Journal of Applied Probability & Statistics Vol. **3**, No. 1, 77–94

Tests for Non-Correlation of Two Infinite-Order Cointegrated Vector Autoregressive Series

Chafik Bouhaddioui
Department of Statistics, United Arab Emirates University, Al Ain, UAE.
Email: ChafikB@uaeu.ac.ae

Jean-Marie Dufour
Department of Economics, McGill University, Montreal, Canada.
Email: jean-marie.dufour@mcgill.ca

Abstract

We propose two approaches for testing non-correlation between the innovations of two nonstationary possibly cointegrated vector processes, in the general case where the processes have infinite-order autoregressive representations. The first approach is based on cross-correlation matrices, while the second approach uses partial cross-correlation matrices. We show that, under the hypothesis of non-correlation, residual cross-correlation matrices follow the same asymptotic Gaussian distribution as the corresponding cross-correlation matrices based on the true innovations, and similarly for partial cross-correlations. Portmanteau tests based on both type of residual cross-correlations are derived. A simulation study is presented to investigate the finite sample properties of the proposed tests.

Keywords: Infinite-order cointegrated vector, partial cross-correlations, test of non-correlation, portmanteau statistics, bootstrap.

2000 Mathematics Subject Classification: 62M10, 62M15.

1 Introduction

In many situations, one needs to study the relationship between two multivariate time series. In econometrics, establishing the relationship between two multivariate time series is an important question for understanding the associated economic mechanisms. In this context, many papers have studied the problem of testing independence (or the absence of serial cross-correlation, in the non-Gaussian case) between two vector processes; see Haugh [12], Koch and Yang [14], Hong [11], El Himdi and Roy [7], Duchesne and Roy [4], Pham, Roy and Cédras [19], Hallin and Saidi [9], and Bouhaddioui and Roy [2, 3]. In this article, we consider this problem when the processes involved have dimensions potentially larger than one and may be nonstationary (integrated) as well as cointegrated with infinite-order autoregressive [IVAR(∞)] representations.

Since the papers by Haugh [12], Koch and Yang [14], Hong [11] and Duchesne and Roy [4] focus on testing non-correlation between two univariate series, the other papers cited above are those most closely related to our setup. El Himdi and Roy [7] extend the procedure developed by Haugh [12] in order to test non-correlation between two time series which follow multivariate stationary invertible VARMA series, while Hallin and Saidi [9] extend the results of Koch and Yang [14] – which takes into account patterns in residual cross-correlations – to obtain more powerful tests under similar assumptions. For nonstationary series, it is usually preferable for power to directly work with the original series (without differencing), but this can create distributional complications and lead to misleading results; see Engle and Granger [8]. To meet this purpose, Pham, Roy and Cédras [19] generalize the main result of El Himdi and Roy [7] to the case of two cointegrated (partially nonstationary) VARMA series.

Since a finite-order VAR process may be a rough approximation to the true data generation process (DGP) of a given multivariate time series, the work of El Himdi and Roy [7] is extended in Bouhaddioui and Roy [2, 3] to the case of two multivariate stationary infinite-order autoregressive series VAR(∞). In [2], for the case of two uncorrelated stationary VAR(∞) time series, the authors show that an arbitrary vector of residual cross-correlation matrices, obtained by approximating the two multivariate series by a finite-order autoregressions, has the same asymptotic distribution as the corresponding vector of cross-correlation matrices between the true (unobservable) innovations, and portmanteau tests for non-correlation are derived from this result. In [3], multivariate extensions of the spectral-type procedures introduced by Hong [11] are supplied.

In this article, we propose two approaches for testing non-correlation between two IVAR(∞) processes. The first approach is a generalization of the asymptotic result of Bouhaddioui and Roy [2] mentioned above to the case of two infinite-order cointegrated autoregressive models IVAR(∞). We first prove the consistency of a residual covariance and correlation vectors. We also show that an arbitrary vector of residual cross-correlation matrices, obtained by approximating the error correction model with a finite-order autoregression, follows asymptotically the same distribution as the corresponding vector of cross-correlation matrices between the two innovation series. Using this result, we develop portmanteau tests for non-correlation between two vector processes, which are based on residual cross-correlation matrices. The second approach is based on the partial cross-correlation matrices. These can easily be computed as parameters of multivariate regression between the two residuals series. Under the null hypothesis, we find that the residual partial cross-correlation vectors follow the same distribution asymptotically as the corresponding vector of partial cross-correlations between the true innovations. Alternative portmanteau tests are derived from this result. The partial cross-correlation approach is computationally simple, reliable form the viewpoint of size, and appears to yield power gains (as indicated by simulation results) with respect to the tests based on usual cross-correlations.

The article is organized as follows. Section 2 contains preliminary results. In section 3, we study the asymptotic distribution of residual covariance and correlation vectors. In section 4, we study the case where the process can be partitioned into two uncorrelated processes and we present two procedures for testing the hypothesis of non-correlation. In section 5, we present a simulation study where we investigate the finite sample properties of the proposed tests and shows that the test based on the partial cross-correlations is more powerful. We conclude in section 6.

2 Framework and Preliminary Results

Following the notations in Saikkonen [21], Saikkonen and Lütkepohl [22], we consider a d-dimensional process $\boldsymbol{X} = \{\boldsymbol{X}_t\ ,\ t \in \mathbb{Z}\}$ partitioned into two subprocesses $\boldsymbol{X}_i = \{\boldsymbol{X}_{it}\ ,\ t \in \mathbb{Z}\}$, $i = 1, 2$, with d_1 and d_2 components respectively ($d_1 + d_2 = d$). The data generation process has the following form

$$\begin{aligned} \boldsymbol{X}_{1t} &= \boldsymbol{C}_1 \boldsymbol{X}_{2t} + \boldsymbol{\varepsilon}_{1t}, && (2.1)\\ \Delta \boldsymbol{X}_{2t} &= \boldsymbol{\varepsilon}_{2t}, && (2.2) \end{aligned}$$

where $\boldsymbol{C}_1$ is a given $(d_1 \times d_2)$ matrix, Δ is the usual difference operator and $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}'_{1t}, \boldsymbol{\varepsilon}'_{2t})'$ is a stationary process with zero mean and continuous spectral density matrix which is positive definite at zero frequency. $\boldsymbol{X}_{2t}$ is an integrated vector process of order one with no cointegrating relationship, while $\boldsymbol{X}_{1t}$ and $\boldsymbol{X}_{2t}$ are cointegrated.

Denoting by $\mathbb{I}_d$ the $(d \times d)$ identity matrix, from (2.1), taking the first differences and rearranging yields the triangular error correction representation

$$\Delta \boldsymbol{X}_t = \begin{bmatrix} -\mathbb{I}_{d_1} & \boldsymbol{C}_1 \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{X}_{t-1} + \boldsymbol{b}_t = \boldsymbol{J}\boldsymbol{\Theta}' \boldsymbol{X}_{t-1} + \boldsymbol{b}_t. \quad (2.3)$$

where $\boldsymbol{J}' = [-\mathbb{I}_{d_1} : \boldsymbol{0}]$, $\boldsymbol{\Theta}' = [\mathbb{I}_{d_1} : -\boldsymbol{C}_1]$, and $\boldsymbol{b}_t = [\boldsymbol{b}'_{1t} : \boldsymbol{b}'_{2t}]'$ is nonsingular transformation of $\boldsymbol{\varepsilon}_t$ defined by $\boldsymbol{b}_{1t} = \boldsymbol{\varepsilon}_{1t} + \boldsymbol{C}_1 \boldsymbol{\varepsilon}_{2t}$ and $\boldsymbol{b}_{2t} = \boldsymbol{\varepsilon}_{2t}$. The notation $\boldsymbol{A} = [\boldsymbol{A}_1 : \boldsymbol{A}_2]$ means that the matrix $\boldsymbol{A}$ is subdivided in matrices $\boldsymbol{A}_1$ consisting of the first columns and $\boldsymbol{A}_2$ consisting of the last columns $\boldsymbol{A}$.

