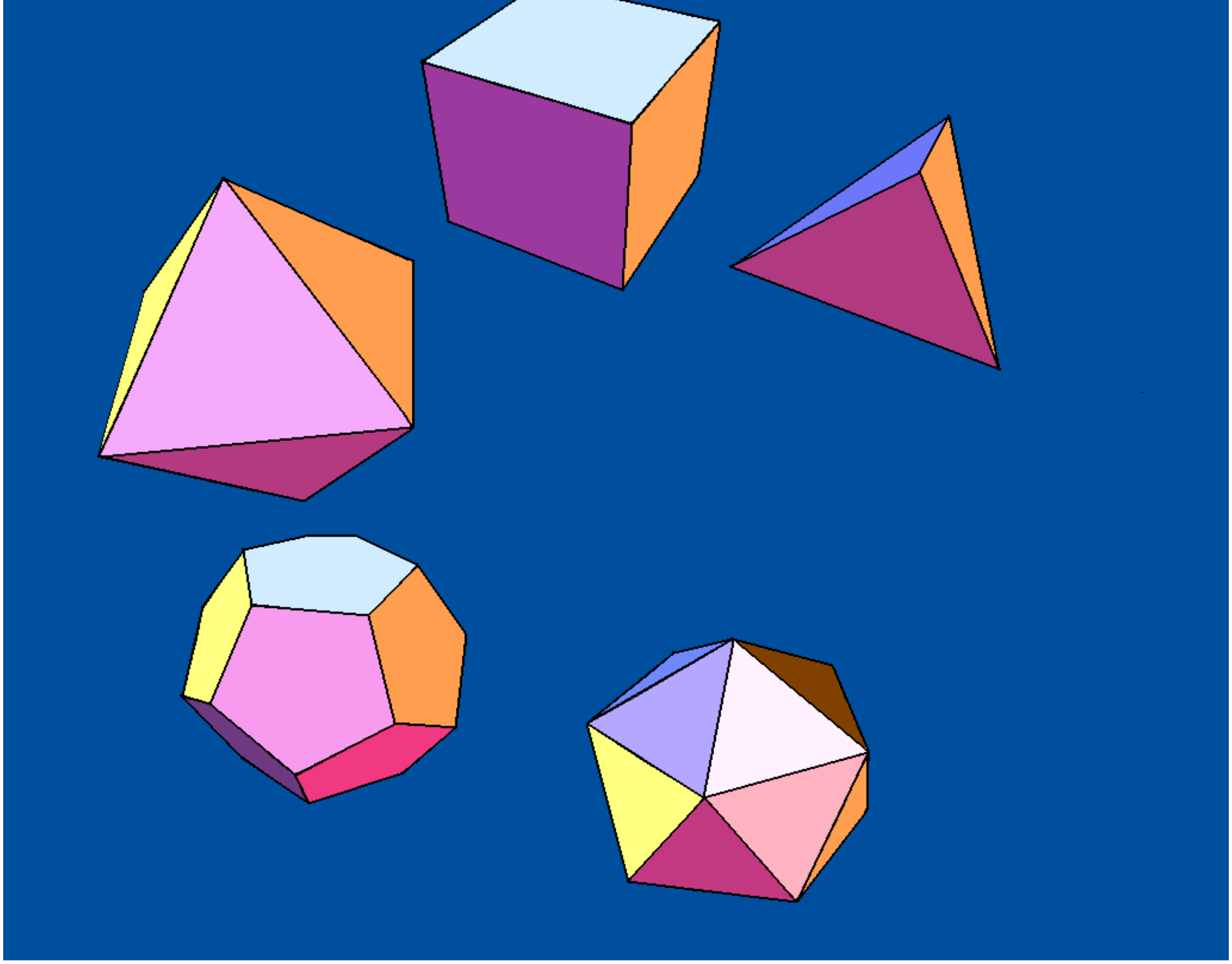
Introduction to Calculus

Volume I

by J.H. Heinbockel



The regular solids or regular polyhedra are solid geometric figures with the same identical regular polygon on each face. There are only five regular solids discovered by the ancient Greek mathematicians. These five solids are the following.

the tetrahedron (4 faces)

the cube or hexadron (6 faces)

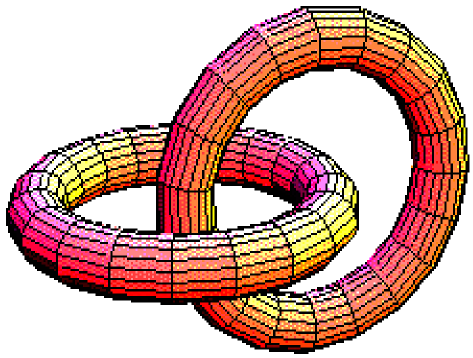
the octahedron (8 faces)

the dodecahedron (12 faces)

the icosahedron (20 faces)

Each figure follows the Euler formula

$$\begin{array}{ccccccc} \text{Number of faces} & + & \text{Number of vertices} & = & \text{Number of edges} & + & 2 \\ F & + & V & = & E & + & 2 \end{array}$$



Introduction to Calculus

Volume I

by J.H. Heinbockel

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Preface

This is the first volume of an introductory calculus presentation intended for future scientists and engineers. Volume I contains five chapters emphasizing fundamental concepts from calculus and analytic geometry and the application of these concepts to selected areas of science and engineering. Chapter one is a review of fundamental background material needed for the development of differential and integral calculus together with an introduction to limits. Chapter two introduces the differential calculus and develops differentiation formulas and rules for finding the derivatives associated with a variety of basic functions. Chapter three introduces the integral calculus and develops indefinite and definite integrals. Rules for integration and the construction of integral tables are developed throughout the chapter. Chapter four is an investigation of sequences and numerical sums and how these quantities are related to the functions, derivatives and integrals of the previous chapters. Chapter five investigates many selected applications of the differential and integral calculus. The selected applications come mainly from the areas of economics, physics, biology, chemistry and engineering.

The main purpose of these two volumes is to (i) Provide an introduction to calculus in its many forms (ii) Give some presentations to illustrate how powerful calculus is as a mathematical tool for solving a variety of scientific problems, (iii) Present numerous examples to show how calculus can be extended to other mathematical areas, (iv) Provide material detailed enough so that two volumes of basic material can be used as reference books, (v) Introduce concepts from a variety of application areas, such as biology, chemistry, economics, physics and engineering, to demonstrate applications of calculus (vi) Emphasize that definitions are extremely important in the study of any mathematical subject (vii) Introduce proofs of important results as an aid to the development of analytical and critical reasoning skills (viii) Introduce mathematical terminology and symbols which can be used to help model physical systems and (ix) Illustrate multiple approaches to various calculus subjects.

If the main thrust of an introductory calculus course is the application of calculus to solve problems, then a student must quickly get to a point where he or she understands enough fundamentals so that calculus can be used as a tool for solving the problems of interest. If on the other hand a deeper understanding of calculus is required in order to develop the basics for more advanced mathematical efforts, then students need to be exposed to theorems and proofs. If the calculus course leans toward more applications, rather than theory, then the proofs presented throughout the text can be skimmed over. However, if the calculus course is for mathematics majors, then one would want to be sure to go into the proofs in greater detail, because these proofs are laying the groundwork and providing background material for the study of more advanced concepts.

If you are a beginner in calculus, then be sure that you have had the appropriate background material of algebra and trigonometry. If you don't understand something then don't be afraid to ask your instructor a question. Go to the library and check out some other calculus books to get a presentation of the subject from a different perspective. The internet is a place where one can find numerous help aids for calculus. Also on the internet one can find many illustrations of the applications of calculus. These additional study aids will show you that there are multiple approaches to various calculus subjects and should help you with the development of your analytical and reasoning skills.

J.H. Heinbockel
September 2012

Introduction to Calculus
Volume I

Chapter 1

Sets, Functions, Graphs and Limits

The study of different types of functions, limits associated with these functions and how these functions change, together with the ability to graphically illustrate basic concepts associated with these functions, is fundamental to the understanding of calculus. These important issues are presented along with the development of some additional elementary concepts which will aid in our later studies of more advanced concepts. In this chapter and throughout this text be aware that definitions and their consequences are the keys to success for the understanding of calculus and its many applications and extensions. Note that appendix B contains a summary of fundamentals from algebra and trigonometry which is a prerequisite for the study of calculus. This first chapter is a preliminary to calculus and begins by introducing the concepts of a function, graph of a function and limits associated with functions. These concepts are introduced using some basic elements from the theory of sets.

Elementary Set Theory

A set can be any collection of objects. A set of objects can be represented using the notation

$$S = \{\, x \mid \text{statement about x}\}$$

and is read,**"S is the set of objects x which make the statement about x true".** Alternatively, a finite number of objects within S can be denoted by listing the objects and writing

$$S = \{S_1, S_2, \ldots, S_n\}$$

For example, the notation

$$S = \{\, x \mid \ x - 4 > 0\}$$

can be used to denote the set of points x which are greater than 4 and the notation

$$T = \{A, B, C, D, E\}$$

can be used to represent a set containing the first 5 letters of the alphabet.

A set with **no elements** is denoted by the symbol $\emptyset$ and is known as the **empty set**. The elements within a set are usually selected from some **universal set** U associated with the elements x belonging to the set. When dealing with real numbers the universal set U is understood to be the set of all real numbers. The universal set is usually defined beforehand or is implied within the context of how the set is being used. For example, the universal set associated with the set T above could be the set of all symbols if that is appropriate and within the context of how the set T is being used.

The symbol $\in$ is read "**belongs to**" or "**is a member of**" and the symbol $\notin$ is read "**not in**" or "**is not a member of**". The statement $x \in S$ is read "x **is a member of** S" or "**x belongs to** S". The statement $y \notin S$ is read "y **does not belong to S**" or "y **is not a member of S**".

Let S denote a non-empty set containing real numbers x. This set is said to be **bounded above** if one can find a number b such that for each $x \in S$, one finds $x \leq b$. The number b is called an **upper bound** of the set S. In a similar fashion the set S containing real numbers x is said to be **bounded below** if one can find a number ℓ such that $\ell \leq x$ for all $x \in S$. The number ℓ is called a **lower bound** for the set S. Note that any number greater than b is also an upper bound for S and any number less than ℓ can be considered a lower bound for S. Let B and C denote the sets

$$B = \{\, x \mid x \text{ is an upper bound of } S\} \quad \text{and} \quad C = \{\, x \mid x \text{ is a lower bound of } S\},$$

then the set B has a **least upper bound** ($\ell.u.b.$) and the set C has a **greatest lower bound** ($g.\ell.b.$). A set which is bounded both above and below is called a **bounded set**.

Some examples of well known sets are the following.

The set of **natural numbers** $N = \{1, 2, 3, \ldots\}$

The set of integers $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

The set of rational numbers $Q = \{\, p/q \mid p \text{ is an integer, } q \text{ is an integer, } q \neq 0\}$

The set of prime numbers $P = \{2, 3, 5, 7, 11, \ldots\}$

The set of complex numbers $C = \{\, x + i\, y \mid \quad i^2 = -1,\ x, y \text{ are real numbers}\}$

The set of real numbers $R = \{\text{All decimal numbers}\}$

The set of 2-tuples $R^2 = \{\, (x, y) \mid x, y \text{ are real numbers }\}$

The set of 3-tuples $R^3 = \{\, (x, y, z) \mid x, y, z \text{ are real numbers }\}$

The set of n-tuples $R^n = \{\, (\xi_1, \xi_2, \ldots, \xi_n) \mid \xi_1, \xi_2, \ldots, \xi_n \text{ are real numbers }\}$

where it is understood that i is an imaginary unit with the property $i^2 = -1$ and decimal numbers represent all terminating and nonterminating decimals.

Example 1-1. **Intervals**

When dealing with real numbers a, b, x it is customary to use the following notations to represent various intervals of real numbers.

Set Notation	Set Definition	Name
$[a, b]$	$\{x \mid a \le x \le b\}$	closed interval
(a, b)	$\{x \mid a < x < b\}$	open interval
$[a, b)$	$\{x \mid a \le x < b\}$	left-closed, right-open
$(a, b]$	$\{x \mid a < x \le b\}$	left-open, right-closed
(a, ∞)	$\{x \mid x > a\}$	left-open, unbounded
$[a, \infty)$	$\{x \mid x \ge a\}$	left-closed,unbounded
$(-\infty, a)$	$\{x \mid x < a\}$	unbounded, right-open
$(-\infty, a]$	$\{x \mid x \le a\}$	unbounded, right-closed
$(-\infty, \infty)$	$R = \{x \mid -\infty < x < \infty\}$	Set of real numbers

■

Subsets

If for every element $x \in A$ one can show that x is also an element of a set B, then the set A is called a **subset of** B or one can say the set A is **contained in the set** B. This is expressed using the mathematical statement $A \subset B$, which is read "A **is a subset of** B". This can also be expressed by saying that B **contains** A, which is written as $B \supset A$. If one can find one element of A which is not in the set B, then A is not a subset of B. This is expressed using either of the notations $A \not\subset B$ or $B \not\supset A$. Note that the above definition implies that every set is a subset of itself, since the elements of a set A belong to the set A. Whenever $A \subset B$ and $A \ne B$, then A is called a **proper subset of** B.

Set Operations

Given two sets A and B, the **union** of these sets is written $A \cup B$ and defined

$$\boldsymbol{A \cup B = \{ x \mid x \in A \textbf{ or } x \in B, \textbf{ or } x \in \textbf{ both } A \textbf{ and } B\}}$$

The **intersection** of two sets A and B is written $A \cap B$ and defined

$$\boldsymbol{A \cap B = \{ x \mid x \in \textbf{ both } A \textbf{ and } B \}}$$

If $A \cap B$ is the empty set one writes $A \cap B = \emptyset$ and then the sets A and B are said to be **disjoint**.

The **difference**[1] between two sets A and B is written $A - B$ and defined

$$A - B = \{\, x \mid x \in A \text{ and } x \notin B \,\}$$

The **equality of two sets** is written $A = B$ and defined

$$A = B \text{ if and only if } A \subset B \text{ and } B \subset A$$

That is, if $A \subset B$ and $B \subset A$, then the sets A and B must have the same elements which implies equality. Conversely, if two sets are equal $A = B$, then $A \subset B$ and $B \subset A$ since every set is a subset of itself.

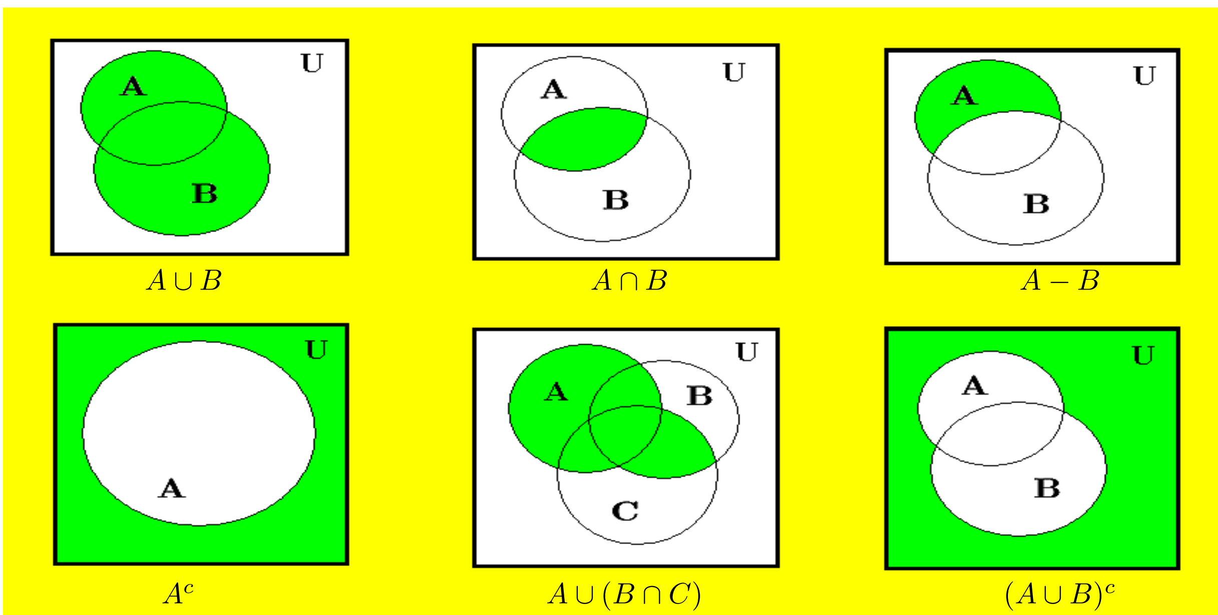

Figure 1-1. Selected Venn diagrams.

The **complement** of set A with respect to the universal set U is written A^c and defined

$$A^c = \{\, x \mid x \in U \text{ but } x \notin A \,\}$$

Observe that the complement of a set A satisfies the **complement laws**

$$A \cup A^c = U, \qquad A \cap A^c = \emptyset, \qquad \emptyset^c = U, \qquad U^c = \emptyset$$

The operations of union $\cup$ and intersection $\cap$ satisfy the **distributive laws**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad\qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

[1] The difference between two sets A and B in some texts is expressed using the notation $A \setminus B$.

and the **identity laws**

$$A \cup \emptyset = A, \qquad A \cup U = U, \qquad A \cap U = A, \qquad A \cap \emptyset = \emptyset$$

The above set operations can be illustrated using circles and rectangles, where the universal set is denoted by the rectangle and individual sets are denoted by circles. This pictorial representation for the various set operations was devised by John Venn[2] and are known as **Venn diagrams**. Selected Venn diagrams are illustrated in the figure 1-1.

Example 1-2. **Equivalent Statements**

Prove that the following statements are equivalent $A \subset B$ and $A \cap B = A$

Solution To show these statements are equivalent one must show

(i) if $A \subset B$, then $A \cap B = A$ and (ii) if $A \cap B = A$, then it follows that $A \subset B$.

(i) Assume $A \subset B$, then if $x \in A$ it follows that $x \in B$ since A is a subset of B. Consequently, one can state that $x \in (A \cap B)$, all of which implies $A \subset (A \cap B)$. Conversely, if $x \in (A \cap B)$, then x belongs to both A and B and certainly one can say that $x \in A$. This implies $(A \cap B) \subset A$. If $A \subset (A \cap B)$ and $(A \cap B) \subset A$, then it follows that $(A \cap B) = A$.

(ii) Assume $A \cap B = A$, then if $x \in A$, it must also be in $A \cap B$ so that one can say $x \in A$ and $x \in B$, which implies $A \subset B$.

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Coordinate Systems

There are many different kinds of coordinate systems most of which are created to transform a problem or object into a simpler representation. The **rectangular coordinate system**[3] with axes labeled x and y provides a way of plotting number pairs (x, y) which are interpreted as points within a plane.

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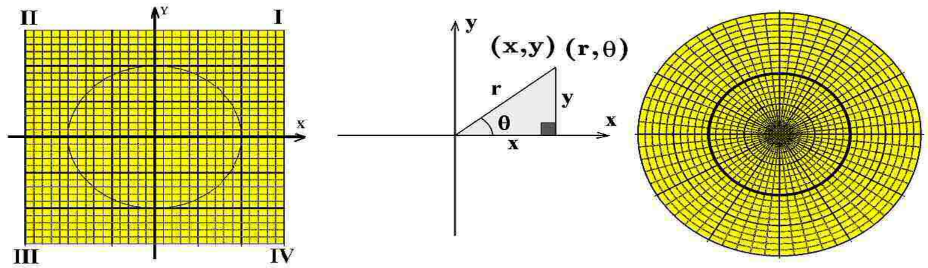
[2] John Venn (1834-1923) An English mathematician who studied logic and set theory.

[3] Also called a cartesian coordinate system and named for René Descartes (1596-1650) a French philosopher who applied algebra to geometry problems.

rectangular coordinates polar coordinates



$$x^2 + y^2 = \rho^2 \qquad x = r\cos\theta, \quad y = r\sin\theta \qquad \tan\theta = \frac{y}{x} \quad \sqrt{x^2+y^2} = r \qquad r = \rho$$

Figure 1-2. Rectangular and polar coordinate systems

A **cartesian or rectangular coordinate system** is constructed by selecting two straight lines intersecting at right angles and labeling the point of intersection as the origin of the coordinate system and then labeling the horizontal line as the x-axis and the vertical line as the y-axis. On these axes some kind of a scale is constructed with positive numbers to the right on the horizontal axis and upward on the vertical axes. For example, by constructing lines at equally spaced distances along the axes one can create a grid of intersecting lines.

A point in the plane defined by the two axes can then be represented by a number pair (x, y). In rectangular coordinates a number pair (x, y) is said to have the **abscissa** x and the **ordinate** y. The point (x, y) is located a distance $r = \sqrt{x^2 + y^2}$ from the origin with x representing distance of the point from the y-axis and y representing the distance of the point from the x-axis. The x axis or abscissa axis and the y axis or ordinate axis divides the plane into four **quadrants** labeled I, II, III and IV.

To construct a **polar coordinate system** one selects an origin for the polar coordinates and labels it 0. Next construct a half-line similar to the x-axis of the rectangular coordinates. This half-line is called the **polar axis** or **initial ray** and the origin is called the **pole** of the polar coordinate system. By placing another line on top of the polar axis and rotating this line about the pole through a positive angle θ, measured in radians, one can create a ray emanating from the origin at an angle θ as illustrated in the figure 1-3. In polar coordinates the rays are illustrated emanating from the origin at equally spaced angular distances around the origin and then concentric circles are constructed representing constant distances from the origin.

A point in polar coordinates is then denoted by the number pair (r,θ) where θ is the angle of rotation associated with the ray and r is a distance outward from the origin along the ray. The polar origin or pole has the coordinates $(0,\theta)$ for any angle θ. All points having the polar coordinates $(\rho,0)$, with $\rho \geq 0$, lie on the polar axis.

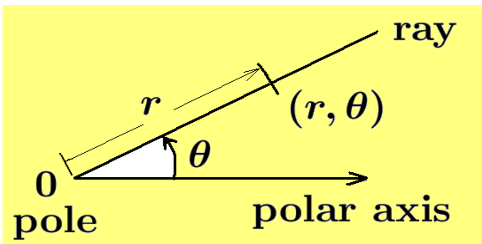


Figure 1-3.
Construction of polar axes

Here angle rotations are treated the same as in trigonometry with a counterclockwise rotation being in the positive direction and a clockwise rotation being in the negative direction. Note that the polar representation of a point is not unique since the angle θ can be increased or decreased by some multiple of 2π to arrive at the same point. That is, $(r,\theta) = (r,\theta \pm 2n\pi)$ where n is an integer.

Also note that a ray at angle θ can be extended to represent negative distances along the ray. Points $(-r,\theta)$ can also be represented by the number pair $(r,\theta+\pi)$. Alternatively, one can think of a rectangular point (x,y) and the corresponding polar point (r,θ) as being related by the equations

$$\begin{aligned} \theta =& \arctan(y/x), \qquad & x =& r\cos\theta \\ r =& \sqrt{x^2+y^2}, \qquad & y =& r\sin\theta \end{aligned} \tag{1.1}$$

An example of a rectangular coordinate system and polar coordinate system are illustrated in the figure 1-2.

Distance Between Two Points in the Plane

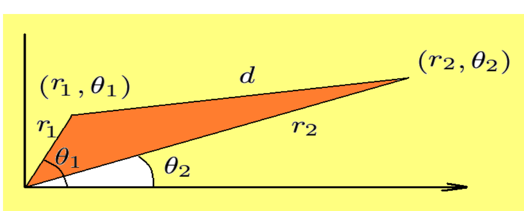


Figure 1-4.
Distance between points
in polar coordinates.

If two points are given in polar coordinates as (r_1,θ_1) and (r_2,θ_2), as illustrated in the figure 1-4, then one can use the law of cosines to calculate the distance d between the points since

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1-\theta_2) \tag{1.2}$$

Alternatively, let (x_1, y_1) and (x_2, y_2) denote two points which are plotted on a cartesian set of axes as illustrated in the figure 1-5. The Greek letter Δ (delta) is used to denote a change in a quantity. For example, in moving from the point (x_1, y_1) to the point (x_2, y_2) the change in x is denoted $\Delta x = x_2 - x_1$ and the change in y is denoted $\Delta y = y_2 - y_1$. These changes can be thought of as the legs of a right-triangle as illustrated in the figure 1-5.

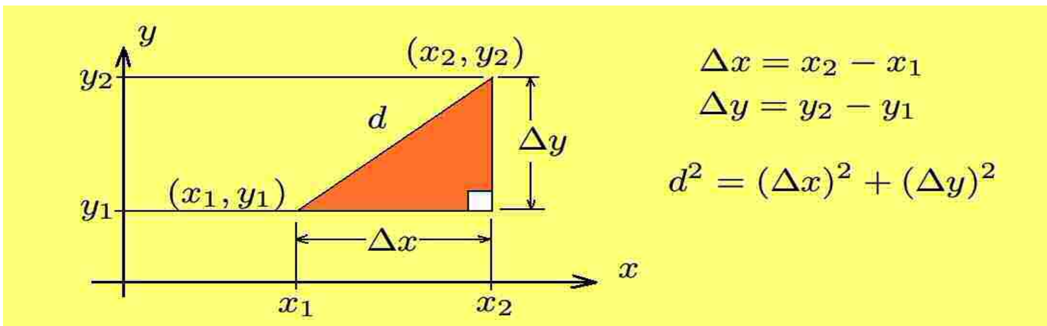


Figure 1-5.
Using a right-triangle to calculate distance between two points in rectangular coordinates.

The figure 1-5 illustrates that by using the Pythagorean theorem the distance d between the two points can be determined from the equations

$$d^2 = (\Delta x)^2 + (\Delta y)^2 \qquad \text{or} \qquad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.3}$$

Graphs and Functions

Let X and Y denote sets which contain some subset of the real numbers with elements $x \in X$ and $y \in Y$. If a rule or relation f is given such that for each $x \in X$ there corresponds exactly one real number $y \in Y$, then y is said to be a **real single-valued function of** x and the relation between y and x is denoted $y = f(x)$ and read as "y **is a function of** x". If for each $x \in X$, there is only one ordered pair (x, y), then a functional relation from X to Y is said to exist. The function is called single-valued if no two different ordered pairs (x, y) have the same first element. A way of representing the set of ordered pairs which define a function is to use one of the notations

$$\{\,(x, y) \mid y = f(x),\ x \in X\,\} \quad \text{or} \quad \{\,(x, f(x)) \mid x \in X\,\} \tag{1.4}$$

The set of values $x \in X$ is called the domain of definition of the function $f(x)$. The set of values $\{y \mid y = f(x)\,, x \in X\}$ is called the **range of the function** or the image of the set X under the mapping or transformation given by f. The set of ordered pairs

$$C = \{\,(x, y) \mid \; y = f(x), \quad x \in X\}$$

is called the **graph of the function** and represents a curve in the x, y-plane giving a pictorial representation of the function. If $y = f(x)$ for $x \in X$, the number x is called the **independent variable** or argument of the function and the image value y is called the **dependent variable** of the function. It is to be understood that the domain of definition of a function contains real values for x for which the relation $f(x)$ is also real-valued. In many physical problems, the domain of definition X must be restricted in order that a given physical problem be well defined. For example, in order that $\sqrt{x-1}$ be real-valued, x must be restricted to be greater than or equal to 1.

When representing many different functions the symbol f can be replaced by any of the letters from the alphabet. For example, one might have several different functions labeled as

$$y = f(x), \qquad y = g(x), \qquad y = h(x), \qquad \ldots, \qquad y = y(x), \qquad \ldots, \quad y = z(x) \tag{1.5}$$

or one could add subscripts to the letter f to denote a set of n-different functions

$$F = \{f_1(x), f_2(x), \ldots, f_n(x)\} \tag{1.6}$$

Example 1-3. (Functions)

(a) Functions defined by a formula over a given domain.

