted] \\
\mathbf{B}^{n+1}\mathrm{U}(1)_{\mathrm{conn}} \arrow[rr, equal] & & \mathbf{B}^{n+1}\mathrm{U}(1)_{\mathrm{conn}} \arrow[rr, equal] & & \mathbf{B}^{n+1}\mathrm{U}(1)_{\mathrm{conn}} \arrow[r, dotted, no head] & {}
\end{tikzcd}$$

prequantum line over point is ground ∞-ring E

moduli stacks of fields

prequantum line bundles and action functionals

∞-group of phases

bulk of cobounding TQFT_{d+1}

The Fourier-Mukai-Hecke-type higher quantum operators are supposed to act on the codimension-2 data, hence quantization here should be this:

$$\begin{aligned}
&\left(\mathbf{Fields}_{\mathrm{out}} \xleftarrow{(-)|_{\mathrm{out}}} \mathbf{Fields}_{\mathrm{traj}} \xrightarrow{(-)|_{\mathrm{in}}} \mathbf{Fields}_{\mathrm{in}} \right) \\
\mapsto &\left(\mathrm{Mod}(\mathbf{Fields}_{\mathrm{out}}) \xleftarrow{\mathrm{out}_!} \mathrm{Mod}(\mathbf{Fields}_{\mathrm{traj}}) \xleftarrow{\mathrm{in}^*} \mathrm{Mod}(\mathbf{Fields}_{\mathrm{in}}) \right)
\end{aligned}$$

The assignment to the boundary data of the $(n+1)$-dimensional cobounding theiry, which are the prequantum line bundles $\mathbf{L}$ of the n-dimensional theory is the choice of superposition principle:

$$
\left(
\begin{tikzcd}
\mathbf{B}^n U(1) \\
\mathbf{Fields} \arrow[u, "\mathbf{L}"] \arrow[d] \\
*
\end{tikzcd}
\right)
\mapsto
\left(
\begin{tikzcd}
\mathrm{Mod}(*) \arrow[d, "1 \mapsto E_{\mathrm{univ}}"'] \\
\mathrm{Mod}(\mathbf{B}^n U(1)) \arrow[d, "\mathbf{L}^*"] \\
\mathrm{Mod}(\mathbf{Fields}) \arrow[d, "\mathbf{Fields}_!"] \\
\mathrm{Mod}(*)
\end{tikzcd}
\right)
$$

Putting this together means that a quantization of the n-dimensional theory amounts to choosing coherent homotopies in the following diagram

$$
\begin{tikzcd}
\mathrm{Mod}(*) \arrow[rr, equal] & & \mathrm{Mod}(*) \arrow[ddll, Rightarrow, "(\mathbf{Fields}_{\mathrm{in}})_![\mathrm{in}]"'] \\
\mathrm{Mod}(\mathbf{Fields}_{\mathrm{out}}) \arrow[u, "(\mathbf{Fields}_{\mathrm{out}})_!"] & & \mathrm{Mod}(\mathbf{Fields})_{\mathrm{in}} \arrow[ll, "\mathrm{out}_! \mathrm{in}^*"'] \arrow[u, "(\mathbf{Fields}_{\mathrm{in}})_!"'] \arrow[ddll, Rightarrow, "\exp(\frac{i}{\hbar} S)", "\sim"'] \\
\mathrm{Mod}(\mathbf{B}^{n+1} U(1)_{\mathrm{conn}}) \arrow[u, "(\mathbf{L}_{\mathrm{out}})^*"] \arrow[rr, equal] & & \mathrm{Mod}(\mathbf{B}^{n+1} U(1)_{\mathrm{conn}}) \arrow[u, "(\mathbf{L}_{\mathrm{in}})^*"'] \\
\mathrm{Mod}(*) \arrow[u, "1 \mapsto E_{\mathrm{univ}}"] \arrow[rr, equal] & & \mathrm{Mod}(*) \arrow[u, "1 \mapsto E_{\mathrm{univ}}"']
\end{tikzcd}
$$

fundamental class [in],
dually: path integral measure $d\mu_{\mathrm{in}}$

primary integral transform
(pull-push of prequantum bundle)

integral kernel
given by action functional

universal
E-line bundle

Analyzing this one finds [Nuiten 13, Schreiber 14]:

Fact 2.54. 1. *The choice of* [in] *is a choice of fiberwise fundamental class which induces a measure* $d\mu$ *for integration in* E*-cohomology;*

2. *the composite transformation filling the diagram is by* E*-linearity determined by its unit component, where it is a morphism in* $\mathrm{Mod}(*)$ *which is a "secondary integral transform"* $\mathbb{D} \int\limits_{\mathbf{Fields}_{\mathrm{traj}}} \exp(\frac{i}{\hbar} S) d\mu :=$

$$
(\mathbf{Fields}_{\mathrm{out}})_! \mathbf{L}_{\mathrm{out}} \xleftarrow{(\mathbf{Fields}_{\mathrm{out}})_! \epsilon_{\mathbf{L}_{\mathrm{out}}}} (\mathbf{Fields}_{\mathrm{out}})_! \mathrm{out}_! \mathrm{out}^* \mathbf{L}_{\mathrm{out}} \xleftarrow{\simeq} (\mathbf{Fields}_{\mathrm{traj}})_! \mathrm{out}^* \mathbf{L}_{\mathrm{out}} \xleftarrow{(\mathbf{Fields}_{\mathrm{traj}})_! \exp(\frac{i}{\hbar} S)} (\mathbf{Fields}_{\mathrm{traj}})_! \mathrm{in}^* \mathbf{L}_{\mathrm{in}} \xleftarrow{\simeq} (\mathbf{Fields}_{\mathrm{in}})_! \mathrm{in}_! \mathrm{in}^* \mathbf{L}
$$

3. *horizontal 2-functoriality of the* $(d+1)$*-theory is the consistency of composition of this path integral, hence its anomaly cancellation.*

Proposition 2.55. *This pull-push path integral transform reproduces pull-push in twisted generalized cohomology as in [Ando-Blumberg-Gepner 11] In particular it reproduces the examples 2.51, 2.52, 2.53.*

2.3.4 Quantization of local topological defect field theory

All of the previous discussion ought to flow to and constitute aspects of what should be the following fully fledged problem of quantization.

The cobordism theorem [Lurie 09] classifies classical/prequantum field theory ([Schreiber 13a])

$$\mathrm{Bord}_n^{\sqcup} \longrightarrow \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}^n\mathbb{G}_m})^{\times}$$

and it classifies quant*ized* topological defect field theories

$$\mathrm{Bord}_n^{\sqcup} \longrightarrow E\mathrm{Mod}_n^{\otimes}\,.$$

Question: What is the process of quantization that takes the former to the latter?

Notice that most field theories of interest in nature and in theory do come with information of how they are obtained from Lagrangian data.

Example 2.56. CS/WZW duality depends on a choice of equivalence to vertex operator algebra representations. This is a remnant of the Lagrangian construcion of Chern-Simons.

Fact 2.57. *The RT construction assigns to a modular tensor category $\mathcal{C}$ a 3d TQFT in codimension ≤ 2.*

This is generally belived to extend to an anomalous 3d TQFT in codimension ≤ 3.

Fact 2.58 (Fuchs-Runkel-Schweigert)**.** *The 3d TQFT defined by a modular tensor category $\mathcal{C}$ holographically determines, for any choice of equivalence $\mathcal{C} \simeq \mathrm{Rep}(V)$ for a rational vertex operator algebra V the full rational 2-dimensional conformal field theory with local chiral sector given by V.*

What higher geometric quantization should ultimately be is a process that reads in higher prequantum data and produces such a cobordism representation.

