Math 497A: Introductory to Ramsey Theory

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1 The Paris-Harrington Theorem

As we have seen, Gödel's first incompleteness theorem has very profound implications: there are true statements about the natural numbers that the Peano axioms alone cannot prove. Gödel's original proof however constructed a very aritifial sentence which essentially boiled down to the liar's paradox, "this statement is false". The Paris-Harrington theorem, a slight modification of the original Ramsey Theorem, was the first theorem to be exhibited which was proven to be true however could not be proved within Peano Arithmetic.

In this section, we will first prove the Paris-Harrington theorem using König's Lemma and then show that it cannot be proven within PA.

We will use the modified Ramsey notation

$$N \stackrel{*}{\to} (m)^p_r$$

to mean that whenever $[N]^p$ is r-colored, there exists a homogeneous subset $Y \subseteq [Y]$ of size at least m where

$$|Y| > \min Y;$$

any set that satisfies this last condition we will call "relatively large".

Theorem 1. (Paris-Harrington Theorem) For all $m, p, r \geq 1$, there exists N such that

$$N \stackrel{*}{\to} (m)^p_r.$$

Proof Suppose that for some m, p, r, there is no such N which satisfies the theorem. For each natural n, let

$$T_n = \{f : [n]^p \to r \text{ such that that there is no homogeneous } Y \text{ which is relatively large.}$$

This set is clearly finite for all n. Moreover, for each $f \in T_{n+1}$ there is a unique $g \in T_n$ such that f is an extension of g, that is $g \subset f$. Therefore, we have that the partially ordered set

$$T := \bigcup T_n$$

is a finite branching tree which, by the assumption that T_n is nonempty for all n, is infinite.

By König's Lemma, we then have an infinite path

$$f_1 \subset f_2 \subset \cdots$$

in T with each $f_i \in T_i$; let $f = \bigcup f_i$ which is an r-coloring of the natural numbers. By Ramsey's Theorem, there exists an infinite homogeneous $X \subset \mathbb{N}$, say

$$\{x_1 < x_2 < x_3 < \ldots\}.$$

If we now restrict this set to the first $s := x_1 + 1$ elements

$$\{x_1, x_2, \ldots, x_s\}$$

the set is a homogeneous subset of $[x_s]$ which is relatively large. This is our contradiction.

2 Provably Recursive Functions

Every Ramsey-type theorem that we have seen has been of the following form:

$$\forall x \exists y\ \varphi(x, y);$$

in the case of the Paris-Harrington theorem, we treat the natural numbers r, p, m as the single variable x and the variable y takes the role of the number we wish to find, N. We will see that Peano Arithmetic can only prove a certain kind of such statemens.

Let $\varphi(x, y)$ be a formula of arithmetic which x and y as free variables; if this formula is true for a given a and b, we write $\mathbb{N} \vDash \varphi[a, b]$. For example, if $\varphi(x, y) = \exists z(x + z = y)$ then we have that $\mathbb{N} \vDash \varphi[3, 5]$ but $\mathbb{N} \nvDash [4, 3]$; this formula defines the $<$ relation in $\mathbb{N}$. In particular, we will consider φ_{PH} which will stand for the statement of the Paris-Harrington Theorem.

It was said that we could condense multiple variables into a single variable but how is this done? The answer lies in coding. Given a sequence of natural numbers $(x_1, \ldots, x_n)$, we can use properties of prime numbers to map this sequence uniquely to the integer $p_1^{x_1} \cdots p_n^{x_n}$ where p_i is the i^{th} prime. By using this form of coding, we can turn sets into sequences of natural numbers, which we can then code into a single natural number. In fact, one can code an entire proof under successive iterations of this process.

In order to know whether or not Peano Arithmetic can prove a certain theorem, we must know what it means to be provable. Note that for any specific k, p, r, N, we can write a computer program which checks if

$$\mathbb{N} \vDash \varphi_{PH}[k, p, r, N].$$

In general, a formula $\varphi(x, y)$ is **provably recursive** if there is an algorithm M that, on input $a, b \in \mathbb{N}$, decides whether or not $\mathbb{N} \models \varphi[a, b]$ and, most importantly, that PA can prove that M always terminates.

We can think of an algorithm as a finite list of instructions which can then be coded. Therefore, the question of whether or not an algorithm terminates is the same as asking the following: does

$$PA \models \forall x \exists s (\text{for the algorithm with code } e,\ s \text{ is a code of a halting computation on input } x).$$

Since the code to check $\varphi_{PH}[a, b]$ simply uses iterated FOR-loops, we know that this formula is provably recursive.