Let $y = f(x) = x^2 + x$ for $x \in R$ be a given rule defining a function which can be represented by a curve in the cartesian x, y-plane. The variable x is a dummy variable used to define the function rule. Substituting the value 3 in place of x in the function rule gives $y = f(3) = 3^2 + 3 = 12$ which represents the height of the curve at $x = 3$. In general, for any given value of x the quantity $y = f(x)$ represents the height of the curve at the point x. By assigning a collection of ordered values to x and calculating the corresponding value $y = f(x)$, using the given rule, one collects a set of (x, y) pairs which can be interpreted as representing a set of points in the cartesian coordinate system. The set of all points corresponding to a given rule is called the locus of points satisfying the rule. The graph illustrated in the figure 1-6 is a pictorial representation of the given rule.

x	$y = f(x) = x^2 + x$
-2.0	2.00
-1.8	1.44
-1.6	0.96
-1.4	0.56
-1.2	0.24
-1.0	0.00
-0.8	-0.16
-0.6	-0.24
-0.4	-0.24
-0.2	-0.16
0.0	0.00
0.2	0.24
0.4	0.56
0.6	0.96
0.8	1.44
1.0	2.00
1.2	2.64
1.4	3.36
1.6	4.16
1.8	5.04
2.0	6.00

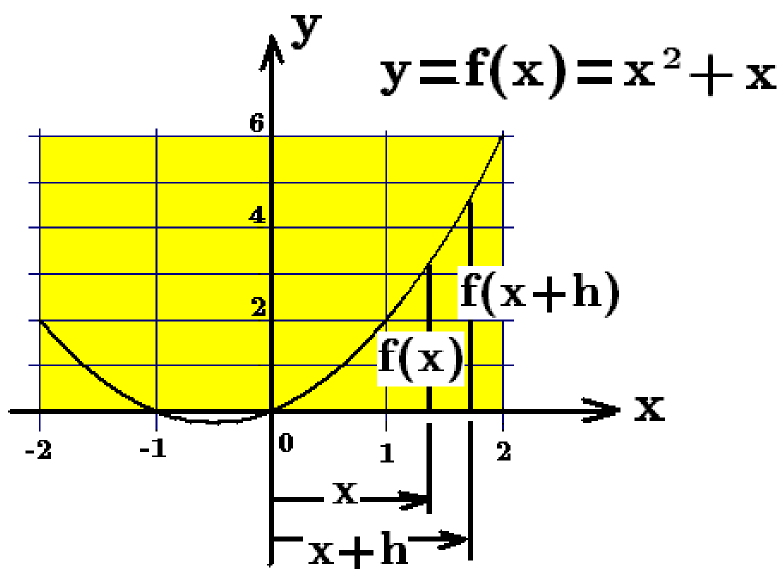


Figure 1-6. A graph of the function $y = f(x) = x^2 + x$

Note that substituting $x + h$ in place of x in the function rule gives

$$f(x+h) = (x+h)^2 + (x+h) = x^2 + (2h+1)x + (h^2 + h).$$

If $f(x)$ represents the height of the curve at the point x, then $f(x+h)$ represents the height of the curve at $x + h$.

(b) Let $r = f(\theta) = 1 + \theta$ for $0 \le \theta \le 2\pi$ be a given rule defining a function which can be represented by a curve in polar coordinates (r, θ). One can select a set of ordered values for θ in the interval $[0, 2\pi]$ and calculate the corresponding values for $r = f(\theta)$. The set of points (r, θ) created can then be plotted on polar graph paper to give a pictorial representation of the function rule. The graph illustrated in the figure 1-7 is a pictorial representation of the given function over the given domain.

Note in dealing with polar coordinates a radial distance r and polar angle θ can have any of the representations

$$((-1)^n r, \theta + n\pi)$$

and consequently a functional relation like $r = f(\theta)$ can be represented by one of the alternative equations $r = (-1)^n f(\theta + n\pi)$

θ	$r = f(\theta) = 1 + \theta$
0	$1 + 0$
$\pi/4$	$1 + \pi/4$
$2\pi/4$	$1 + 2\pi/4$
$3\pi/4$	$1 + 3\pi/4$
$4\pi/4$	$1 + 4\pi/4$
$5\pi/4$	$1 + 5\pi/4$
$6\pi/4$	$1 + 6\pi/4$
$7\pi/4$	$1 + 7\pi/4$
$8\pi/4$	$1 + 8\pi/4$

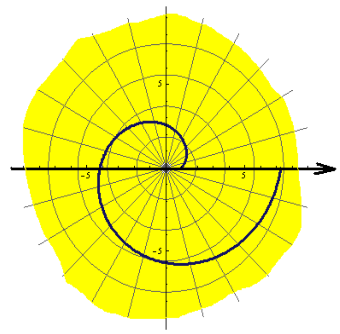
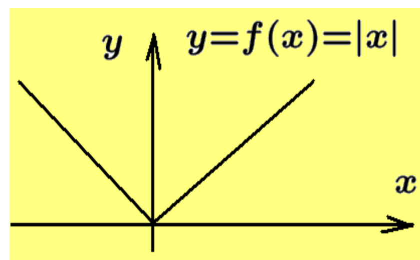


Figure 1-7.
A polar plot of the function $r = f(\theta) = 1 + \theta$

(c) The absolute value function

The absolute value function is defined

$$y = f(x) = \mid x \mid = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$

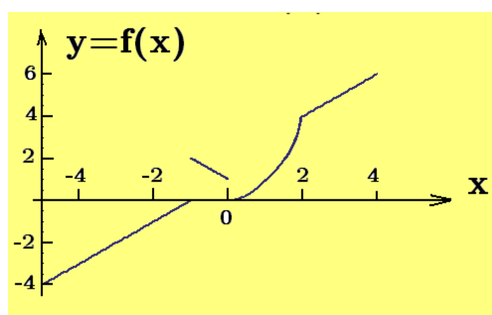


Substituting in a couple of specific values for x one can form a set of (x, y) number pairs and then sketch a graph of the function, which represents a pictorial image of the functional relationship between x and y.

(d) Functions defined in a piecewise fashion.

A function defined by

$$f(x) = \begin{cases} 1 + x, & x \leq -1 \\ 1 - x, & -1 \leq x \leq 0 \\ x^2, & 0 \leq x \leq 2 \\ 2 + x, & x > 2 \end{cases} \qquad x \in R$$



is a collection of rules which defines the function in a piecewise fashion. One must examine values of the input x to determine which portion of the rule is to be used in evaluating the function. The above example illustrates a function having jump discontinuities at the points where $x = -1$ and $x = 0$.

(e) Numerical data.

If one collects numerical data from an experiment such as recording temperature T at different times t, then one obtains a set of data points called number pairs. If these number pairs are labeled (t_i, T_i), for $i = 1, 2, \ldots, n$, one obtains a table of values such as

Time t	t_1	t_2	$\cdots$	t_n
Temperature T	T_1	T_2	$\cdots$	T_n

It is then possible to plot a t, T-axes graph associated with these data points by plotting the points and then drawing a smooth curve through the points or by connecting the points with straight line segments. In doing this, one is assuming that the curve sketched is a graphical representation of an unknown functional relationship between the variables.

(e) Other representation of functions

Functions can be represented by different methods such as using equations, graphs, tables of values, a verbal rule, or by using a machine like a pocket calculator which is programmable to give some output for a given input. Functions can be **continuous** or they can have **discontinuities**. Continuous functions are recognized by their graphs which are smooth unbroken curves with **continuously turning tangent lines at each point on the curve.** Discontinuities usually occur when **functional values or tangent lines are not well defined at a point.**

■

Increasing and Decreasing Functions

One aspect in the study of calculus is to examine how functions change over an interval. A function is said to be **increasing over an interval** (a, b) if for every pair of points (x_0, x_1) within the interval (a, b), satisfying $x_0 < x_1$, the height of the curve at x_0 is less than the height of the curve at x_1 or $f(x_0) < f(x_1)$. A function is called **decreasing over an interval** (a, b) if for every pair of points (x_0, x_1) within the interval (a, b), satisfying $x_0 < x_1$, one finds the height of the curve at x_0 is greater than the height of the curve at x_1 or $f(x_0) > f(x_1)$.

Linear Dependence and Independence

A **linear combination** of a set of functions $\{f_1, f_2, \ldots, f_n\}$ is formed by taking arbitrary constants $c_1, c_2, \ldots, c_n$ and forming the sum

$$y = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n \tag{1.7}$$

One can then say that y is a linear combination of the set of functions $\{f_1, f_2, \ldots, f_n\}$.

If a function $f_1(x)$ is some constant c times another function $f_2(x)$, then one can write $f_1(x) = c f_2(x)$ and under this condition the function f_1 is said to be **linearly dependent** upon f_2. If no such constant c exists, then the functions are said to be **linearly independent**. Another way of expressing **linear dependence** and **linear independence** applied to functions f_1 and f_2 is as follows. One can say that, if there are nonzero constants c_1, c_2 such that the linear combination

$$c_1 f_1(x) + c_2 f_2(x) = 0 \tag{1.8}$$

for all values of x, then the set of functions $\{f_1, f_2\}$ is called a set of **dependent functions**. This is due to the fact that if $c_1 \neq 0$, then one can divided by c_1 and express the equation (1.8) in the form $f_1(x) = -\frac{c_2}{c_1} f_2(x) = c f_2(x)$. If the only constants which make equation (1.8) a true statement are $c_1 = 0$ and $c_2 = 0$, then the set of functions $\{f_1, f_2\}$ is called a set of **linearly independent functions**.

An immediate generalization of the above is the following. If there exists constants $c_1, c_2, \ldots, c_n$, not all zero, such that the linear combination

$$c_1 f_1(x) + c_2 f_2(x) + \cdots c_n f_n(x) = 0, \tag{1.9}$$

for all values of x, then the set of functions $\{f_1, f_2, \ldots, f_n\}$ is called a linearly dependent set of functions. If the only constants, for which equation (1.9) is true, are when $c_1 = c_2 = \cdots = c_n = 0$, then the set of functions $\{f_1, f_2, \ldots, f_n\}$ is called a linearly independent set of functions. Note that if the set of functions are linearly dependent, then one of the functions can be made to become a linear combination of the other functions. For example, assume that $c_1 \neq 0$ in equation (1.9). One can then write

$$f_1(x) = -\frac{c_2}{c_1} f_2(x) - \cdots - \frac{c_n}{c_1} f_n(x)$$

which shows that f_1 is some linear combination of the other functions. That is, f_1 is dependent upon the values of the other functions.

Single-valued Functions

Consider plane curves represented in rectangular coordinates such as the curves illustrated in the figures 1-8. These curves can be considered as a set of ordered pairs (x, y) where the x and y values satisfy some specified condition.

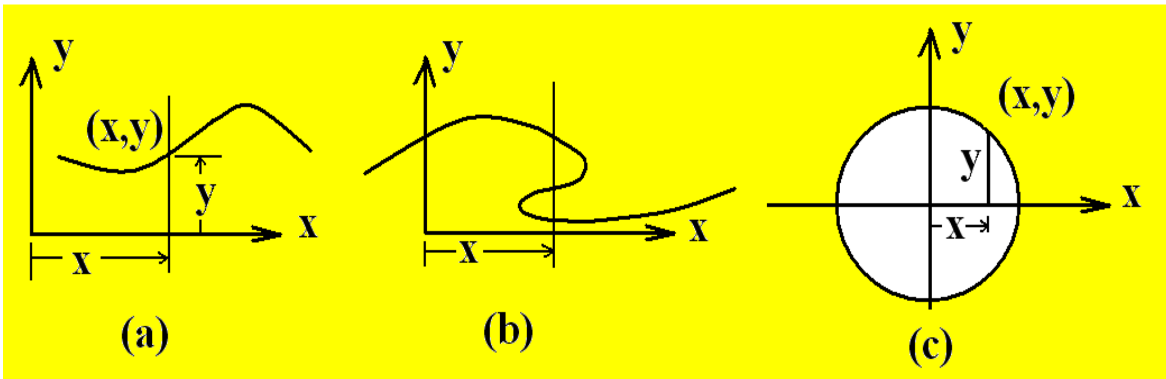


Figure 1-8. Selected curves sketched in rectangular coordinates

In terms of a set representation, these curves can be described using the set notation

$$C = \{ (x, y) \mid \text{relationship satisfied by } x \text{ and } y \text{ with } x \in X \}$$

This represents a collection of points (x, y), where x is restricted to values from some set X and y is related to x in some fashion. A graph of the function results when the points of the set are plotted in rectangular coordinates. If for all values x_0 a vertical line $x = x_0$ cuts the graph of the function in only a single point, then the function is called **single-valued**. If the vertical line intersects the graph of the function in more than one point, then the function is called **multiple-valued**.

Similarly, in polar coordinates, a graph of the function is a curve which can be represented by a collection of ordered pairs (r, θ). For example,

$$C = \{ (r, \theta) \mid \text{ relationship satisfied by } r \text{ and } \theta \text{ with } \theta \in \Theta \}$$

where Θ is some specified domain of definition of the function. There are available many plotting programs for the computer which produce a variety of specialized graphs. Some computer programs produce not only cartesian plots and polar plots, but also many other specialized graph types needed for various science and engineering applications. These other graph types give an alternative way of representing functional relationships between variables.

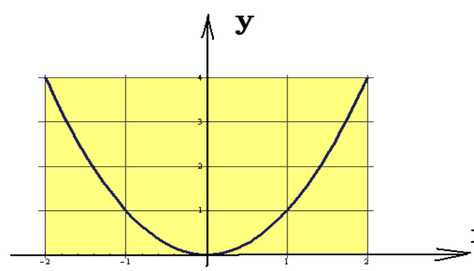
Example 1-4. Rectangular and Polar Graphs

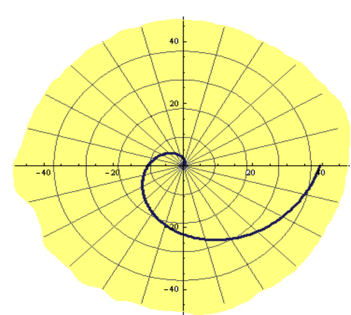
Plotting the same function in both rectangular coordinates and polar coordinates gives different shaped curves and so the graphs of these functions have different properties depending upon the coordinate system used to represent the function. For example, plot the function $y = f(x) = x^2$ for $-2 \le x \le 2$ in rectangular coordinates and then plot the function $r = g(\theta) = \theta^2$ for $0 \le \theta \le 2\pi$ in polar coordinates. Show that one curve is a parabola and the other curve is a spiral.

Solution

x	$y = f(x) = x^2$
-2.00	4.0000
-1.75	3.0625
-1.50	2.2500
-1.25	1.5625
-1.00	1.0000
-0.75	0.5625
-0.50	0.2500
-0.25	0.0625
0.00	0.0000
0.25	0.0625
0.50	0.2500
0.75	0.5625
1.00	1.0000
1.25	1.5625
1.50	2.2500
1.75	3.0625
2.00	4.0000

θ	$r = g(\theta) = \theta^2$
0.00	0.0000
$\pi/4$	$\pi^2/16$
$\pi/2$	$\pi^2/4$
$3\pi/4$	$9\pi^2/16$
π	π^2
$5\pi/4$	$25\pi^2/16$
$3\pi/2$	$9\pi^2/4$
$7\pi/4$	$49\pi^2/16$
2π	$4\pi^2$





$C_1 = \{ (x,y) \mid y = f(x) = x^2, \ -2 \le x \le 2 \}$ $\quad C_2 = \{ (r,\theta) \mid r = g(\theta) = \theta^2, \ 0 \le \theta \le 2\pi \}$

Figure 1-9. Rectangular and polar graphs give different pictures of function.

Select some points x from the domain of the function and calculate the image points under the mapping $y = f(x) = x^2$. For example, one can use a spread sheet and put values of x in one column and the image values y in an adjacent column to obtain a table of values for representing the function at a discrete set of selected points.

Similarly, select some points θ from the domain of the function to be plotted in polar coordinates and calculate the image points under the mapping $r = g(\theta) = \theta^2$. Use a spread sheet and put values of θ in one column and the image values r in an adjacent column to obtain a table for representing the function as a discrete set of selected points. Using an x spacing of 0.25 between points for the rectangular graph and a θ spacing of $\pi/4$ for the polar graph, one can verify the table of values and graphs given in the figure 1-9.

■

Some well known cartesian curves are illustrated in the following figures.

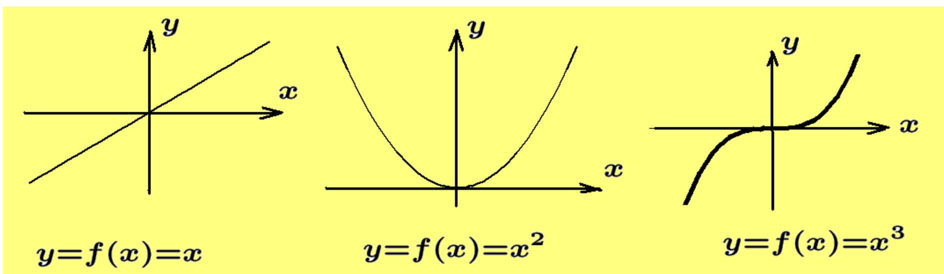


Figure 1-10. Polynomial curves $y = x$, $y = x^2$ and $y = x^3$

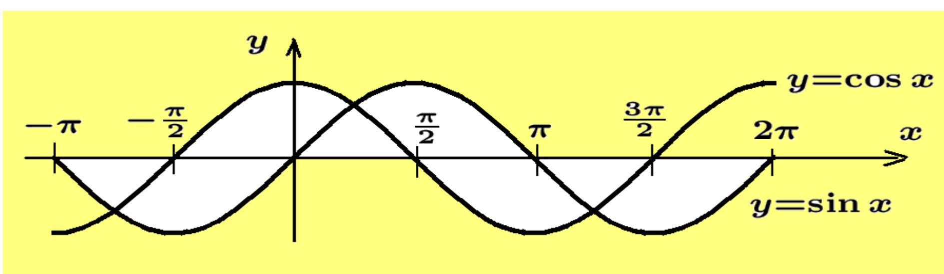
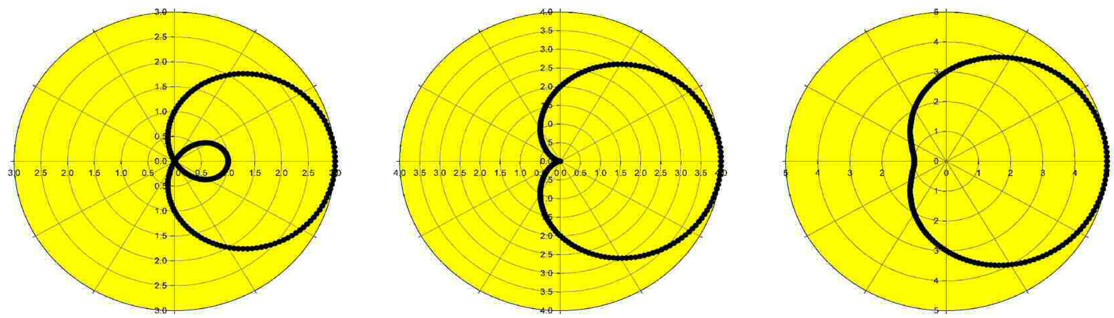


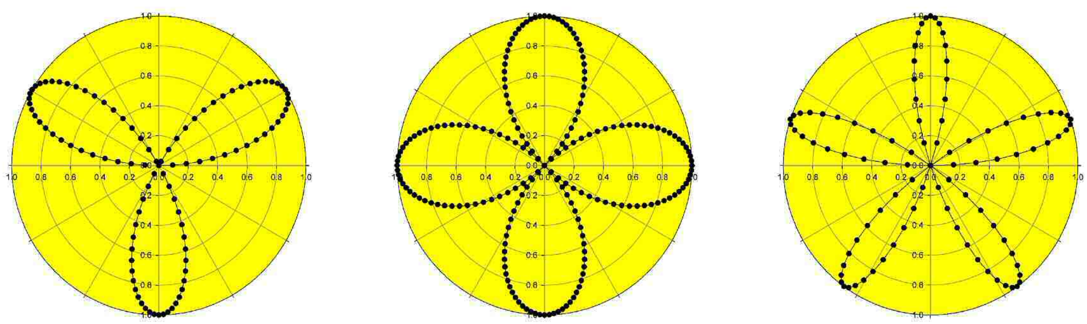
Figure 1-11. The trigonometric functions $y = \sin x$ and $y = \cos x$ for $-\pi \le x \le 2\pi$.

Some well known polar curves are illustrated in the following figures.



b= -a **b=2a** **b=3a**

Figure 1-12. The limaçon curves $r = 2a\cos\theta + b$



$\boldsymbol{r = a\sin 3\theta}$ $\boldsymbol{r = a\cos 2\theta}$ $\boldsymbol{r = a\sin 5\theta}$

Figure 1-13. The rose curves $r = a\cos n\theta$ and $r = a\sin n\theta$
If n odd, curve has n-loops and if n is even, curve has $2n$ loops.

Parametric Representation of Curve

Examine the graph in figure 1-8(b) and observe that it does not represent a single-valued function $y = f(x)$. Also the circle in figure 1-8(c) does not define a single valued function. An alternative way of graphing a function is to represent it in a parametric form.[4] In general, a graphical representation of a function or a

[4] The parametric form for representing a curve is not unique and the parameter used may or may not have a physical meaning.

section of a function, be it single-valued or multiple-valued, can be defined by a **parametric representation**

$$C = \{(x,y) \mid \ x = x(t),\ y = y(t),\ a \le t \le b\} \tag{1.10}$$

where both $x(t)$ and $y(t)$ are single-valued functions of the parameter t. The relationship between x and y is obtained by eliminating the parameter t from the representation $x = x(t)$ and $y = y(t)$. For example, the parametric representation $x = x(t) = t$ and $y = y(t) = t^2$, for $t \in R$, is one parametric representation of the parabola $y = x^2$.

The Equation of a Circle

A circle of radius ρ and centered at the point (h,k) is illustrated in the figure 1-14 and is defined as the set of all points (x,y) whose distance from the point (h,k) has the constant value of ρ. Using the distance formula (1.3), with (x_1,y_1) replaced by (h,k), the point (x_2,y_2) replaced by the variable point (x,y) and replacing d by ρ, one can show the equation of the circle is given by one of the formulas

$$\begin{aligned} &(x-h)^2 + (y-k)^2 = \rho^2 \\ \text{or} \qquad &\sqrt{(x-h)^2 + (y-k)^2} = \rho \end{aligned} \tag{1.11}$$

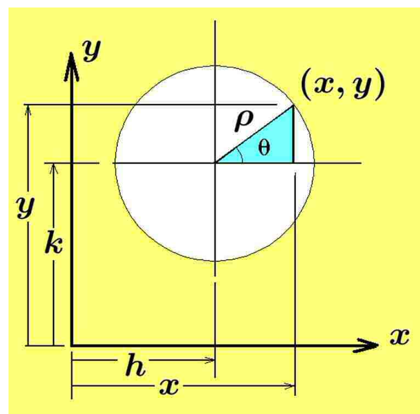


Figure 1-14.
Circle centered at (h,k).

Equations of the form

$$x^2 + y^2 + \alpha x + \beta y + \gamma = 0, \qquad \alpha,\ \beta,\ \gamma \text{ constants} \tag{1.12}$$

can be converted to the form of equation (1.11) by completing the square on the x and y terms. This is accomplished by taking 1/2 of the x-coefficient, squaring and adding the result to both sides of equation (1.12) and then taking 1/2 of the y-coefficient, squaring and adding the result to both sides of equation (1.12). One then obtains

$$\left(x^2+\alpha x+\frac{\alpha^2}{4}\right)+\left(y^2+\beta y+\frac{\beta^2}{4}\right)=\frac{\alpha^2}{4}+\frac{\beta^2}{4}-\gamma$$

which simplifies to
$$\left(x+\frac{\alpha}{2}\right)^2+\left(y+\frac{\beta}{2}\right)^2=r^2$$

where $r^2=\frac{\alpha^2}{4}+\frac{\beta^2}{4}-\gamma$. Completing the square is a valid conversion whenever the right-hand side $\frac{\alpha^2}{4}+\frac{\beta^2}{4}-\gamma\geq 0$.

An alternative method of representing the equation of the circle is to introduce a parameter θ such as the angle illustrated in the figure 1-14 and observe that by trigonometry

$$\sin\theta=\frac{y-k}{\rho}\qquad\text{and}\qquad\cos\theta=\frac{x-h}{\rho}$$

These equations are used to represent the circle in the alternative form

$$C=\{\,(x,y)\mid\ x=h+\rho\cos\theta,\quad y=k+\rho\sin\theta,\quad 0\leq\theta\leq 2\pi\,\}\tag{1.13}$$

This is called a parametric representation of the circle in terms of a parameter θ.

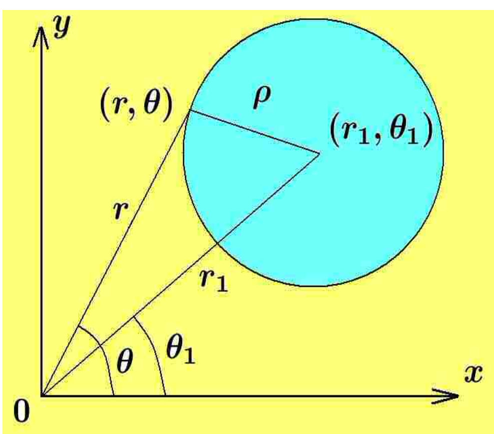


Figure 1-15.
Circle centered at (r_1,θ_1).

The equation of a circle in polar form can be constructed as follows. Let (r,θ) denote a variable point which moves along the circumference of a circle of radius ρ which is centered at the point (r_1,θ_1) as illustrated in the figure 1-15. Using the distances r, r_1 and ρ, one can employ the law of cosines to express the polar form of the equation of a circle as

$$r^2+r_1^2-2rr_1\cos(\theta-\theta_1)=\rho^2\tag{1.14}$$

Functions can be represented in a variety of ways. Sometimes functions are represented in the **implicit form** $G(x,y)=0$, because it is not always possible to solve for one variable explicitly in terms of another. In those cases where it is possible to solve for one variable in terms of another to obtain $y=f(x)$ or $x=g(y)$, the function is said to be represented in **an explicit form.**

For example, the circle of radius ρ can be represented by any of the relations

$$G(x,y) = x^2 + y^2 - \rho^2 = 0,$$

$$y = f(x) = \begin{cases} +\sqrt{\rho^2 - x^2}, & -\rho \le x \le \rho \\ -\sqrt{\rho^2 - x^2}, & -\rho \le x \le \rho \end{cases}, \qquad x = g(y) = \begin{cases} +\sqrt{\rho^2 - y^2}, & -\rho \le y \le \rho \\ -\sqrt{\rho^2 - y^2}, & -\rho \le y \le \rho \end{cases} \tag{1.15}$$

Note that the circle in figure 1-8(c) does not define a single valued function. The circle can be thought of as a graph of two single-valued functions $y = +\sqrt{\rho^2 - x^2}$ and $y = -\sqrt{\rho^2 - x^2}$ for $-\rho \le x \le \rho$ if one treats y as a function of x. The other representation in equation (1.15) results if one treats x as a function of y.

Types of functions

One can define a functional relationships between the two variables x and y in different ways.

A **polynomial function** in the variable x has the form

$$y = p_n(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n \tag{1.16}$$

where $a_0, a_1, \ldots, a_n$ represent constants with $a_0 \ne 0$ and n is a positive integer. The integer n is called the degree of the polynomial function. The fundamental theorem of algebra states that a polynomial of degree n has n-roots. That is, the polynomial equation $p_n(x) = 0$ has n-solutions called the roots of the polynomial equation. If these roots are denoted by $x_1, x_2, \ldots, x_n$, then the polynomial can also be represented in the form

$$p_n(x) = a_0(x - x_1)(x - x_2)\cdots(x - x_n)$$

If x_i is a number, real or complex, which satisfies $p_n(x_i) = 0$, then $(x - x_i)$ is a factor of the polynomial function $p_n(x)$. Complex roots of a polynomial function must always occur in conjugate pairs and one can say that if $\alpha + i\beta$ is a root of $p_n(x) = 0$, then $\alpha - i\beta$ is also a root. Real roots x_i of a polynomial function give rise to linear factors $(x - x_i)$, while complex roots of a polynomial function give rise to quadratic factors of the form $\left[(x - \alpha)^2 + \beta^2\right]$, α, β constant terms.

A **rational function** is any function of the form

$$y = f(x) = \frac{P(x)}{Q(x)} \tag{1.17}$$

where both $P(x)$ and $Q(x)$ are polynomial functions in x and $Q(x) \ne 0$. If $y = f(x)$ is a root of an equation of the form

$$b_0(x) y^n + b_1(x) y^{n-1} + b_2(x) y^{n-2} + \cdots + b_{n-1}(x) y + b_n(x) = 0 \tag{1.18}$$

where $b_0(x), b_1(x), \ldots, b_n(x)$ are polynomial functions of x and n is a positive integer, then $y = f(x)$ is called an **algebraic function**. Note that polynomial functions and rational functions are special types of algebraic functions. Functions which are built up from a finite number of operations of the five basic operations of addition, subtraction, multiplication, division and extraction of integer roots, usually represent algebraic functions. Some examples of algebraic functions are

1. Any polynomial function.
2. $f_1(x) = (x^3+1)\sqrt{x+4}$
3. $f_2(x) = \dfrac{x^2 + \sqrt[3]{6+x^2}}{(x-3)^{4/3}}$

The function $f(x) = \sqrt{x^2}$ is an example of a function which is not an algebraic function. This is because the square root of x^2 is the absolute value of x and represented

$$f(x) = \sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}$$

and the absolute value operation is not one of the five basic operations mentioned above.

