Math 497A: Introductory to Ramsey Theory

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1 First Order Logic

In order to make mathematics rigorous, we need to introduce a language and rules with which we can manupulate the language so that the sentences we demonstrate actually correspond to mathematical truth. The language of arithmetic is made of the following components:

- logical symbols: $\forall, \exists, \wedge, \vee, \neg, \Rightarrow, (,), =$;
- variable symbols: $x, y, z, \ldots$;
- non-logical symbols: $+, \cdot, S, \underline{0}$.

Here, the nonlogical symbols don't have any real meaning however S is some sort of successor function and $\underline{0}$ stands for the concept of a constant.

Formulas are built out of symbols via rules. For example,

$$\begin{aligned}
&\forall x \forall y (x + y = y + x),\\
&\forall x (x = x),\\
&\underline{0} + x = x,\\
&\exists x (y + x = \underline{0}),
\end{aligned}$$

are all formulas however $()X = 00)$ is not.

A variable is free if it is not accounted for by a quantifier. In the example

$$\exists x (y + x = 0),$$

the variable y is free while the variable x is not. A **sentence** is a formula with no free variables. In the examples of formulas above, the first two are sentences while the last two are not. Formulas like

$$\underline{0} + \underline{0} = \underline{0}$$

are also sentences because they have no variables.

Depending on what mathematical structure we are interested in, we can choose a different language, i.e. different non-logical symbols. For group theory, we would need to make use of the symbols $\cdot$ and $\underline{\mathrm{e}}$; for orders, we would need $<$; for set theory, $\in$. We can then give interpretations to each of the chosen non-logical symbols via **structures**. One example of a structure is that of arithmetic,

$$(\mathbb{N}, +, \cdot, +1, 0),$$

which are the universe, and interpretations of the non-logical symbols $+, \cdot, S$, and $\underline{0}$, respectively. As this is an important structure, we will often just denote it by $\mathbb{N}$.

For the theory of groups, we have various structures, where the universes range over the various groups. For example, we could have the structure

$$(\mathbb{Z}, +, 0)$$

where $+$ and 0 are the interpretations of $\cdot$ and $\underline{\mathrm{e}}$ or we could have

$$(GL_n(\mathbb{R}), \cdot, I_n)$$

where $\cdot$ is interpreted as matrix multiplication and $\underline{\mathrm{e}}$ as the $n \times n$ identitiy matrix.

One interesting structure is

$$(\mathscr{C}^\infty(\mathbb{R}), +, \cdot, \frac{\partial}{\partial x}, e^x);$$

here, the universe is the space of smooth functions while the successor operation is that of differentiation.

We can then interpret sentences in a structure. If we have a language, $\mathscr{L}$, i.e. collection of non-logical symbols, a structure, $\mathscr{M}$, in this language is called an $\mathscr{L}$-structure and a sentence, φ, is called an $\mathscr{L}$-sentence. We write $\mathscr{M} \vDash \varphi$ to mean that φ is true in $\mathscr{M}$ under the interpretation of all symbols given in $\mathscr{M}$.

For example, in the language of arithmetic, if

$$\varphi = \forall x \forall y (x + y = y + x),$$

then we have that $\mathbb{N} \vDash \varphi$ however if we have the sentence

$$\psi = \forall x \exists y (x + y = \underline{0})$$

then $\mathbb{N} \nvDash \psi$. For examples in the theory of groups, if $\varphi = \forall x \exists y (xy = \underline{\mathrm{e}})$ and $\psi = \forall x \forall y (x \cdot y = y \cdot x)$, then $\mathbb{Z} \vDash \varphi$ but $GL_n(\mathbb{R}) \nvDash \psi$.

To show how all theorems can be formulated into sentences, consider the following

$$\varphi = \forall n \forall x \forall z [(n \geq 3 \ \wedge \ x^n + y^n = z^n) \Rightarrow (x = 0 \vee y = 0)];$$

this is Fermat's Last Theorem.

2 Peano Arithmetic

Natural questions about arithmetic arise. Which statements of arithmetic are true in $\mathbb{N}$? Can we tell which statements are true in $\mathbb{N}$? Shat is $\mathbb{N}$?

If a langauage, $\mathscr{L}$, is fixed, then an **$\mathscr{L}$-theory** is a collection of $\mathscr{L}$-sentences. One theory of arithmetic is the Peano Arithmetic axioms (PA), which were thought of around 1890 by Giuseppe Peano in order to axiomitize the natural numbers.

(P1) $\forall x(Sx \neq \underline{0})$

(P2) $\forall x \forall y(Sx = Sy \Rightarrow x = y)$

(P3) $\forall x(x + \underline{0} = x)$

(P4) $\forall x \forall y(x + Sy = S(x + y))$

(P5) $\forall x(x \cdot \underline{0} = \underline{0})$

(P6) $\forall x \forall y(x \cdot Sy = xy + y)$

(PIND) For any formula φ, $[\varphi(\underline{0}) \wedge \forall x(\varphi(x) \Rightarrow \varphi(Sx))] \Rightarrow \forall x \varphi(x)$

The last axiom, namely the induction axiom, is actually an axiom scheme as it produces a different axiom for every formula.