Suppose that the process $\boldsymbol{b}_t$ (and hence $\boldsymbol{\varepsilon}_t$) has an infinite order autoregressive representation

$$\sum_{l=0}^{\infty} \boldsymbol{G}_j \boldsymbol{b}_{t-j} = \boldsymbol{a}_t, \qquad \boldsymbol{G}_0 = \mathbb{I}_m, \quad (2.4)$$

where $\boldsymbol{a}_t$ is a sequence of continuous white noise with $\mathbb{E}(\boldsymbol{a}_t) = \boldsymbol{0}$ and $\mathbb{E}(\boldsymbol{a}_t \boldsymbol{a}'_t) = \boldsymbol{\Sigma}_{\boldsymbol{a}}$ is a definite positive matrix, and the fourth moment exists. Denoting by $\boldsymbol{G}(z) = \boldsymbol{I}_d -$

$\sum_{l=1}^{\infty} \boldsymbol{G}_l z^l$, the stationarity hypothesis of the process $\boldsymbol{b}_t$ implies that the zeros of the equation $\det\{\boldsymbol{G}(z)\} = 0$ all lie outside the unit circle $|z| = 1$, where $\det\{\boldsymbol{A}\}$ denotes the determinant of the square matrix $\boldsymbol{A}$. A further assumption is that the coefficient matrices $\boldsymbol{G}_l$ satisfy the summability condition so that

$$\sum_{l=1}^{\infty} l^n \|\boldsymbol{G}_l\| < \infty, \qquad n \geq 1,$$

where $\|.\|$ is the Euclidean matrix norm defined by $\|\boldsymbol{A}\|^2 = tr(\boldsymbol{A}'\boldsymbol{A})$. This is a standard condition for weakly stationary processes. It ensures, for instance, that the process is well defined. Depending on n, it imposes weak restrictions on the autocorrelation structure of the process $\boldsymbol{b}_t$. Also, it implies that the process $\boldsymbol{b}_t$ and, consequently, $\boldsymbol{X}_t$ can be approximate by a finite-order autoregression. The order p of the fitted autoregression is a function of the sample size; i.e., $p = p(N)$, where p increases, at some rate, simultaneously with realization length N. In the sequel, we assume the following assumption on the finite autoregressive order.

Assumption 2.1. $N^{-1/3}p \to 0$ and $\sqrt{N}\sum_{l=p+1}^{\infty} \|\boldsymbol{G}_l\| \to 0$ as $N \to \infty$.

Using the equations (2.3) - (2.4) and rearranging terms then gives the autoregressive *error correction model* (ECM) representation

$$\Delta \boldsymbol{X}_t = \boldsymbol{\Psi}\boldsymbol{\Theta}'\boldsymbol{X}_{t-1} + \sum_{l=1}^{p} \boldsymbol{\Pi}_l \boldsymbol{\Delta X}_{t-l} + \boldsymbol{e}_t, \qquad t = p+1, p+2, \ldots, \tag{2.5}$$

where $\boldsymbol{e}_t = \boldsymbol{a}_t - \sum_{l=p+1}^{\infty} \boldsymbol{G}_l \boldsymbol{b}_{t-l}$, $\boldsymbol{\Psi} = -\sum_{l=0}^{p} \boldsymbol{G}_l \boldsymbol{J}$. Details for this derivation can be found in Saikkonen and Lütkepohl [22]. The $(d \times d_1)$ matrix $\boldsymbol{\Psi}$ is of full column rank (at least for p large enough). Note that the coefficient matrices $\boldsymbol{\Pi}_l (l = 1, \ldots, p)$ are functions of $\boldsymbol{\Theta}$ and $\boldsymbol{G}_l (l = 1, 2, \ldots)$, and they depend on p. Furthermore, the sequence $\boldsymbol{\Pi}_l (l = 1, \ldots, p)$ is absolutely summable as $p \to \infty$.

The autoregressive ECM in (2.5) can also be rewritten in a pure vector autoregressive (VAR) form

$$\boldsymbol{X}_t = \sum_{l=1}^{p+1} \boldsymbol{\Phi}_l \boldsymbol{X}_{t-l} + \boldsymbol{e}_t \tag{2.6}$$

where $\boldsymbol{\Phi}_1 = \mathbb{I}_d + \boldsymbol{\Psi}\boldsymbol{\Theta}' + \boldsymbol{\Pi}_1$, $\boldsymbol{\Phi}_l = \boldsymbol{\Pi}_l - \boldsymbol{\Pi}_{l-1}$, $l = 2, \ldots, p$ and $\boldsymbol{\Phi}_{p+1} = -\boldsymbol{\Pi}_p$. Although the parameters $\boldsymbol{\Pi}_l$ depend on p, the same is not true for the $\boldsymbol{\Phi}_l$ except for $\boldsymbol{\Phi}_{p+1}$.

Saikkonen and Lütkepohl [22] derive the asymptotic properties of the multivariate least square (LS) estimators of the VAR coefficients under a standard assumption. Let $\boldsymbol{\Phi}(p) = (\boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_p)$ be the matrix of the first p autoregressive parameter matrices in the representation (2.5) and denote by $\hat{\boldsymbol{\Phi}}(p) = (\hat{\boldsymbol{\Phi}}_1, \ldots, \hat{\boldsymbol{\Phi}}_p)$ the corresponding LS estimator. The following proposition gives a direct result on the asymptotic properties of the estimator $\hat{\boldsymbol{\Phi}}(p)$. It can be proved using the same straightforward techniques that in part (i) of Saikkonen [21, Theorem 3.2] (see also Saikkonen and Lütkepohl [22, Theroem 2]).

Proposition 2.1. *Let* $\{\boldsymbol{X}_t\}$ *a process given by* (2.6) *and assume that* $\mathbb{E}|a_{i,t}a_{j,t}a_{k,t}a_{l,t}| < \gamma_4 < \infty$*;* $1 \le i,j,k,l \le d$*. Then, under the assumption 2.1,*

$$\|\hat{\boldsymbol{\Phi}}(p) - \boldsymbol{\Phi}(p)\| \quad = \quad O_p(\frac{p^{1/2}}{N^{1/2}}).$$

Note that this proposition is formulated for the first p coefficient matrices, whereas the underlying process fitted to the data is a VAR($p+1$), where p goes to infinity with the sample size N. A details of the estimates of the $\boldsymbol{\Phi}_l$ are given in Saikkonen and Lütkepohl [22]. This result can be considered as a generalization of Lewis and Reinsel [15, Theorem 1] in the infinite order stationary vector autoregressive case. Also, in the stationary case, Paparoditis [18] established this result under the same assumption when the estimators of the parameters are based on a bootstrap procedure.

3 Asymptotic Distribution of Residual Correlation Vector

Let $\boldsymbol{X} = \{\boldsymbol{X}_t,\ t \in \mathbb{Z}\}$ be a multivariate process of dimension m which follows an infinite order cointegrated autoregressive model IVAR(∞) given by (2.3). The innovation process $\boldsymbol{a} = \{\boldsymbol{a}_t,\ t \in \mathbb{Z}\}$ whose covariance and correlation matrices are denoted respectively by $\boldsymbol{\Sigma}_{\boldsymbol{a}}$ and $\boldsymbol{\rho}_{\boldsymbol{a}}$. In the sequel, we suppose that the process $\boldsymbol{a}$ verify the following assumption

Assumption 3.1. (i) $\{\boldsymbol{a}_t\}$ is a strong white-noise process of dimension m whose matrix of covariance $\boldsymbol{\Sigma}_{\boldsymbol{a}}$ and matrix of correlation $\boldsymbol{\rho}_{\boldsymbol{a}}$.
(ii) The fourth-order moments of the vector $\boldsymbol{a}_t$ components $\{a_{i,t}, i = 1,\ldots,m\}$ exist, i.e., for all $t \in \mathbb{Z}$,

$$\mathbb{E}|a_{i,t}a_{j,t}a_{k,t}a_{l,t}| < \infty\ ,\ i,j,k,l = 1,\ldots,m.$$

Given a realization of process $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_N$ of length N, we approximate the series by a finite order autoregression VAR(p) given by (2.6). The autoregressive order p depends on the realization length N. The resultant residuals are given by

$$\hat{\boldsymbol{a}}_t = \begin{cases} \boldsymbol{X}_t - \sum_{l=1}^{p+1} \hat{\boldsymbol{\Phi}}_l \boldsymbol{X}_{t-l} & \text{if } t = p+2,\ldots,N\ , \\ \boldsymbol{0} & \text{if } t \le p, \end{cases}$$

where $\hat{\boldsymbol{\Phi}}_l$ are the OLS estimators of $\boldsymbol{\Phi}_l$. The residual covariance matrix $\boldsymbol{C}_{\hat{\boldsymbol{a}}}(j) = (\hat{c}_{ls}(j))_{m\times m}$ is given by

$$\hat{c}_{ls}(j) = \begin{cases} N^{-1}\sum_{t=j+1}^{N} \hat{a}_{l,t}\hat{a}_{s,t-j} & \text{if } 0 \le j \le N-1\ , \\ N^{-1}\sum_{t=-j+1}^{N} \hat{a}_{l,t+j}\hat{a}_{s,t} & \text{if } -N+1 \le j \le 0\ . \end{cases}$$