A **transcendental function** is any function which is not an algebraic function. The exponential functions, logarithmic functions, trigonometric functions, inverse trigonometric functions, hyperbolic functions and inverse hyperbolic functions are examples of transcendental functions considered in this calculus text.

The Exponential and Logarithmic Functions

The **exponential functions** have the form $y = b^x$, where $b > 0$ is a positive constant and the variable x is an exponent. If $x = n$ is a positive integer, one defines

$$b^n = \underbrace{b \cdot b \cdots b}_{n \text{ factors}} \qquad \text{and} \qquad b^{-n} = \frac{1}{b^n} \tag{1.19}$$

By definition, if $x = 0$, then $b^0 = 1$. Note that if $y = b^x$, then $y > 0$ for all real values of x.

Logarithmic functions and exponential functions are related. By definition,

$$\textbf{if } \boldsymbol{y = b^x} \quad \textbf{then} \quad \boldsymbol{x = \log_b y} \tag{1.20}$$

and x, the exponent, is called the logarithm of y to the base b. Consequently, one can write

$$\boldsymbol{\log_b(b^x) = x} \textbf{ for every } \boldsymbol{x \in R} \qquad \textbf{and} \qquad \boldsymbol{b^{\log_b x} = x} \textbf{ for every } \boldsymbol{x > 0} \tag{1.21}$$

Recall that logarithms satisfy the following properties

$$
\begin{aligned}
&\log_b(xy) = \log_b x + \log_b y, \qquad x > 0 \text{ and } y > 0 \\
&\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y, \qquad x > 0 \text{ and } y > 0 \\
&\log_b(y^x) = x \log_b y, \quad x \text{ can be any real number}
\end{aligned}
\tag{1.22}
$$

Of all the numbers $b > 0$ available for use as a base for the logarithm function the base $b = 10$ and base $b = e = 2.71818\cdots$ are the most often seen in engineering and scientific research. The number e is a physical constant[5] like π. It can not be represented as the ratio of two integer so is an irrational number. It can be defined as the limiting sum of the infinite series

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots + \frac{1}{n!} + \cdots$$

Using a computer[6] one can verify that the numerical value of e to 50 decimal places is given by

$$e = 2.71828182845904523536028747135266249775724709369\,99\ldots$$

The irrational number e can also be determined from the limit $e = \lim_{h\to 0}(1+h)^{1/h}$.

In the early part of the seventeenth century many mathematicians dealt with and calculated the number e, but it was Leibnitz[7] in 1690 who first gave it a name and notation. His notation for the representation of e didn't catch on. The value of the number represented by the limit $\lim_{h\to 0}(1+h)^{1/h}$ is used so much in mathematics it was represented using the symbol e by Leonhard Euler[8] sometime around 1731 and his notation for representing this number has been used ever since. The number e is sometimes referred to as Euler's number, **the base of the natural logarithms**. The **number** e and the **exponential function** e^x **will occur frequently in our study of calculus.**

---

[5] There are many physical constants in mathematics. Some examples are e, π, i,(imaginary component), γ (Euler-Mascheroni constant). For a listing of additional mathematical constants go to the web site *http://en.wikipedia.org/wiki/Mathematical_constant*.

[6] One can go to the web site *http://www.numberworld.org/misc_runs/e-500b.html* to see that over 500 billion digits of this number have been calculated.

[7] Gottfried Wilhelm Leibnitz (1646-1716) a German physicist, mathematician.

[8] Leonhard Euler (1707-1783) a famous Swiss mathematician.

The logarithm to the base e, is called the **natural logarithm** and its properties are developed in a later chapter. The natural logarithm is given a special notation. Whenever the base $b = e$ one can write either

$$y = \log_e x = \ln x \qquad or \qquad y = \log x \tag{1.23}$$

That is, if the notation ln is used or whenever the base is not specified in using logarithms, it is to be understood that the base $b = e$ is being employed. In this special case one can show

$$y = e^x = \exp(x) \qquad \Longleftrightarrow \qquad x = \ln y \tag{1.24}$$

which gives the identities

$$\ln(e^x) = x, \quad x \in R \qquad \text{and} \qquad e^{\ln x} = x, \quad x > 0 \tag{1.25}$$

In our study of calculus it will be demonstrated that the natural logarithm has the special value $\ln(e) = 1$.

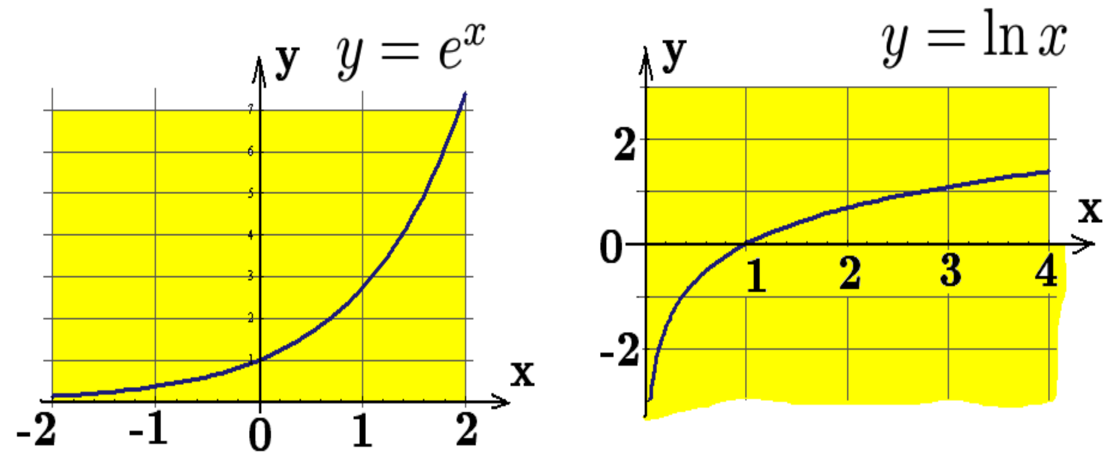


Figure 1-16. The exponential function $y = e^x$ and logarithmic function $y = \ln x$

Note that if $y = \log_b x$, then one can write the equivalent statement $b^y = x$ since a logarithm is an exponent. Taking the natural logarithm of both sides of this last equation gives

$$\ln(b^y) = \ln x \qquad \text{or} \qquad y \ln b = \ln x \tag{1.26}$$

Consequently, for any positive number b different from one

$$y = \log_b x = \frac{\ln x}{\ln b}, \qquad b \neq 1 \tag{1.27}$$

The exponential function $y = e^x$, together with the natural logarithm function can then be used to define all exponential functions by employing the identity

$$y = b^x = (e^{\ln b})^x = e^{x \ln b} \tag{1.28}$$

Graphs of the exponential function $y = e^x = \exp(x)$ and the natural logarithmic function $y = \ln(x) = \log(x)$ are illustrated in the figure 1-16.

The Trigonometric Functions

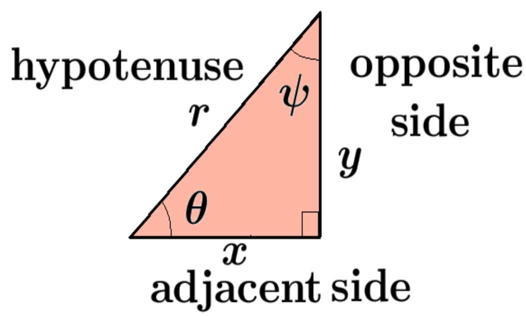
The ratio of sides of a right triangle are used to define the six trigonometric functions associated with one of the acute angles of a right triangle. These definitions can then be extended to apply to positive and negative angles associated with a point moving on a unit circle.

The six trigonometric functions associated with a right triangle are

sine	**tangent**	**secant**
cosine	**cotangent**	**cosecant**

which are abbreviated respectively as

sin, tan, sec, cos, cot, and csc.



Let θ and ψ denote complementary angles in a right triangle as illustrated above. The six trigonometric functions associated with the angle θ are

$$\sin\theta = \frac{y}{r} = \frac{\text{opposite side}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{y}{x} = \frac{\text{opposite side}}{\text{adjacent side}}, \qquad \sec\theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\cos\theta = \frac{x}{r} = \frac{\text{adjacent side}}{\text{hypotenuse}}, \qquad \cot\theta = \frac{x}{y} = \frac{\text{adjacent side}}{\text{opposite side}}, \qquad \csc\theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{opposite side}}$$

Graphs of the Trigonometric Functions

Graphs of the trigonometric functions $\sin\theta$, $\cos\theta$ and $\tan\theta$, for θ varying over the domain $0 \le \theta \le 4\pi$, can be represented in rectangular coordinates by the point sets

$$S = \{\, (\theta, y) \mid y = \sin\theta, \quad 0 \le \theta \le 4\pi \,\},$$

$$C = \{\, (\theta, x) \mid x = \cos\theta, \quad 0 \le \theta \le 4\pi \,\}$$

$$T = \{\, (\theta, y) \mid y = \tan\theta, \quad 0 \le \theta \le 4\pi \,\}$$

and are illustrated in the figure 1-17.

Using the periodic properties

$$\sin(\theta+2\pi)=\sin\theta, \qquad \cos(\theta+2\pi)=\cos\theta \quad \text{and} \quad \tan(\theta+\pi)=\tan\theta$$

these graphs can be extend and plotted over other domains.

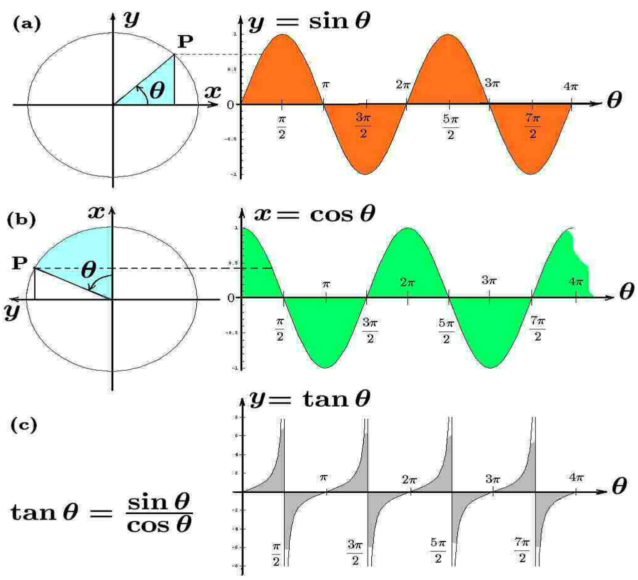


Figure 1-17. Graphs of the trigonometric functions $\sin\theta$, $\cos\theta$ and $\tan\theta$

The function $y=\sin\theta$ can also be interpreted as representing the motion of a point P moving on the circumference of a unit circle. The point P starts at the point $(1,0)$, where the angle θ is zero and then moves in a counterclockwise direction about the circle. As the point P moves around the circle its ordinate value is plotted against the angle θ. The situation is as illustrated in the figure 1-17(a). The function $x=\cos\theta$ can be interpreted in the same way with the point P moving on a circle but starting at a point which is shifted $\pi/2$ radians clockwise. This is the equivalent to rotating the $x,y-$axes for the circle by $\pi/2$ radians and starting the point P at the coordinate $(1,0)$ as illustrated in the figure 1-17(b).

The Hyperbolic Functions

Related to the exponential functions e^x and e^{-x} are the **hyperbolic functions**

hyperbolic sine written sinh,	**hyperbolic cotangent written coth**
hyperbolic cosine written cosh,	**hyperbolic secant written sech**
hyperbolic tangent written tanh,	**hyperbolic cosecant written csch**

These functions are defined

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2}, & \operatorname{csch} x &= \frac{1}{\sinh x} \\ \cosh x &= \frac{e^x + e^{-x}}{2}, & \operatorname{sech} x &= \frac{1}{\cosh x} \\ \tanh x &= \frac{\sinh x}{\cosh x}, & \coth x &= \frac{1}{\tanh x} \end{aligned} \tag{1.29}$$

As the trigonometric functions are related to the circle and are sometimes referred to as circular functions, it has been found that the hyperbolic functions are related to equilateral hyperbola and hence the name hyperbolic functions. This will be explained in more detail in the next chapter.

Symmetry of Functions

The expression $y = f(x)$ is the representation of a function in an explicit form where one variable is expressed in terms of a second variable. The set of values given by

$$S = \{(x, y) \mid y = f(x),\ x \in X\}$$

where X is the domain of the function, represents a graph of the function. The notation $f(x)$, read "f of x", has the physical interpretation of representing y which is the **height of the curve at the point** x**.** Given a function $y = f(x)$, one can replace x by any other argument. For example, if $f(x)$ is a **periodic function** with least period T, one can write $f(x) = f(x+T)$ for all values of x. One can interpret the equation $f(x) = f(x+T)$ for all values of x as stating that the height of the curve at any point x is the same as the height of the curve at the point $x+T$. As another example, if the notation $y = f(x)$ represents the height of the curve at the point x, then $y + \Delta y = f(x + \Delta x)$ would represent the height of the given curve at the point $x+\Delta x$ and $\Delta y = f(x+\Delta x) - f(x)$ would represent the change in the height of the curve y in moving from the point x to the point $x+\Delta x$. If the argument x of the function is replaced by $-x$, then one can compare the height of the curve at the points x and $-x$. If $f(x) = f(-x)$ for all values of x, then the height of the curve at x equals the height of the curve at $-x$ and when this happens the function $f(x)$ is called an **even function of** x and one can state that $f(x)$ is a **symmetric function about the** y**-axis**. If $f(x) = -f(-x)$ for all values of x, then the height of the curve at x equals the negative of the height of the curve at $-x$ and in this case the function $f(x)$ is called an **odd function of** x and one can state that the function $f(x)$ is **symmetric about the origin**. By interchanging the roles of x and y and **shifting or rotation of axes**, other symmetries can be discovered. The figure 1-18 and 1-19 illustrates some examples of symmetric functions.

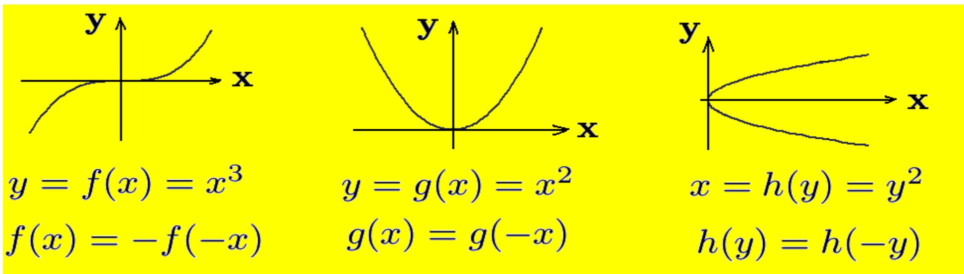
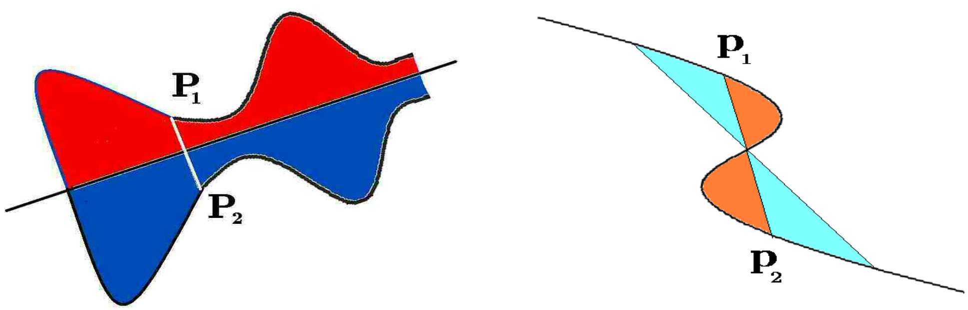


Figure 1-18. Examples of symmetric functions



Symmetry about a line. Symmetry about a point.

Figure 1-19. Examples of symmetric functions

In general, two points P_1 and P_2 are said to be **symmetric to a line** if the line is the perpendicular bisector of the line segment joining the two points. In a similar fashion a graph is said to symmetric to a line if all points of the graph can be grouped into pairs which are symmetric to the line and then the line is called the **axis of symmetry of the graph**. A **point of symmetry** occurs if all points on the graph can be grouped into pairs so that all the line segments joining the pairs are then bisected by the same point. See for example the figure 1-19. For example, one can say that a curve is **symmetric with respect to the x-axis** if for each point (x, y) on the curve, the point $(x, -y)$ is also on the curve. A curve is **symmetric with respect to the y-axis** if for each point (x, y) on the curve, the point $(-x, y)$ is also on the curve. A curve is said to be **symmetric about the origin** if for each point (x, y) on the curve, then the point $(-x, -y)$ is also on the curve.

Example 1-5. A polynomial function $p_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ of degree n has the following properties.

(i) If only even powers of x occur in $p_n(x)$, then the polynomial curve is symmetric about the y-axis, because in this case $p_n(-x) = p_n(x)$.

(ii) If only odd powers of x occur in $p_n(x)$, then the polynomial curve is symmetric about the origin, because in this case $p_n(-x) = -p_n(x)$.

(iii) If there are points $x = a$ and $x = c$ such that $p_n(a)$ and $p_n(c)$ have opposite signs, then there exists at least one point $x = b$ satisfying $a < b < c$, such that $p_n(b) = 0$. This is because polynomial functions are continuous functions and they must change continuously from the value $p_n(a)$ to the value $p_n(c)$ and so must pass, at least once, through the value zero.

■

Translation and Scaling of Axes

Consider two sets of axes labeled (x, y) and $(\bar{x}, \bar{y})$ as illustrated in the figure 1-20(a) and (b). Pick up the $(\bar{x}, \bar{y})$ axes and keep the axes parallel to each other and place the barred axes at some point P having the coordinates (h, k) on the (x, y) axes as illustrated in the figure 1-20(c). One can now think of the barred axes as being a **translated set of axes** where the new origin has been translated to the point (h, k) of the old set of unbarred axes. How is an arbitrary point (x, y) represented in terms of the new barred axes coordinates? An examination of the figure 1-20 shows that a general point (x, y) can be represented as

$$x = \bar{x} + h \quad \text{and} \quad y = \bar{y} + k \qquad \text{or} \qquad \bar{x} = x - h \quad \text{and} \quad \bar{y} = y - k \tag{1.30}$$

Consider a curve $y = f(x)$ sketched on the (x, y) axes of figure 1-20(a). Change the symbols x and y to $\bar{x}$ and $\bar{y}$ and sketch the curve $\bar{y} = f(\bar{x})$ on the $(\bar{x}, \bar{y})$ axes of figure 1-20(b). The two curves should look exactly the same, the only difference being how the curves are labeled. Now move the $(\bar{x}, \bar{y})$ axes to a point (h, k) of the (x, y) coordinate system to produce a situation where the curve $\bar{y} = f(\bar{x})$ is now to be represented with respect to the (x, y) coordinate system.

The new representation can be determined by using the transformation equations (1.30). That is, the new representation of the curve is obtained by replacing $\bar{y}$ by $y - k$ and replacing $\bar{x}$ by $x - h$ to obtain

$$y - k = f(x - h) \tag{1.31}$$

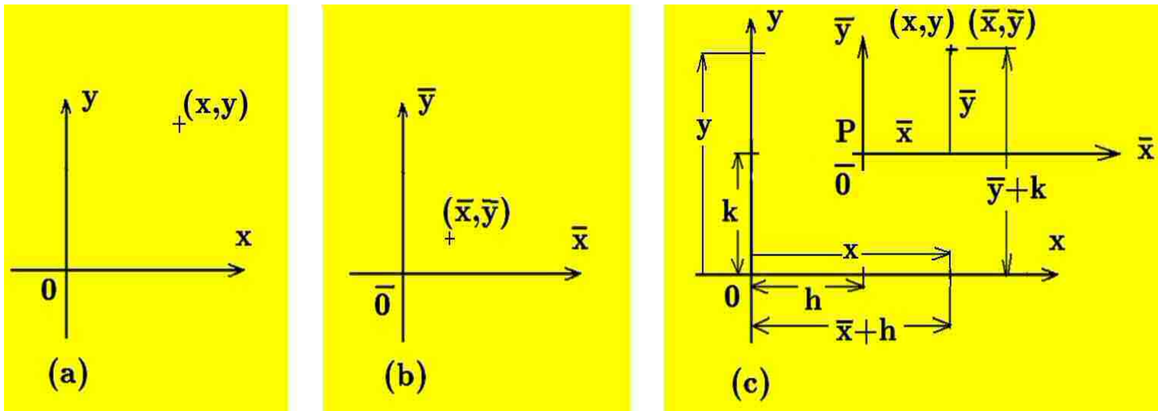


Figure 1-20. Shifting of axes.

In the special case $k = 0$, the curve $y = f(x - h)$ represents a **shifting of the curve** $y = f(x)$ a distance of h units to the right. Replacing h by $-h$ and letting $k = 0$ one finds the curve $y = f(x + h)$ is a **shifting of the curve** $y = f(x)$ a distance of h units to the left. In the special case $h = 0$ and $k \neq 0$, the curve $y = f(x) + k$ represents a shifting of the graph $y = f(x)$ a distance of k units upwards. In a similar fashion, the curve $y = f(x) - k$ represents a shifting of the graph of $y = f(x)$ a distance of k units downward. The figure 1-21 illustrates the shifting and translation of axes for the function $y = f(x) = x^2$.

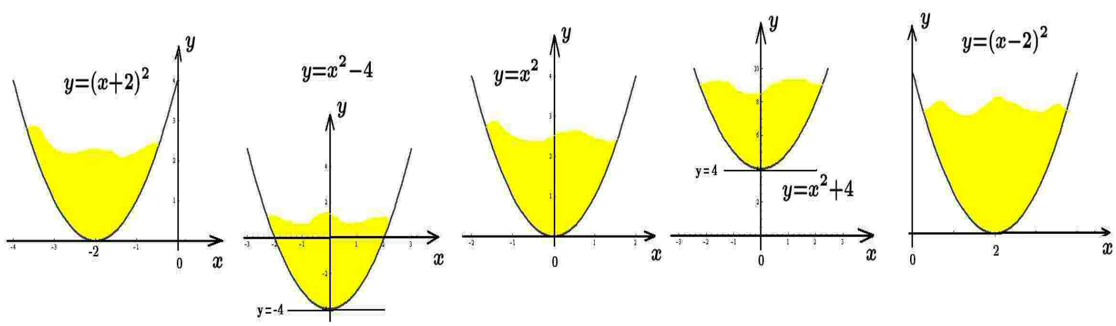


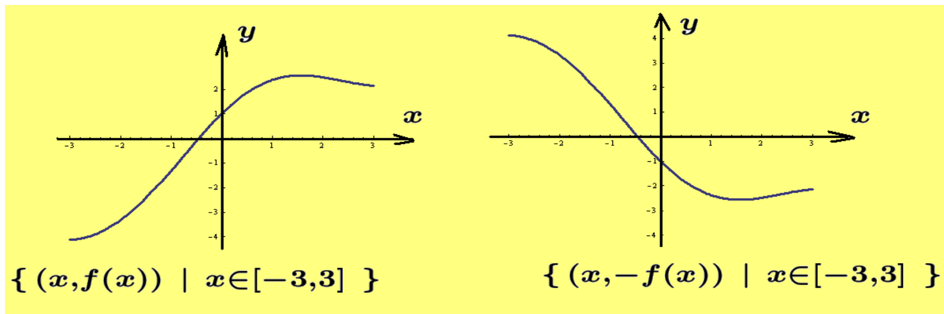
Figure 1-21. Translation and shifting of axes.

Introducing a constant scaling factor $s > 0$, by replacing y by y/s one can create the scaled function $y = sf(x)$. Alternatively one can replace x by sx and obtain the scaled function $y = f(sx)$. These functions are interpreted as follows.

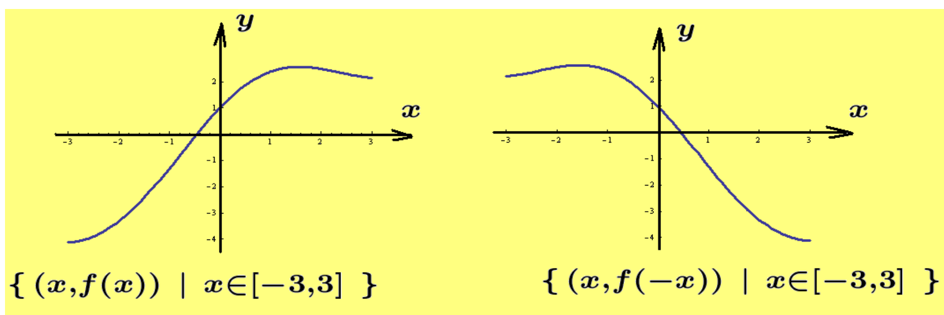
(1) Plotting the function $y = sf(x)$ has the effect of expanding the graph of $y = f(x)$ in the vertical direction if $s > 1$ and compresses the graph if $s < 1$. This is equivalent to changing the **scaling of the units** on the y-axis in plotting a graph. As an exercise plot graphs of $y = \sin x$, $y = 5\sin x$ and $y = \frac{1}{5}\sin x$.

(2) Plotting the function $y = f(sx)$ has the effect of expanding the graph of $y = f(x)$ in the horizontal direction if $s < 1$ and compresses the graph in the x-direction if $s > 1$. This is equivalent to changing the scaling of the units on the x-axis in plotting a graph. As an exercise plot graphs of $y = \sin x, y = \sin(\frac{1}{3}x)$ and $y = \sin(3x)$.

(3) A plot of the graph $(x, -f(x))$ gives a **reflection of the graph** $y = f(x)$ with respect to the x-axis.



(4) A plot of the graph $(x, f(-x))$ gives a reflection of the graph $y = f(x)$ with respect to the y-axis.



Rotation of Axes

Place the $(\bar{x}, \bar{y})$ axes from figure 1-20(b) on top of the (x, y) axes of figure 1-20(a) and then rotate the $(\bar{x}, \bar{y})$ axes through an angle θ to obtain the figure 1-22. An arbitrary point (x, y), a distance r from the origin, has the coordinates $(\bar{x}, \bar{y})$ when referenced to the $(\bar{x}, \bar{y})$ coordinate system. Using basic trigonometry one can find the relationship between the rotated and unrotated axes. Examine the figure 1-22 and verify the following trigonometric relationships.

The projection of r onto the $\bar{x}$ axis produces $\bar{x} = r\cos\phi$ and the projection of r onto the $\bar{y}$ axis produces $\bar{y} = r\sin\phi$. In a similar fashion consider the projection of r onto the y-axis to show $y = r\sin(\theta + \phi)$ and the projection of r onto the x-axis produces $x = r\cos(\theta + \phi)$.

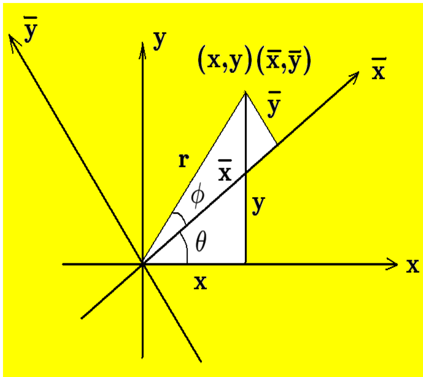


Figure 1-22.
Rotation of axes.

Expressing these projections in the form

$$\begin{aligned} \cos(\theta+\phi) &= \frac{x}{r} \\ \sin(\theta+\phi) &= \frac{y}{r} \end{aligned} \tag{1.32}$$

$$\begin{aligned} \cos\phi &= \frac{\bar{x}}{r} \\ \sin\phi &= \frac{\bar{y}}{r} \end{aligned} \tag{1.33}$$

one can expand the equations (1.32) to obtain

$$\begin{aligned} x = r\cos(\theta+\phi) =& r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ y = r\sin(\theta+\phi) =& r(\sin\theta\cos\phi + \cos\theta\sin\phi) \end{aligned} \tag{1.34}$$

Substitute the results from the equations (1.33) into the equations (1.34) to obtain the transformation equations from the rotated coordinates to the unrotated coordinates. One finds these transformation equations can be expressed

$$\begin{aligned} x =& \bar{x}\cos\theta - \bar{y}\sin\theta \\ y =& \bar{x}\sin\theta + \bar{y}\cos\theta \end{aligned} \tag{1.35}$$

Solving the equations (1.35) for $\bar{x}$ and $\bar{y}$ produces the inverse transformation

$$\begin{aligned} \bar{x} =& x\cos\theta + y\sin\theta \\ \bar{y} =& -x\sin\theta + y\cos\theta \end{aligned} \tag{1.36}$$

Inverse Functions

If a function $y = f(x)$ is such that it never takes on the same value twice, then it is called a **one-to-one function**. One-to-one functions are such that if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. One can test to determine if a function is a one-to-one function by using the **horizontal line test** which is as follows. If there exists a horizontal line $y = a$ *constant*, which intersects the graph of $y = f(x)$ in more than one point, then there will exist numbers x_1 and x_2 such that the height of the curve at x_1 is the same as the height of the curve at x_2 or $f(x_1) = f(x_2)$. This shows that the function $y = f(x)$ is not a one-to-one function.