For 3d DW theory, this is done by Freed, ... Morton.

For higher DW theory this is a grand project vaguely indicated in [FHLT09] with the first installment of details in [HopkinsLurie14].

for geometric higher prequantum theory as above discussion is in [Nuiten 13, Schreiber 14].

In summary, the quantization of pre-quantum correspondences in the slice of a cohesive ∞-topos via fiber integration in twisted stable cohomology corresponds to lifts of the original pre-quantum field theory as shown in the following diagram:

$$\begin{tikzcd}
 & & \mathrm{Corr}_n^{\mathrm{or}}(\mathbf{H}_{/\mathbf{B}\mathrm{GL}_1(\mathbf{E})})^{\otimes} \arrow[r, "\int_{(-)}(-)"] \arrow[d] & E\mathrm{Mod}_n\ , \\
\mathrm{Bord}_n^{\mathrm{sing}\,\otimes} \arrow[urrr, bend left, "\underset{\phi\in\mathbf{Fields}}{\int} \exp\left(\frac{i}{\hbar}S(\phi)\right)D\phi"] \arrow[urr, "\exp\left(\frac{i}{\hbar}S\right)D(-)"] \arrow[r, "\exp\left(\frac{i}{\hbar}S\right)"] \arrow[dr, "\mathbf{Fields}"'] & \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})^{\otimes} \arrow[r, "\rho"] \arrow[d] & \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}\mathrm{GL}_1(\mathbf{E})})^{\otimes} & \\
 & \mathrm{Corr}_n(\mathbf{H})^{\otimes} & &
\end{tikzcd}$$

Here

- $\mathbf{Fields}$ is the higher moduli stack of pre-quantum fields;
- $\exp\left(\frac{i}{\hbar}S\right)$ is the specified local action functional on $\mathbf{Fields}$, defining the given pre-quantum field theory;
- ρ is the chosen higher superposition principle, linearizing in E-cohomology;

- $\exp\left(\frac{i}{\hbar}S\right) D(-)$ is a lift of the local action functional to consistently twisted E-oriented correspondences, hence is a choice of *cohomological path integral measure* on **Fields**;
- $\int\limits_{\phi \in \mathbf{Fields}} \exp\left(\frac{i}{\hbar}S(\phi)\right) D(\phi)$ is the composition of the latter the previous item with the pull-push operation, this is the cohomological realization of the *path integral*.

Conclusion:

References

[Amati 7x] D. Amati, as cited in
E. Witten, *Universe on a String*, American Scientist Astronomy Issue 2002, www.sns.ias.edu/ witten/papers/string.pdf

[Ando-Blumberg-Gepner 11] M. Ando, A. Blumberg, D. Gepner, *Parametrized spectra, multiplicative Thom spectra, and the twisted Umkehr map* arXiv:1112.2203

[Ando-Hopkins-Strickland 01] M. Ando, M. Hopkins, N. Strickland, *Elliptic spectra, the Witten genus and the theorem of the cube*, Invent. Math. 146 (2001) 595687 MR1869850, www.math.rochester.edu/people/faculty/doug/otherpapers/musix.pdf

[Ando-Hopkins-Rezk 10] M. Ando, M. Hopkins, C. Rezk, *Multiplicative orientations of KO-theory and the spectrum of topological modular forms*, 2010, www.math.uiuc.edu/ mando/papers/koandtmf.pdf

[Arnold 89] V. Arnold, *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics (1989)

[Bates-Weinstein 97] S. Bates, A. Weinstein, *Lectures on the geometry of quantization*, Berkeley Mathematics Lecture Notes series, AMS (1997) `http://www.math.berkeley.edu/~alanw/GofQ.pdf`

[BFFLS 78] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation Theory and Quantization*, Annals of Physics (N.Y.) 111, 61, 111 (1978)

[Bongers 14] S. Bongers, *Geometric quantization of symplectic and Poisson manifolds*, MSc thesis, Utrecht, January 2014, ncatlab.org/schreiber/show/master+thesis+Bongers