If we denote by $\boldsymbol{D}\{b_i\}$ a diagonal matrix whose elements are $b_1, \ldots, b_m$, we define the residual correlation matrix by

$$\boldsymbol{R}_{\hat{\boldsymbol{a}}}(j) = \boldsymbol{D}\{c_{ii}(0)^{-1/2}\}\boldsymbol{C}_{\hat{\boldsymbol{a}}}(j)\boldsymbol{D}\{c_{ii}(0)^{-1/2}\}, \quad |j| \leq N-1. \tag{3.1}$$

Denote by $\boldsymbol{c}_{\hat{\boldsymbol{a}}} = (vec(\boldsymbol{C}_{\hat{\boldsymbol{a}}}(j_1))', \ldots, vec(\boldsymbol{C}_{\hat{\boldsymbol{a}}}(j_n))')'$, where the symbol *vec* stands for the usual operator that transforms a matrix into a vector by stacking its columns. $\boldsymbol{r}_{\hat{\boldsymbol{a}}}$ is defined as above by replacing the covariance matrix by the correlation matrix. These vectors were used by El Himdi and Roy [7] in order to construct a non-correlation test between two multivariate stationary VARMA. In the nonstationary case, we also need to study the sample covariance and the sample correlation of the innovation process instead of the sample covariance of $\{\boldsymbol{X}_t\}$, because $\mathbb{E}[\boldsymbol{X}_t\boldsymbol{X}'_{t-j}]$ depends not only on the lag j but also on t.

The following theorem gives a consistent properties on the sample covariance matrix of the residual series $\hat{\boldsymbol{a}}_t$. The proof is given in the appendix of a technical report available from the author.

Theorem 3.1. *Let $\boldsymbol{X}$ be a linear process which satisfies the multivariate infinite order cointegrated autoregressive model* (2.5)*. Suppose also that the corresponding innovation process satisfies Assumption 3.1. If the Assumption 2.1 is verified, then*

$$\sqrt{N}(\boldsymbol{c}_{\hat{\boldsymbol{a}}} - \boldsymbol{c}_{\boldsymbol{a}}) \quad = \quad o_p(1). \tag{3.2}$$

In the stationary case, this result can be deduced from the main result in Hannan [10] which is also proved by Roy [20]. Thus, $\sqrt{N}\boldsymbol{c}_{\hat{\boldsymbol{a}}}$ has asymptotically a multivariate normal distribution with a complex formula for the covariance matrix which depends on the fourth-order cumulants, see again Hannan [10]. If all of these cumulants are zero, the covariance matrix formula still complex but has a more reduced form than the general case.

Using the same techniques as El Himdi and Roy [7, Theorem 1], we can deduce from Theorem 3.1 the asymptotic distribution of the residual correlation vector.

Corollary 3.1. *Let $\{\boldsymbol{X}_t, t \in \mathbb{Z}\}$ satisfy* (2.5)*. Under the same assumptions as in Theorem 3.1, we have*

$$\sqrt{N}(\boldsymbol{r}_{\hat{\boldsymbol{a}}} - \boldsymbol{r}_{\boldsymbol{a}}) \quad = \quad o_p(1). \tag{3.3}$$

4 Tests for Non-Correlation Between Subprocesses

Let $\boldsymbol{X} = \{\boldsymbol{X}_t,\ t \in \mathbb{Z}\}$ be a multivariate process of dimension m. Suppose that the process $\boldsymbol{X}$ is partitioned into two subprocesses $\boldsymbol{X}^{(h)} = \{\boldsymbol{X}_t^{(h)},\ t \in \mathbb{Z}\}$, $h = 1, 2$, with m_1 and m_2 components respectively ($m_1 + m_2 = m$). In the sequel, we suppose that for $h = 1, 2$, $\boldsymbol{X}^{(h)}$ follows an infinite order cointegrated vector autoregressive model IVAR(∞)

given by (2.3) and are uncorrelated. The innovation process $\boldsymbol{a} = \{[\boldsymbol{a}_t^{(1)'} : \boldsymbol{a}_t^{(2)'}]' , \ t \in \mathbb{Z}\}$ whose covariance and correlation matrices are given respectively by

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix}, \ \boldsymbol{\rho} = \begin{pmatrix} \boldsymbol{\rho}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\rho}_2 \end{pmatrix}.$$

4.1 Method based on cross-correlations

Given a realization of process $\boldsymbol{X}$ of length N, the residual covariance matrix is partitioned as

$$\boldsymbol{C}_{\hat{\mathbf{a}}}(j) = \begin{pmatrix} \boldsymbol{C}_{\hat{\mathbf{a}}}^{(11)}(j) & \boldsymbol{C}_{\hat{\mathbf{a}}}^{(12)}(j) \\ \boldsymbol{C}_{\hat{\mathbf{a}}}^{(21)}(j) & \boldsymbol{C}_{\hat{\mathbf{a}}}^{(22)}(j) \end{pmatrix}, \ j \in \mathbb{Z}, \tag{4.1}$$

where $\boldsymbol{C}_{\hat{\mathbf{a}}}^{(hh)}(j)$ is the residual covariance matrix of process $\boldsymbol{a}^{(h)}$, for $h = 1, 2$, and $\boldsymbol{C}_{\hat{\mathbf{a}}}^{(12)}(j)$ is the residual cross-covariance matrix given by

$$\boldsymbol{C}_{\hat{\mathbf{a}}}^{(12)}(j) = N^{-1} \sum_{t=j+1}^{N} \hat{\boldsymbol{a}}_t^{(1)} \hat{\boldsymbol{a}}_{t-j}^{(2)'}, \ 0 \le j \le N-1.$$

Also, for $-N+1 \le j \le 0$, $\boldsymbol{C}_{\hat{\mathbf{a}}}^{(12)}(-j) = \boldsymbol{C}_{\hat{\mathbf{a}}}^{(21)}(j)'$ and $\boldsymbol{C}_{\hat{\mathbf{a}}}^{(12)}(j) = 0$ for $|j| \ge N$. If we denote by $\boldsymbol{D}\{b_i\}$ a diagonal matrix whose elements are $b_1, \ldots, b_m$, the sample cross-correlation matrix at lag j is given by

$$\boldsymbol{R}_{\hat{\mathbf{a}}}^{(12)}(j) = \boldsymbol{D}_1\{c_{ii}^{-1/2}(0)\} \boldsymbol{C}_{\hat{\mathbf{a}}}^{(12)}(j) \boldsymbol{D}_2\{c_{ii}^{-1/2}(0)\}. \tag{4.2}$$

Let $j_1, \ldots, j_L$ a finite set of lags such that $|j_i| < N$, $i = 1, \ldots, L$. We denote by $\boldsymbol{r}_{\boldsymbol{a}}^{(12)}$ the cross-correlation vector of dimension Lm_1m_2 associated with the innovation series $\{\boldsymbol{a}_t^{(1)}\}$ and $\{\boldsymbol{a}_t^{(2)}\}$, that is

$$\boldsymbol{r}_{\boldsymbol{a}}^{(12)} = \Big(vec(\boldsymbol{R}_{\boldsymbol{a}}^{(12)}(j_1))', \ldots, vec(\boldsymbol{R}_{\boldsymbol{a}}^{(12)}(j_L))' \Big)',$$

Note that the non-correlation between $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$ is equivalent to the non-correlation between the corresponding innovation processes $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$, see the Proposition 2.1 in Pham *et al.* [19]. The processes $\boldsymbol{a}^{(h)}$ satisfy Assumption 3.1. The following theorem provides the asymptotic distribution of cross-correlation vector $\boldsymbol{r}_{\hat{\mathbf{a}}}^{(12)}$ under the non-correlation assumption between the two processes. The proof is also presented in the appendix of the technical report available from the corresponding author.