Let $y = f(x)$ be a **single-valued function** of x which is a one-to-one function such as the function sketched in the figure 1-23(a). On this graph interchange the values of x and y everywhere to obtain the graph in figure 1-23(b). To represent the function

$x = f(y)$ with y in terms of x define the inverse operator f^{-1} with the property that $f^{-1}f(x) = x$. Now apply this operator to both sides of the equation $x = f(y)$ to obtain $f^{-1}(x) = f^{-1}f(y) = y$ or $y = f^{-1}(x)$. The function $y = f^{-1}(x)$ is called the **inverse function** associated with the function $y = f(x)$. Rearrange the axes in figure 1-23(b) so the x-axis is to the right and the y-axis is vertical so that the axes agree with the axes representation in figure 1-23(a). This produces the figure 1-23(c). Now place the figure 1-23(c) on top of the original graph of figure 1-23(a) to obtain the figure 1-23(d), which represents a comparison of the original function and its inverse function. This figure illustrates the function $f(x)$ and its inverse function $f^{-1}(x)$ are one-to-one functions which are symmetric about the line $y = x$.

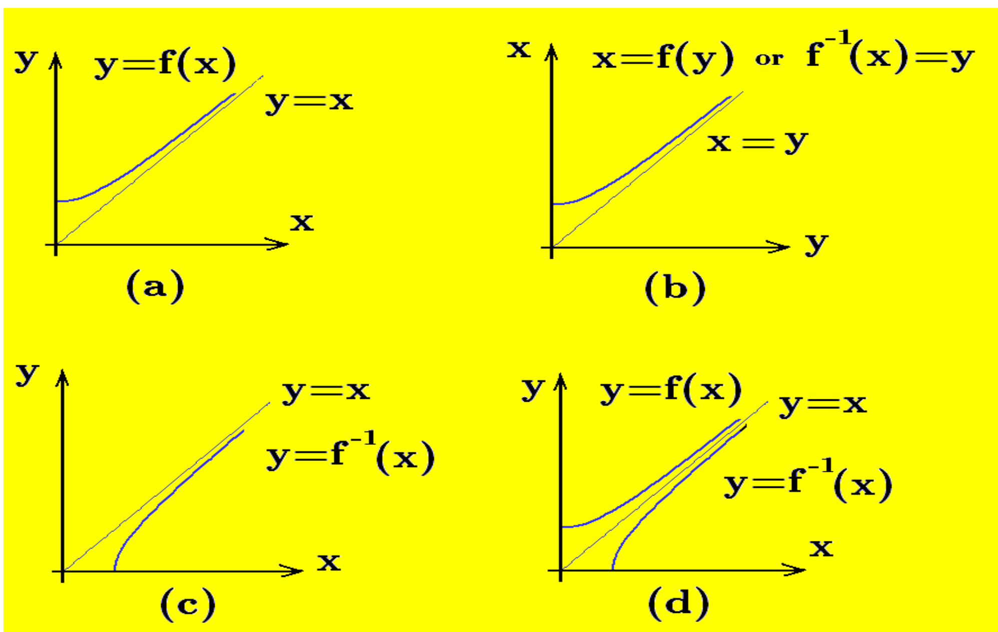


Figure 1-23. Sketch of a function and its inverse function.

Note that some functions **do not have an inverse function**. This is usually the result of the original function not being a one-to-one function. By selecting a **domain of the function** where it is **a one-to-one function** one can define a **branch of the function** which has an inverse function associated with it.

Lets examine what has just been done in a slightly different way. If $y = f(x)$ is a one-to-one function, then the graph of the function $y = f(x)$ is a set of **ordered pairs** (x, y), where $x \in X$, the domain of the function and $y \in Y$ the range of the function.

Now if the function $f(x)$ is such that no two ordered pairs have the same second element, then the function obtained from the set

$$S = \{\, (x,y) \mid y = f(x),\ x \in X\}$$

by interchanging the values of x and y is called the inverse function of f and it is denoted by f^{-1}. Observe that the inverse function has the domain of definition Y and its range is X and one can write

$$y = f(x) \qquad \Longleftrightarrow \qquad f^{-1}(y) = x \tag{1.32}$$

Still another way to approach the problem is as follows. Two functions $f(x)$ and $g(x)$ are said to be inverse functions of one another if $f(x)$ and $g(x)$ have the properties that

$$g(f(x)) = x \qquad \text{and} \qquad f(g(x)) = x \tag{1.33}$$

If $g(x)$ is an inverse function of $f(x)$, the notation f^{-1}, (read f-inverse), is used to denote the function g. That is, an inverse function of $f(x)$ is denoted $f^{-1}(x)$ and has the properties

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(x)) = x \tag{1.34}$$

Given a function $y = f(x)$, then by interchanging the symbols x and y there results $x = f(y)$. This is an equation which defines the inverse function. If the equation $x = f(y)$ can be solved for y in terms of x, to obtain a single valued function, then this function is called the inverse function of $f(x)$. One then obtains the equivalent statements

$$x = f(y) \qquad \Longleftrightarrow \qquad y = f^{-1}(x) \tag{1.35}$$

The process of interchanging x and y in the representation $y = f(x)$ to obtain $x = f(y)$ implies that geometrically the graphs of f and f^{-1} are **mirror images of each other** about the line $y = x$. In order that the inverse function be **single valued and one-to-one**, it is necessary that there are no horizontal lines, $y = constant$, which intersect the graph $y = f(x)$ more than once. Observe that one way to find the inverse function is the following.

(1.) Write $y = f(x)$ and then interchange x and y to obtain $x = f(y)$

(2.) Solve $x = f(y)$ for the variable y to obtain $y = f^{-1}(x)$

(3.) Note the inverse function $f^{-1}(x)$ sometimes turns out to be **a multiple-valued function**. Whenever this happens it is customary to break the function up into a collection of single-valued functions, called **branches**, and then **one of these branches is selected** to be called the **principal branch of the function**. That is, if multiple-valued functions are involved, then select **a branch of the function which is single-valued** such that the range of $y = f(x)$ is the domain of $f^{-1}(x)$. An example of a function and its inverse is given in the figure 1-22.

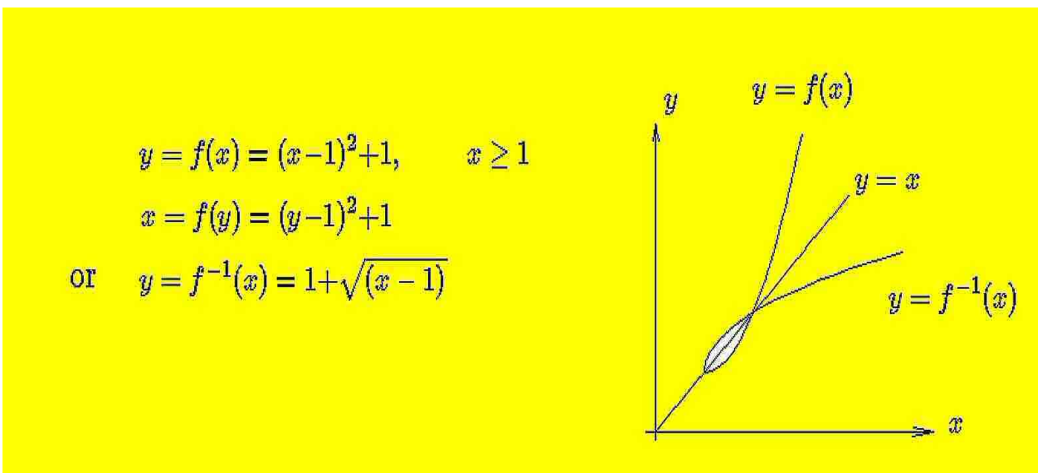


Figure 1-22. An example of a function and its inverse.

Example 1-6. (Inverse Trigonometric Functions)

The **inverse trigonometric functions** are defined in the table 1-1. The inverse trigonometric functions can be graphed by interchanging the axes on the graphs of the trigonometric functions as illustrated in the figures 1-23 and 1-24. Observe that these inverse functions are **multi-valued functions** and consequently one must define an interval associated with each inverse function such that the inverse function becomes a single-valued function. This is called selecting a **branch of the function** such that it is single-valued.

Table 1-1			
Inverse Trigonometric Functions			
Function	Alternate notation	Definition	Interval for single-valuedness
$\arcsin x$	$\sin^{-1} x$	$\sin^{-1} x = y$ if and only if $x = \sin y$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\arccos x$	$\cos^{-1} x$	$\cos^{-1} x = y$ if and only if $x = \cos y$	$0 \le y \le \pi$
$\arctan x$	$\tan^{-1} x$	$\tan^{-1} x = y$ if and only if $x = \tan y$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$\text{arccot}\, x$	$\cot^{-1} x$	$\cot^{-1} x = y$ if and only if $x = \cot y$	$0 < y < \pi$
$\text{arcsec}\, x$	$\sec^{-1} x$	$\sec^{-1} x = y$ if and only if $x = \sec y$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
$\text{arccsc}\, x$	$\csc^{-1} x$	$\csc^{-1} x = y$ if and only if $x = \csc y$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$

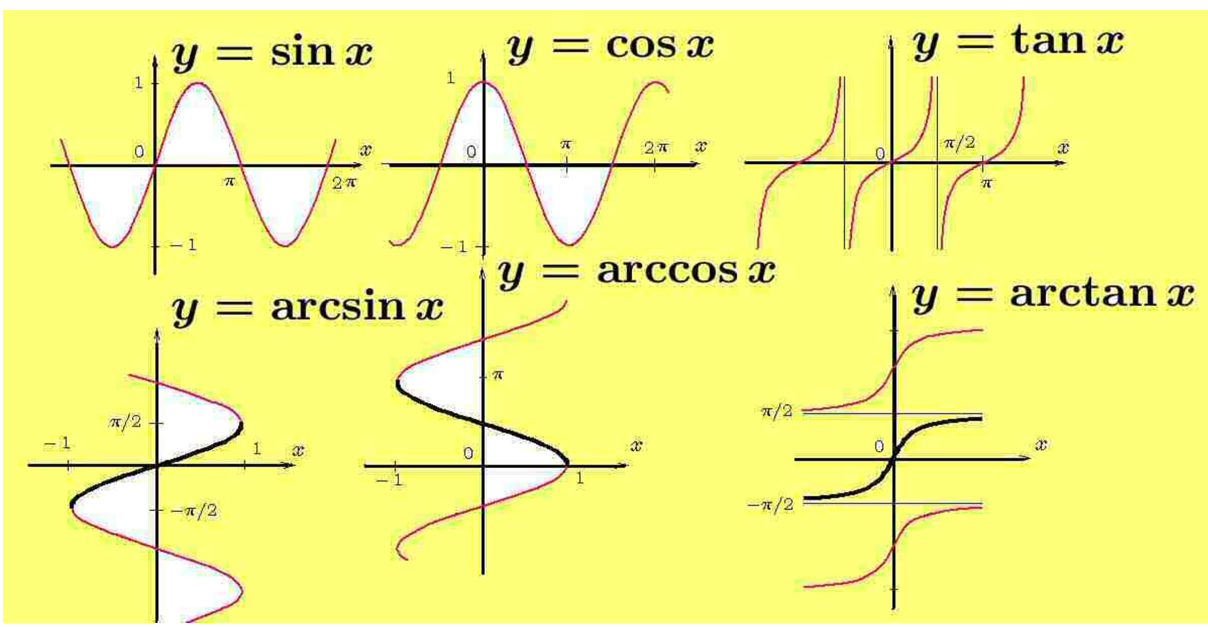


Figure 1-23. The inverse trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$.

There are many different intervals over which each inverse trigonometric function can be made into **a single-valued function.** These different intervals are referred to as **branches of the inverse trigonometric functions**. Whenever a particular branch is required for certain problems, then by agreement these branches are called **principal branches** and are always used in doing calculations. The following table gives **one way** of defining principal value branches for the inverse trigonometric functions. These branches are highlighted in the figures 1-23 and 1-24.

Principal Values for Regions Indicated	
$x<0$	$x\ge 0$
$-\frac{\pi}{2}\le \sin^{-1}x<0$	$0\le \sin^{-1}x\le \frac{\pi}{2}$
$\frac{\pi}{2}\le \cos^{-1}x\le \pi$	$0\le \cos^{-1}x\le \frac{\pi}{2}$
$-\frac{\pi}{2}\le \tan^{-1}x<0$	$0\le \tan^{-1}x<\frac{\pi}{2}$
$\frac{\pi}{2}<\cot^{-1}x<\pi$	$0<\cot^{-1}x\le \frac{\pi}{2}$
$\frac{\pi}{2}\le \sec^{-1}x\le \pi$	$0\le \sec^{-1}x<\frac{\pi}{2}$
$-\frac{\pi}{2}\le \csc^{-1}x<0$	$0<\csc^{-1}x\le \frac{\pi}{2}$

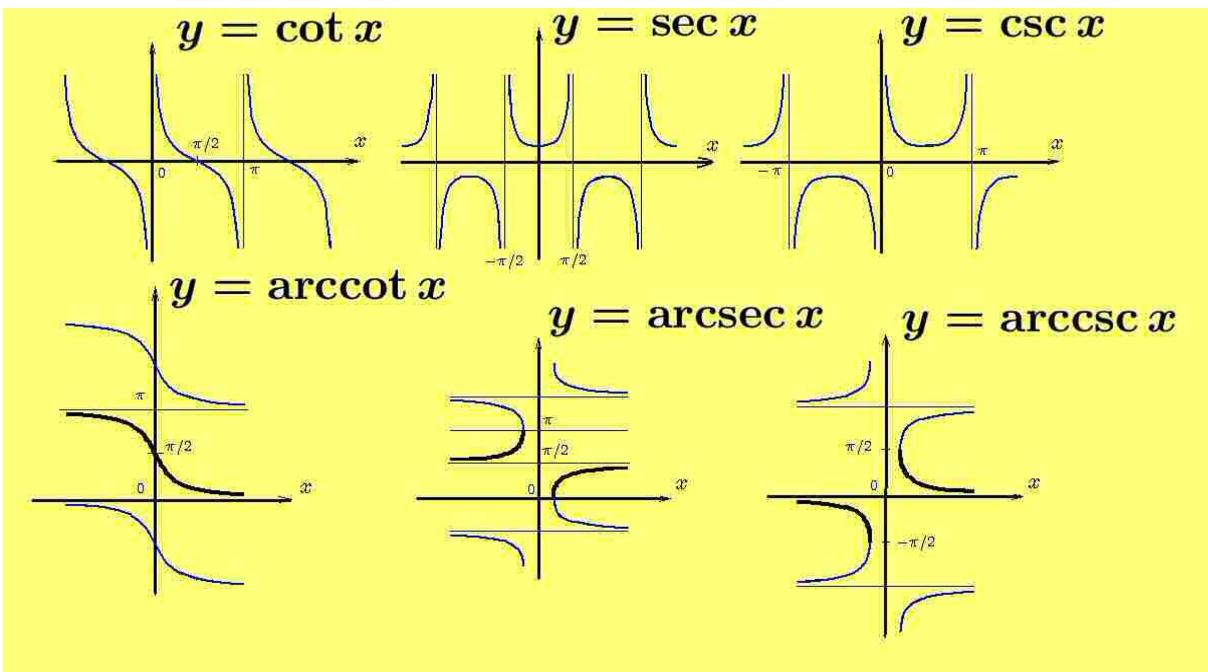


Figure 1-24. The inverse trigonometric functions $\cot^{-1}x$, $\sec^{-1}x$ and $\csc^{-1}x$.

■

Equations of lines

Given two points (x_1,y_1) and (x_2,y_2) one can plot these points on a rectangular coordinate system and then draw a straight line ℓ through the two points as illustrated in the sketch given in the figure 1-25. By definition the **slope of the line** is defined as the tangent of the angle α which is formed where the line ℓ intersects the x-axis.

Move from point (x_1, y_1) to point (x_2, y_2) along the line and let Δy denote a change in y and let Δx denote a change in x, then the slope of the line, call it m, is calculated

$$\text{slope of line} = m = \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} \tag{1.36}$$

If (x, y) is used to denote a variable point which moves along the line, then one can make use of similar triangles and write either of the statements

$$m = \frac{y - y_1}{x - x_1} \qquad \text{or} \qquad m = \frac{y - y_2}{x - x_2} \tag{1.37}$$

The first equation representing the change in y over a change in x relative to the first point and the second equation representing a change in y over a change in x relative to the second point on the line.

This gives the **two-point formulas** for representing a line

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = m \qquad \text{or} \qquad \frac{y - y_2}{x - x_2} = \frac{y_2 - y_1}{x_2 - x_1} = m \tag{1.38}$$

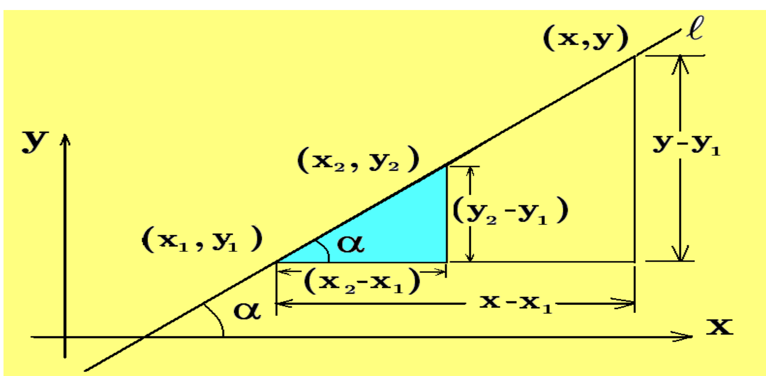


Figure 1-25. The line $y - y_1 = m(x - x_1)$

Once the slope $m = \dfrac{\text{change in } y}{\text{change in } x} = \tan\alpha$ of the line is known, one can represent the line using either of the **point-slope** forms

$$y - y_1 = m(x - x_1) \qquad \text{or} \qquad y - y_2 = m(x - x_2) \tag{1.39}$$

Note that lines parallel to the x-axis have **zero slope** and are represented by equations of the form $y = y_0 = a\ constant$. For lines which are perpendicular to the x-axis or parallel to the y-axis, the slope is not defined. This is because the slope tends toward a + **infinite slope** or - **infinite slope** depending upon how the angle of intersection α approaches $\pi/2$. Lines of this type are represented by an equation having the form $x = x_0 =$ a constant. The figure 1-26 illustrates the general shape of a straight line which has a positive, zero and negative slope.

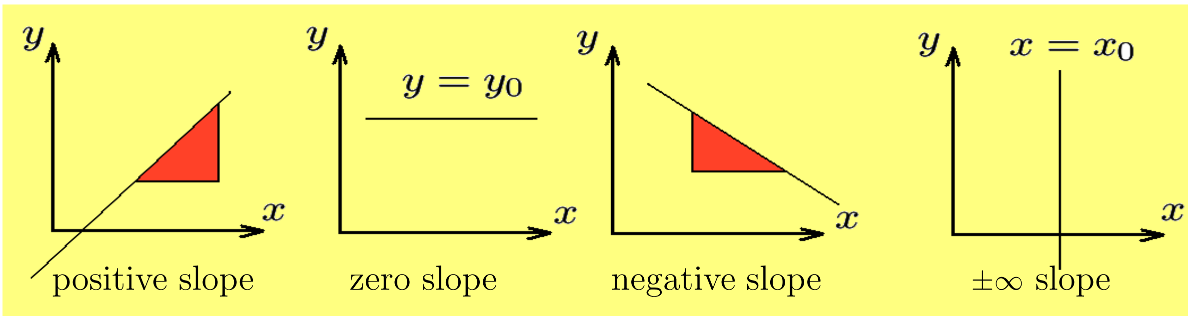


Figure 1-26. The slope of a line.

The **general equation of a line** is given by

$$Ax + By + C = 0, \quad \text{where } A, B, C \text{ are constants.} \tag{1.40}$$

The **slope-intercept** form for the equation of a line is given by

$$y = mx + b \tag{1.41}$$

where m is the slope and b is the $y-$intercept. Note that when $x = 0$, then the point $(0, b)$ is where the line intersects the y axis. If the line intersects the y-axis at the point $(0, b)$ and intersects the x-axis at the point $(a, 0)$, then the **intercept form** for the equation of a straight line is given by

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad a \neq 0, \quad b \neq 0 \tag{1.42}$$

One form[1] for the **parametric equation of a straight line** is given by the set of points

$$\ell = \{\, (x, y) \mid x = t \text{ and } y = mt + b, \quad -\infty < t < \infty \,\}$$

Another parametric form such as

$$\ell = \{\, (x, y) \mid x = \sin t \text{ and } y = m \sin t + b, \quad -\infty < t < \infty \,\}$$

gives only **a segment of the total line**.

[1] The parametric representation of a line or curve is not unique and depends upon the representation.

The **polar form for the equation of a straight line** can be obtained from the intercept form for a line, given by equation (1.42), by using the transformation equations (1.1) previously considered. For example, if the intercept form of the line ℓ is $\frac{x}{a}+\frac{y}{b}=1$, then the transformation equations $x=r\cos\theta$ and $y=r\sin\theta$ change this equation to the form

$$\frac{r\cos\theta}{a}+\frac{r\sin\theta}{b}=1 \tag{1.43}$$

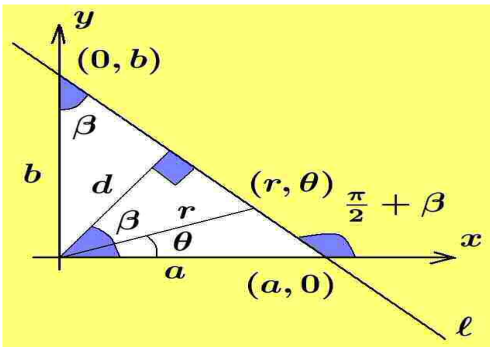


Figure 1-27.
Determining polar form for line.

Let d denote **the perpendicular distance from the line ℓ to the origin** of the rectangular x,y coordinate system as illustrated in the figure 1-27. This perpendicular line makes an angle β such that

$$\cos\beta=\frac{d}{a} \quad \text{and} \quad \cos(\frac{\pi}{2}-\beta)=\sin\beta=\frac{d}{b}$$

Solve for $\frac{1}{a}$ and $\frac{1}{b}$ and substitute the results into the equation (1.43) to show

$$\frac{r}{d}\cos\theta\cos\beta+\frac{r}{d}\sin\theta\sin\beta=1$$

Use trigonometry to simplify the above equation and show the **polar form for the equation of the line** is

$$r\cos(\theta-\beta)=d \tag{1.44}$$

Here (r,θ) is a general point in polar coordinates which moves along the line ℓ described by the polar equation (1.44) and β is the angle that the x-axis makes with the line which passes through the origin and is perpendicular to the line ℓ.

Perpendicular Lines

Consider a line ℓ_2 which is perpendicular to a given line ℓ_1 as illustrated in the figure 1-28. The slope of the line ℓ_1 is given by $m_1=\tan\alpha_1$ and the slope of the line ℓ_2 is given by $m_2=\tan\alpha_2$ where α_1 and α_2 are the positive angles made when the lines ℓ_1 and ℓ_2 intersect the x-axis.

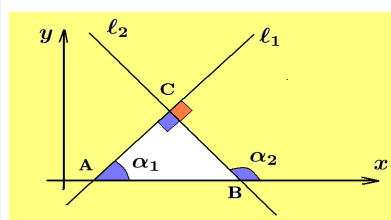


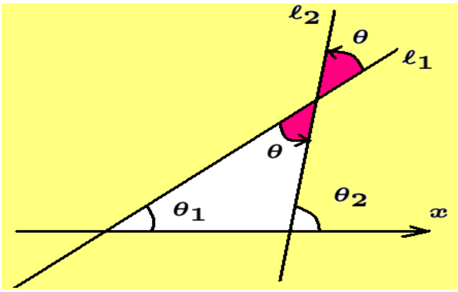
Figure 1-28.
Perpendicular lines.

Two lines are said to **intersect orthogonally** when they intersect to form two right angles. Note that α_2 is an **exterior angle** to a right triangle ABC and so one can write $\alpha_2 = \alpha_1 + \pi/2$. If the two lines are perpendicular, then the product of the slopes m_1 and m_2 must satisfy

$$m_1 m_2 = \tan\alpha_1 \tan\alpha_2 = \left(\frac{\sin\alpha_1}{\cos\alpha_1}\right)\frac{\sin(\alpha_1+\pi/2)}{\cos(\alpha_1+\pi/2)} = \frac{\sin\alpha_1}{\cos\alpha_1}\cdot\frac{\cos\alpha_1}{(-\sin\alpha_1)} = -1 \tag{1.45}$$

which shows that if the two lines are perpendicular, then **the product of their slopes must equal -1**, or alternatively, one slope must be the **negative reciprocal of the other slope**. This relation breaks down if one of the lines is parallel to the x-axis, because then a zero slope occurs.

In general, if line ℓ_1 with slope $m_1 = \tan\theta_1$ intersects line ℓ_2 with slope $m_2 = \tan\theta_2$ and θ denotes the angle of intersection as measured from line ℓ_1 to ℓ_2, then $\theta = \theta_2 - \theta_1$ and



$$\boldsymbol{\tan\theta = \tan(\theta_2-\theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_1\tan\theta_2}}$$

$$\boldsymbol{\tan\theta = \frac{m_2 - m_1}{1+m_1 m_2}}$$

Note that as m_2 approaches $-1/m_1$, then the angle θ approaches $\pi/2$.

Limits

The notation $x \to \infty$ is used to denote the value x increasing without bound in the positive direction and the notation $x \to -\infty$ is used to denote x increasing without bound in the negative direction. The limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, if they exist, are used to denote the values of a function $f(x)$ as the variable x is allowed to increase without bound in the positive and negative directions. For example one can write

$$\lim_{x\to\infty}(2+\frac{1}{x}) = 2 \qquad \text{and} \qquad \lim_{x\to-\infty}(2+\frac{1}{x^2}) = 2$$

The notation $x \to x_0$ denotes the variable x approaching the finite value x_0, but **it never gets there**. If x and x_0 denote real numbers, then x **can approach** x_0 **from any direction** and get as close to x_0 as you desire, but **it cannot equal the value** x_0. For $\epsilon > 0$ a small real number, sketch on a piece of paper the difference between the x-values defined by the sets

$$N = \{x \mid \; |x - x_0| < \epsilon\} \qquad \text{and} \qquad N_0 = \{x \mid \; 0 < |x-x_0| < \epsilon\} \tag{1.46}$$

Observe that the set N **contains all the points between** $x_0-\epsilon$ **and** $x_0+\epsilon$ **together with the point** $x_0 \in N$, while the set N_0 is the same as N but **has the point** x_0 **excluded** so one can write $x_0 \notin N_0$. The set N is called a **neighborhood of the point** x_0 while the set N_0 is called a **deleted neighborhood of the point** x_0. The notation $x \to x_0$ emphasizes the requirement that x approach the value x_0, but x is **restricted to taking on the values in the set** N_0. The situation is illustrated in the figures 1-29(a) and 1-29(b). The notation $x \to x_0^+$ is used to denote x approaching x_0 from the **right-hand side of** x_0, where x is restricted such that $x > x_0$. The notation $x \to x_0^-$ is used to denote x approaching x_0 from the **left-hand side of** x_0, where x is restricted such that $x < x_0$. In general, the notation $x \to x_0$ means x can approach x_0 from any direction, but x can never equal x_0.

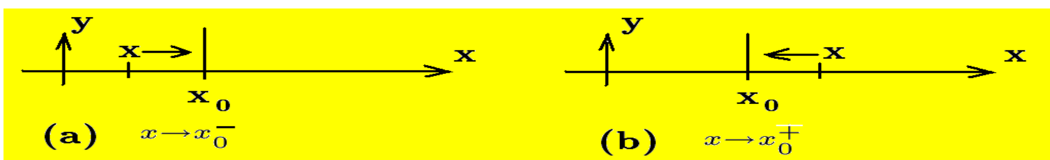


Figure 1-29. Left and right-hand approaches of x to the value x_0.