[Bunke-Nikolaus-Völkl 13] U. Bunke, T. Nikolaus, M. Völkl, *Differential cohomology theories as sheaves of spectra*, arXiv:1311.3188

[Brown 73] K. Brown, *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Transactions of the American Mathematical Society, Vol. 186 (1973), 419-458

[Cattaneo-Felder 00] A. Cattaneo, G. Felder, *A path integral approach to the Kontsevich quantization formula*, Commun.Math.Phys. 212 (2000) 591-611, arXiv:math/9902090

[Cohen-Godin] Cohen, Godin *A polarized view of string topology*, in G. Segal, U. Tillmann (eds.) *Topology, Geometry and Quantum Field Theory*, LMS, Lecture Notes Series 308, arXiv:math/0303003

[Deligne-Morgan 99] P. Deligne, J. Morgen et al. *Quantum fields and strings – A course for mathematicians*, Amer. Math. Soc. Providence 1999, ncatlab.org/nlab/show/Quantum+Fields+and+Strings

[DFM 03] E. Diaconescu, D. Freed, G. Moore, *The M-theory 3-form and E8 gauge theory*, [`arXiv:hep-th/0312069`]

[Distler-Freed-Moore 11] J. Distler, D. Freed, G. Moore, *Orientifold précis*, arXiv:0906.0795, in H. Sati, U. Schreiber (eds.), *Mathematical foundations of quantum field theory and perturbative string theory*, Proceedings of Symposia in Pure Mathematics, AMS (2011)

[Dütsch-Fredenhagen 00] M. Dütsch, K. Fredenhagen, *Algebraic Quantum Field Theory, Perturbation Theory, and the Loop Expansion*, Commun.Math.Phys. 219 (2001) 5-30, arXiv:hep-th/0001129, surveyed in *Perturbative algebraic quantum field theory and deformation quantization* Proceedings of the Conference on Mathematical Physics in Mathematics and Physics, Siena June 20-25 (2000) arXiv:hep-th/0101079

[Esnault-Viehweg 88] H. Esnault, E. Vieweg, *Deligne-Beilinson cohomolog* in Rapoport, Schappacher, Schneider (eds.) *Beilinsons Conjectures on Special Values of L-Functions* Perspectives in Math. 4, Academic Press (1988), `http://www.uni-due.de/~mat903/preprints/ec/deligne_beilinson.pdf`

[Fiorenza-Rogers-Schreiber 11] D. Fiorenza, C. Rogers, U. Schreiber *A higher Chern-Weil derivation of AKSZ s-models* Int. J. Geom. Methods Mod. Phys. 10 (2013) 1250078, arXiv:1108.4378

[Fiorenza-Rogers-Schreiber 13] D. Fiorenza, C. L. Rogers, U. Schreiber, *Higher geometric prequantum theory*, arXiv:1304.0236; *L_∞-algebras of local observables from higher prequantum bundles*, Homology, Homotopy and Applications (2014) arXiv:1304.6292

[Fiorenza-Sati-Schreiber 09] D. Fiorenza, H. Sati, U. Schreiber, *Twisted differential string and Fivebrane structures*, Comm. Math. Phys. Volume 315 (2012) arXiv:0910.4001

[Fiorenza-Sati-Schreiber 12a] D- Fiorenza, H. Sati, U. Schreiber, *Multiple M5-branes, String 2-connections, and 7d nonabelian Chern-Simons theory* Advances in Theoretical and Mathematical Physics, Volume 18-2 (March-April 2014) arXiv:1201.5277

[Fiorenza-Sati-Schreiber 12b] D. Fiorenza, H. Sati, U. Schreiber, *The E_8-moduli 3-stack of the C-field in M-theory*, apparently to appear in Communications of Mathematical Physics, arXiv:1202.2455