Theorem 4.1. *Let $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$ be two linear processes which satisfy the multivariate infinite order cointegrated autoregressive model* (2.3)*. Suppose also that the corresponding innovation processes satisfy Assumption 3.1 and that all their fourth-order cumulants are zero. If the two processes are uncorrelated and Assumption 2.1 is verified, then $\sqrt{N}\boldsymbol{r}_{\hat{\mathbf{a}}}^{(12)}$ asymptotically follow a multivariate normal distribution with mean* $\mathbf{0}$ *and covariance matrix $\mathbb{I}_L \otimes (\boldsymbol{\rho}_2 \otimes \boldsymbol{\rho}_1))$.*

4.2 Method based on partial cross-correlations

Now, one can define the partial cross-correlation as an ordinary regression coefficient matrix for the regression of $\boldsymbol{a}_t^{(1)}$ on $\boldsymbol{a}_t^{(2)}$. The regression model is such that

$$\boldsymbol{a}_t^{(1)} = \sum_{l=1}^{j} \boldsymbol{P}_l^{(12)} \boldsymbol{a}_{t-l}^{(2)} + \boldsymbol{u}_t.$$

where $\boldsymbol{u} = \{\boldsymbol{u}_t, t \in \mathbb{Z}\}$ is an *i.i.d errors* with mean $\mathbf{0}$ and regular covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{u}}$. If we denote by

$$\boldsymbol{A}^{(1)} = (\boldsymbol{a}_1^{(1)}, \ldots, \boldsymbol{a}_N^{(1)}),\ \boldsymbol{A}_{t-1}^{(2)} = \begin{bmatrix} \boldsymbol{a}_{t-1}^{(2)} \\ \vdots \\ \boldsymbol{a}_{t-j}^{(2)} \end{bmatrix},\ \boldsymbol{A}^{(2)} = (\boldsymbol{A}_0^{(2)}, \ldots, \boldsymbol{A}_{N-1}^{(2)})$$

and $\boldsymbol{U} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N)$, the regression model can be written compactly as

$$\boldsymbol{A}^{(1)} = \boldsymbol{\mathcal{P}}_{(j)}^{(12)} \boldsymbol{A}^{(2)} + \boldsymbol{U},$$

where $\boldsymbol{\mathcal{P}}_{(j)}^{(12)} = [\boldsymbol{P}_1^{(12)}, \ldots, \boldsymbol{P}_j^{(12)}]$. The OLS estimator of $\boldsymbol{\mathcal{P}}_{(j)}^{(12)}$ is given by

$$\tilde{\boldsymbol{\mathcal{P}}}_{(j)}^{(12)} = \boldsymbol{A}^{(1)} \boldsymbol{A}^{(2)'} (\boldsymbol{A}^{(2)} \boldsymbol{A}^{(2)'})^{-1}. \tag{4.3}$$

If we denote by $\tilde{\boldsymbol{p}}_{(j)}^{(12)} := vec(\tilde{\boldsymbol{\mathcal{P}}}_{(j)}^{(12)}) = ((\boldsymbol{A}^{(2)} \boldsymbol{A}^{(2)'})^{-1} \boldsymbol{A}^{(2)} \otimes \mathbb{I}_{m_1}) vec(\boldsymbol{A}^{(1)})$ and $\boldsymbol{p}_{(j)}^{(12)} = vec(\boldsymbol{\mathcal{P}}_{(j)}^{(12)})$, it has been proven, see Lütkepohl [17], that

$$\sqrt{N}(\tilde{\boldsymbol{p}}_{(j)}^{(12)} - \boldsymbol{p}_{(j)}^{(12)}) \xrightarrow{L} N(\mathbf{0}, \boldsymbol{\Gamma}_j^{-1} \otimes \Sigma_{\boldsymbol{u}}) \tag{4.4}$$

where the $jm_2 \times jm_2$ matrix $\boldsymbol{\Gamma}_j = \mathbb{E}(\boldsymbol{A}^{(2)} \boldsymbol{A}^{(2)'})$ has the (i, l)th block matrix equal to $\Gamma(i-l) = \mathbb{E}(\hat{\boldsymbol{a}}_{t-i}^{(2)} \hat{\boldsymbol{a}}_{t-l}^{(2)'})$. Thus, the residual partial cross-correlation matrices can be defined by replacing the unknown innovations by the residuals series $\hat{\boldsymbol{a}}_t^{(1)}$ and $\hat{\boldsymbol{a}}_t^{(2)}$. We can write

$$\hat{\boldsymbol{\mathcal{P}}}_{(j)}^{(12)} = \hat{\boldsymbol{A}}^{(1)} \hat{\boldsymbol{A}}^{(2)'} (\hat{\boldsymbol{A}}^{(2)} \hat{\boldsymbol{A}}^{(2)'})^{-1}, \tag{4.5}$$

where $\hat{\boldsymbol{A}}^{(1)}$ and $\hat{\boldsymbol{A}}^{(2)}$ are defined as $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ by replacing their components by the corresponding residuals series.

Proposition 4.1. *Let $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$ be two linear processes which satisfy the multivariate infinite order cointegrated autoregressive model* (2.3). *If the Assumption 2.1 is verified, then*

$$\sqrt{N}(\hat{\boldsymbol{p}}_{(j)}^{(12)} - \tilde{\boldsymbol{p}}_{(j)}^{(12)}) \xrightarrow{p} 0. \tag{4.6}$$

The proof is also presented in the appendix of the technical report available from the corresponding author.

4.3 Portmanteau tests

In order to test that two infinite-order cointegrated autoregressive processes $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$ are uncorrelated, we consider two approaches. In the first one, we consider a test statistic based on the residual cross-correlation matrices. For a finite set of lags $\{j_1, \ldots, j_n\}$, the asymptotic distribution of $\boldsymbol{r}_{\hat{\boldsymbol{a}}}^{(12)}$ given by Theorem 4.1 is particularly simple to use in the construction of tests. Since $\hat{\boldsymbol{\rho}}_h = \mathbf{R}_{\hat{\boldsymbol{a}}}^{(hh)}(0)$, $h = 1, 2$, is a consistent estimator of $\boldsymbol{\rho}_h$, we can define a test based on the cross-correlation at individual lags. One considers the test statistic

$$Q_{\hat{\boldsymbol{a}}}(j) = N vec(\boldsymbol{R}_{\hat{\boldsymbol{a}}}^{(12)}(j))' \left(\hat{\boldsymbol{\rho}}_2^{-1} \otimes \hat{\boldsymbol{\rho}}_1^{-1}\right) vec(\boldsymbol{R}_{\hat{\boldsymbol{a}}}^{(12)}(j)),$$

and under the hypothesis of non-correlation, $Q_{\hat{\boldsymbol{a}}}(j)$ is asymptotically distributed as $\chi^2_{m_1 m_2}$. Thus, for a given significance level α, $\mathcal{H}_0$ is rejected if $Q_{\hat{\boldsymbol{a}}}(j) > \chi^2_{m_1 m_2, 1-\alpha}$, where $\chi^2_{m,p}$ denotes the p-quantile of the χ^2_m distribution.

Also, the Theorem 4.1 may permit to construct another type of test statistic which depends on many lags. This type of test is a generalization of the global test proposed by Haugh [12]. The test statistic is given by

$$Q_{\hat{\boldsymbol{a}},M} = N \boldsymbol{r}_{\boldsymbol{a},\hat{M}}^{(12)'} \left(\boldsymbol{I}_{2M+1} \otimes \hat{\boldsymbol{\rho}}_2^{-1} \otimes \hat{\boldsymbol{\rho}}_1^{-1}\right) \boldsymbol{r}_{\hat{\boldsymbol{a}},M}^{(12)} = \sum_{j=-M}^{M} Q_{\hat{\boldsymbol{a}}}(j),$$

where $\boldsymbol{r}_{\hat{\boldsymbol{a}},M}^{(12)} = (vec(\boldsymbol{R}_{\hat{\boldsymbol{a}}}(-M))', \ldots, vec(\boldsymbol{R}_{\hat{\boldsymbol{a}}}(M))')'$, and $M \le N-1$ is fixed with respect to N. Under the null hypothesis, $Q_{\hat{\boldsymbol{a}},M}$ follows asymptotically a $\chi^2_{(2M+1)m_1 m_2}$ distribution. This statistic can be also expressed by using the residual autocovariances $\boldsymbol{C}_{\hat{\boldsymbol{a}}}^{(hh)}(j)$, $h = 1, 2$, and the residual cross-covariances $\boldsymbol{C}_{\hat{\boldsymbol{a}}}^{(12)}(j)$.

