Final exam preparation list for
MATH 497A, Introduction to Ramsey Theory

Pool 1

Problem 1

Ordinal numbers. Explain why every well-ordering is isomorphic to a unique ordinal.

Problem 2

The Hales-Jewett theorem. Outline Shelah's proof of it.

Problem 3

The Recurrence Principle for topological dynamical systems. Explain why it holds. Explain the different kinds of recurrence for points in $2^{\mathbb{Z}}$ with respect to the shift map.

Problem 4

Van der Waerden's Theorem. Explain the 'color-focusing' method for 3-AP's and two colors.

Problem 5

The infinite Ramsey theorem. Prove it for pairs and 2 colors.

Problem 6

Zorn's Lemma. Explain why it is equivalent to the axiom of choice.

Problem 7

The general finite Ramsey theorem. Show how it can be inferred from the infinite Ramsey theorem.

Problem 8

The Erdös-Rado Theorem. Explain the basic idea of the proof for the case $\kappa = \aleph_0$.

Problem 9

König's Lemma and compactness. Explain the proof.

Problem 10

The Probabilistic Method. Explain how it can be used to give a lower bound for $R(k)$.

Problem 11

The shift map on $2^{\mathbb{Z}}$. State the three principles a topological dynamical system has to satisfy to be considered *chaotic*. Argue that the shift map on $2^{\mathbb{Z}}$ has all three properties.

Problem 12

Turán's Theorem. Explain how the proof works.

Problem 13

The Paris-Harrington Theorem. Explain why Paris and Harrington's modification of the finite Ramsey theorem produces rapidly growing Ramsey functions.

Problem 14

The Axiom of Choice. Explain how it is used to produce a non-measurable set.

Problem 15

The concept of cardinality and cardinal numbers. Sketch a proof of the Cantor-Schröder-Bernstein Theorem.

Pool 2

Problem 1

Deduce the infinite Pigeonhole Principle from the recurrence principle for sets.

Problem 2

Use the Cantor Normal Form of an ordinal to construct for every $\alpha < \varepsilon_0$ a fundamental sequence α_n, i.e. an increasing sequence of ordinals such that $\alpha_n \to \alpha$.

Problem 3

Use the Hales-Jewett Theorem to show that if $\mathbb{Z}^d$ is r-colored, and $F \subseteq \mathbb{Z}^d$ is finite, then there exist $v \in \mathbb{Z}^d$ and $t \neq 0$ such that all points of the form

$$v + tw, \qquad w \in F$$

are of the same color.

Problem 4

Show that if $\mathbb{N}$ is finitely colored then there exist infinitely many *distinct* integers x and y such that $\{x, y, x + y\}$ are monochromatic.

Problem 5

The following is a "multidimensional" recurrence theorem:

> Let X be a compact topological space, and let $T_1, \ldots, T_n$ be commuting homeomorphisms on X. Let $(U_i)_{i \in I}$ be an open cover of X. Then there exists U_i such that
>
> $$T_1^{-r} U_i \cap \cdots \cap T_n^{-r} U_i \neq \emptyset.$$
>
> for infinitely many r.

Use it to show that if $\mathbb{Z}^d$ is r-colored, and $F \subseteq \mathbb{Z}^d$ is finite, then there exist $v \in \mathbb{Z}^d$ and $t \neq 0$ such that all points of the form

$$v + tw, \qquad w \in F$$

are of the same color.

Problem 6

Let $S \subseteq \mathbb{R}^2$ with d the usual Euclidean distance. The *diameter* of S is given by

$$d(S) = \sup\{d(x, y) \colon x, y \in S\}.$$

Suppose now $S = \{x_1, x_2, \ldots, x_n\}$ and $d(S) \leq 1$. Show that the maximum number of pairs of points x, y in S with $d(x, y) > 1/\sqrt{2}$ is $\lfloor n^2/3 \rfloor$.

Show further that this bound is sharp by exhibiting, for each n, a set of diameter 1 with exactly $\lfloor n^2/3 \rfloor$ pairs of points at distance $> 1/\sqrt{2}$.

Problem 7

Infer the following statement form Van der Waerden's Theorem, without using Szemeredi's Theorem:

> Let (a_n) be a strictly increasing sequence of natural numbers so that for some $c \geq 1$, $a_{n+1} - a_n < c$ for all n. Show that the set $S = \{a_n : n \in \mathbb{N}\}$ contains arbitrarily long (finite) arithmetic progressions.

Give a counterexample that shows that S need not contain infinitely long arithmetic progressions.

Problem 8

Show that Ramsey's theorem is not true for ω-subsets by producing a 2-coloring of $[\mathbb{N}]^\omega$ with no infinite homogeneous set.

Problem 9

Say that a set of points in $\mathbb{R}^2$ is in *general position* if no three of them lie on the same straight line.

Show that given five points in $\mathbb{R}^2$ in general position, some four of them form the corners of a convex quadrilateral.

Use this and Ramsey's Theorem to show that

> for all n there exists an N such that for any N points in $\mathbb{R}^2$ in general position, there are n points that form a convex n-gon.

Problem 10

Show that for any cardinal κ, $2^\kappa \not\longrightarrow (\kappa^+)^2_2$.

Problem 11

Let $(X, <)$ be a linearly ordered set. Let $G_X = (V, E)$ be the following graph: The set of nodes is $V = [X]^2$, i.e. pairs of (distinct) elements of X. There is an edge between $\{x < y\}$ and $\{x' < y'\}$ if and only if $y = x'$. (Strictly speaking, this is a directed graph. But we interpret it as undirected.)

Show that G does not contain a K_3 subgraph.

Show further that for any n, there exists a finite X such that G_X has chromatic number $\geq n$, i.e. one needs at least n colors to color the vertices of G_X so that no two adjacent vertices have the same color.

Problem 12

Let $r \geq 2$, $p \geq 1$, and let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. Assume that for every r-coloring of $[\mathbb{N}]^p$ there exists a member $A \in \mathcal{F}$ such that $[A]^p$ is monochromatic. Show that there exists an $N > 0$ such that for every r-coloring of $[N]^p = [\{1, \ldots, N\}]^p$, there exists an $A \in \mathcal{F}$ such that $A \subseteq [1, N]$ and $[A]^p$ is monochromatic.

Problem 13

A family $\mathcal{S}$ of sets is a Δ-*system* if there exists a set X such that for all $A, B \in \mathcal{S}$, $A \cap B = X$.

Show that if $\mathcal{S}$ is an infinite family of sets, each set of cardinality $k \in \mathbb{N}$, then it contains an infinite Δ-system.

Problem 14

Show that for any $k, r \geq 1$ there exists a set S of positive integers such that S does not contain a $(k+1)$-AP, but for every r-coloring of S there exists a monochromatic k-AP in S.

Problem 15

Assuming Szemeredi's Theorem, prove the following finite version of Szemeredi's Theorem: For every $k \geq 1$ and $\varepsilon > 0$ there is some N with the property that any subset of $\{1, ..., N\}$ with more than $\lfloor \varepsilon N \rfloor$ elements contains an arithmetic progression of length k.