Infinitesimals

You cannot compare two quantities which are completely different in every way. To measure related quantities you must have

(i) A basic unit of measurement applicable to the quantity being measured.

(ii) A number representing the ratio of the measured quantity to the basic unit.

The concept of largeness or smallness of a quantity is relative to the basic unit selected for use in the measurement. For example, if a quantity Q is divided up into some fractional part f, then fQ is smaller than Q because the ratio $fQ/Q = f$ is small. For $f < 1$, quantities like $fQ, f^2Q, f^3Q, \ldots$ are called small quantities of the first, second, third orders of smallness, since each quantity is a small fraction f of the previous quantity. If the fraction f is allowed to approach zero, then the quantities $fQ, f^2Q, f^3Q, \ldots$ are very, very small and are called infinitesimals of the first, second, third, ..., orders. Thus, if Δx is a small change in x, then $(\Delta x)^2$ would be an infinitesimal of the second order, $(\Delta x)^3$ would be an infinitesimal of the third order, etc.

In terms of limits, if α and β are infinitesimals and $\lim_{\alpha\to 0}\frac{\beta}{\alpha}$ is some constant different from zero, then α and β are called **infinitesimals of the same order.** However, if $\lim_{\alpha\to 0}\frac{\beta}{\alpha}=0$, then β is called **an infinitesimal of higher order than** α**.**

If you are dealing with an equation involving infinitesimals of different orders,* you only need to retain those infinitesimals of lowest order*, since the higher order infinitesimals are significantly smaller and will not affect the results when these infinitesimals approach zero .

This concept is often used in **comparing the ratio of two small quantities** which approach zero. Consider the problem of finding the volume of a hollow cylinder, as illustrated in the figure 1-30, as the thickness of the cylinder sides approaches zero.

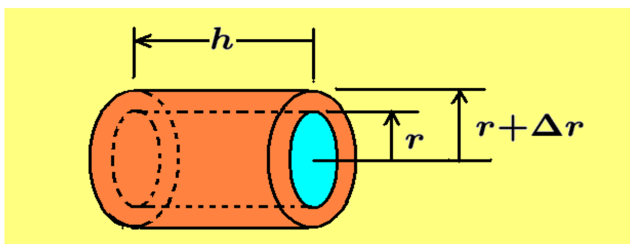


Figure 1-30. Volume of hollow cylinder.

Let ΔV denote the volume of the hollow cylinder with r the inner radius of the hollow cylinder and $r+\Delta r$ the outer radius. One can write

$$\begin{aligned}\Delta V =& \text{Volume of outer cylinder} - \text{Volume of inner cylinder}\\ \Delta V =& \pi(r+\Delta r)^2h - \pi r^2 h = \pi[r^2+2r\Delta r+(\Delta r)^2]h - \pi r^2 h\\ \Delta V =& 2\pi r h\Delta r + \pi h(\Delta r)^2\end{aligned}$$

This relation gives the exact volume of the hollow cylinder. If one takes the limit as Δr tends toward zero, then the Δr and $(\Delta r)^2$ terms become infinitesimals and the infinitesimal of the second order can be neglected since one is only interested in comparison of ratios when dealing with small quantities. For example

$$\lim_{\Delta r\to 0}\frac{\Delta V}{\Delta r} = \lim_{\Delta r\to 0}(2\pi r h + \pi h\Delta r) = 2\pi r h$$

Limiting Value of a Function

The notation $\lim_{x\to x_0} f(x) = \ell$ is used to denote **the limiting value of a function** $f(x)$ **as** x **approaches the value** x_0**, but** $x \neq x_0$. Note that the limit statement $\lim_{x\to x_0} f(x)$ is dependent upon values of $f(x)$ for x near x_0, **but not for** $x = x_0$. One must examine the values of $f(x)$ both for x_0^+ values (values of x slightly greater than x_0) and for x_0^- values (values of x slightly less than x_0). These type of limiting statements are written

$$\lim_{x\to x_0^+} f(x) \qquad \text{and} \qquad \lim_{x\to x_0^-} f(x)$$

and are called **right-hand and left-hand limits** respectively. There may be situations where (a) $f(x_0)$ is not defined (b) $f(x_0)$ is defined but does not equal the limiting value ℓ (c) the limit $\lim_{x\to x_0} f(x)$ might become unbounded, in which case one can write a statement stating that "no limit exists as $x \to x_0$ ".

Some limits are easy to calculate, for example $\lim_{x\to 2}(3x+1) = 7$, is a limit of the form $\lim_{x\to x_0} f(x) = f(x_0)$, where $f(x_0)$ is the value of $f(x)$ at $x = x_0$, if the value $f(x_0)$ exists. This method is fine if the graph of the function $f(x)$ is a smooth unbroken curve in the neighborhood of the point x_0.

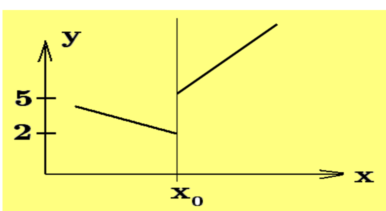


Figure 1-31. Sectionally-continuous function

The limiting value $\lim_{x\to x_0} f(x)$ cannot be calculated by evaluating the function $f(x)$ at the point x_0 if the function is not defined at the point x_0. A function $f(x)$ is called **a sectionally continuous function** if its graph can be represented by sections of unbroken curves. The function $f(x)$ defined

$$f(x) = \begin{cases} 2-(x-x_0), & x < x_0 \\ 5+2(x-x_0), & x > x_0 \end{cases}$$

is an example of **a sectionally continuous function**. Note this function is not defined at the point x_0 and the **left-hand limit** $\lim_{x\to x_0^-} f(x) = 2$ and the **right-hand limit** given by $\lim_{x\to x_0^+} f(x) = 5$ are not equal and $f(x_0)$ is not defined. The graph of $f(x)$ is sketched in the figure 1-31. The function $f(x)$ is said to have a **jump discontinuity at the point** $x = x_0$ and one would write $\lim_{x\to x_0} f(x)$ does not exit.

Some limiting values produce the **indeterminate forms** $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and these resulting forms **must be analyzed further to determine their limiting value.** For example, the limiting value $\frac{0}{0}$ may reduce to a zero value, a finite quantity or it may become an infinite quantity as the following examples illustrate.

$$\lim_{x\to 0}\frac{3x^2}{2x} = \lim_{x\to 0}\frac{3}{2}x = 0, \quad \text{a zero limit}$$

$$\lim_{x\to 0}\frac{3x}{2x} = \lim_{x\to 0}\frac{3}{2} = \frac{3}{2}, \quad \text{a finite limit}$$

$$\lim_{x\to 0}\frac{3x}{2x^2} = \lim_{x\to 0}\frac{3}{2x} = \infty, \quad \text{an infinite limit}$$

$$\lim_{x\to x_0}\frac{x^2-x_0^2}{x-x_0} = \lim_{x\to x_0}\frac{(x-x_0)(x+x_0)}{(x-x_0)} = 2x_0, \quad \text{a finite limit}$$

Example 1-7. (Geometry used to determine limiting value)

Consider the function $f(x) = \dfrac{\sin x}{x}$ and observe that this function is not defined at the value $x = 0$, because $f(0) = \dfrac{\sin x}{x}\Big|_{x=0} = \dfrac{0}{0}$, **an indeterminate form.** Let us investigate the limit $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$ using the geometry of the figure 1-32 as the angle[2] x gets very small, but with $x \neq 0$. The figure 1-32 illustrates part of a circle of radius r sketched in the first quadrant along with a ray from the origin constructed at the angle x. The lines AD and BC perpendicular to the polar axis are constructed along with the line BD representing a chord. These constructions are illustrated in the figure 1-32. From the geometry of figure 1-32 verify the following values.

$$AD = r\sin x \qquad BC = r\tan x$$

$$\text{Area } \triangle\, 0BD = \frac{1}{2}0B\cdot AD = \frac{1}{2}r^2\sin x$$

$$\text{Area sector 0BD} = \frac{1}{2}r^2 x$$

$$\text{Area } \triangle\, 0BC = \frac{1}{2}0B\cdot BC = \frac{1}{2}r^2\tan x$$

One can compare the areas of triangles $\triangle 0BD$, $\triangle 0BC$ and sector $0BD$ to come up with the inequalities

$$\begin{array}{lcccc} \text{Area}\,\triangle\, 0BD & \leq & \text{Area sector } 0BD & \leq & \text{Area } \triangle\, 0BC \\ \text{or} \quad \frac{1}{2}r^2\sin x & \leq & \frac{1}{2}r^2 x & \leq & \frac{1}{2}r^2\tan x \end{array} \tag{1.47}$$

Divide this inequality through by $\frac{1}{2}r^2\sin x$ to obtain the result $1 \leq \dfrac{x}{\sin x} \leq \dfrac{1}{\cos x}$. Taking the reciprocals one can write

$$1 \geq \frac{\sin x}{x} \geq \cos x \tag{1.48}$$

[2] Radian measure is always used.

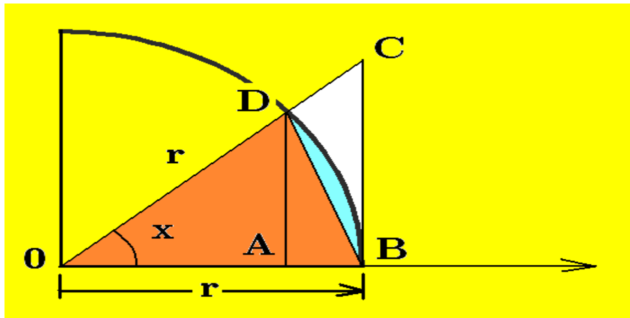


Figure 1-32. Construction of triangles and sector associated with circle.

Now take the limit as x approaches zero to show

$$1 \geq \lim_{x \to 0} \frac{\sin x}{x} \geq \lim_{x \to 0} \cos x \tag{1.49}$$

The function $\dfrac{\sin x}{x}$ is **squeezed or sandwiched** between the values 1 and $\cos x$ and since the cosine function approaches 1 as x approaches zero, one can say the limit of the function $\dfrac{\sin x}{x}$ must also approach 1 and so one can write

$$\boldsymbol{\lim_{x \to 0} \frac{\sin x}{x} = 1} \tag{1.50}$$

In our study of calculus other methods are developed to verify the above limiting value associated with the indeterminate form $\dfrac{\sin x}{x}$ as x approaches zero.

■

Example 1-8. **(Algebra used to determine limiting value)**

Algebra as well as geometry can be used to aid in evaluating limits. For example, to calculate the limit

$$\lim_{x \to 1} \frac{x^n - 1}{x - 1} = \left. \frac{x^n - 1}{x - 1} \right|_{x=1} = \frac{0}{0} \quad \text{an indeterminate form} \tag{1.51}$$

one can make the change of variables $z = x - 1$ and express the limit given by equation (1.51) in the form

$$\lim_{z \to 0} \frac{(1+z)^n - 1}{z} \tag{1.52}$$

The numerator of this limit expression can be expanded by using the binomial theorem

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \frac{n(n-1)(n-2)}{3!} z^3 + \cdots \tag{1.53}$$

Substituting the expansion (1.53) into the equation (1.52) and simplifying reduces the given limit to the form

$$\lim_{z\to 0}\left[n+\frac{n(n-1)}{2!}z+\frac{n(n-1)(n-2)}{3!}z^2+\cdots\right]=n \tag{1.54}$$

This shows that $\displaystyle\lim_{x\to 1}\frac{x^n-1}{x-1}=n$

■

Example 1-9. **(Limits)**

The following are some examples illustrating limiting values associated with functions.

$$\lim_{x\to 3}x^2=9 \qquad \lim_{x\to 0^-}\frac{1}{x}=-\infty \qquad \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1$$

$$\lim_{x\to 0^+}\frac{1}{x}=+\infty \qquad \lim_{x\to\infty}\left(3+\frac{1}{x}\right)=3 \qquad \lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\frac{1}{2}$$

■

Formal Definition of Limit

In the early development of mathematics the concept of a limit was very vague. The calculation of a limit was so fundamental to understanding certain aspects of calculus, that it required a precise definition. A more formal $\epsilon-\delta$ (read epsilon-delta) definition of a limit was finally developed around the 1800's. This formalization resulted from the combined research into limits developed by the mathematicians Weierstrass,[3] Bolzano[4] and Cauchy.[5]

Definition 1: Limit of a function

Let $f(x)$ be defined and single-valued for all values of x in some deleted neighborhood of the point x_0. A number ℓ is called a limit of $f(x)$ as x approaches x_0, written $\lim_{x\to x_0} f(x)=\ell$, if for every small positive number $\epsilon>0$ there exists a number δ such that[6]

$$|f(x)-\ell|<\epsilon \qquad \textit{whenever} \qquad 0<|x-x_0|<\delta \tag{1.55}$$

Then one can write $f(x)\to\ell$ ($f(x)$ approaches ℓ) as $x\to x_0$ (x approaches x_0). Note that $f(x)$ need not be defined at the point x_0 in order for a limit to exist.

[3] Karl Theodor Wilhelm Weierstrass (1815-1897) A German mathematician.
[4] Bernard Placidus Johan Nepomuk Bolzano (1781-1848) A Bohemian philosopher and mathematician.
[5] Augustin Louis Cauchy (1789-1857) A French mathematician.
[6] The number δ usually depends upon how ϵ is selected.

The above definition must be modified if restrictions are placed upon how x approaches x_0. For example, the limits $\lim_{x\to x_0^+} f(x) = \ell_1$ and $\lim_{x\to x_0^-} f(x) = \ell_2$ are called the **right-hand and left-hand limits** associated with the function $f(x)$ as x approaches the point x_0. Sometimes the right-hand limit is expressed $\lim_{x\to x_0^+} f(x) = f(x_0^+)$ and the left-hand limit is expressed $\lim_{x\to x_0^-} f(x) = f(x_0^-)$. The $\epsilon - \delta$ definitions associated with these left and right-hand limits is exactly the same as given above with the understanding that for right-hand limits x is restricted to the set of values $x > x_0$ and for left-hand limits x is restricted to the set of values $x < x_0$.

Definition 2: Limit of a function $f(x)$ as $x \to \infty$

Let $f(x)$ be defined over the unbounded interval $c < x < \infty$, then a number ℓ is called a limit of $f(x)$ as x increases without bound, written as $\lim_{x\to\infty} f(x) = \ell$, if for every $\epsilon > 0$, there exists a number $N_1 > 0$ such that $|f(x) - \ell| < \epsilon$, whenever $x \ge N_1$.

In a similar fashion, if $f(x)$ is defined over the unbounded interval $-\infty < x < c$, then the number ℓ is called a limit of $f(x)$ as x decreases without bound, written $\lim_{x\to-\infty} f(x) = \ell$, if for every $\epsilon > 0$, there exists a number $N_2 > 0$ such that $|f(x) - \ell| < \epsilon$, whenever $x \le -N_2$.

In terms of the graph $\{ (x, y) \mid y = f(x), x \in R \}$ one can say that for x sufficiently large, larger than N_1 or less than $-N_2$, the y values of the graph would get as close as you want to the line $y = \ell$.

Definition 3: Limit of a function becomes unbounded

In the cases where the limit $\lim_{x\to x_0} f(x)$ either increases or decreases without bound and does not approach a limit, then the notation $\lim_{x\to x_0} f(x) = +\infty$ is used to denote that there exists a number $N_3 > 0$, such that $f(x) > N_3$, whenever $0 < |x - x_0| < \delta$ and the notation $\lim_{x\to x_0} f(x) = -\infty$ is used to denote that there exists a number $N_4 > 0$, such that $f(x) < -N_4$, whenever $0 < |x - x_0| < \delta$.

In the above notations the symbols $+\infty$ (plus infinity) and $-\infty$ (minus infinity) are used to denote unboundedness of the functions. These symbols **are not numbers**. Also observe that there are situations where to use the above notation one might have to replace the limit subscript $x \to x_0$ by either $x \to x_0^+$ or $x \to x_0^-$ in order to denote right or left-handed limits.

In using the $\epsilon-\delta$ methods to prove limit statements, observe that the statement "$f(x)$ is near ℓ" is expressed mathematically by the statement $|f(x)-\ell|<\epsilon$ where $\epsilon>0$ is very small. By selecting ϵ very small you can force $f(x)$ to be very near ℓ, but what must δ be in order that $f(x)$ be that close to ℓ? The selection of δ, in most cases, will depend upon how ϵ is specified. The statement that "x is near x_0, but x is not equal to x_0" is expressed mathematically by the statement $0<|x-x_0|<\delta$. The real number δ which is selected to achieve the smallness specified by ϵ, is not a unique number. Once one value of δ is found, then any other value $\delta_1<\delta$ would also satisfy the definition.

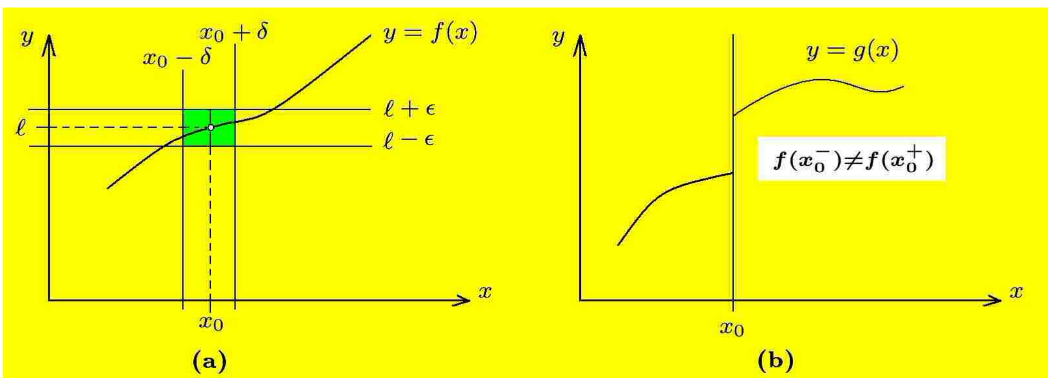


Figure 1-33.
(a) Graphical sketch of $\epsilon-\delta$ limit.
(b) Function having jump discontinuity at the point x_0

A sketch of the $\epsilon-\delta$ definition of a limit is given in the figure 1-33(a). Here $x_0, \ell, \epsilon>0, \delta>0$ are all real numbers and the given function $y=f(x)$ is understood to be well defined for both $x<x_0$ and for $x>x_0$, while the function value $f(x_0)$ may or may not be defined. That portion of the graph inside the shaded rectangle is given by the set of values

$$G=\{\,(x,y)\ |\ \ 0<|x-x_0|<\delta \text{ and } y=f(x)\ \}$$

which is a subset of all the points inside the shaded rectangle.

The shaded rectangle consists of the set of values

$$S = \{\, (x,y) \mid 0 < |x - x_0| < \delta \text{ and } |y - \ell| < \epsilon \,\}$$

Note that the line where $x = x_0$ and $\ell - \epsilon < y < \ell + \epsilon$ is excluded from the set. The problem is that for every $\epsilon > 0$ that is specified, one must know how to select the δ to insure the curve stays within the shaded rectangle. If this can be done then ℓ is defined to be the $\lim_{x \to x_0} f(x)$. In order to make $|f(x) - \ell|$ small, as $x \to x_0$, one must restrict the values of x to some small deleted neighborhood of the point x_0. If only points near x_0 are to be considered, **it is customary to always select δ to be less than or equal to 1.** Thus if $|x - x_0| < 1$, then x is restricted to the interval $[x_0 - 1, x_0 + 1]$.

Example 1-10. ($\epsilon - \delta$ proof)

Use the $\epsilon - \delta$ definition of a limit to prove that $\lim_{x \to 3} x^2 = 9$

Solution

Here $f(x) = x^2$ and $\ell = 9$ so that

$$|f(x) - \ell| = |x^2 - 9| = |(x+3)(x-3)| = |x+3| \cdot |x-3| \tag{1.56}$$

To make $|f(x) - \ell|$ small one must control the size of $|x - 3|$. Recall that by agreement δ is to be selected such that $\delta < 1$ and as a consequence of this the statement "x is near 3" is to mean x is restricted to the interval $[2, 4]$. This information allows us to place bounds upon the factor $(x + 3)$. That is, $|x + 3| < 7$, since x is restricted to the interval $[2, 4]$. One can now use this information to change equation (1.56) into an inequality by noting that if $|x - 3| < \delta$, one can then select δ such that

$$|f(x) - \ell| = |x^2 - 9| = |x+3| \cdot |x-3| < 7\delta < \epsilon \tag{1.57}$$

where $\epsilon > 0$ and less than 1, is as small as you want it to be. The inequality (1.57) tells us that if $\delta < \epsilon/7$, then it follows that

$$|x^2 - 9| < \epsilon \quad \text{whenever} \quad |x - 3| < \delta$$

■

Special Considerations

1. The quantity ϵ used in the definition of a limit is often replaced by some scaled value of ϵ, such as $\alpha\epsilon$, ϵ^2, $\sqrt{\epsilon}$, etc. in order to make the algebra associated with some theorem or proof easier.

2. The limiting process has the property that for $f(x) = c$, a constant, for all values of x, then
$$\lim_{x\to x_0} c = c \tag{1.58}$$
This is known as the **constant function rule for limits**.
3. The limiting process has the property that for $f(x) = x$, then $\lim\limits_{x\to x_0} x = x_0$. This is sometimes called the **identity function rule for limits**.

Properties of Limits

If $f(x)$ and $g(x)$ are functions and the limits $\lim\limits_{x\to x_0} f(x) = \ell_1$ and $\lim\limits_{x\to x_0} g(x) = \ell_2$ both exist and are finite, then

(a) **The limit of a constant times a function equals the constant times the limit of the function.**
$$\lim_{x\to x_0} cf(x) = c \lim_{x\to x_0} f(x) = c\ell_1 \quad \text{for all constants } c$$
(b) **The limit of a sum is the sum of the limits.**
$$\lim_{x\to x_0} [f(x)+g(x)] = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = \ell_1 + \ell_2$$
(c) **The limit of a difference is the difference of the limits.**
$$\lim_{x\to x_0} [f(x)-g(x)] = \lim_{x\to x_0} f(x) - \lim_{x\to x_0} g(x) = \ell_1 - \ell_2$$
(d) **The limit of a product of functions equals the product of the function limits.**
$$\lim_{x\to x_0} [f(x)\cdot g(x)] = \left(\lim_{x\to x_0} f(x)\right)\cdot\left(\lim_{x\to x_0} g(x)\right) = \ell_1\cdot\ell_2$$
(e) **The limit of a quotient is the quotient of the limits provided that the denominator limit is nonzero.**
$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)} = \frac{\ell_1}{\ell_2}, \quad \text{provided } \ell_2 \neq 0$$
(f) **The limit of an nth root is the nth root of the limit.**
$$\lim_{x\to x_0} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to x_0} f(x)} = \sqrt[n]{\ell_1} \begin{cases} \text{if } n \text{ is an odd positive integer or} \\ \text{if } n \text{ is an even positive integer and } \ell_1 > 0 \end{cases}$$
(g) **Repeated applications of the product rule with $g(x) = f(x)$ gives the extended product rule.**
$$\lim_{x\to x_0} f(x)^n = \left(\lim_{x\to x_0} f(x)\right)^n$$

(h) **The limit theorem for composite functions is as follows.**
If $\lim\limits_{x\to x_0} g(x) = \ell$, then $\lim\limits_{x\to x_0} f(g(x)) = f\left(\lim\limits_{x\to x_0} g(x)\right) = f(\ell)$

Example 1-11. (Limit Theorem)

Use the $\epsilon - \delta$ definition of a limit to prove the limit of a sum is the sum of the limits $\lim\limits_{x\to x_0}(f(x)+g(x)) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x) = \ell_1 + \ell_2$

Solution By hypothesis, $\lim\limits_{x\to x_0} f(x) = \ell_1$ and $\lim\limits_{x\to x_0} g(x) = \ell_2$, so that for a small number $\epsilon_1 > 0$, there exists numbers δ_1 and δ_2 such that

$$|f(x) - \ell_1| < \epsilon_1 \quad \text{when} \quad 0 < |x - x_0| < \delta_1$$
$$|g(x) - \ell_2| < \epsilon_1 \quad \text{when} \quad 0 < |x - x_0| < \delta_2$$

where $\epsilon_1 > 0$ is a small quantity to be specified at a later time. Select δ to be the smaller of δ_1 and δ_2, then using the triangle inequality, one can write

$$\begin{aligned} |(f(x)+g(x)) - (\ell_1+\ell_2)| =& |(f(x)-\ell_1) + (g(x)-\ell_2)| \\ & \le |f(x)-\ell_1| + |g(x)-\ell_2| \\ & \le \epsilon_1 + \epsilon_1 = 2\epsilon_1 \quad \text{when} \quad 0 < |x - x_0| < \delta \end{aligned}$$

Consequently, if ϵ_1 is selected as $\epsilon/2$, then one can say that

$$|(f(x)+g(x)) - (\ell_1+\ell_2)| < \epsilon \quad \text{when} \quad 0 < |x - x_0| < \delta$$

which implies

$$\lim_{x\to x_0}(f(x)+g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = \ell_1 + \ell_2$$

■

Example 1-12. (Limit Theorem)

Use the $\epsilon - \delta$ definition of a limit to prove the limit of a product of functions equals the product of the function limits. That is, if $\lim\limits_{x\to x_0} f(x) = \ell_1$ and $\lim\limits_{x\to x_0} g(x) = \ell_2$, then $\lim\limits_{x\to x_0} f(x)g(x) = \left(\lim\limits_{x\to x_0} f(x)\right)\left(\lim\limits_{x\to x_0} g(x)\right) = \ell_1\ell_2$.