[Fiorenza-Sati-Schreiber 12c] D. Fiorenza, H. Sati, U. Schreiber, *Extended higher cup-product Chern-Simons theories*, Journal of Geometry and Physics, Volume 74, 2013, Pages 130163, arXiv:1207.5449

[Fiorenza-Sati-Schreiber 13a] D. Fiorenza, H. Sati, U. Schreiber, *A higher stacky perspective on Chern-Simons theory*, in D. Calaque et al. (eds.) *Mathematical Aspects of Quantum Field Theories*, Springer (2014), arXiv:1301.2580

[Fiorenza-Sati-Schreiber 13b] D. Fiorenza, H. Sati, U. Schreiber *Super Lie n-algebra extensions, higher WZW models and super p-branes with tensor multiplet fields*, arXiv:1308.5264

[Fiorenza-Schreiber-Stasheff 10] D. Fiorenza, U. Schreiber, J. Stasheff, *Cech cocycles for differential characteristic classes*, Advances in Theoretical and Mathematical Physics, Volume 16 Issue 1 (2012), pages 149-250, arXiv:1011.4735

[Freed 92] D. Freed, *Higher algebraic structures and quantization*, arXiv:hep-th/9212115

[Freed 00] , *Dirac charge quantization and generalized differential cohomology*, Surveys in Differential Geometry, Int. Press, Somerville, MA, 2000, pp. 129194, arXiv:hep-th/0011220

[Frenkel 09] E. Frenkel, *Gauge Theory and Langlands Duality*, Séminaire Bourbaki, no 1010, June 2009 arXiv:0906.2747

[FHLT09] D. Freed, M. Hopkins, J. Lurie, C. Teleman, *Topological Quantum Field Theories from Compact Lie Groups*, in P. R. Kotiuga (ed.) *A celebration of the mathematical legacy of Raoul Bott* AMS (2010), arXiv:0905.0731

[Grothendieck 83] A. Grothendieck, Pursuing Stacks (1983) ncatlab.org/nlab/show/Pursuing+Stacks

[Hélein 02] F. Hélein, *Hamiltonian formalisms for multidimensional calculus of variations and perturbation theory*, contribution to *Conference on Noncompact Variational Problems and General Relativity* in Honor of H. Brezis and F.E. Browder, Oct. 2001, arXiv:math-ph/0212036

[Henneaux-Teitelboim 92] M. Henneaux, C. Teitelboim, *Quantization of gauge systems* Princeton University Press, 1992

[Hilbert 1900] D. Hilbert, Problems of mathematics

[Hopkins-Singer 02] M. Hopkins, I. Singer, *Quadratic Functions in Geometry, Topology,and M-Theory* J. Differential Geom. Volume 70, Number 3 (2005), 329-452, arXiv:math.AT/0211216

[Hopkins-Quick 12] M. Hopkins, G.Quick, *Hodge filtered complex bordism*, arXiv:1212.2173

[Jaffe-Witten-Douglas] A. Jaffe, E. Witten, M. Douglas, *Clay Millennium Problem description: Yang-Mills and Mass Gap*
www.claymath.org/millenium-problems/yangmills-and-mass-gap

[Jardine 87] J. F. Jardine, *Simplicial presheaves*, Journal of Pure and Applied Algebra 47 (1987), www.math.uwo.ca/$\tilde{\ }$jardine/papers/Fields-01.pdf

[Kapustin 10] A. Kapustin, *Topological Field Theory, Higher Categories, and Their Applications*, Proceedings of the International Congress of Mathematicians, 2010 arXiv:1004.2307

[Kapustin-Witten 06] A. Kapustin, E. Witten, *Electric-Magnetic Duality And The Geometric Langlands Program*, Communications in Number Theory and Physics, Volume 1, Number 1, 1236 (2007) arXiv:hep-th/0604151

[Langlands 14] R. Langlands, *Problems in the theory of automorphic forms: 45 years later* talk at *Symmetries and Correspondences* 2014, nlab/show/LanglandsOxford2014