Now we can turn our sights onto the formulas of the form

$$\forall x \exists y\ \varphi(x, y).$$

Statements like this induce a function; for example, φ_{PH} induces the function $f_{PH} : \mathbb{N} \to \mathbb{N}$ where

$$f_{PH}(a) = \text{least } b \text{ such that } \mathbb{N} \models \varphi_{PH}[a, b].$$

If a formula is provably recursive, and

$$PA \models \forall a \exists b\ \varphi[a, b]$$

then the induced function is also called provably recursive.

One last thing to note is that every primitive recursive function is provably recursive.

3 Unprovablity of Paris-Harrington

Kreisel showed that formulas whose corresponding functions grew "too fast" could not be shown in Peano Arithmetic. To understand the neccesary growth rate a function has to take in order to be unprovable, we must extend the Ackermann Hierarchy into the transfinite.

Recall that we defined

$$f_1(n) = 2n$$

and then for each positive integer, we inductively defined

$$f_{k+1}(n) = f_k^{(n)}(1).$$

Upon going to the limit ordinal ω, we used the diagonal function

$$f_\omega(n) = f_n(n).$$

We can continue this process by defining

$$f_{\omega+1}(n) = f_\omega^{(n)}(1);$$

in general, if α is an ordinal whose corresponding Ackermann function is $f_\alpha(n)$, then we can define the Ackermann function of the successor ordinal as

$$f_{\alpha+1} = f_\alpha^{(n)}(1).$$

Once we have defined $f_{\omega+n}$ for all natural n, we can define the limit Ackermann function $f_{\omega\cdot 2}(n)$ as

$$f_{\omega\cdot 2}(n) = f_{\omega+n}(n).$$

In general, if we have a limit ordinal α, and a fundamental sequence of ordinals α_n which converge to α, then we can let

$$f_\alpha(n) = f_{\alpha_n}(n).$$

In particular, we will be interested in the limit ordinal ε_0 which has

$$\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$$

as a fundamental sequence. Therefore, we have that

$$f_{\varepsilon_0}(n) = f_{\underbrace{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}_{n \text{ copies}}}(n).$$

We can now state Kreisel's theorem.

Theorem 2. (Kreisel) If f_φ is provably recursive then f_φ is eventually dominated by f_α for some $\alpha < \varepsilon_0$.

To finish demonstrating that the Paris-Harrington theorem cannot be proved in PA, we will show that its corresponding function grows faster than f_{ε_0}. To help with the computations, we will consider the function of a simplified statement: for all $p, r \geq 1$, there exists and N such that for any r-coloring of $[p+1, N]^p$ there exists and relatively large homogeneous subset $Y \subset [p+1, N]$. Let $PH(p, r)$ denote the least such N where this is true. This is the same as the usual Paris-Harrington principle where $m = p$.

We will first get a lower bound on $PH(2, r)$. To do so, split the set $[3, \infty)$ into intervals $[x, 2x)$:

$$[3, 6) \cup [6, 12) \cup [12, 24) \cup \cdots;$$

call these the type 1 subsets. If i and j lie in the same type 1 subset, color the pair $\{i, j\}$ by a 1. Now, if A is a homogeneous subset of color 1, then $A \subset [x, 2x)$ which means that $|A| \leq x$ and hence A is not relatively large.

We can define the type 2 block structure in a similar way; split $[3, \infty)$ into sets $[x, x \cdot 2^x)$:

$$[3, 24) \cup [24, 24 \cdot 2^{24}) \cup \cdots.$$

Now, color the pair $\{i, j\}$ with color 2 if i and j are in the same type 2 block but not the same type 1 block. If $A \subset [x, x \cdot 2^x)$ but this interval is further split into type 1 subintervals,

$$[x, 2x) \cup [2x, 4x) \cup \cdots \cup [x \cdot 2^{x-1}, x \cdot 2^x).$$

Since A is homogeneous, no two elements can lie in one of these type 1 subintervals and therefore $|A| \leq x$; again, A cannot be relatively large.

In general, define $g_1(x) = 2x$ and $g_{s+1}(x) = g_s^{(x)}(x)$. Then, the type $s+1$ blocks have the form $[x, g_{s+1}(x))$ and we color $\{i, j\}$ with color $s+1$ if they lie in the same type $s+1$ block but not the same type s block. However, if our homogeneous A has color $s+1$, then no two elements can lie in a single type s block, of which there are x, and so $|A| \leq x$. Therefore, we have that $PH(2, r) \geq g_r$ but these functions grow like $f_\omega(r)$.

If we continue arguing in this fashion, we see that

$$\begin{aligned} PH(3, r) &\geq f_{\omega^\omega}(r), \\ PH(4, r) &\geq f_{\omega^{\omega^\omega}}(r), \\ &\vdots \end{aligned}$$

and and therefore, the function $PH(r, r)$ grows faster than $f_{\varepsilon_0}(r)$. By Kreisel, this proves that, although the Paris-Harrington theorem is a true statement about the natural numbers, it cannot be proven in Peano Arithmetic.