Assignment 1

Problem 1. *Let B_t be a standard Brownian motion, $s, t \in [0, T]$. Prove that $E(B_t B_s) = t \wedge s$.*

Problem 2. *Define the partition τ_n on $[0, T]$ as $\tau_n : \; 0 = t_0 < t_1 < \ldots t_{n-1} < t_n = T$ and $\delta = \max_{0 \le i \le n-1}(t_{i+1} - t_i)$. Prove that the quadratic variation accumulated until T of the standard Brownian process is given by*

$$[B, B](T) = \lim_{\delta \to 0} \sum_{i=0}^{n-1} \left(B_{t_{i+1}} - B_{t_i}\right)^2 = T.$$

Problem 3. *Consider that B_{1_t} and B_{2_t} are standard Brownian process on $0 \le t \le T$. Let $0 \le \rho \le 1$. Define*

$$A_t := \rho B_{1_t} + \sqrt{1 - \rho^2} B_{2t}$$

Compute:

(i) $E(A_t)$ *and* $Var(A_t)$,

(ii) $Var(A_{t+u} - A_t)$,

(iii) $Cov(A_t, B_{1_t})$ *and* $Cov(A_t, B_{2_t})$.

Problem 4. *Let B_t the standard Brownian process and its natural filtration $\mathcal{F}_s = \sigma(B_x, x \le s)$. Compute:*

(i) $E(B_t | \mathcal{F}_s)$,

(ii) $E(B_t^2 | \mathcal{F}_s) \quad s < t$.

Problem 5. *The Itô integral $I_t = \int_0^T \sigma_u dB_u$ is the mean square convergence when $n \to \infty$ of $\sum_{i=1}^n \sigma_{t_{i-1}} \left(B_{t_i} - B_{t_{i-1}}\right)$ where σ_t is adaptad to $\mathcal{F}_t = \sigma(B_s, s \le t)$ and is writen as*

$$\lim_{n \to \infty} E\left[\left(\sum_{i=1}^n \sigma_{t_{i-1}} \left(B_{t_i} - B_{t_{i-1}}\right) - \int_0^T \sigma_u dB_u\right)^2\right] = 0.$$

Prove

(i) that the quadratic variation of I_t is $[I, I](t) = \int_0^t \sigma_u^2 du$,

(ii) that I_t is martingale with respect to $\mathcal{F}_s$.

Problem 6. *Consider the Itô process on* $0 \leq t \leq T$

$$X_t = x + \int_0^t \mu\left(u, X_u\right) du + \int_0^t \sigma\left(u, X_u\right) dB_u,$$

where $X_0 = x$ *is the initial value* $\mu\left(t, X_t\right)$ *and* $\sigma\left(t, X_t\right)$ *are adapted to the natural filtration of* B_t*,* $\mathcal{F}_t$*. Find* $\left[X, X\right]\left(T\right)$*.*