Math 497A: Introductory to Ramsey Theory

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1 Dynamical Systems

Let X be a set and let T be a map

$$T : X \to X$$

which we will call a **transformation** of the set; the pair (X, T) is called a **dynamical system**. We can consider the evolution of the system in discrete time by looking at the iterates of T,

$$T^n : X \to X.$$

Note that if T is a bijection of X, we can also go back in time. In this way, we have $\mathbb{Z}$ acting on X by

$$\begin{aligned} \alpha : \mathbb{Z} \times X &\to X \\ n \times x &\mapsto T^n(x), \end{aligned}$$

where

$$\begin{aligned} \alpha(0, x) &= x, \\ \alpha(m + n, x) &= \alpha(m, \alpha(n, x)). \end{aligned}$$

In the same way, we could also consider continuous actions by allowing $\mathbb{R}$ to act on X.

We are often considered about the dynamics of a single point, that is, the trajectory of that point under interations of T. Given $x \in X$, define the **orbit** of x to be the set

$$\{T^n(x) : n \in \mathbb{Z}\}.$$

2 Topological Dynamical Systems

In most cases, we want T to preserve more than just set structure. A **topological dynamical system** is a pair (X, T) where X is now a compact, metrizable, topological space and T is a homeomorphism on X.

In particular, we will consider the system where

$$X = 2^{\mathbb{Z}} := \{0,1\}^{\mathbb{Z}},$$

the set of bi-infinite binary sequences. The topology on this set is the product topology of the discrete topology. By Tychonoff's Theorem, this space is compact and we will later exhibit a metric on this space.

Given $a \in \{0,1\}^n$, a finite binary sequence, consider the cylinder sets

$$C_t[a] := \{x \in 2^{\mathbb{Z}} : x_t = a_0, \dots, x_{t+n-1} = a_{n-1}\};$$

these form a basis of the topology described above.

The transformation of interest will be ths shift map

$$T : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$$
$$(x_n)_{n\in\mathbb{Z}} \mapsto (x_{n-1})_{n\in\mathbb{Z}}.$$

This is the map which takes a bi-infinite sequence and shifts every term one space to the right.

One metric we can prescribe to X is the following: for two bi-infinite binary sequences (x_n) and (y_n), define

$$d((x_n),(y_n)) = \begin{cases} 2^{-n}, n = \min\{|i| : x_i \neq y_i\} \\ 0, \text{if } x = y \end{cases}.$$

Note that this metric only depends on the cylinder $C_0[a]$; two points are closer together if they lie in the same $C_0[a]$ where the length of a is larger. Therefore, the shift map is not an isometry with this metric.

We can also look at the set $2^{\mathbb{Z}}$ as characteristic functions of sets in $\mathbb{Z}$. That is, given a point $x \in X$, we have a set

$$A_x := \{n \in \mathbb{Z}, x_n = 1\}.$$

The shift map then corresponds to adding 1 to all the values in the set:

$$A_{Tx} = \{n + 1 : n \in A_x\}.$$

We can of course look at this instead as a 2-coloring of $\mathbb{Z}$. In this way, we can similarly consider sets $A^{\mathbb{Z}}$, where $|A| = r$, as r-colorings.

Specifically, given $i \in \{0,1\}$, $C_0[i]$ is the set of sequences shose 0^{th} position is either 0 or 1, respectively. Then, $T^n C_0[i]$ is the set whose n^{th} position is 0 or 1. Using this, we can go back and forth between the language of sets and dynamical systems. If S is a set of integers with characteristic function $x^s \in 2^{\mathbb{Z}}$ then it is clear that the following are equivalent:

$$T^n x^s \in C_0[1] \Leftrightarrow (T^n x^s)_0 = 1 \Leftrightarrow x^s_n = 1 \Leftrightarrow n \in S;$$

we call this the *transfer principle*. Therefore, we can talk about colorings of arithmetic progressions of modulus m, by looking at what cylinders the orbit of a point under $T^m, T^{2m}, \ldots$ lie in.

3 Properties of Dynamical Systems

It is easy to see that the shift map on $2^{\mathbb{Z}}$ has only two fixed points, points such that $Tx = x$. These are the points

$$(\ldots, 0, 0, 0, 0, 0, \ldots) \qquad \text{and} \qquad (\ldots, 1, 1, 1, 1, 1, \ldots).$$

The shift map also has many periodic points, points such that $T^n x = x$ for some n. For example, the point

$$(\ldots, 0, 1, 0, 1, 0, \ldots)$$

has a period of 2.