Let us note that, as in the univariate case (see Haugh [12]), computation of $Q_{\hat{\boldsymbol{a}},M}$ in fact uses $1/N$ as the asymptotic variance of each component $\hat{c}_{uv}^{(12)}(j)$ required in the computation. The exact variances can be considerably smaller, and a better approximation is provided by $(N-j)/N^2$. The corresponding modified statistic $\tilde{Q}_{\hat{\boldsymbol{a}},M}$ is defined by

$$\tilde{Q}_{\hat{\boldsymbol{a}},M} = \sum_{j=-M}^{M} \tilde{Q}_{\hat{\boldsymbol{a}}}(j) = \sum_{j=-M}^{M} \frac{N}{N-|j|} Q_{\hat{\boldsymbol{a}}}(j). \tag{4.7}$$

Simulation results presented by Pham et al. [19] indicate that the upper quantiles of the exact distribution of $\tilde{Q}_{\hat{\boldsymbol{a}},M}$ are better approximated by the asymptotic chi-square distribution, particularly when M is large enough in comparison with N. See also Hosking [13] and Li and McLeod [16].

The second approach is based on the residual partial cross-correlation matrices defined by (4.3). The null hypothesis of non-correlation at lag j is equivalent to $\boldsymbol{P}_j^{(12)} = \mathbf{0}$. Then, as defined in (4.4), we consider the following decomposition of $\boldsymbol{\Gamma}_j$

$$\boldsymbol{\Gamma}_j = \begin{bmatrix} \boldsymbol{\Gamma}_{j-1} & \boldsymbol{\Gamma}^*_{(j-1)} \\ \boldsymbol{\Gamma}^{*'}_{(j-1)} & \boldsymbol{\Gamma}(0) \end{bmatrix},$$

where $\boldsymbol{\Gamma}^*_{(j-1)} = [\boldsymbol{\Gamma}(m-1), \ldots, \boldsymbol{\Gamma}(1)]$. It follows from a standard matrix inversion properties that $\boldsymbol{\Gamma}_j^{-1}$ has the lower right $m_2 \times m_2$ block elements of the form $\boldsymbol{H}^{-1}$, where $\boldsymbol{H} = \boldsymbol{\Gamma}(0) - \boldsymbol{\Gamma}^{*'}_{(j-1)}\boldsymbol{\Gamma}^{-1}_{j-1}\boldsymbol{\Gamma}^*_{(j-1)}$. Now, since $\hat{\boldsymbol{P}}_j^{(12)}$ is the OLS estimator of $\boldsymbol{P}_j^{(12)}$, it can be easily deduced from (4.4) and Proposition 4.1 that

$$\sqrt{N}[vec(\hat{\boldsymbol{P}}_j^{(12)}) - vec(\boldsymbol{P}_j^{(12)})] \xrightarrow{L} N(\mathbf{0}, \boldsymbol{H}^{-1} \otimes \boldsymbol{\Sigma}_{\boldsymbol{u}}).$$

Then, under the null hypothesis, $\boldsymbol{P}_j^{(12)} = \mathbf{0}$, the statistic test

$$N vec(\boldsymbol{P}_j^{(12)})'(\boldsymbol{\Sigma}_{\hat{\boldsymbol{u}}}^{-1} \otimes \hat{\boldsymbol{H}}) vec(\boldsymbol{P}_j^{(12)}) \xrightarrow{L} \chi^2_{m_1 m_2},$$

where $\boldsymbol{\Sigma}_{\hat{\boldsymbol{u}}}$ and $\hat{\boldsymbol{H}}$ are a consistent estimators respectively of $\boldsymbol{\Sigma}_{\boldsymbol{u}}$ and $\boldsymbol{H}$. This test is equivalent to the *likelihood ratio* (LR) test where the LR statistic is given by

$$\mathcal{L}_j = -(N - jm_1m_2 - 1 - 1/2)ln[det(\hat{\boldsymbol{\Sigma}}_{\boldsymbol{u}}^{(j)})/det(\hat{\boldsymbol{\Sigma}}_{\boldsymbol{u}}^{(j-1)})], \tag{4.8}$$

where $\boldsymbol{\Sigma}_{\hat{\boldsymbol{u}}}^{(j)} = N^{-1}\sum_{t=j+1}^{N} \hat{\boldsymbol{u}}_t\hat{\boldsymbol{u}}_t'$. $\boldsymbol{\Sigma}_{\hat{\boldsymbol{u}}}^{(j-1)}$ can be viewed as the residual sum of square matrix obtained from fitting a VAR model of order $j-1$ to the same set of vector observations as used in fitting the VAR(j) model. Under the null hypothesis of non-correlation, when $\boldsymbol{P}_j^{(12)} = \mathbf{0}$, we have

$$\mathcal{L}_j \xrightarrow{L} \chi^2_{m_1 m_2}.$$

We can also consider a global test which depends on many lags. The test statistic is given by

$$\mathcal{L}_M = N\hat{\boldsymbol{p}}^{(12)'}_{(M)}(\boldsymbol{I}_M \otimes \boldsymbol{\Sigma}_{\hat{\boldsymbol{u}}}^{-1} \otimes \hat{\boldsymbol{H}})\hat{\boldsymbol{p}}^{(12)}_{(M)}, \tag{4.9}$$

and under the null hypothesis, $\mathcal{L}_M$ follows asymptotically $\chi^2_{Mm_1m_2}$ distribution.

5 Simulation Study

In the previous sections, we have studied a few asymptotic results of the distribution of the cross-correlation and partial cross-correlation vectors between two infinite cointegrated vector. For the finite sample properties, we can apply the technique of Monte Carlo (MC) tests to calculate the empirical frequencies of rejection of the null hypothesis. This procedure can be interpreted as a parametric bootstrap, see Dufour [5] and Dufour *et al.* [6]. We consider two different global models of dimension four which are described in Table 5.1. The submodels $\{\boldsymbol{X}_t^{(1)}\}$ and $\{\boldsymbol{X}_t^{(2)}\}$ are bivariate. Also, with the considered values for

Table 5.1: Time series models used in the simulation study.

MODELS	EQUATIONS	Σ_a
$\boldsymbol{AR}(1)$	$\begin{bmatrix} \boldsymbol{X}_t^{(1)} \\ \boldsymbol{X}_t^{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}^{(1)} & 0 \\ 0 & \boldsymbol{\Phi}^{(2)} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{t-1}^{(1)} \\ \boldsymbol{X}_{t-1}^{(2)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{a}_t^{(1)} \\ \boldsymbol{a}_t^{(2)} \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{\Sigma_a}^{(1)} & 0 \\ 0 & \boldsymbol{\Sigma_a}^{(2)} \end{bmatrix}$
$\boldsymbol{AR}(2)$	$\begin{bmatrix} \boldsymbol{X}_t^{(1)} \\ \boldsymbol{X}_t^{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_1^{(1)} & 0 \\ 0 & \boldsymbol{\Phi}_1^{(2)} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{t-1}^{(1)} \\ \boldsymbol{X}_{t-1}^{(2)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Phi}_2^{(1)} & 0 \\ 0 & \boldsymbol{\Phi}_2^{(2)} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{t-2}^{(1)} \\ \boldsymbol{X}_{t-2}^{(2)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{a}_t^{(1)} \\ \boldsymbol{a}_t^{(2)} \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{\Sigma_a}^{(1)} & 0 \\ 0 & \boldsymbol{\Sigma_a}^{(2)} \end{bmatrix}$
$\boldsymbol{VAR}_\delta$	$\begin{bmatrix} \boldsymbol{X}_t^{(1)} \\ \boldsymbol{X}_t^{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}^{(1)} & 0 \\ 0 & \boldsymbol{\Phi}^{(2)} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{t-1}^{(1)} \\ \boldsymbol{X}_{t-1}^{(2)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{a}_t^{(1)} \\ \boldsymbol{a}_t^{(2)} \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{\Sigma_a}^{(1)} & \boldsymbol{\Sigma_{a,\delta}}^{(12)} \\ \boldsymbol{\Sigma_{a,\delta}}^{(12)} & \boldsymbol{\Sigma_a}^{(2)} \end{bmatrix}$
	PARAMETERS VALUES	
$\boldsymbol{\Phi}^{(1)} = \begin{bmatrix} 0.4 & 0.0 \\ -1.0 & 1.0 \end{bmatrix}$	$\boldsymbol{\Phi}^{(2)} = \begin{bmatrix} 1.0 & 0.0 \\ -0.8 & 0.5 \end{bmatrix}$	$\boldsymbol{\Phi}_1^{(1)} = \begin{bmatrix} 0.6 & -0.5 \\ 0.3 & 0.4 \end{bmatrix}$
$\boldsymbol{\Phi}_1^{(2)} = \begin{bmatrix} -0.5 & -0.8 \\ -0.4 & 0.2 \end{bmatrix}$	$\boldsymbol{\Phi}_2^{(1)} = \begin{bmatrix} 0.4 & 0.5 \\ 0.6 & 0.3 \end{bmatrix}$	$\boldsymbol{\Phi}_2^{(2)} = \begin{bmatrix} -0.5 & 0.3 \\ -0.8 & 0.5 \end{bmatrix}$
$\boldsymbol{\Sigma_a}^{(1)} = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix}$	$\boldsymbol{\Sigma_a}^{(2)} = \begin{bmatrix} 1.0 & 0.75 \\ 0.75 & 1.0 \end{bmatrix}$	$\boldsymbol{\Sigma_{a,\delta}}^{(12)} = \begin{bmatrix} 1.0\delta & 0 \\ 0 & 0.05\delta \end{bmatrix}$