Clearly, $\mathbb{N} \vDash PA$; we say that $\mathbb{N}$ is a **model** of PA. Is every structure $\mathscr{M}$ for which $\mathscr{M} \vDash PA$ isomorphic to $\mathbb{N}$? The answer is, in fact, no. However, Dedekind proved that if we replace (PIND) with a second order induction axiom, then these uniquely define $\mathbb{N}$.

If a langauage, $\mathscr{L}$, is fixed, then an **$\mathscr{L}$-theory** is a collection of $\mathscr{L}$-sentences. PA is an example of a theory of arithmetic. The Peano Arithmetic is just one of many theories. In the language of groups, one theory is that of Group Theory (GT).

(G1) $\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$

(G2) $\forall x(x \cdot \underline{e} = x)$

(G3) $\forall x \exists y(x \cdot y = \underline{e})$

So we have that $\mathbb{Z} \vDash GT$, $GL_n(\mathbb{R}) \vDash GT$, that is, these groups are models of GT, however $\mathbb{N} \nvDash GT$.

Since $\mathbb{N} \vDash PA$, we have that every statement which is true in PA is true for $\mathbb{N}$. Does this mean, however, that every statement true about $\mathbb{N}$ is implied by PA? The answer to this question is no, proved by Gödel in his First Incompleteness Theorem.

3 Theories and Proof

If T is a theory, we write $T \vDash \varphi$ to indicate that if $\mathscr{M} \vDash T$ then $\mathscr{M} \vDash \varphi$. This means that if $T = GT$, then $T \vDash \varphi$ means that φ is a statement true about all groups. In other words, the study of group theory is the study of sentences which are true in all structures satisfying GT. For example, it is easy to show from the axioms that

$$GT \vDash (\forall a(a \cdot a = \underline{\text{e}})) \Rightarrow (\forall a \forall b(ab = ba)).$$

Can this deduction from the axioms be made rigorous?

Suppose that T is a set of $\mathscr{L}$-sentences where φ is in T. We say that T **proves** φ, $T \vdash \varphi$, if there exists a finite sequence of formulas $\langle \varphi_1, \ldots, \varphi_n = \varphi \rangle$ where, for all $i \leq n$, of the following is true:

(i) φ_i is an axiom in T;

(ii) φ_i is a logical axiom, i.e. one of a list of tautologies of which we have a decidable list;

(iii) modus ponens: there exists $j_i, j_2 < i$ such that $\varphi_{j_2} = (\varphi_{j_1} \Rightarrow \varphi_i)$.

In normal mathematical practice, we usually "approximate" this structure of a proof but with some effort one can formalize any proof.

However can we be sure of the following? When we prove statements for T, are they true for every model of T? Anything implied by T is provable by T?

The first answer is clearly yes; for example, everything proved in GT is true for every group. Interestingly, the answer to the second question is also yes.

Theorem 1. (Gödel Completeness Theorem) For any first order T,

$$T \vdash \varphi \Leftrightarrow T \vDash \varphi.$$

4 Consistency and Completeness

We say that a theory T is **consistent** if for no φ, we have that $T \vdash \varphi$ and $T \vdash \neg\varphi$. We say that T is **complete** if for any sentence φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

For example, GT is incomplete because the sentence

$$\varphi = \forall x \forall y (x \cdot y = y \cdot x)$$

is true for some groups but not others. A natural example of a complete theory is a model, say $\mathbb{N}$, together with the theory

$$T := T_{\mathbb{N}} = \{\varphi : \mathbb{N} \vDash \varphi\}.$$

This theory is complete and consistent however it is not decidable, that is, we cannot effectively tell whether a given formula is in T. This set of sentences has many models, even uncountable ones. However, the important question is still: is PA complete?

Theorem 2. (Gödel's First Incompleteness Theorem) If T is a decidable and consistent extension of PA, meaning it contains all the sentences in PA and possibly more, then T is incomplete.

It is important to note that even if we take any such true statement and include it in the set of axioms, this theory will still be incomplete. In fact, we don't need all the sentences in PA; the First Incompleteness Theorem still holds if T is an extension of just the first four sentences in PA.

Corollary 3. Since every statement about $\mathbb{N}$ is either true or false, there exists true statements about $\mathbb{N}$ that PA cannot prove.

To prove the theorem, Gödel constructed a sentence φ which is coded from the symbols in the language and is self-referential, which leads to a contradiction to either a proof of it or its negation. This Gödel sentence is, however, quite artificial. In the next section, we will see a theorem, the Paris-Harrington theorem, which is true about the natural numbers but cannot be proved in PA.

Gödel's other famous theorem deals with the ability to prove the consistency of a theory.

Theorem 4. (Gödel's Second Incompleteness Theorem) If T is a decidable extension of PA, then T cannot prove its own consistency.

Again, we can consider an extension of T, say T', which would be able to prove the consistency of T. However, now T' would not be able to be proven consistent from itself so we would again need to go to a larger extension.