[Lurie 06] J. Lurie, *Higher topos theory*, Princeton University Press 2009 arXiv:math.CT/0608040

[Lurie 09] J. Lurie, *On the Classification of Topological Field Theories* Current Developments in Mathematics Volume 2008 (2009), 129-280 arXiv:0905.0465

[Lurie 1x] J. Lurie, *Higher algebra*,
www.math.harvard.edu/ lurie/papers/HigherAlgebra.pdf

[HopkinsLurie14] M. Hopkins, J. Lurie, *Ambidexterity in $K(n)$-Local Stable Homotopy Theory*, `http://www.math.harvard.edu/~lurie/papers/Ambidexterity.pdf`

[Freed-Hopkins 00] D. Freed, M. Hopkins, *On Ramond-Ramond fields and K-theory*, JHEP 0005 (2000) 044, arXiv:hep-th/0002027

[Moore 14] G. Moore, *Physical mathematics and the future*, notes supplementing a "vision talk" at *Strings2014* www.physics.rutgers.edu/ gmoore/PhysicalMathematicsAndFuture.pdf

[MS-P-S 11] S. Müller-Stach, C. Peters, V. Srinivas, *Abelian varieties and theta functions associated to compact Riemannian manifolds; constructions inspired by superstring theory*, Journal de mathématiques pures et appliquées 98.5 (2012): 542-573. arXiv:1105.4108

[Nastase 07] H. Nastase, *Introduction to AdS-CFT*, arxiv.org/abs/0712.0689

[Nikolaus-Schreiber-Stevenson 12] T. Nikolaus, U. Schreiber, D. Stevenson, *Principal ∞-bundles – General theory and Presentations*, Journal of Homotopy Theory and Related Structures (2014), arXiv:1207.0248, arXiv:1207.0249

[Nuiten 13] J. Nuiten, *Cohomological quantization of boundary field theory* , MSc thesis, Utrecht, 2013, ncatlab.org/schreiber/show/master+thesis+Nuiten

[Rogers 10] C. Rogers, *L_∞-algebras from multisymplectic geometry*, Lett. Math. Phys. **100** (2012), 29–50. arXiv:1005.2230

[Rogers 11] C. Rogers, *Higher symplectic geometry*, PhD thesis (2011), arXiv:1106.4068

[Román-Roy 09] N. Román-Roy, *Multisymplectic Lagrangian and Hamiltonian Formalisms of Classical Field Theories*, SIGMA 5 (2009), 100, arXiv:math-ph/0506022

[Sati 11] H. Sati, *Corners in M-theory*, J. Phys. **A 44** (2011), 255402, arXiv:1101.2793

[Sati-Schreiber 11] H. Sati, U. Schreiber, *Survey of mathematical foundations of QFT and perturbative string theory.* arXiv:1109.0955 in H. Sati, U. Schreiber (eds.), *Mathematical foundations of quantum field theory and perturbative string theory*, Proceedings of Symposia in Pure Mathematics, AMS (2011), ncatlab.org/schreiber/show/AMSVolume2011

[Schaefer-Shuryak 96] T. Schaefer, E. Shuryak, *Instantons in QCD*, Rev. Mod. Phys.70:323-426 1998 arXiv:hep-ph/9610451

[Schreiber 13a] U. Schreiber, *Differential cohomology in a cohesive ∞-topos*, based on Habilitation thesis, Hamburg (October 2011), arXiv:1310.7930

[Schreiber 13b] U. Schreiber, *Synthetic quantum field theory*

[Schreiber 13c] U. Schreiber, *Classical field theory via Cohesive homotopy types*, Research Perspectives CRM Barcelona in *Trends in Mathematics*, Birkhäuser 2014 ncatlab.org/schreiber/show/Classical+field+theory+via+Cohesive+homotopy+types