the autoregressive parameters as well as for the covariance matrix of the innovations, the subprocesses $\{\boldsymbol{X}_t^{(1)}\}$ and $\{\boldsymbol{X}_t^{(2)}\}$ are uncorrelated.

The bootstrap procedure to test the hypothesis $\mathcal{H}_0$ is used in the following way.

1. The Gaussian white noise of dimension four $\{\boldsymbol{a}_t\}$ is generated and the $\boldsymbol{X}_t$ values were obtained by directly solving the model difference equation. The initial values were set at zero.

2. For both series $\{\mathbf{X}_t^{(h)}\}$, $h = 1, 2$, the true models were individually estimated by conditional least square method. The autoregressive order was obtained by minimizing the AIC criterion for $p \leq P$, where P was fixed to 12. First, the residual series $\{\hat{\boldsymbol{a}}_t^{(h)}\}$, $h = 1, 2$, were cross-correlated by computing $\boldsymbol{R}_{\hat{\boldsymbol{a}}}^{(12)}(j)$ as defined by (4.2). The parameters of a regression of the residual series $\{\hat{\boldsymbol{a}}_t^{(1)}\}$ on $\{\hat{\boldsymbol{a}}_{t-M}^{(2)}\}$, $M = 1, \ldots 12$, were estimated.

3. The values of the test statistic $\tilde{Q}_{\hat{\boldsymbol{a}}}(j)$ were computed for $j = -12, \ldots, 12$ and those for $\tilde{Q}_{\hat{\boldsymbol{a}},M}$ and $\mathcal{L}_M$ for $M = 1, \ldots, 12$. We denote by $Q_{\hat{\boldsymbol{a}},j}(0)^*$ and $Q_{\hat{\boldsymbol{a}},M}(0)^*$ the test statistic based on this first simulated data.

4. Using the model estimate in 2, we generate $n = 99$ simulated sample by Monte Carlo methods. We repeat the step 2 and 3 and we denote by $Q_{\hat{\boldsymbol{a}},j}(k)^*$ and $Q_{\hat{\boldsymbol{a}},M}(k)^*$ the test statistic under $\mathcal{H}_0$ based on the k-th simulated sample ($1 \leq k$).

5. The simulated p-value $\hat{p}[Q_{\hat{\boldsymbol{a}},j}(0)^*]$ is obtained, where

$$\hat{p}[x] \quad = \quad \{1 + \sum_{k=1}^{n} \mathcal{I}[Q_{\hat{\boldsymbol{a}},j}(k)^* - x]\}/(n+1), \tag{5.1}$$

where $\mathcal{I}[x] = 1$ if $x \geq 0$ and $\mathcal{I}[x] = 0$ if $x < 0$.

6. The null hypothesis is rejected at level α if $\hat{p}[Q_{\hat{\boldsymbol{a}},j}(0)^*] \leq \alpha$.

7. Finally, for each nominal level $\alpha = 1\%, 5\%$ and 10%, and for each series length $N = 100, 200$, we obtained from the 2000 realizations, the empirical frequencies of rejection of the null hypothesis of non-correlation. The power analysis was conducted in the similar way using the model VAR$_\delta$ for different values of δ.

The empirical levels of tests based on an individual lags, $|k| = 0, 1, 2, 4, 6, 8, 10$ and 12, and a global test for different values of $M = 1, \ldots, 12$ are presented in Table 5.2. As expected, the approximation of the exact distribution by the asymptotic one seems good for both models. For the series length $N = 100$, the chi-square distribution provides a relatively good approximation for all lags at the three significance levels. At level 1%, we remark a little underrejection, especially for small lags. As expected, the chi-square approximation is improving as the length series N increases from 100 to 200 and is better when M becomes larger. Table 5.3 shows the results of the bootstrap procedure. This procedure gives similar results but slightly better than the asymptotic distribution especially for the levels 5% and 10%. In Table 5.4, we remark that we get a much better size control by using the statistic based on the partial cross-correlations vectors. We also see that at level 1%, the statistic $\mathcal{L}_M$ performs better than $\tilde{\mathcal{Q}}_M$. For the series length $N = 200$, the

Table 5.2: Empirical level of test at individual lags based on $\tilde{Q}_{\hat{a}}(j)$ and global test based on $\tilde{Q}_{\hat{a},M}$ defined by (4.7) for the VAR(1) and VAR(2) models.

	$\tilde{Q}_{\hat{a}}(j)$						$\tilde{Q}_{\hat{a}}(j)$					
	VAR(1)						VAR(2)					
	N=100			N=200			N=100			N=200		
α	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
k=-12	0.60	4.10	8.40	0.75	4.30	9.10	0.70	4.40	8.90	0.65	4.20	8.9
-10	0.65	4.10	8.30	0.65	4.40	9.30	0.80	4.70	8.80	0.85	4.60	9.20
-8	1.20	4.20	9.10	0.90	4.20	9.20	1.10	5.10	9.20	0.90	4.90	11.00
-6	0.80	5.60	10.70	1.00	5.40	10.20	0.75	4.50	10.50	0.85	4.40	10.30
-4	0.70	4.60	9.20	0.85	4.50	9.70	0.90	4.90	9.40	1.20	4.60	9.40
-2	1.10	4.50	9.30	0.90	5.10	9.60	0.95	5.20	9.50	0.90	4.70	9.80
0	1.20	5.20	11.10	1.10	5.20	10.20	0.95	5.00	10.60	0.95	5.10	10.50
2	0.90	4.20	9.10	1.30	4.50	9.50	0.90	4.70	9.20	0.85	4.30	10.20
4	0.90	4.40	9.30	0.90	5.90	11.2	1.10	5.10	10.70	1.30	5.60	11.00
6	1.30	4.50	9.50	1.40	4.10	9.90	1.20	4.70	9.75	1.40	5.30	10.10
8	1.20	4.80	9.30	0.90	4.40	9.80	1.30	5.60	10.80	0.90	5.40	10.80
10	0.90	4.85	10.60	1.10	4.90	10.40	0.90	5.10	10.85	0.85	4.80	10.70
12	1.10	5.10	10.30	1.30	4.60	11.00	1.10	4.70	11.20	0.90	4.80	10.30
	$\tilde{Q}_{\hat{a},M}$						$\tilde{Q}_{\hat{a},M}$					
$M=1$	0.70	5.20	10.50	0.80	5.40	10.60	0.70	4.90	9.80	1.10	5.60	11.00
2	0.70	5.10	9.70	0.90	5.50	9.40	0.70	5.20	9.20	0.80	4.00	10.50
3	0.90	4.60	9.30	1.00	5.20	10.10	0.90	4.50	9.60	0.90	5.40	10.20
4	0.85	5.10	10.40	0.80	5.30	10.60	0.70	5.40	9.20	0.75	5.60	10.80
5	0.60	4.30	8.40	0.70	4.50	8.70	0.55	4.30	8.30	0.60	4.70	8.65
6	0.65	5.20	9.40	0.70	5.40	11.00	0.75	4.50	9.30	0.75	5.10	10.85
7	0.70	4.55	9.60	0.70	6.00	11.10	0.60	4.75	9.50	0.70	5.60	10.50
8	0.65	4.55	9.40	0.80	5.10	9.50	0.70	4.40	9.45	0.80	4.80	9.70
9	0.55	3.90	9.30	0.70	4.60	9.70	0.65	4.20	9.50	0.65	4.80	9.65
10	0.80	4.65	10.30	0.70	4.80	10.50	0.70	5.70	9.90	0.80	5.10	10.45
11	0.90	4.55	9.80	0.70	4.50	9.70	0.80	5.50	9.90	0.80	4.80	11.00
12	0.85	4.40	10.25	0.80	4.90	9.80	0.80	5.60	10.85	0.90	5.20	10.80

results are better than $N = 100$. The rejection rates are much closer to nominal level for all levels. Also, for this size, the statistic $\mathcal{L}_M$ performs very good.