Solution

By hypothesis, $\lim\limits_{x\to x_0} f(x) = \ell_1$ and $\lim\limits_{x\to x_0} g(x) = \ell_2$, so that for every small number ϵ_1, there exists numbers ϵ_1, δ_1 and ϵ_1, δ_2 such that

$$\begin{aligned} |f(x) - \ell_1| < \epsilon_1, & \text{ whenever } 0 < |x - x_0| < \delta_1 \\ \text{and} \quad |g(x) - \ell_2| < \epsilon_1 & \text{ whenever } 0 < |x - x_0| < \delta_2 \end{aligned}$$

where $\epsilon_1 > 0$ is some small number to be specified later. Let δ equal the smaller of the numbers δ_1 and δ_2 so that one can write

$$|f(x) - \ell_1| < \epsilon_1, \quad |g(x) - \ell_2| < \epsilon_1, \text{ whenever } 0 < |x - x_0| < \delta$$

To prove the above limit one must specify how to select δ associated with a given value of ϵ such that

$$|f(x)g(x) - \ell_1\ell_2| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

One can select ϵ_1 above as a small number which is some scaled version of ϵ. Observe that the function $f(x)$ is bounded, since by the triangle inequality one can write

$$|f(x)| = |f(x) - \ell_1 + \ell_1| < |f(x) - \ell_1| + |\ell_1| < \epsilon_1 + |\ell_1| < 1 + |\ell_1|$$

where ϵ_1 is assumed to be less than unity. Also note that one can write

$$\begin{aligned}
|f(x)g(x) - \ell_1\ell_2| =& |f(x)g(x) - \ell_2 f(x) + \ell_2 f(x) - \ell_1\ell_2| \\
\leq& |f(x)(g(x) - \ell_2)| + |\ell_2(f(x) - \ell_1)| \\
\leq& |f(x)||g(x) - \ell_2| + |\ell_2||f(x) - \ell_1| \\
\leq& (1 + |\ell_1|)\epsilon_1 + |\ell_2|\epsilon_1 = (1 + |\ell_1| + |\ell_2|)\epsilon_1
\end{aligned}$$

Consequently, if the quantity ϵ_1 is selected to satisfy the inequality $(1+|\ell_1|+|\ell_2|)\epsilon_1 < \epsilon$, then one can say that

$$|f(x)g(x) - \ell_1\ell_2| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

so that

$$\lim_{x\to x_0} f(x)g(x) = \left(\lim_{x\to x_0} f(x)\right)\left(\lim_{x\to x_0} g(x)\right) = \ell_1\ell_2$$

■

Example 1-13. (Limit Theorem)

If $\ell_2 \neq 0$, prove that if $\lim_{x\to x_0} g(x) = \ell_2$, then $\lim_{x\to x_0} \dfrac{1}{g(x)} = \dfrac{1}{\lim_{x\to x_0} g(x)} = \dfrac{1}{\ell_2}$

Solution By hypothesis $\lim_{x\to x_0} g(x) = \ell_2$, with $\ell_2 \neq 0$. This means that for every $\epsilon_1 > 0$ there exists a δ_1 such that $|g(x) - \ell_2| < \epsilon_1$ whenever $|x - x_0| < \delta_1$. How can this information be used to show that for every $\epsilon > 0$ there exists a δ such that

$$\left|\frac{1}{g(x)} - \frac{1}{\ell_2}\right| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta\ ? \tag{1.59}$$

The left-hand side of the inequality (1.59) can be expressed

$$\left|\frac{1}{g(x)} - \frac{1}{\ell_2}\right| = \frac{|\ell_2 - g(x)|}{|g(x)\ell_2|} = \frac{|g(x) - \ell_2|}{|\ell_2|} \cdot \frac{1}{|g(x)|} \tag{1.60}$$

For a given ϵ_1, one can find a δ_1 such that the quantity $|g(x) - \ell_2| < \epsilon_1$ whenever $|x - x_0| < \delta_1$. What can be constructed as an inequality concerning the quantity $\dfrac{1}{|g(x)|}$? If $\ell_2 \neq 0$ one can employ the triangle inequality and write

$$|\ell_2| = |\ell_2 - g(x) + g(x)| \leq |\ell_2 - g(x)| + |g(x)|$$

By the definition of a limit, one can select values ϵ_3 and δ_3 such that

$$|g(x) - \ell_2| < \epsilon_3 \quad \text{when} \quad |x - x_0| < \delta_3$$

This gives the inequalities

$$|\ell_2| < \epsilon_3 + |g(x)| \quad \text{or} \quad |\ell_2| - \epsilon_3 < |g(x)| \quad \text{or} \quad \frac{1}{|g(x)|} < \frac{1}{|\ell_2| - \epsilon_3} \tag{1.61}$$

provided $|\ell_2| - \epsilon_3$ is not zero. Recall that the values of ϵ_1 and ϵ_3 have not been specified and their values can be selected to have any small values that we desire. The inequality given by equation (1.60) can be expressed in the form

$$\left|\frac{1}{g(x)} - \frac{1}{\ell_2}\right| < \frac{|g(x) - \ell_2|}{|\ell_2|} \cdot \frac{1}{|g(x)|} < \frac{\epsilon_1}{|\ell_2|} \cdot \frac{1}{|\ell_2| - \epsilon_3} \tag{1.62}$$

and is valid for all x values satisfying $|x - x_0| < \delta$, where δ is selected as the smaller of the values δ_1 and δ_3. Let us now specify an ϵ_1 and ϵ_3 value so that with some algebra the right-hand side of equation (1.62) can be made less than ϵ for $|x - x_0| < \delta$. One way to accomplish this is as follows. After ϵ is selected, one can select δ_1 above to go with $\epsilon_1 = \epsilon(1 - \beta)|\ell_2|^2$ and select δ_3 above to go with $\epsilon_3 = \beta|\ell_2|$, where β is some small fraction less than 1. Then δ is selected as the smaller of the values δ_1 and δ_3 and the product on the right-hand side of equation (1.62) is less than ϵ for $|x - x_0| < \delta$.

The above result can now be combined with the limit of a product rule $\lim\limits_{x \to x_0} f(x)h(x) = \lim\limits_{x \to x_0} f(x) \cdot \lim\limits_{x \to x_0} h(x)$ with $h(x) = \frac{1}{g(x)}$ to establish the quotient rule

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} \frac{1}{g(x)}\right) = \frac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)} = \frac{\ell_1}{\ell_2}, \quad \text{provided } \ell_2 \neq 0$$

■

The Squeeze Theorem

Assume that for x near x_0 there exists three functions $f(x)$, $g(x)$ and $h(x)$ which can be shown to satisfy the inequalities $f(x) \leq g(x) \leq h(x)$. If one can show that

$$\lim_{x \to x_0} f(x) = \ell \qquad \textit{and} \qquad \lim_{x \to x_0} h(x) = \ell,$$

then one can conclude

$$\lim_{x \to x_0} g(x) = \ell$$

This result is known as the squeeze theorem.

Continuous Functions and Discontinuous Functions

A function $f(x)$ is called a **continuous function** over the interval $a \le x_0 \le b$ if for all points x_0 within the interval

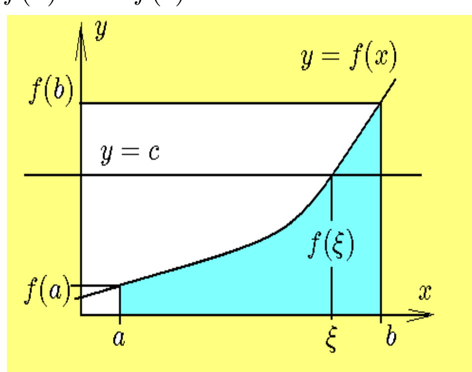
$$\begin{aligned} &(i) \quad f(x_0) \text{ is well defined}\\ &(ii) \quad \lim_{x\to x_0} f(x) \text{ exists}\\ &(iii) \quad \lim_{x\to x_0} f(x) = f(x_0) \end{aligned} \tag{1.63}$$

Polynomial functions are continuous functions. Rational algebraic functions, represented by the quotient of two polynomials, are continuous except for those points where the denominator becomes zero. The trigonometric functions, exponential functions and logarithmic functions are continuous functions over appropriate intervals.

Alternatively, one can assume the right-hand limit $\lim_{x\to x_0+} f(x) = \ell_1$ and the left-hand limit $\lim_{x\to x_0^-} f(x) = \ell_2$ both exist, then if $\ell_1 = \ell_2$ and $f(x_0) = \ell_1$, then the function $f(x)$ is said to be continuous at the point x_0. A function continuous at all points x_0 within an interval is said to be continuous over the interval.

If any of the conditions given in equation (1.63) are not met, then $f(x)$ is called **a discontinuous function.** For example, if $\lim_{x\to x_0+} f(x) = \ell_1$ and $\lim_{x\to x_0^-} f(x) = \ell_2$ both exist and $\ell_1 \ne \ell_2$, the function $f(x)$ is said to have a jump discontinuity at the point x_0. An example of a function with a jump discontinuity is given in the figure 1-33(b). If $\lim_{x\to x_0} f(x)$ does not exist, then $f(x)$ is said to be discontinuous at the point x_0.

The **Intermediate Value Property** states that a function $f(x)$ which is continuous on a closed interval $a \le x \le b$ is such that when x moves from the point a to the point b **the function takes on every intermediate value between** $f(a)$ **and** $f(b)$ **at least once**. An alternative version of the intermediate value property is the following. If $y = f(x)$ denotes a continuous function on the interval $a \le x \le b$, where $f(a) < c < f(b)$, and the line $y = c = \textit{constant}$ is constructed , then the Intermediate Value Theorem states that there must exist at least one number ξ satisfying $a < \xi < b$ such that $f(\xi) = c$.



Example 1-14. **(Discontinuities)**

(a) $f(x)=\frac{x^2-1}{x-1}$ is not defined at the point $x=1$, so $f(x)$ is said to be discontinuous at the point $x=1$. The limit $\lim\limits_{x\to 1}\dfrac{(x-1)(x+1)}{x-1}=2$ exists and so by defining the function $f(x)$ to have the value $f(1)=2$, the function can be made continuous. In this case the function is said to have a **removable discontinuity** at the point $x=1$.

(b) If $\lim\limits_{x\to x_0} f(x)=\pm\infty$, then obviously $f(x)$ is not defined at the point x_0. Another way to spot an infinite discontinuity is to set the denominator of a function equal to zero and solve for x. For example, if $f(x)=\dfrac{1}{(x-1)(x-2)(x-3)}$, then $f(x)$ is said to have infinite discontinuities at the points $x=1$, $x=2$ and $x=3$.

(c) The function $f(x)=\begin{cases}1, & x<2\\ 5, & x>2\end{cases}$ is not defined at the point $x=2$. The left and right-hand limits as $x\to 2$ are not the same and so the function is said to have a **jump discontinuity** at the point $x=2$. The limit $\lim_{x\to 2} f(x)$ does not exist. At a point where a jump discontinuity occurs, it is sometimes convenient to define the value of the function as the **average value of the left and right-hand limits**.

■

Asymptotic Lines

A **graph** is a set of **ordered pairs** (x,y) which are well defined over some region of the x,y-plane. If there exists one or more straight lines such that the graph approaches one of these lines as x or y increases without bound, then the lines are called **asymptotic lines**.

Example 1-15. **(Asymptotic Lines)**

Consider the curve $C=\{\,(x,y)\mid\ y=f(x)=1+\dfrac{1}{x-1},\ x\in R\,\}$ and observe that the curve passes through the origin since when $x=0$ one finds $y=f(0)=0$. Also note that as x increases, $\lim\limits_{x\to\infty} f(x)=1$ and that as x approaches the value 1, $\lim\limits_{x\to 1} f(x)=\pm\infty$. If one plots some selected points one can produce the illustration of the curve C as given by the figure 1-34. In the figure 1-34 the line $y=1$ is a **horizontal asymptote** associated with the curve and the line $x=1$ is a **vertical asymptote** associated with the curve.

Consider a curve defined by one of the equations

$$G(x,y)=0, \qquad y=f(x), \qquad x=g(y)$$

If a line ℓ is an asymptotic line associated with one of the above curves, then the following properties must be satisfied. Let d denote the perpendicular distance from a point (x,y) on the curve to the line ℓ. If one or more of the conditions

$$\lim_{x\to\infty} d=0, \qquad \lim_{x\to-\infty} d=0, \qquad \lim_{y\to\infty} d=0, \qquad \lim_{y\to-\infty} d=0,$$

is satisfied, then the line ℓ is called an **asymptotic line** or **asymptote** associated with the given curve.

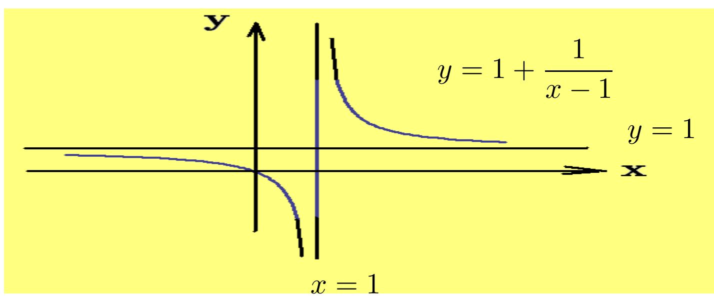


Figure 1-34. The graph of $y=f(x)=1+\dfrac{1}{x-1}$

■

Finding Asymptotic Lines

One can determine an asymptotic line associated with a curve $y=f(x)$ by applying one of the following procedures.

1. Solve for y in terms of x and set the denominator equal to zero and solve for x. The resulting values for x represent the vertical asymptotic lines.

2. Solve for x in terms of y and set the denominator equal to zero and solve for y. The resulting values for y represent the horizontal asymptotic lines.

3. The line $x=x_0$ is called a vertical asymptote if one of the following conditions is true.

$$\lim_{x\to x_0} f(x)=\infty, \qquad \lim_{x\to x_0^-} f(x)=\infty, \qquad \lim_{x\to x_0^+} f(x)=\infty$$

$$\lim_{x\to x_0} f(x)=-\infty, \qquad \lim_{x\to x_0^-} f(x)=-\infty, \qquad \lim_{x\to x_0^+} f(x)=-\infty$$

4. The line $y = y_0$ is called a horizontal asymptote if one of the following conditions is true.

$$\lim_{x\to\infty} f(x) = y_0, \qquad \lim_{x\to-\infty} f(x) = y_0$$

5. The line $y = mx + b$ is called a **slant asymptote** or **oblique asymptote** if

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0$$

Example 1-16. Asymptotic Lines

Consider the curve $y = f(x) = 2x + 1 + \frac{1}{x}$, where $x \in R$. This function has the properties that

$$\lim_{x\to\infty} [f(x) - (2x + 1)] = \lim_{x\to\infty} \frac{1}{x} = 0 \qquad \text{and} \qquad \lim_{x\to 0} f(x) = \pm\infty$$

so that one can say the line $y = 2x + 1$ is an oblique asymptote and the line $x = 0$ is a vertical asymptote. A sketch of this curve is given in the figure 1-35.

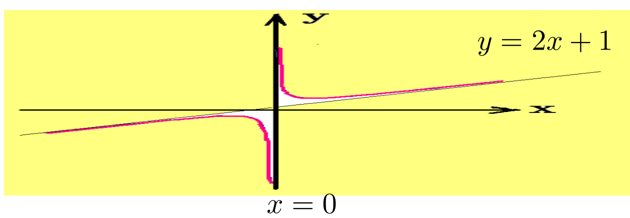


Figure 1-35. Sketch of curve $y = f(x) = 2x + 1 + \frac{1}{x}$

■

Conic Sections

A general equation of the second degree has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1.64}$$

where A, B, C, D, E, F are constants. All curves which have the form of equation (1.64) can be obtained by cutting a right circular cone with a plane. The figure 1-36(a) illustrates a right circular cone obtained by constructing a circle in a horizontal plane and then moving perpendicular to the plane to a point V above or below the center of the circle. The point V is called the **vertex of the cone**. All the lines through the point V and points on the circumference of the circle are called **generators of the**

cone. The set of all generators produces a right circular cone. The figure 1-36(b) illustrates a horizontal plane intersecting the cone in a **circle**. The figure 1-36(c) illustrates a nonhorizontal plane section which cuts two opposite generators. The resulting curve of intersection is called an **ellipse**. Figure 1-36(d) illustrates a plane parallel to a generator of the cone which also intersects the cone. The resulting curve of intersection is called a **parabola**. Any plane cutting both the upper and lower parts of a cone will intersect the cone in a curve called a **hyperbola** which is illustrated in the figure 1-36(e).

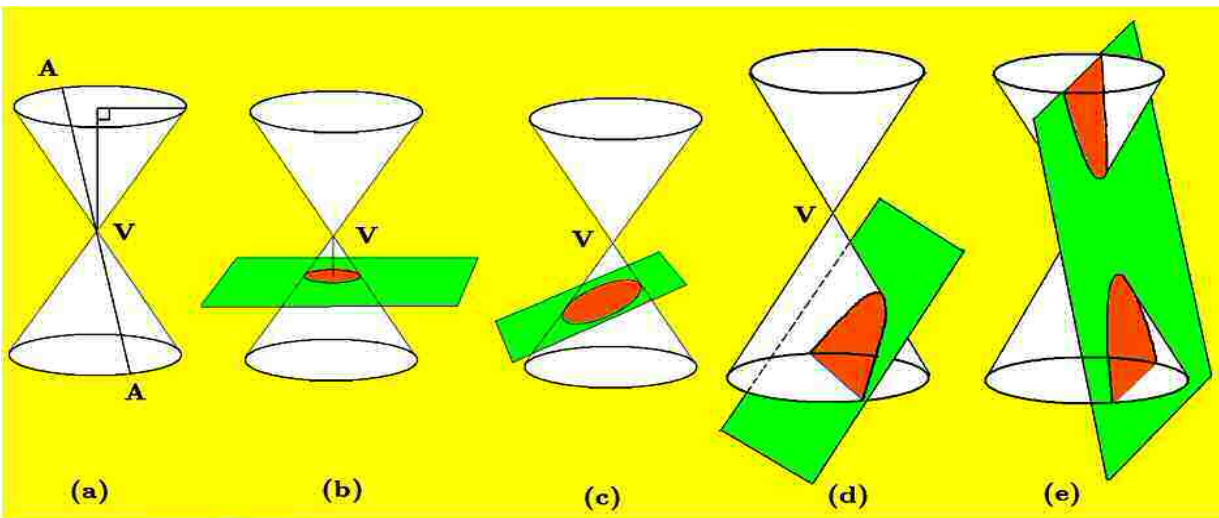


Figure 1-36. The intersection of right circular cone with a plane.

Conic sections were studied by the early Greeks. Euclid[7] supposedly wrote four books on conic sections. The Greek geometer Appollonius[8] wrote eight books on conic sections which summarized Greek knowledge of conic sections and his work has survived the passage of time.

Conic sections can be defined as follows. In the xy-plane select a point f, called the **focus**, and a line D not through f. This line is called the **directrix**. The set of points P satisfying the condition that the distance from f to P, call it $r = \overline{Pf}$, is some multiple e times the distance $d = \overline{PP'}$, where d represents the perpendicular distance from the point P to the line D. The resulting equation for the conic section is obtained from the equation $r = ed$ with the geometric interpretation of this equation illustrated in the figure 1-37.

[7] Euclid of Alexandria (325-265 BCE)
[8] Appollonius of Perga (262-190 BCE)

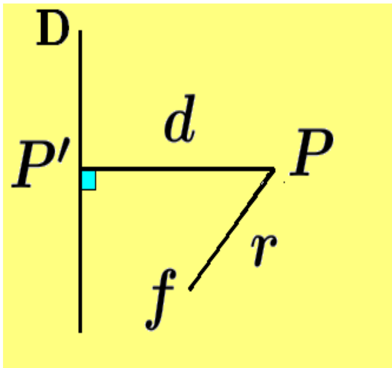


Figure 1-37.
Defining a conic section.

The plane curve resulting from the equation $r = ed$ is called a **conic section with eccentricity** e, **focus** f **and directrix** D and if the eccentricity e satisfies

$0 < e < 1$, the conic section is an ellipse.

$e = 1$, the conic section is a parabola.

$e > 1$, the conic section is a hyperbola.

In addition to the **focus** and **directrix** there is associated with each conic section the following quantities.

The vertex V The vertex V of a conic section is the midpoint of a line from the focus perpendicular to the directrix.

Axis of symmetry The line through the focus and perpendicular to the directrix is called an axis of symmetry.

Focal parameter $2p$ This is the perpendicular distance from the focus to the directrix, where p is the distance from the focus to the vertex or distance from vertex to directrix.

Latus rectum 2ℓ This is a chord parallel to a directrix and perpendicular to a focus which passes between two points on the conic section. The latus rectum is used as a measure associated with **the spread of a conic section**. If ℓ is the semi-latus rectum intersecting the conic section at the point where $x = p$, one finds $r = \ell = ed$ and so it follows that $2\ell = 2ed$.

Circle

A circle is the locus of points (x, y) in a plane equidistant from a fixed point called the center of the circle. Note that no real locus occurs if the radius r is negative or imaginary. It has been previously demonstrated how to calculate the equation of a circle. The figure 1-38 is a summary of these previous results. The circle $x^2 + y^2 = r^2$ has eccentricity zero and latus rectum of $2r$. Parametric equations for the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$, centered at (x_0, y_0), are

$$x = x_0 + r\cos t, \qquad y = y_0 + r\sin t, \quad 0 \le t \le 2\pi$$

When dealing with second degree equations of the form $x^2 + y^2 + \alpha x + \beta y = \gamma$, where α, β and γ are constants, it is customary to **complete the square** on the x and y terms to obtain

$$(x^2 + \alpha x + \frac{\alpha^2}{4}) + (y^2 + \beta y + \frac{\beta^2}{4}) = \gamma + \frac{\alpha^2}{4} + \frac{\beta^2}{4} \implies (x + \frac{\alpha}{2})^2 + (y + \frac{\beta}{2})^2 = r^2$$

where it is assumed that $r^2 = \gamma + \frac{\alpha^2}{4} + \frac{\beta^2}{4} > 0$. This produces the equation of a circle with radius r which is centered at the point $\left(-\frac{\alpha}{2}, -\frac{\beta}{2}\right)$.

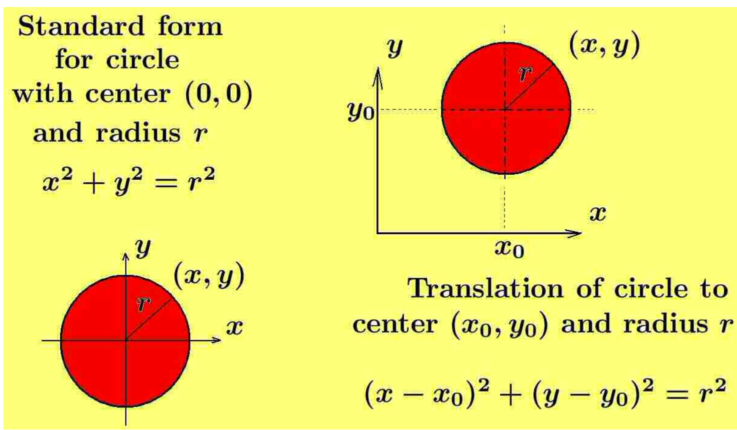


Figure 1-38.
Circle about origin and circle translated to point (x_0, y_0)

Parabola

The parabola can be defined as the locus of points (x, y) in a plane, such that (x, y) moves to **remain equidistant from a fixed point** (x_0, y_0) **and fixed line** ℓ. The fixed point is called **the focus of the parabola** and the fixed line is called **the directrix of the parabola**. The midpoint of the perpendicular line from the focus to the directrix is called **the vertex of the parabola**.

In figure 1-39(b), let the point $(0, p)$ denote the focus of the parabola symmetric about the y-axis and let the line $y = -p$ denote the directrix of the parabola. If (x, y) is a general point on the parabola, then

$$\begin{aligned} d_1 =& \text{distance from } (x, y) \text{ to focus} = \sqrt{x^2 + (y-p)^2} \\ d_2 =& \text{distance from } (x, y) \text{ to directrix} = y + p \end{aligned}$$

If $d_1 = d_2$ for all values of x and y, then

$$\sqrt{x^2 + (y-p)^2} = y + p \quad \text{or} \quad x^2 = 4py, \quad p \neq 0 \tag{1.65}$$

This parabola has its vertex at the origin, an eccentricity of 1, a semi-latus rectum of length $2p$, latus rectum of $2\ell = 4p$ and focal parameter of $2p$.

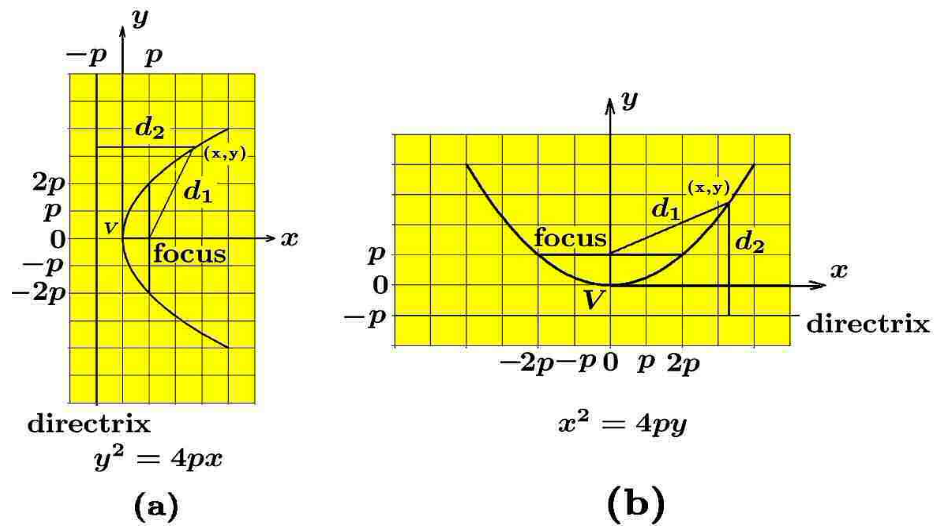


Figure 1-39.

Parabolas symmetric about the x and y axes.

Other forms for the equation of a parabola are obtained by replacing p by $-p$ and interchanging the variables x and y. For $p > 0$, other standard forms for the equation of a parabola are illustrated in the figure 1-40. In the figure 1-40 observe the upward/downward and left/right opening of the parabola depend upon the sign before the parameter p, where $p > 0$ represents the distance from the origin to the focus. By replacing x by $-x$ and y by $-y$ one can verify the various symmetries associated with these shapes.

Using the translation of axes equations (1.30), the vertex of the parabolas in the figure 1-40 can be translated to a point (h, k). These translated equations have the representations

$$\begin{aligned} (x-h)^2 =& 4p(y-k) \qquad & (y-k)^2 =& 4p(x-h) \\ (x-h)^2 =& -4p(y-k) \qquad & (y-k)^2 =& -4p(x-h) \end{aligned} \tag{1.66}$$

Also note that the lines of symmetry are also shifted.

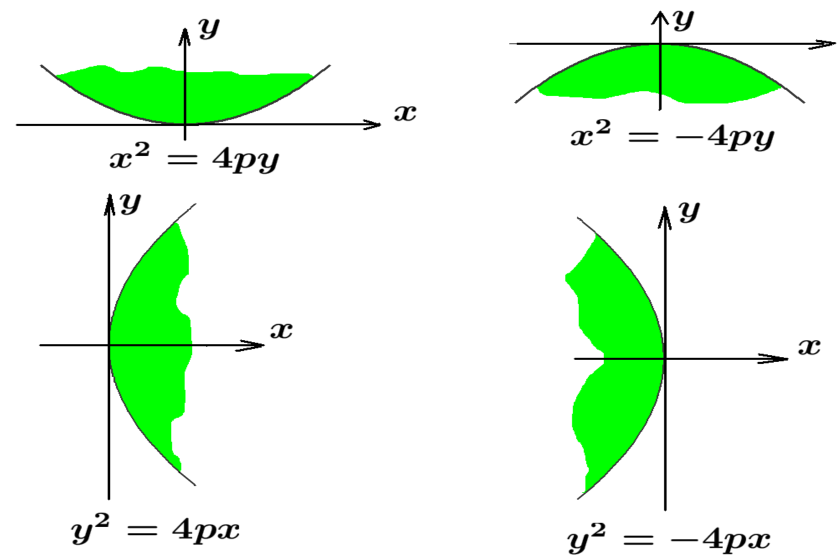


Figure 1-40.
Other forms for representing a parabola.

One form for the parametric representation of the parabola $(x-h)^2 = 4p(y-k)$ is given by

$$P = \{\, (x,y) \mid\ x = h+t,\ y = k + t^2/4p,\ -\infty < t < \infty \,\} \tag{1.67}$$

with similar parametric representations for the other parabolas.

Use of determinants

The equation of the parabola passing through the three distinct points (x_1, y_1), (x_2, y_2) and (x_3, y_3) can be determined by evaluating the determinant[9]

$$\begin{vmatrix} y & x^2 & x & 1 \\ y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \end{vmatrix} = 0$$

provided the following determinants are different from zero.

$$\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} \neq 0, \qquad \text{and} \qquad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \neq 0$$

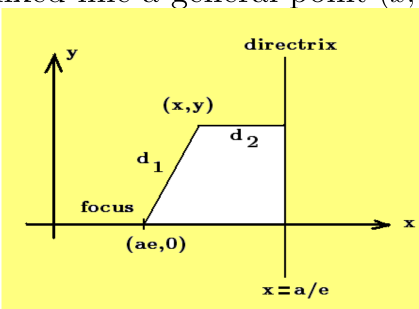
[9] Determinants and their properties are discussed in chapter 10.

Ellipse

The eccentricity e of an ellipse satisfies $0 < e < 1$ so that for any given positive number a one can state that

$$ae < \frac{a}{e}, \qquad 0 < e < 1 \tag{1.68}$$

Consequently, if the point $(ae, 0)$ is selected as the focus of an ellipse and the line $x = a/e$ is selected as the directrix of the ellipse, then in relation to this fixed focus and fixed line a general point (x, y) will satisfy



$$d_1 = \text{distance of } (x, y) \text{ to focus} = \sqrt{(x - ae)^2 + y^2}$$

$$d_2 = \perp \text{ distance of } (x, y) \text{ to directrix} = |x - a/e|$$

The ellipse can then be defined as the set of points (x, y) satisfying the constraint condition $d_1 = e\,d_2$ which can be expressed as the set of points

$$E_1 = \{\, (x, y) \mid \sqrt{(x - ae)^2 + y^2} = e|x - a/e|,\ 0 < e < 1 \,\} \tag{1.69}$$

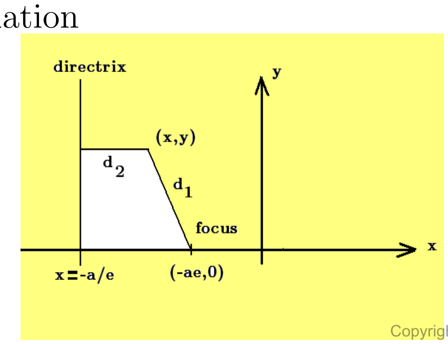
Applying some algebra to the constraint condition on the points (x, y), the ellipse can be expressed in a different form. Observe that if $d_1 = e\,d_2$, then

$$\begin{aligned} (x - ae)^2 + y^2 &= e^2(x - a/e)^2 \\ \text{or} \qquad x^2 - 2aex + a^2e^2 + y^2 &= e^2x^2 - 2aex + a^2 \end{aligned}$$

which simplifies to the condition

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b^2 = a^2(1 - e^2) \tag{1.70}$$

where the eccentricity satisfies $0 < e < 1$. In the case where the focus is selected as $(-ae, 0)$ and the directrix is selected as the line $x = -a/e$, there results the following situation



$$d_1 = \text{distance of } (x, y) \text{ to focus} = \sqrt{(x + ae)^2 + y^2}$$

$$d_2 = \perp \text{ distance of } (x, y) \text{ to directrix} = |x + a/e|$$

The condition that $d_1 = e\,d_2$ can be represented as the set of points

$$E_2 = \{\, (x,y) \mid \sqrt{(x+ae)^2+y^2} = e|x+a/e|,\ 0<e<1\,\} \tag{1.71}$$

As an exercise, show that the simplification of the constraint condition for the set of points E_2 also produces the equation (1.70).