There are other points which behave similarly to periodic points in a topological manner. A point is said to be **recurrent** if for all $\varepsilon > 0$, the set

$$\{x \in \mathbb{Z} : d(T^n x, x) \leq \varepsilon\}$$

is infinite. In other words, a recurrent point will return arbitrarily close infinitely often. A point is **almost periodic** if for all $\varepsilon > 0$, the set

$$\{x \in \mathbb{Z} : d(T^n x, x) \leq \varepsilon\}$$

has bounded gaps, which depend on ε. Clearly, we have that every period point is almost periodic and every almost periodic point is recurrent.

An example of a point which is recurrent is the following infinite sequence in $10^{\mathbb{N}}$:

$$(1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 1, 1, 2, 1, 3, \ldots).$$

Since compact topological sets share properties similar to those of finite sets, it is not surprising to know that the following recurrence principle is equivalent to the pidgeonhole principle.

Theorem 1. Given a topological dynamical system (X, T) where $(U_i)_{i \in I}$ is an open cover, there exists $i \in I$ such that U_i intersects $T^n U_i$ infinitely many times.

Proof. By compactness, we can assume that I is finite. Consider the orbit of an arbitrary point $x \in X$. By the pidgeonhole principle, some U_i must contain infinitely many elements of the orbit of x, that is, the set

$$\{n \in \mathbb{Z} : T^n x \in U_i\}$$

is infinite. If we pick some $n_0 \in S$, then the set

$$U_i \cap T^{n - n_0} U_i$$

contains $T^{n_0} x$ for all $n \in S$. □

A **subsystem** of a topological system (X, T) is a closed, and therefore compact, subset of X which is T-invariant. We say that the system is **minimal** if it contains no proper subsystems. Equivalently, a system is minimal if every orbit is dense in X. Indeed, if some orbit is not dense, we can take its closure to obtain a proper subsystem. Conversely, if Y is a proper subset which is compact, then $X \setminus Y$ is non-empty and open. Since any given orbitof a point in Y is dense, it visits both Y and $X \setminus Y$ and therefore, Y is not T-invariant.

Theorem 2. (Birkhoff) Every topological dynamical system contains a point which is almost periodic.

Proof. We will prove this by showing that in a minimal system every point is almost periodic. However, we must first show that every system contains a minimal system. We will do this using Zorn's Lemma.

Consider descending chains of closed invariant subsets. Our partial ordering is reverse containment, that is $U_1 < U_2$ if $U_1 \supset U_2$. Take the intersection of a chain; this is non-empty by compactness. The intersection is also closed and invariant. Hence, we have obtained a bound. By Zorn's Lemma, there exists a "maximal" element, i.e. a subsystem which contains no smaller subsystems, which means that is is minimal.

Now, suppose there is a point in our minimal system which is not almost periodic. That is, there is a point $x \in X$ and $\varepsilon > 0$ such that for all gap sizes $m > 0$ there is some n_m where $d(T^n x, x) \geq \varepsilon$ for all $n \in [n_m, n_m + m]$. Since X is compact, the sequence $(T^{n_m} x)$ has a limit point $y \in X$. By continuity of T,

$$\begin{aligned} d(T^k y, x) &= \lim_{m \to \infty} (T^k (T^{n_m} y), x) \\ &= \lim_{m \to \infty} (T^{k+n_m} y), x). \end{aligned}$$

For large enough m, $n_m + k \in [n, n_m + m]$ and so every point will be at least ε away from x. Therefore, this orbit is not dense, which contradicts the fact that this is a minimal system. □

4 Multiple Recurrence and Van der Waerden's Theorem

We are now set to prove Van der Waerden's Theorem via toplolgical dynamics. This is done by extending the above recurrence principle to get the Multiple Recurrence Principle.

Theorem 3. (Multiple Recurrence Principle) Let $\{U_i\}_{i \in I}$ be an open cover of (X, T). Then there exists some U_i such that, for every $k \geq 1$, there exists an integer m where

$$U_i \cap T^m U_i \cap T^{2m} U_i \cap \cdots \cap T^{(k-1)m} U_i \neq \emptyset.$$

In fact, there exists infinitely many m with this property.