To study the power of the two tests based on the statistics $\tilde{\mathcal{Q}}_M$ and $\mathcal{L}_M$ respectively, we consider the models $\boldsymbol{VAR}_\delta(1)$ where the cross-covariance matrix is different from zero and depends on a parameter δ which controls the instantaneous dependence between the two innovation processes. For the large values of δ, the correlation is stronger and the test is more powerful. Results in Table 5.5 shows that the test based on partial cross-correlation

Table 5.3: Empirical level of test at individual lags and global test based on bootstrap proccedure for the VAR(1) and VAR(2) models.

	$\tilde{Q}_{\hat{\mathbf{a}}}(j)$						$\tilde{Q}_{\hat{\mathbf{a}}}(j)$					
	VAR(1)						VAR(2)					
	N=100			N=200			N=100			N=200		
α	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
k=-12	0.90	5.15	10.15	1.20	5.80	10.90	0.70	4.80	10.15	1.10	5.60	10.20
-10	0.95	4.90	10.60	0.90	4.80	10.50	0.85	5.05	10.60	0.80	5.20	9.90
-8	0.85	4.40	9.00	0.50	4.60	9.90	0.85	5.20	9.00	0.95	4.80	10.10
-6	0.95	5.20	10.55	1.00	5.20	10.20	0.75	4.60	10.55	1.20	5.00	10.20
-4	1.10	5.20	9.70	1.10	4.40	9.60	1.00	4.80	9.70	1.20	4.60	9.30
-2	0.85	4.20	9.15	0.90	5.20	9.70	0.65	5.10	9.15	0.70	4.40	9.80
0	1.05	5.15	11.00	1.20	5.90	10.90	1.15	5.15	11.00	0.80	5.20	10.50
2	0.80	5.10	9.55	1.10	4.90	10.00	0.90	4.90	9.55	0.80	4.40	10.00
4	0.80	5.20	10.30	1.10	6.30	11.7	0.90	4.80	10.30	1.10	5.10	10.30
6	1.00	4.75	9.75	0.90	4.20	9.70	1.20	4.75	9.75	0.90	5.30	9.70
8	0.95	4.80	10.30	0.70	4.30	9.10	0.80	5.20	10.30	0.75	5.30	9.80
10	1.20	5.20	11.00	1.30	5.80	10.60	1.40	5.50	10.70	0.90	5.90	10.50
12	1.40	5.90	10.50	1.70	5.50	11.00	1.60	5.70	10.30	1.30	5.60	10.60
	$\tilde{Q}_{\hat{\mathbf{a}},M}$						$\tilde{Q}_{\hat{\mathbf{a}},M}$					
$M=1$	0.60	5.05	10.50	0.90	5.50	11.10	0.70	4.65	9.60	0.80	5.40	10.40
2	0.90	4.55	9.35	0.60	5.20	11.10	0.75	4.80	9.80	0.80	4.80	9.50
3	0.95	4.40	9.40	1.00	5.60	11.10	0.70	4.40	9.70	0.90	5.30	11.20
4	0.80	4.80	10.10	0.80	5.40	10.60	0.60	5.10	9.60	0.85	5.30	10.60
5	0.50	5.05	10.15	0.60	5.70	11.40	0.55	5.30	9.40	0.60	4.90	9.45
6	0.55	5.10	9.75	0.70	6.30	11.10	0.50	4.30	9.30	0.55	5.10	9.85
7	0.60	4.45	9.70	0.70	6.10	11.10	0.55	4.75	9.50	0.70	5.50	10.30
8	0.65	4.35	9.45	0.80	5.10	9.30	0.80	5.40	9.45	0.85	4.70	9.70
9	0.55	3.90	9.30	0.70	4.60	9.70	0.65	4.20	9.50	0.65	4.80	9.65
10	0.85	4.55	10.00	0.60	4.70	10.10	0.70	5.60	9.90	0.80	5.10	10.45
11	0.90	4.35	9.80	0.60	4.30	9.80	0.70	4.50	9.90	0.90	4.70	11.00
12	0.80	4.45	9.85	0.80	4.90	9.50	0.60	4.85	9.65	0.95	5.10	10.60

Table 5.4: Empirical level of the global tests based on $\tilde{Q}_{\hat{\mathbf{a}},M}$ and $\mathcal{L}_M$ defined by (4.7) and (4.8) for the VAR(1) and VAR(2) models.

	$N = 100$						$N = 200$					
VAR(1)	$\tilde{Q}_{\hat{\mathbf{a}}}(j)$			$\mathcal{L}_M$			$\tilde{Q}_{\hat{\mathbf{a}}}(j)$			$\mathcal{L}_M$		
α	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
M=1	1.15	5.00	9.40	1.35	5.10	10.60	0.80	5.50	10.50	0.90	5.60	9.70
2	1.10	5.60	9.90	0.95	5.90	9.20	0.90	5.10	10.70	1.00	5.20	9.50
3	0.80	6.00	11.10	1.10	5.70	9.60	1.00	4.70	8.90	1.15	4.70	9.30
4	0.85	5.50	8.10	0.80	4.70	10.40	0.85	4.10	9.90	0.85	5.90	8.50
5	0.90	4.90	8.70	1.40	5.40	8.60	1.45	5.20	9.70	1.35	5.40	10.50
6	1.30	4.20	10.80	1.15	5.50	9.40	1.20	5.80	9.10	1.45	4.50	8.90
7	1.20	4.60	8.80	1.30	4.90	11.00	1.40	5.00	8.70	1.20	5.30	10.70
8	1.45	4.00	9.80	1.20	4.40	8.80	1.30	5.70	10.10	1.10	4.20	8.70
9	0.70	5.80	9.30	0.90	5.00	10.80	1.15	4.40	11.30	1.50	5.50	10.10
10	1.05	4.80	9.70	0.75	4.30	10.00	0.90	4.30	10.30	1.30	5.70	9.90
11	0.75	4.40	10.30	1.25	5.20	8.40	1.10	5.30	9.30	1.05	5.80	9.10
12	1.50	5.10	11.40	1.45	4.80	9.00	1.50	4.90	11.50	1.40	4.90	10.30
VAR(2)	$\tilde{Q}_{\hat{\mathbf{a}}}(j)$			$\mathcal{L}_M$			$\tilde{Q}_{\hat{\mathbf{a}}}(j)$			$\mathcal{L}_M$		
M=1	1.05	5.90	9.40	1.05	5.40	10.80	0.80	5.10	10.50	1.50	4.30	10.30
2	0,85	5.30	8.20	0.90	4.60	9.80	1.40	5.20	11.30	1.35	5.30	9.30
3	1.00	4.00	8.80	1.15	4.50	9.60	1.30	4.30	9.10	1.25	5.40	9.10
4	0.80	4.60	10.60	1.40	5.60	9.00	1.00	4.90	10.50	1.20	5.80	10.90
5	1.25	4.70	10.20	1.10	6.00	10.40	1.35	5.00	10.70	0.90	4.60	10.10
6	0.95	4.50	10.80	0.95	5.80	10.60	1.50	5.70	11.10	1.05	6.00	8.50
7	0.90	5.50	11.00	1.30	5.50	9.20	1.45	4.60	9.50	1.15	5.00	8.90
8	1.40	4.40	8.60	1.35	5.30	8.80	1.05	4.40	10.90	1.45	4.50	10.70
9	1.10	5.60	11.40	0.85	4.90	8.60	1.15	6.00	8.50	1.30	4.90	9.50
10	1.15	4.10	9.00	1.20	5.7	9.40	1.20	4.80	10.10	0.95	4.70	9.90
11	0.70	5.80	10.00	1.25	5.00	10.20	1.10	4.70	9.90	1.00	5.10	9.70
12	1.20	5.20	8.00	0.90	4.30	8.40	0.85	5.90	10.30	1.40	5.20	10.50

vectors seems more powerful than the one based on cross-correlation vectors. For the series length $N = 100$ and $\delta = 2$, the $\mathcal{L}_M$-test is more powerful. When $N = 200$, for all values of δ, the power of the $\mathcal{L}_M$-test is significantly better. Finally, we note that the power decreases as M increases since the test statistics use more lags and the considered processes have a short memory.