[Schreiber 14] U. Schreiber, *Quantization via Linear homotopy types*, to appear in Proceedings of *Philosophy of Mechanics: Mathematical Foundations*, Paris-Diderot, Feb 2014 arXiv:1402.7041

[Seiberg-Witten 94] N. Seiberg, E. Witten, *Monopole Condensation, And Confinement In $N = 2$ Supersymmetric Yang-Mills Theory*, Nucl.Phys.B426:19-52,1994; Erratum-ibid.B430:485-486,1994 arXiv:hep-th/9407087

[Simons-Sullivan 07] J. Simons, D. Sullivan, *Axiomatic Characterization of Ordinary Differential Cohomology*, Journal of Topology 1.1 (2008): 45-56, arXiv:math/0701077

[Toën-Vezzosi 02] B. Toën, G. Vezzosi, *Homotopical Algebraic Geometry I: Topos theory*, Advances in Mathematics 193.2 (2005): 257-372 arXiv:math.AG/0207028

[Weinstein 71] A. Weinstein, *Symplectic manifolds and their lagrangian submanifolds*, Advances in Math. 6 (1971)

[Weinstein 83] A. Weinstein, *Lectures on Symplectic Manifolds*, volume 29 of CBMS Regional Conf. Series in Math. Amer. Math. Soc., 1983. third printing.

[Witten 87] E. Witten, *Elliptic Genera And Quantum Field Theory*, Commun.Math.Phys. 109 525 (1987) projecteuclid.org/euclid.cmp/1104117076

[Witten 89] E. Witten, *Quantum Field Theory and the Jones Polynomial* Commun. Math. Phys. 121 (3) (1989) 351399, projecteuclid.org/euclid.cmp/1104178138

[Witten 95] E. Witten, *String Theory Dynamics In Various Dimensions*, Nucl.Phys.B443:85-126 (1995) arXiv:hep-th/9503124

[Witten 96] E Witten, *Five-Brane Effective Action In M-Theory* J. Geom. Phys.22:103-133, 1997, hep-th/9610234

[Witten 98a] E. Witten, *Anti-de Sitter space and holography*, Advances in Theoretical and Mathematical Physics 2: 253291 (1998) arXiv:hep-th/9802150

[Witten 98] E. Witten, *Magic, mystery, and matrix*, Notices Amer. Math. Soc. **45** (1998), no. 9, 1124–1129, `http://www.sns.ias.edu/~witten/papers/mmm.pdf`

[Witten 98b] E. Witten, *AdS/CFT Correspondence And Topological Field Theory*, Journal of High Energy Physics 9812:012 (1998), arXiv:hep-th/9812012

[Witten 99] E. Witten, *Duality Relations Among Topological Effects In String Theory*, Journal of High Energy Physics, 0005:031 (2000) arXiv:hep-th/9912086

[Witten 04] E. Witten, *Conformal field theory in four and six dimensions*, in U. Tillmann (ed.) *Topology, geometry and quantum field theory* LMS Lecture Note Series (2004) arXiv:0712.0157,

[Witten 08] *Mirror Symmetry, Hitchins Equations, And Langlands Duality*, in O. Garcia-Prada, et al. (eds.), *The many faces of geoemtry: A tribute to Nigel Hitchin*, Oxford Scholarship Online, arXiv:0802.0999

[Witten 09] E. Witten *Geometric Langlands From Six Dimensions*, in P. Kotiuga (ed.) *A Celebration of the Mathematical Legacy of Raoul Bott*, AMS 2010 arXiv:0905.2720

[Witten 11] E. Witten, *Fivebranes and Knots*, arXiv:1101.3216

[Zuckerman 87] G. J. Zuckerman, *Action Principles and Global Geometry*, in S. T. Yau (ed.), *Mathematical Aspects of String Theory*, World Scientific, Singapore, 1987, pp. 259-284
`http://ncatlab.org/nlab/files/ZuckermanVariation.pdf`