This theorem implies Van der Waerden's theorem: let c be an r-coloring and let $\bar{\mathcal{O}}_c$ denote the orbit closure of c. Consider the topological dynamical system $(\bar{\mathcal{O}}_c, T)$. The open cover we will choose is the set $\{C_0[0], C_0[1], \dots, C_0[r-1]\}$. By the Multiple Recurrence Principle, one of these open sets, say $C_0[i]$ such that, for every k, there exists an integer m such that

$$C_0[i] \cap T^m C_0[i] \cap T^{2m} C_0[i] \cap \cdots \cap T^{(k-1)m} C_0[i]$$

is non-empty. By the transfer principle, any element which lies in this intersection has the property that $0, m, 2m, \dots, (k-1)m$ all have the same color i; this proves the existance of a monochromatic k-AP.

The proof of the Multiple Recurrence Princple we will give was first demonstrated by Furstenberg and Weiss and mimics the original color focusing proof of Van der Waerden's theorem. We will actually be proving a weaker statement which implies the Multiple Recurrence Princple.

Proposition 4. Let $\{U_i\}_{i \in I}$ be an open cover of (X, T). Then for every $k \geq 1$, there exists some U_i, $x \in U_i$, and integer m where

$$\{x, T^m x, T^{2m} x, \dots, T^{(k-1)m} x\} \subset U_i.$$

This proposition implies the Multiple Recurrence Principle by the following argument: by compactness, we can assume the cover is finite. We then apply the proposition for every k and by, the infinite pidgeonhole principle, one of the U_i must work for k infinitely often. Therefore, this U_i will work because it allows arbitrarily long arithmetic progressions and therefore arithmetic progressions of any shorter length.

We will prove the proposition for minimal systems by proving the following asssertion: in a minimal system, for every non-empty open set U and every k, U contains a k-AP

$$x, T^m x, \dots, T^{(k-1)m} x.$$

This is equivalent to the proposition. The fact that it implies the proposition is clear and it is implied by the proposition because any minimal system X can be covered by a finite number of translates $T^n U$ of any non-empty open set U. Otherwise, the union of all the translates $\bar{U} = \bigcup T^n U$ would be a T-invariant proper open susbset of X whose compliment would be a T-invariant proper closed subset of X, thus contradicting the minimality of X.

The next lemma, which is used in the proof of the previous assertion, is the part of this proof that uses the focusing arguments.

Lemma 5. For any $J \geq 0$, there exist $x_0, \dots, x_J \in X$ such that $x_a \in U_{i_a}$, where these sets are members of an open cover which are not necessarily distinct, and there exist $m_1, \dots, m_J \geq 1$ such that

$$T^{l(m_{a+1} + \cdots + m_b)} x_b \in U_{i_a}$$

for all $0 \leq a < b \leq J$ and all $1 \leq l \leq k-1$.

Proof. We will prove this lemma by induction on J. When $J = 0$, the statement is trivially true. Suppose then that we have constructed $x_0, \dots, x_{J-1}$, $U_{i_0}, \dots, U_{i_{J-1}}$ and $r_1, \dots, r_{J-1}$ with the desired property.

It should be noted that although we will be using this lemma to prove the assertion, the proof can be completed via simultaneous induction. That is, we can also assume we have proved the assertion for $k-1$ and use it to prove the inductive step of this argument.

Let V be a small neighborhood of x_{J-1}. Then by the assertion for $k-1$, we can find a $k-1$-AP in V:

$$y, T^{m_J} y, \dots, T^{(k-2)m_J} y \in V.$$

Then let $x_J := T^{-m_J} y$ and U_{i_J} to be the element of the covering containing this point. Then it must be that

$$T^{l(m_{a+1}+\cdots+m_J)} x_J = T^{l(m_{a+1}+\cdots+m_{J-1})} T^{(l-1)m_J} y \in U_{i_a}$$

for all $a < J$ because we have that $T^{(l-1)m_J} y \in V$ and so by continuity, and picking a small enough V, we can ensure that

$$T^{l(m_{a+1}+\cdots+m_{J-1})} V \in U_{i_a}.$$

□

We can now prove the assertion by letting J equal the number of sets in our open cover and noticing that, by the pidgeonhole principle, there is some $a < b$ such that $U_{i_a} = U_{i_b}$. We then let $x := x_b$ and $m = m_{a+1} + \cdots + m_b$ and the lemma gives us the desired arithmetic progression. This proves not only the assertion but also finishes the proofs of the Multiple Recurrence Principle and Van der Waerden's Theorem.