Table 5.5: Power level of the global tests based on $\tilde{Q}_{\hat{\mathbf{a}},M}$ and $\mathcal{L}_M$ defined by (4.7) and (4.8) under the alternative model $VAR_{\delta=1}$.

			$\tilde{Q}_{\hat{\mathbf{a}},M}$			$\mathcal{L}_M$		
		M	1%	5%	10%	1%	5%	10%
$N=100$	$\delta=1$	5	38.3	40.2	44.3	47.6	41.6	43.6
		10	34.2	37.4	38.6	36.4	40.3	43.4
		15	32.6	35.3	34.8	35.4	34.2	36.8
	$\delta=1.5$	5	44.2	48.1	50.3	46.2	48.3	52.1
		10	42.2	44.5	49.6	45.4	43.2	50.4
		15	34.4	38.2	42.4	36.5	38.8	46.2
	$\delta=2$	5	52.6	58.3	67.3	62.2	68.3	85.8
		10	46.8	54.4	60.5	54.3	62.6	72.3
		15	44.2	48.2	54.6	48.2	54.8	64.4
$N=200$	$\delta=1$	5	46.4	50.3	56.2	52.6	54.8	62.6
		10	38.1	42.3	48.8	46.3	50.2	59.2
		15	28.6	33.2	38.6	40.1	46.1	54.5
	$\delta=1.5$	5	54.4	60.3	68.2	62.8	68.4	76.8
		10	48.3	52.6	60.4	56.2	62.1	70.2
		15	40.8	44.2	51.7	53.1	58.3	66.8
	$\delta=2$	5	62.2	68.1	72.2	74.1	79.6	91.2
		10	60.8	63.5	66.2	70.5	74.2	85.8
		15	52.3	57.4	60.6	62.2	70.8	79.6

6 Conclusion

In this paper, we were interested to test the non-correlation between two infinite order cointegrated vector autoregressive series. We thus generalized the case of two cointegrated VARMA series which is considered by Pham and al. [19]. First, at modeling stage, a possible high order autoregression is fitted to each series. As mentioned by Saikkonen [21] and Saikkonen and Lütkepohl [22], the VAR modeling protects us against misspecifications of the true underlying VARMA models that may invalidate the asymptotic theory and consequently lead to possible wrong conclusions. Second, at testing stage, the methods proposed permit us to be able to draw conclusions on the original variables than on the differenced variables as used in the stationary case, see Bouhaddioui and Roy [2].

Thus, we have proposed two different methods to test the non-correlation between two infinite order cointegrated vector autoregressive series. The first one is a generalization of the method developed by Haugh [12] and El Himdi and Roy [7] for checking the non-correlation of two univariate or multivariate stationary ARMA which was also used by Pham and al. [19] in the case of two cointegrated VARMA series. We have shown that this method still valid in our more general model. The test statistic depends on the vector of residual cross-correlation matrices and we showed that asymptotically it can be approximate by a chi-square distribution. We also proposed a global portmanteau test statistic which is based on the lags j such that $|j| \leq M$. For the second method, we proposed a test statistic based on partial cross-correlation matrices. We showed that this test statistic follows asymptotically a chi-square distribution. We noted that this statistic appears more simple and easier to implement than the portmanteau statistic. Using a bootstrap procedure, we also proposed a small Monte Carlo experiment to study the size and power of the two tests. Finally, we showed that the test based on the partial cross-correlation matrices is usually at least as powerful as the statistic based on cross-correlation matrices.

Acknowledgements

The authors wish to thank a referee for helpful comments that led to a more concise and improved presentation. This work was supported by William Dow Chair in Economics (McGill University), the bank of Canada (Research Fellowship), the Canada Research Chair Program (Chair in Econometrics, Université de Montréal, Canada), the Alexander-von-Humboldt Foundation (Germany), the Institut de finance mathmatique de Montral (IFM2), the Canadian Network of Centres of Excellence [program on Mathematics of Information Technology and Complex Systems (MITACS)], the Canada Council for the Arts (Killam Fellowship), the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, the Fonds de recherche sur la société et la culture (Québec), and the Fonds de recherche sur la nature et les technologies (Québec), and the Fonds de recherche sur la nature et les technologies (Québec).

References

[1] C. Bouhaddioui and J.-M. Dufour (2008). Tests for Non-Correlation of Two Infinite-Order Cointegrated Vector Autoregressive Series. Technical Report, Dept. of Economics, McGill University, and CIRANO, Montréal, Canada.

[2] C. Bouhaddioui and R. Roy, (2006a). On the distribution of the residual cross-correlations between two uncorrelated infinite order vector autoregressive series, *Statistics and Probability Letters*, **76(1)**, 58–68.

[3] C. Bouhaddioui and R. Roy, (2006b). A generalized portmanteau test for independence of two infinite order vector autoregressive series, *Journal of Time Series Analysis*, **27(4)**, 505–544.

[4] P. Duchesne and R. Roy, (2003). Robust tests of independence between two time series, *Statistica Sinica*, **13**, 827–852.

[5] J.-M. Dufour, (2006). Monte Carlo tests with nuisance parameters: A general approach to finite-sample inference and nonstandard asymptotics in econometrics, *Journal of Econometrics*, **133(2)**, 443–477.

[6] J.-M. Dufour, D. Pelletier and É. Renault, (2006). Short run and long run causality in time series: Inference, *Journal of Econometrics*, **133(2)**, 337–362.

[7] K. El Himdi and R. Roy, (1997). Tests for non-correlation of two multivariate ARMA time series, *Canadian Journal of Statistics* **25**, 233–256.

[8] R. Engle and C. W. J. Granger, (1987). Co-integration and error correction: representation, estimation and testing, *Econometrica*, **55**, 251–276.

[9] M. Hallin and A. Saidi, (2005). Testing independance and causality between multivariate ARMA times series, *Journal of Time Series Analysis* **26**, 83–106.

[10] E. J. Hannan, (1976). The asymptotic distribution of serial covariances, *The Annals of Statistics*, **4**(2), 396–399.

[11] Y. Hong, (1996). Testing for independence between two covariance stationary time series, *Biometrika*, **83(3)**, 615–625.

[12] L. D. Haugh, (1976). Checking the independence of two covariance-stationary time series: A univariate residual cross-correlation approach, *Journal of the American Statistical Association*, **71**, 378–385.

[13] J. R. M. Hosking, (1980). The multivariate portemanteau statistic, *Journal of the American Statistical Association*, **75**, 602–608.

[14] P. Koch and S.-S. Yang, (1986). A method for testing the independance of two time series that accounts for a potential pattern in the crosscorrelation function, *Journal of the American Statistical Analysis*, **81**, 533-544.

[15] R. Lewis and G. C. Reinsel, (1985). Prediction of multivariate time series by autoregressive model fitting, *Journal of Multivariate Analysis*, **16**, 393–411.

[16] W. K. Li and A. I. McLeod, (1981). Distribution of the residual autocorrelation in multivariate ARMA time series models, *Journal of the Royal Statistical Society*, Series B, **43**, 231–239.

[17] H. Lütkepohl, (1993). Testing for causation between two variables in higher dimensional VAR models, in: Studies in Applied Econometrics, H. Schneeweiss and K. Zimmermann, Eds (1993), Springer-Verlag, Heidelberg.

[18] E. Paparoditis, (1996). Bootstrapping autoregressive and moving average parameter estimates of infinite order vector autoregressive processes, *Journal of Multivariate Analysis*, **57**, 277–296.

[19] D. T. Pham, R. Roy and L. Cédras, (2003). Tests for non-correlation of two cointegrated ARMA time series, *Journal of Time Series Analysis*, **24**, 553–577.

[20] R. Roy (1989). Asymptotic covariance structure of serial correlation in multivariate time series, *Biometrika*, **76**, 824–827.

[21] P. Saikkonen, (1992). Estimation and testing of cointegrated systems by an autoregressive approximation, *Econometric Theory*, **8**, 1–27.

[22] P. Saikkonen and H. Lütkepohl, (1996). Infinite-order cointegrated vector autoregressive processes: Estimation and inference, *Econometric Theory*, **12**, 814–844.