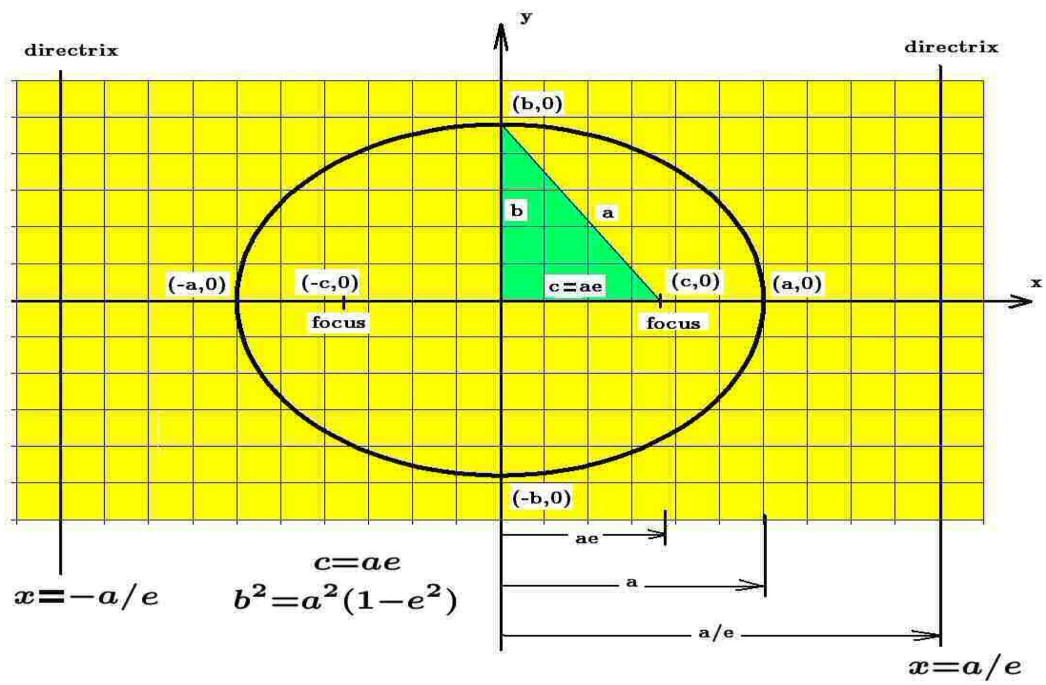


Figure 1-41. The ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Define the constants

$$c = ae \quad \text{and} \quad b^2 = a^2(1-e^2) = a^2 - c^2 \tag{1.72}$$

and note that $b^2 < a^2$, then from the above discussion one can conclude that an ellipse is defined by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \quad 0<e<1, \quad b^2 = a^2(1-e^2), \quad c = ae \tag{1.73}$$

and has the points $(ae, 0)$ and $(-ae, 0)$ as foci and the lines $x = -a/e$ and $x = a/e$ as directrices. The resulting graph for the ellipse is illustrated in the figure 1-41. This ellipse has vertices at $(-a,0)$ and $(a,0)$, a latus rectum of length $2b^2/a$ and eccentricity given by $\sqrt{1-b^2/a^2}$.

In the figure 1-41 a right triangle has been constructed as a mnemonic device to help remember the relations given by the equations (1.72). The distance $2a$ between $(-a,0)$ and $(a,0)$ is called the **major axis of the ellipse** and the distance $2b$ from $(0,-b)$ to $(0,b)$ is called the **minor axis of the ellipse**. The origin $(0,0)$ is called the **center of the ellipse**.

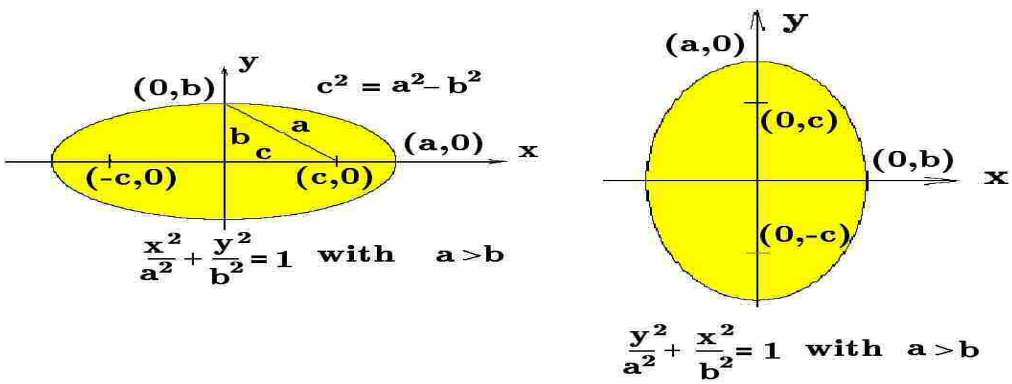


Figure 1-42. Symmetry of the ellipse.

Some algebra can verify the following property satisfied by a general point (x,y) on the ellipse. Construct the distances

$$\begin{aligned} d_3 =& \text{distance of } (x,y) \text{ to focus } (c,0) = \sqrt{(x-c)^2+y^2} \\ d_4 =& \text{distance of } (x,y) \text{ to focus } (-c,0) = \sqrt{(x+c)^2+y^2} \end{aligned} \tag{1.74}$$

and show

$$d_3 + d_4 = \sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a \tag{1.75}$$

One can use this property to define the ellipse as the locus of points (x,y) such that the sum of its distances from two fixed points equals a constant.

The figure 1-42 illustrates that when the roles of x and y are interchanged, then the major axis and minor axis of the ellipse are reversed. A shifting of the axes so that the point (x_0,y_0) is the center of the ellipse produces the equations

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \qquad \text{or} \qquad \frac{(y-y_0)^2}{a^2} + \frac{(x-x_0)^2}{b^2} = 1 \tag{1.76}$$

These equations represent the ellipses illustrated in the figure 1-42 where the centers are shifted to the point (x_0,y_0).

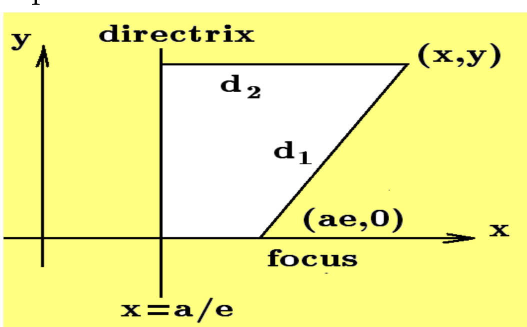
The ellipse given by $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ which is centered at the point (h,k) can be represented in a parametric form[10]. One parametric form is to represent the ellipse as the set of points

$$E = \{\, (x,y) \mid x = h + a\cos\theta,\ y = k + b\sin\theta, \quad 0 \le \theta \le 2\pi \,\} \tag{1.77}$$

involving the parameter θ which varies from 0 to 2π.

Hyperbola

Let $e > 1$ denote the eccentricity of a hyperbola. Again let $(ae, 0)$ denote the focus of the hyperbola and let the line $x = a/e$ denote the directrix of the hyperbola. The hyperbola is defined such that points (x,y) on the hyperbola satisfy $d_1 = ed_2$ where d_1 is the distance from (x,y) to the focus and d_2 is the perpendicular distance from the point (x,y) to the directrix. The hyperbola can then be represented by the set of points

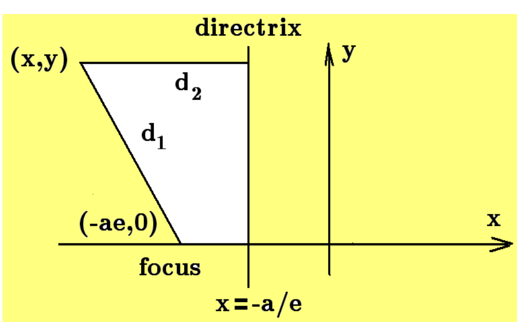


$$H_1 = \{\, (x,y) \mid \sqrt{(x-ae)^2 + y^2} = e|x - a/e|,\ e > 1 \,\}$$

A simplification of the constraint condition on the set of points (x,y) produces the alternative representation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1, \qquad e > 1 \tag{1.78}$$

Placing the focus at the point $(-ae, 0)$ and using as the directrix the line $x = -a/e$, one can verify that the hyperbola is represented by the set of points



$$H_2 = \{\, (x,y) \mid \sqrt{(x+ae)^2 + y^2} = e|x + a/e|,\ e > 1 \,\}$$

and it can be verified that the constraint condition on the points (x,y) also simplifies to the equation (1.78).

[10] The parametric representation of a curve or part of a curve is not unique.

Define $c = ae$ and $b^2 = a^2(e^2 - 1) = c^2 - a^2 > 0$ and note that for an eccentricity $e > 1$ there results the inequality $c > a$. The hyperbola can then be described as having the foci $(c, 0)$ and $(-c, 0)$ and directrices $x = a/e$ and $x = -a/e$. The hyperbola represented by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad b^2 = a^2(e^2 - 1) = c^2 - a^2 \tag{1.79}$$

is illustrated in the figure 1-43.

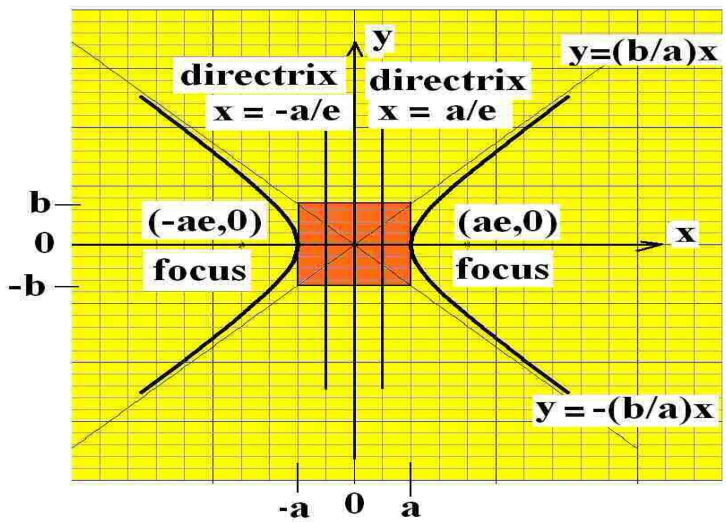


Figure 1-43. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

This hyperbola has vertices at $(-a, 0)$ and $(a, 0)$, a latus rectum of length $2b^2/a$ and eccentricity of $\sqrt{1 + b^2/a^2}$. The origin is called the center of the hyperbola. The line containing the two foci of the hyperbola is called the principal axis of the hyperbola. Setting $y = 0$ and solving for x one can determine that the hyperbola intersects the principal axis at the points $(-a, 0)$ and $(a, 0)$ which are called the vertices of the hyperbola. The line segment between the vertices is called the **major axis of the hyperbola** or **transverse axis of the hyperbola**. The distance between the points $(b, 0)$ and $(-b, 0)$ is called the **conjugate axis of the hyperbola**. The chord through either focus which is perpendicular to the transverse axis is called a **latus rectum**. One can verify that the latus rectum intersects the hyperbola at the points $(c, b^2/a)$ and $(c, -b^2/a)$.

Write the equation (1.79) in the form

$$y = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}} \tag{1.80}$$

and note that for very large values of x the right-hand side of this equation approaches 1. Consequently, for large values of x the equation (1.80) becomes the lines

$$y = \frac{b}{a}x \qquad \text{and} \qquad y = -\frac{b}{a}x \tag{1.81}$$

These lines are called the asymptotic lines associated with the hyperbola and are illustrated in the figure 1-43. Note that the hyperbola has two branches with each branch approaching the asymptotic lines for large values of x.

Let (x, y) denote a general point on the above hyperbola and construct the distances

$$\begin{aligned} d_3 =& \text{distance from } (x,y) \text{ to the focus } (c,0) \ = \ \sqrt{(x-c)^2+y^2} \\ d_4 =& \text{distance from } (x,y) \text{ to the focus } (-c,0) \ = \ \sqrt{(x+c)^2+y^2} \end{aligned} \tag{1.82}$$

Use some algebra to verify that

$$d_4 - d_3 = 2a \tag{1.83}$$

This property of the hyperbola is sometimes used to define the hyperbola as the locus of points (x, y) in the plane such that the difference of its distances from two fixed points is a constant.

The hyperbola with transverse axis on the x-axis have the asymptotic lines $y = +\frac{b}{a}x$ and $y = -\frac{b}{a}x$. Any hyperbola with the property that the conjugate axis has the same length as the transverse axis is called a rectangular or equilateral hyperbola. Rectangular hyperbola are such that the asymptotic lines are perpendicular to each other.

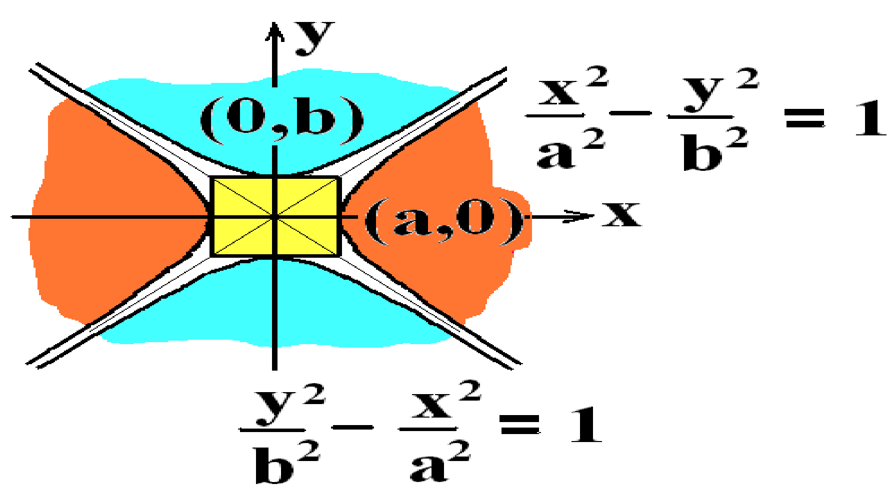


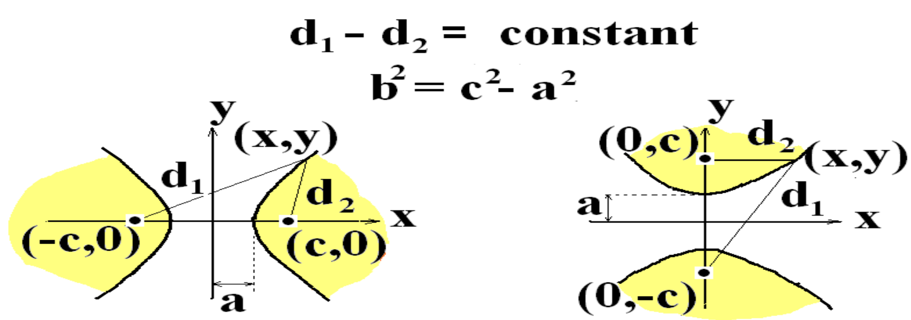
Figure 1-44. Conjugate hyperbola.

If two hyperbola are such that the transverse axis of either is the conjugate axis of the other, then they are called conjugate hyperbola. Conjugate hyperbola will have the same asymptotic lines. Conjugate hyperbola are illustrated in the figure 1-44.

The figure 1-45 illustrates that when the roles of x and y are interchanged, then the transverse axis and conjugate axis of the hyperbola are reversed. A shifting of the axes so that the point (x_0, y_0) is the center of the hyperbola produces the equations

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1 \qquad \text{or} \qquad \frac{(y-y_0)^2}{a^2} - \frac{(x-x_0)^2}{b^2} = 1 \tag{1.84}$$

The figure 1-45 illustrates what happens to the hyperbola when the values of x and y are interchanged.



If the foci are on the x-axis at $(c,0)$ and $(-c,0)$, then $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

If the foci are on the y-axis at $(0,c)$ and $(0,-c)$, then $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

Figure 1-45.
Symmetry of the hyperbola .

The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ can also be represented in a parametric form as the set of points

$$\begin{aligned} & H = H_1 \cup H_2 \quad \text{where} \\ & H_1 = \{\, (x,y) \mid x = a\cosh t,\ y = b\sinh t, \quad -\infty < t < \infty \,\} \\ \text{and} \quad & H_2 = \{\, (x,y) \mid x = -a\cosh t,\ y = b\sinh t, \quad -\infty < t < \infty \,\} \end{aligned} \tag{1.85}$$

which represents a union of the right-branch and left-branch of the hyperbola. Similar parametric representations can be constructed for those hyperbola which undergo a translation or rotation of axes. Remember that the parametric representation of a curve is not unique.

Conic Sections in Polar Coordinates

Place the origin of the polar coordinate system at the focus of a conic section with **the y-axis parallel to the directrix** as illustrated in the figure 1-46. If the point $(x,y) = (r\cos\theta, r\sin\theta)$ is a point on the conic section, then the distance d from the point (x,y) to the directrix of the conic section is given by either

$$d = p + r\cos\theta \qquad \text{or} \qquad d = p - r\cos\theta \tag{1.86}$$

depending upon whether the directrix is to the left or right of the focus. The conic section is defined by $r = ed$ so there results two possible equations $r = e(p - r\cos\theta)$ or $r = e(p + r\cos\theta)$. Solving these equations for r demonstrates that the equations

$$r = \frac{ep}{1 - e\cos\theta} \qquad \text{or} \qquad r = \frac{ep}{1 + e\cos\theta} \tag{1.87}$$

represent the basic forms associated with representing a conic section in polar coordinates.

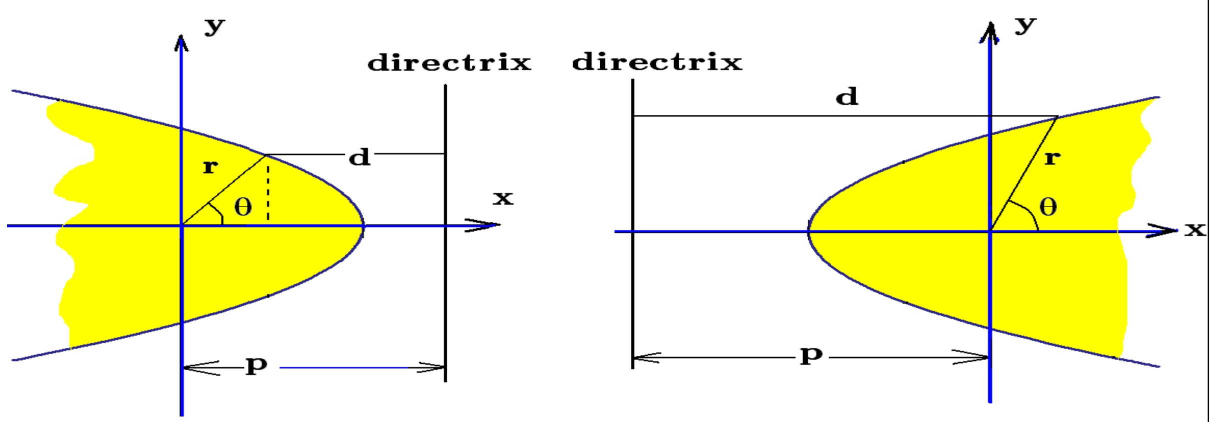


Figure 1-46. Representing conic sections using polar coordinates.

If the directrix is **parallel to the x-axis** at $y = p$ or $y = -p$, then the general forms for representing a conic section in polar coordinates are given by

$$r = \frac{ep}{1 - e\sin\theta} \qquad \text{or} \qquad r = \frac{ep}{1 + e\sin\theta} \tag{1.88}$$

If the eccentricity satisfies $e = 1$, then the conic section is a parabola, if $0 < e < 1$, an ellipse results and if $e > 1$, a hyperbola results.

General Equation of the Second Degree

Consider the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{1.89}$$

where a, b, c, d, e, f are constants, which is **a general equation of the second degree**. If one performs a rotation of axes by substituting the rotation equations

$$x = \bar{x}\cos\theta - \bar{y}\sin\theta \quad \text{and} \quad y = \bar{x}\sin\theta + \bar{y}\cos\theta \tag{1.90}$$

into the equation (1.89), one obtains the new equation

$$\bar{a}\bar{x}^2 + \bar{b}\bar{x}\bar{y} + \bar{c}\bar{y}^2 + \bar{d}\bar{x} + \bar{e}\bar{y} + \bar{f} = 0 \tag{1.91}$$

with new coefficients $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ defined by the equations

$$\begin{aligned} \bar{a} =& a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta & \bar{d} =& d\cos\theta + e\sin\theta \\ \bar{b} =& b(\cos^2\theta - \sin^2\theta) + 2(c-a)\sin\theta\cos\theta & \bar{e} =& -d\sin\theta + e\cos\theta \\ \bar{c} =& a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta & \bar{f} =& f \end{aligned} \tag{1.92}$$

As an exercise one can show the quantity $b^2 - 4ac$, called **the discriminant**, is **an invariant** under a rotation of axes. One can show $b^2 - 4ac = \bar{b}^2 - 4\bar{a}\bar{c}$. The discriminant is used to predict the conic section from the equation (1.89). For example, if $b^2 - 4ac < 0$, then an ellipse results, if $b^2 - 4ac = 0$, then a parabola results, if $b^2 - 4ac > 0$, then a hyperbola results.

In the case where the original equation (1.89) has a cross product term xy, so that $b \neq 0$, then **one can always find a rotation angle θ such that in the new equations (1.91) and (1.92) the term $\bar{b} = 0$**. If the cross product term $\bar{b}$ is made zero, then one can complete the square on the $\bar{x}$ and $\bar{y}$ terms which remain. This completing the square operation converts the new equation (1.90) into one of the standard forms associated with a conic section. By setting the $\bar{b}$ term in equation (1.92) equal to zero one can determine the angle θ such that $\bar{b}$ vanishes. Using the trigonometric identities

$$\cos^2\theta - \sin^2\theta = \cos 2\theta \qquad \text{and} \qquad 2\sin\theta\cos\theta = \sin 2\theta \tag{1.93}$$

one can determine the angle θ which makes the cross product term vanish by solving the equation

$$\bar{b} = b\cos 2\theta + (c-a)\sin 2\theta = 0 \tag{1.94}$$

for the angle θ. One finds the new term $\bar{b}$ is zero if θ is selected to satisfy

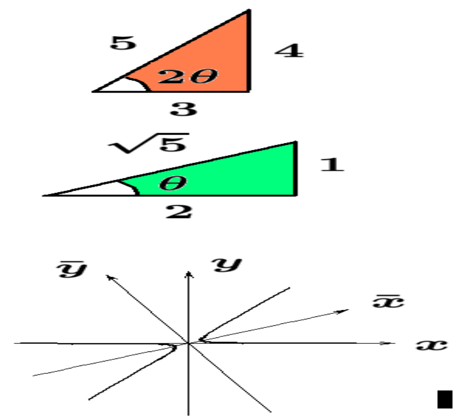
$$\cot 2\theta = \frac{a-c}{b} \quad \text{(recall our hypothesis that } b \neq 0) \tag{1.95}$$

Example 1-17. **(Conic Section)** Sketch the curve $4xy - 3y^2 = 64$

Solution

To remove the product term xy from the general equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ of a conic, the axes must be rotated through an angle θ determined by the equation $\cot 2\theta = \frac{a-c}{b} = \frac{3}{4}$. For the given conic $a = 0,\ b = 4,\ c = -3$.

This implies that $\cos 2\theta = 3/5 = 1 - 2\sin^2\theta$ or $2\sin^2\theta = 2/5$ giving $\sin\theta = 1/\sqrt{5}$ and $\cos\theta = 2/\sqrt{5}$. The rotation equations (1.90) become



$$x = \frac{1}{\sqrt{5}}(2\bar{x} - \bar{y}) \qquad \text{and} \qquad y = \frac{1}{\sqrt{5}}(\bar{x} + 2\bar{y})$$

The given equation then becomes

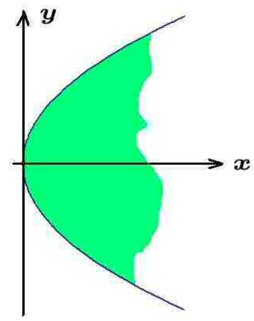
$$4\left(\frac{2\bar{x} - \bar{y}}{\sqrt{5}}\right)\left(\frac{\bar{x} + 2\bar{y}}{\sqrt{5}}\right) - 3\left(\frac{\bar{x} + 2\bar{y}}{\sqrt{5}}\right)^2 = 64$$

which simplifies to the hyperbola $\frac{\bar{x}^2}{8^2} - \frac{\bar{y}^2}{4^2} = 1$ with respect to the $\bar{x}$ and $\bar{y}$ axes. ∎

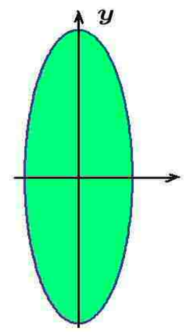
Example 1-18. The parametric forms for representing conic sections are not unique. For a, b constants and θ, t used as parameters, the following are some representative parametric equations which produce conic sections.

Parametric form for conic sections			
Conic Section	x	y	parameter
Circle	$a\cos\theta$	$a\sin\theta$	θ
Parabola	at^2	$2at$	t
Ellipse	$a\cos\theta$	$b\sin\theta$	θ
Hyperbola	$a\sec\theta$	$b\tan\theta$	θ
Rectangular Hyperbola	at	a/t	t
The symbol $a > 0$ denotes a nonzero constant.			

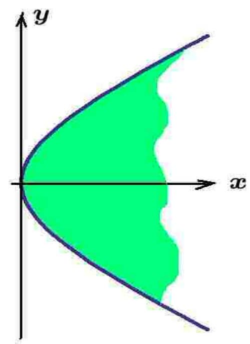
The shape of the curves depends upon the range of values assigned to the parameters representing the curve. Because of this restriction, the parametric representation usually only gives a portion of the total curve. Sample graphs using the parameter values indicated are given below.



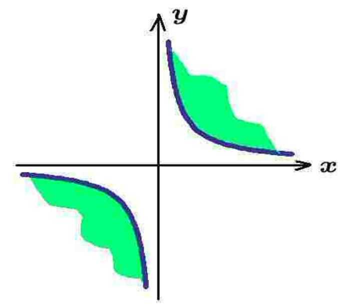
$\{ \ (x,y) \ | \ x=t^2, \ y=2t, \ -2 \le t \le 2 \ \}$



$\{ \ (x,y) \ | \ x=\cos\theta, \ y=3\sin\theta, \ 0 \le \theta \le 2\pi \ \}$



$\{ \ (x,y) \ | \ x=\sec\theta, \ y=\frac{1}{2}\tan\theta, \ -\frac{\pi}{3} \le \theta \le \frac{\pi}{3} \ \}$



$\{ \ (x,y) \ | \ x=3t, \ y=3/t$ where $t \in N_0 \ \}$
$N_0 = \{ \ x \ | \ 0 < |x| \le 2 \ \}$

■

Computer Languages

There are many computer languages and apps that can do graphics and mathematical computations to aid in the understanding of calculus. Many of these programming languages can be used to perform specific functions on a computing device such as a desk-top computer, a lap-top computer, a touch-pad, or hand held calculator. The following is **a partial list**[11] of some computer languages that you might want to investigate. In alphabetical order:
Ada, APL, C, C++, C#, Cobol, Fortran, Java, Javascript, Maple, Mathcad, Mathematica, Matlab, Pascal, Perl, PHP, Python, Visual Basic.