5 Szemeredi's Theorem

By adding topological structure to the theory of dynamical systems we were able to prove Van der Waerden's theorem. In this section, we will again be adding structure, this time measure-theoretic structure, to give a proof of Szemeredi's Theorem.

Let $S \subset \mathbb{N}$. We say that S has **positive upper densitiy** if

$$\bar{d}(S) := \limsup \frac{|S \cap [1,n]|}{n} > 0;$$

equivalently, if there exists a sequence $n_i \to \infty$ of natural numbers such that

$$\lim \frac{|S \cap [1,n_i]|}{n_i} > 0.$$

We can generalize this density in the following way: we say that S has **positive upper banach density** if there exist sequences m_i and n_i such that $\lim n_i - m_i = \infty$ and

$$\bar{d}_B(S) := \lim \frac{|S \cap [m_i, n_i]|}{n_i - m_i} > 0.$$

Theorem 6. (Szemeredi's Theorem) If $S \subset \mathbb{Z}$ has positive upper banach density then S contains arithmetic progressions of arbitrary length.

The proof of this comes from the Furstenberg Multiple Recurrence Principle, a statement very similar to the Multiple Recurrence Principle which also adds measure-theoretic data which is needed when discussing this density. These notes will not, however, contain a proof of the Furstenberg Multiple Recurrence Principle.

Theorem 7. For any measure-theoretic dynamical system $(X, \mathcal{F}, \mu, T)$ and any set $A \in \mathcal{F}$ such that $\mu(A) > 0$ and any $k \in nn$, there exists $m \geq 1$ such that

$$\mu(A \cap T^m A \cap \cdots \cap T^{(k-1)m} A) > 0.$$

Before continuing, let us recall a couple of definitions which occur in this theorem. First, a measure μ is a function defined on a σ-algebra $\mathcal{F} \subset \mathcal{P}(X)$, that is, a collection of so-called measurable sets which is closed under compliments, finite unions, and contains the empty set. A map $T : X \to X$ is **measureable** if for every $A \in \mathcal{F}$, $T^{-1}A \in \mathcal{F}$. We then say that T is **measure preserving** if for all $A \in \mathcal{F}$, $\mu(T^{-1}A) = \mu(A)$. The measure-theoretic dynamical system in the theorem above is a 4-tuple including a topological space X together with a σ-algebra $\mathcal{F}$, a measure μ, and a measure-preserving map T.

Suppose as above that $S \subset \mathbb{Z}$ has positive upper banach density. Let $x^S \in 2^{\mathbb{Z}}$ be the characteristic sequence of S, i.e. $x^S_n = 1$ if $n \in S$ and 0 otherwise. Our space of consideration will be $X = \bar{\mathcal{O}}_{x^S}$, the orbit closure of x^S.

We desire our map to be the shift map T so that we can employ the same transfer principle as above, however this means we need to find an appropriate measure so that T becomes measure-preserving. To construct such a measure, we will invoke the Dirac measure: given $y \in 2^{\mathbb{Z}}$,

$$\delta_y(E) := \begin{cases} 1 & \text{if } y \in E \\ 0 & \text{if } y \notin E \end{cases}.$$

In other words, the measure of a set only depends on whether or not the specified point y is in that set. We can now consider the following sequence of measures

$$\mu_j = \frac{1}{n_j - m_j} \sum_{l=m_j}^{n_j} \delta_{T^l x^S},$$

where m_j and n_j were given from the upper banach density. In particular, if we take the measure of $C_0[1]$, then we only add 1 in the sum when $T^l x^S \in C_0[1]$ which, by the transfer principle, only occurs when $l \in S$ and so

$$\mu_j(C_0[1]) = \frac{|S \cap [m_j, n_j]|}{n_j - m_j}.$$

Since measures are just linear functionals, we can take the weak* limit point of the μ_j, which exists because the space of probability measures is compact. This gives us a measure μ^* under which the shift map is measure-preserving and such that

$$\mu^*(C_0[1]) = \bar{d}_B(C_0[1]) > 0.$$

After applying the Furstenberg Multiple Recurrence Principle, we get an arithmetic progression of arbitrary length in a manner completely analogous to the proof of Van der Waerden's theorem. In fact, this gives a stronger result, that is, the set of such points in X which give us arithmetic progressions has positive upper banach density.