[11] For a more detailed list of programming languages go to
en.Wikipedia.org/wiki/List_of_programming_Languages

Exercises

▶ **1-1.** Find the union $A \cup B$ and intersection $A \cap B$ if

$$(a) \quad A = \{x | -2 < x \le 4\} \text{ and } B = \{x | 2 \le x < 7\}$$
$$(b) \quad A = \{x | -2 < x \le 4\} \text{ and } B = \{x | 4 < x \le 7\}$$
$$(c) \quad A = \{x | x^3 < 8\} \text{ and } B = \{x | x^2 < 16\}$$

▶ **1-2.** Sketch a Venn diagram to illustrate the following statements.

$(a)\quad A \subset B \qquad (c)\quad B - A \qquad (e)\quad (A \cap B) \cup (A \cap C)$

$(b)\quad A \cap B = \phi \qquad (d)\quad A \cap (B \cup C) \qquad (f)\quad A \cap B^c$

▶ **1-3.** If any set operation involving $\emptyset, U, \cap, \cup$ is an identity, then the principle of duality states that the replacements $\emptyset \to U, \quad U \to \emptyset, \quad \cap \to \cup, \quad \cup \to \cap$ in the identity produces a dual statement which is also an identity. Determine the dual statements associated with the given identities.

$(a)\quad (U \cap A) \cup (B \cap A) = A \quad (b)\quad (B \cup A) \cap (\emptyset \cup A) = A \quad (c)\quad A \cup (A \cap B) = A$

▶ **1-4.** Show that the following are equivalent.

$(a)\quad A \subset B$ if and only if $A \cup B = B$

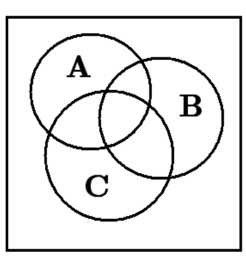
$(b)\quad A \subset B$ if and only if $A \cap B^c = \emptyset$

$(c)\quad A \subset B$ if and only if$A^c \cup B = U$

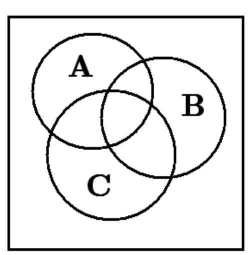
▶ **1-5.** Prove the absorption laws

$(a)\quad A \cup (A \cap B) = A \qquad (b)\quad A \cap (A \cup B) = A$

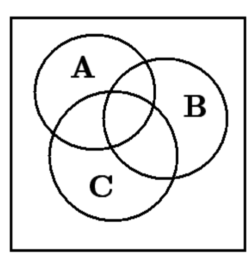
▶ **1-6.** Shade the Venn diagram to represent the statement underneath.



$A \cap B \cap C$



$A \cup (B \cap C)$



$A^c \cap B^c \cap C$

▶ **1-7.** Sketch a Venn diagram to illustrate the following set operations.

$$(a)\quad A\cup(B\cap C) \qquad (b)\quad (A\cup B)^c \qquad (c)\quad (A\cup B\cup C)^c$$

▶ **1-8.** Determine if the given sets are bounded. If a set is bounded above find the least upper bound ($\ell.u.b.$), if the set is bounded below, find its greatest lower bound ($g.\ell.b.$).

$$(a)\quad S_a=\{x|\, x^2<16\} \qquad (c)\quad S_c=\{x|\,\sqrt{x}<5\}$$
$$(b)\quad S_b=\{x|x^3<27\} \qquad (d)\quad S_d=\{x|\,\sqrt[3]{x}>3\}$$

▶ **1-9.** Find the general equation of the line satisfying the given conditions.

(a) The line passes through the point $(2,4)$ with slope -2.

(b) The line has zero slope and passes through the point $(2,4)$

(c) The line is parallel to $2x+3y=4$ and passes through the point $(2,4)$

(d) The line is parallel to the y-axis and passes through the point $(2,4)$

▶ **1-10.** Express the line $3x+4y=12$ in the following forms.

(a) The slope-intercept form and then find the slope and y-intercept.
(b) The intercept form and then find the x-intercept and y-intercept.
(c) Polar form.
(d) The point-slope form using the point $(1,1)$

▶ **1-11.** Determine conditions that x must satisfy if the following inequalities are to be satisfied.

$$(a)\quad \alpha x-\beta<0 \qquad (c)\quad x+1-\frac{12}{x}<0$$
$$(b)\quad \frac{2x+3}{x+4}<0 \qquad (d)\quad \frac{x+2}{x-3}\le 0$$

▶ **1-12.** For each function state how the domain of the function is to be restricted?

(a) $y=f(x)=\sqrt{8-x}$

(b) $y=f(x)=\dfrac{1}{(x-a)(x-b)(x-c)}$, a,b,c are real constants.

(c) Area of a circle is given by $A=f(r)=\pi r^2$

(d) $y=f(x)=\dfrac{x-1}{\sqrt{x^3+1}}$

(e) Volume of a sphere $V=f(r)=\dfrac{4}{3}\pi r^3$

▶ **1-13.** Sketch a graph of the given functions.

$$
\begin{array}{lllll}
(a) & y = \frac{1}{2}x, & y = x, & y = 2x, & -4 \le x \le 4 \\
(b) & y = \frac{1}{4}x^2, & y = x^2, & y = 4x^2, & -4 \le x \le 4 \\
(c) & y = \frac{1}{2}\sin x, & y = \sin x, & y = 2\sin x, & 0 \le x \le 2\pi \\
(d) & y = \frac{1}{2}\cos x, & y = \cos x, & y = 2\cos x, & 0 \le x \le 2\pi
\end{array}
$$

▶ **1-14.** Sketch the graphs defined by the parametric equations.

$$
\begin{array}{ll}
(a) & C_a = \{\, (x,y) \mid x = t^2, \quad y = 2t+1, \quad 0 \le t \le 4 \,\} \\
(b) & C_b = \{\, (x,y) \mid x = t, \quad y = 2t+1, \quad -2 \le t \le 2 \,\} \\
(c) & C_c = \{\, (x,y) \mid x = \cos t, \quad y = \sin t, \quad \frac{\pi}{2} \le t \le \frac{3\pi}{2} \,\} \\
(d) & C_d = \{\, (x,y) \mid x = \sin t, \quad y = \cos t, \quad 0 \le t \le \pi \,\} \\
(e) & C_e = \{\, (x,y) \mid x = t, \quad y = -\sqrt{9-t^2}, \quad -3 \le t \le 3 \,\}
\end{array}
$$

Note that the part of the curve represented depends on (i) the form of the parametric representation and (ii) the values assigned to the parameters.

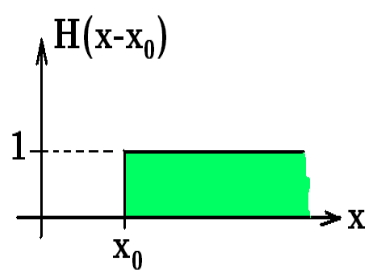
▶ **1-15.** Sketch a graph of the given polynomial functions for $x \in R$.

$$(a) \quad y = x - 1 \qquad (b) \quad y = x^2 - 2x - 3 \qquad (c) \quad y = (x-1)(x-2)(x-3)$$

(d) Show the function $y = (x-1)(x-2)(x-3)$ is skew-symmetric about the line $x = 2$.

▶ **1-16.** The Heaviside[12] step function is defined

$$H(\xi) = \begin{cases} 0, & \xi < 0 \\ 1, & \xi > 0 \end{cases}$$



Sketch the following functions.

$$
\begin{array}{llll}
(a) & y = H(x) & (d) & y = H(x-1) - H(x-2) \\
(b) & y = H(x-1) & (e) & y = H(x) + H(x-1) - 2H(x-2) \\
(c) & y = H(x-2) & (f) & y = \frac{1}{\epsilon}\left[H(x-x_0) - H(x-(x_0+\epsilon))\right], \quad \epsilon > 0 \text{ is small.}
\end{array}
$$

[12] Oliver Heaviside (1850-1925) An English engineer.

▶ **1-17.** Sketch the given curves.

(a) $\{(x,y) \mid y = x^2,\ -2 \le x \le 2\}$

(b) $\{(x,y) \mid y = 1 + x^2,\ -2 \le x \le 2\}$

(c) $\{(x,y) \mid y = -1 + x^2,\ -2 \le x \le 2\}$

(d) $\{(x,y) \mid y = (x-1)^2,\ -1 \le x \le 3\}$

(e) $\{(x,y) \mid y = (x+1)^2,\ -3 \le x \le 1\}$

(f) $\{(x,y) \mid y = 1 + (x-1)^2,\ -1 \le x \le 3\}$

▶ **1-18.** In polar coordinates the equation of a circle with radius ρ and center at the point (r_1, θ_1) is given by

$$r^2 + r_1^2 - 2rr_1\cos(\theta - \theta_1) = \rho^2$$

Write the equation of the circle and sketch its graph in polar coordinates for the following special cases.

(a) $r_1 = \rho, \quad \theta_1 = 0$

(b) $r_1 = \rho, \quad \theta_1 = \pi/2$

(c) $r_1 = \rho, \quad \theta_1 = \pi$

(d) $r_1 = \rho, \quad \theta_1 = 3\pi/2$

(e) $r_1 = 0, \quad \theta_1 = 0$

(f) $r_1 = 3, \quad \theta_1 = \pi/4$ in the cases $\rho < 3,\ \rho = 3,\ \rho > 3$

▶ **1-19.** In rectangular coordinates the equation of a circle with radius $\rho > 0$ and center (h, k) is given by the equation

$$(x-h)^2 + (y-k)^2 = \rho^2$$

Write the equation of the circle and sketch its graph in the following special cases.

(a) $h = \rho, \quad k = 0$

(b) $h = 0, \quad k = \rho$

(c) $h = -\rho, \quad k = 0$

(d) $h = 0, \quad k = -\rho$

(e) $h = 3, \quad k = 4,$ in the cases $\rho < 5$, $\rho = 5$ and $\rho > 5$

▶ **1-20.** Show that each trigonometric function of an acute angle θ is equal to the co-function of the complementary angle $\psi = \frac{\pi}{2} - \theta$.

$$\sin\theta = \cos\psi \qquad \tan\theta = \cot\psi \qquad \sec\theta = \csc\psi$$
$$\cos\theta = \sin\psi \qquad \cot\theta = \tan\psi \qquad \csc\theta = \sec\psi$$

▶ **1-21.** If $f(x) = x,\ 0 \le x < 1$, and $f(x+1) = f(x)$ for all values of x, sketch a graph of this function over the domain $X = \{\, x \mid 0 \le x < 5 \,\}$.

▶ **1-22.** If $f(x)=x^2$ and $g(x)=3-2x$, calculate each of the following quantities.

$(a)\ f(3)$ $\quad(c)\ f(x+h)$ $\quad(e)\ \dfrac{f(x+h)-f(x)}{h}$ $\quad(g)\ f(g(x))$

$(b)\ g(3)$ $\quad(d)\ g(x+h)$ $\quad(f)\ \dfrac{g(x+h)-g(x)}{h}$ $\quad(h)\ g(f(x))$

▶ **1-23.** Sketch the given curves.

$(a)\quad \{\,(x,y)\mid y=\sin x,\ 0\le x\le 2\pi\,\}$

$(b)\quad \{\,(x,y)\mid y=\sin\left(\frac{1}{2}x\right),\ 0\le x\le 2\pi\,\}$

$(c)\quad \{\,(x,y)\mid y=\sin(2x),\ 0\le x\le 2\pi\,\}$

$(d)\quad \{\,(x,y)\mid y=\sin(x-\pi),\ 0\le x\le 2\pi\,\}$

▶ **1-24.** Sketch the given curves.

$(a)\quad \{\,(x,y)\mid y=\cos x,\ 0\le x\le 2\pi\,\}$

$(b)\quad \{\,(x,y)\mid y=\cos\left(\frac{1}{2}x\right),\ 0\le x\le 2\pi\,\}$

$(c)\quad \{\,(x,y)\mid y=\cos(2x),\ 0\le x\le 2\pi\,\}$

$(d)\quad \{\,(x,y)\mid y=\cos(x-\pi),\ 0\le x\le 2\pi\,\}$

▶ **1-25.** Graph the functions and then find the inverse functions.

$(a)\quad y=f_1(x)=x^2,$

$(b)\quad y=f_2(x)=x^3$

$(c)\quad y=f_3(x)=5x-1$

$(d)\quad y=f_4(x)=\sqrt[3]{x+4}$

$(e)\quad y=f_5(x)=\dfrac{2x-3}{5x-2}$

▶ **1-26.** Test for symmetry, asymptotes and intercepts and then sketch the given curve.

$(a)\quad y=1-\dfrac{1}{x^2}$

$(b)\quad y=1+\dfrac{1}{(x-1)(x-3)^2}$

$(c)\quad x^2y=1$

$(d)\quad y^2-x^2y^2=1$

$(e)\quad x^2y-2y=1$

$(f)\quad xy=x^2-1$

▶ **1-27.** Sketch the given curves.

$(a)\quad \{\,(x,y)\mid x=3\cosh t,\ y=4\sinh t,\quad 0\le t\le 3\,\}$

$(b)\quad \{\,(x,y)\mid x=3\cos t,\ y=4\sin t,\quad 0\le t\le 2\pi\,\}$

$(c)\quad \{\,(x,y)\mid x=3+t,\ y=4+t^2/4,\quad -3\le t\le 3\,\}$

$(d)\quad \{\,(r,\theta)\mid r=\dfrac{4}{1+2\cos\theta},\quad 0\le\theta\le 2\pi\,\}$

▶ **1-28.** Test for symmetry and sketch the given curves.

$(a)\quad y=x^2$

$(b)\quad x=y^2$

$(c)\quad y^2=-x$

$(d)\quad \dfrac{x^2}{9}+\dfrac{y^2}{16}=1$

$(e)\quad \dfrac{y^2}{4}+\dfrac{x^2}{16}=1$

$(f)\quad x^2+y^2=25$

$(g)\quad \dfrac{x^2}{9}-\dfrac{y^2}{16}=1$

$(h)\quad \dfrac{y^2}{4}-\dfrac{x^2}{16}=1$

$(i)\quad \dfrac{y^2}{9}-\dfrac{x^2}{4}=1$

▸ **1-29.** Translate axes, then sketch the given curves.

$(a)\quad (x-2)^2 = 4(y-1)$

$(b)\quad (y-2)^2 = -4(x-1)$

$(c)\quad (x-3)^2 = -8(y-2)$

$(a)\quad \dfrac{(x-1)^2}{9} + \dfrac{(y-2)^2}{16} = 1$

$(b)\quad \dfrac{(x-1)^2}{9} - \dfrac{(x-2)^2}{16} = 1$

$(c)\quad \dfrac{(y-1)^2}{9} + \dfrac{(x-2)^2}{16} = 1$

▸ **1-30. The number e**

Consider two methods for estimating the limit $e = \lim\limits_{h\to 0}(1+h)^{1/h}$

Method 1

(a) Make the substitution $n = 1/h$ and show $e = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$ and then use the binomial theorem to show

$$e = \lim_{n\to\infty}\left[1 + \frac{1}{1!} + \frac{1-\frac{1}{n}}{2!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})}{3!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})}{4!} + \cdots\right]$$

(b) Show that as n increases without bound that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

Find the sum of 5,6 and 7 terms of the series to estimate the number e.

Method 2

Use a calculator to fill in the given table to estimate both $(1+h)^{1/h}$ and $\dfrac{1}{h}\ln(1+h)$ for small values of h. Your results should show

$$e = \lim_{h\to 0}(1+h)^{1/h}$$

and $$\ln e = \lim_{h\to 0}\frac{1}{h}\ln(1+h) = 1$$

h	$e \approx (1+h)^{1/h}$	$1 \approx \frac{1}{h}\ln(1+h)$
1.0		
0.5		
0.1		
0.01		
0.001		
0.0001		
0.00001		
0.000001		

▸ **1-31.** If $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$, then make appropriate substitutions and find the following limits.

$(a)\quad \lim\limits_{x\to\infty}\left(1+\dfrac{\alpha}{x}\right)^x \qquad (b)\quad \lim\limits_{x\to-\infty}\left(1+\dfrac{\beta}{x}\right)^x$

where α and β are positive constants.

▸ **1-32.** Use the $\epsilon - \delta$ definition of a limit to prove that $\lim\limits_{x\to 3}(4x+2) = 14$

▶ **1-33.** Sketch a graph of the following straight lines. State the slope of each line and specify the x or y-intercept if it exists.

(a)	$x = 5$	(e)	$\frac{x}{3} + \frac{y}{4} = 1$
(b)	$y = 5$	(f)	$\ell = \{\, (x, y) \mid x = t + 2,\ y = 2t + 3 \,\}$
(c)	$y = x + 1$	(g)	$r\cos(\theta - \pi/4) = 2$
(d)	$3x + 4y + 5 = 0$	(h)	$3x + 4y = 0$

▶ **1-34.** For each line in the previous problem construct the perpendicular bisector which passes through the origin.

▶ **1-35.** Consider the function $y = f(x) = \dfrac{x^2 - 1}{x - 1}$, for $-2 \le x \le 2$.

(a) Is $f(1)$ defined?

(b) Is the function continuous over the interval $-2 \le x \le 2$?

(c) Find $\lim\limits_{x \to 1} f(x)$

(d) Can $f(x)$ be made into a continuous function?

(e) Sketch the function $f(x)$.

▶ **1-36.** Assume $\lim\limits_{x \to x_0} f(x) = \ell_1$ and $\lim\limits_{x \to x_0} g(x) = \ell_2$. Use the $\epsilon - \delta$ proof to show that

$$\lim_{x \to x_0} [f(x) - g(x)] = \ell_1 - \ell_2$$

▶ **1-37.**

(a) Find the equation of the line with slope 2 which passes through the point $(3, 4)$.

(b) Find the equation of the line perpendicular to the line in part (a) which passes through the point $(3, 4)$.

▶ **1-38.** Find the following limits if the limit exists.

(a) $\lim\limits_{x \to 1} \left(x^2 + \dfrac{1}{x} \right)$

(b) $\lim\limits_{x \to 0} \left(2^x + \dfrac{1}{2^x} \right)$

(c) $\lim\limits_{h \to 0} \dfrac{\sqrt{x + h} - \sqrt{x}}{h\sqrt{x(x + h)}}, \quad x \ne 0$

(d) $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}, \quad f(x) = x^2$

(e) $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 + x - 6}$

(f) $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 - 4}$

▶ **1-39.** Find the following limits if the limit exists.

$$(a)\quad \lim_{x\to\infty}\frac{3x^2+2x+1}{x^2+3x+2} \qquad (d)\quad \lim_{x\to 0} x\sin\frac{1}{x}$$

$$(b)\quad \lim_{x\to\infty}\frac{\sin x}{x} \qquad (e)\quad \lim_{x\to 4^-}\frac{7}{x-4}$$

$$(c)\quad \lim_{x\to 0}\sin\frac{1}{x} \qquad (f)\quad \lim_{x\to 1}\frac{x^3-1}{x-1}$$

▶ **1-40.** Show that

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right)=0$$

▶ **1-41.** Evaluate the following limits.

$$(a)\ \lim_{h\to 0}\frac{\sin 4h}{h} \qquad (d)\ \lim_{h\to 0}\frac{\sin(h/2)}{h}$$

$$(b)\ \lim_{h\to 0}\frac{\sin^2(2h)}{h^2} \qquad (e)\ \lim_{x\to 0}\frac{\tan x}{x}$$

$$(c)\ \lim_{h\to 0}\frac{1-\cos h}{h^2} \qquad (f)\ \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$$

▶ **1-42.** Evaluate the limit $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, if $f(x)=\sqrt{x}$ and $x\neq 0$.

▶ **1-43.** Determine if the following limits exist. State why they exist or do not exist.

$$(a)\quad \lim_{x\to\infty}\sin x \qquad (d)\quad \lim_{x\to\infty}\frac{1-\cos(mx)}{\sin(nx)}$$

$$(b)\quad \lim_{x\to 0}\frac{\sin(mx)}{\sin(nx)} \qquad (e)\quad \lim_{x\to 0}\frac{\sin^2 x}{1-\cos x}$$

$$(c)\quad \lim_{x\to 0} x\sin(\frac{1}{x}) \qquad (f)\quad \lim_{x\to 0}\frac{x\sin x}{1-\cos x}$$

▶ **1-44.** Given the line $3x+4y+5=0$

(a) Find the slope of the line.

(b) Find the x and y-intercepts.

(c) Write the equation of the line in intercept form.

(d) Find the line perpendicular to the given line which passes through the point $(0,1)$.

▶ **1-45.**

(a) Show that if $p>0$, and $r=\dfrac{1}{1+p}$, then r is such that $0<r<1$.

(b) Write $r^n=\dfrac{1}{(1+p)^n}$ and use the binomial theorem to show that if $0<r<1$, then $\displaystyle\lim_{n\to\infty} r^n=0$.

(c) Show that if $p>0$, and $x=(1+p)$, then x is such that $x>1$.

(d) Write $x^n=(1+p)^n$ and use the binomial theorem to show that if $x>1$, then $\displaystyle\lim_{n\to\infty} x^n=\infty$

▶ **1-46.** Use the $\epsilon - \delta$ method to prove that if $\lim_{x\to x_0} f(x) = f(x_0)$, then $\lim_{x\to x_0} cf(x) = c \lim_{x\to x_0} f(x) = cf(x_0)$ where c is a constant.

▶ **1-47.** The equation of a line passing through two points on a curve is called a secant line.

(a) Given the parabola $y = x^2$ find the equation of the secant line passing through the points $(1,1)$ and $(2,4)$. Sketch a graph of the curve and the secant line.

(b) Find the equation of the secant line which passes through the points $(1,1)$ and $(3/2, 9/4)$. Sketch this secant line on your graph from part (a).

(c) Discuss how one can determine the equation of the tangent line to the curve $y = x^2$ at the point $(1,1)$.

(d) Can you find the equation of the tangent line to the curve $y = x^2$ at the point $(1,1)$?

▶ **1-48.** Sketch the given parabola and find (i) the focus (ii) the vertex (iii) the directrix and (iv) the latus rectum.

$(a)\ y^2 - 8y - 8x + 40 = 0$ $\qquad$ $(d)\ y^2 - 6y + 12x - 3 = 0$

$(b)\ x^2 = 12y$ $\qquad$ $(e)\ y^2 = -8x$

$(c)\ y^2 - 8y + 4x + 8 = 0$ $\qquad$ $(f)\ x^2 - 6x + 12y - 15 = 0$

▶ **1-49.** Sketch the given ellipse and find (i) the foci (ii) the directrices (iii) the latus rectum and (iv) the eccentricity and (v) center.

$(a)\ 4y^2 + 9x^2 - 16y - 18x - 11 = 0$ $\qquad$ $(d)\ 25y^2 + 16x^2 - 150y - 64x - 689 = 0$

$(b)\ \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ $\qquad$ $(e)\ \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

$(c)\ 16y^2 + 25x^2 - 64y - 150x - 111 = 0$ $\qquad$ $(f)\ 4y^2 + 9x^2 + 8y + 18x - 23 = 0$

▶ **1-50.** Sketch the given hyperbola and find (i) the foci (ii) the vertices (iii) the directrices (iv) the eccentricity and (v) the asymptotes.

$(a)\ 9x^2 - 4y^2 - 36x + 24y - 36 = 0$ $\qquad$ $(d)\ x^2 - 4y^2 + 32y - 2x - 67 = 0$

$(b)\ \dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ $\qquad$ $(e)\ \dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$

$(c)\ 4y^2 - 9x^2 - 16y + 54x - 101 = 0$ $\qquad$ $(f)\ 4x^2 - y2 + 4y - 24x + 28 = 0$

▶ **1-51.** Given the parabola $y^2 = 4x$ and the line $y = x + b$. What condition(s) must be satisfied in order for the line to be a tangent line to the parabola?

▶ **1-52.** Examine the general equation of the second degree, given by equation (1.89). When this equation is transformed using a rotation of axes there results the equation (1.91) with coefficients defined by equation (1.92).

(a) Show that the quantity $a+c$ is an invariant. That is, show $a+c=\bar{a}+\bar{c}$.

(b) Show that the discriminant is an invariant. That is, show $b^2-4ac=\bar{b}^2-4\bar{a}\bar{c}$ Note that these two invariants are used as a check for numerical errors when one performs the algebra involved in the rotation of axes.

▶ **1-53.** Show that the equation $xy=a^2$, with a constant is a hyperbola.

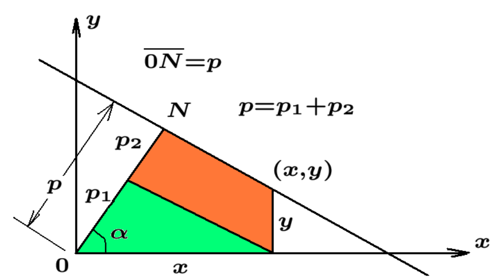
▶ **1-54.** Find the parabola symmetric about the x-axis which passes through the points $(-1,0)$, $(0,1)$ and $(0,-1)$.

▶ **1-55.**

(a) Sketch the hyperbola $\dfrac{y^2}{4}-\dfrac{x^2}{9}=1$ and label the y-intercepts, and the asymptotes.

(b) Find the equation of the conjugate hyperbola.

▶ **1-56. Normal form for equation of line**



Let $p=\overline{0N}>0$ denote the perpendicular distance of the line from the origin. Assume the line $\overline{0N}$ makes an angle α with the x-axis and let (x,y) denote a variable point on the line. Write $p=p_1+p_2$, where $p_1=$ projection of x on $\overline{ON}$ and $p_2=$ projection of y on $\overline{ON}$.

(a) Show $p_1+p_2=p \implies x\cos\alpha+y\sin\alpha-p=0$ which is called the normal form for the equation of a line.

(b) Show that the line $Ax+By+C=0$ has the normal form $\dfrac{Ax+By+C}{\pm\sqrt{A^2+B^2}}=0$ where the correct sign is selected so that $p>0$.

(c) Find the normal form for the line $3x+4y-5=0$

(d) Show that the distance d of a point (x_0,y_0) from the line described by $Ax+By+C=0$ is given by $d=\left|\dfrac{Ax_0+By_0+C}{\sqrt{A^2+B^2}}\right|$

▶ **1-57.** The boiling point of water is 100° Celsius or 212° Fahrenheit and the freezing point of water is 32° Fahrenheit and 0° Celsius. If there is a linear relationship between degrees Celsius and degrees Fahrenheit, then find this relationship.

▸ **1-58.** Graph the given equations by selecting θ such that r is well defined.

$$(a) \quad r\cos\theta = 3 \qquad (b) \quad r\sin\theta = 3 \qquad (c) \quad r = 4\cos\theta \qquad (d) \quad r = 4\sin\theta$$

▸ **1-59.** Graph the given equations by selecting θ so that r is well defined.

$$(a) \quad r = \frac{3}{1-\cos\theta} \qquad (b) \quad r = \frac{3}{1+\cos\theta} \qquad (c) \quad r = \frac{6}{2-\cos\theta}$$

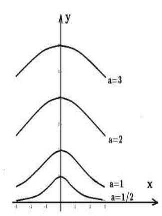
▸ **1-60.** Graph the following equations by selecting θ so that r is well defined.

$$(a) \quad r = \frac{3}{1-\sin\theta} \qquad (b) \quad r = \frac{3}{2-3\cos\theta} \qquad (c) \quad r = \frac{3}{2-\cos\theta}$$

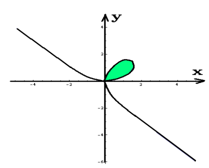
▸ **1-61.** Find the horizontal and vertical asymptotes of the curve

$$xy - 3x + 2y - 10 = 0$$

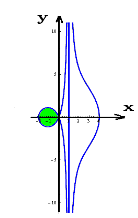
▸ **1-62.** Verify the following graphs by any method you wish.



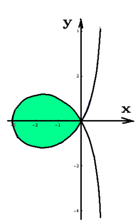
Witch of Agnesi
$\boldsymbol{x^2y + 4a^2y - 8a^3 = 0}$



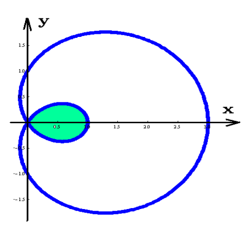
Folium of Descartes
$\boldsymbol{x^3 + y^3 - 3axy = 0}$



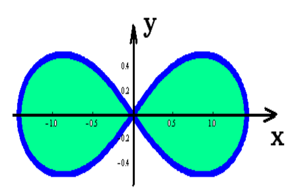
Conchoid of Nicomedes with $b > a$
$\boldsymbol{(x-a)^2(x^2+y^2) = b^2x^2}$



Trisectrix of Maclaurin
$\boldsymbol{y^2(a-x) = x^2(3a+x)}$



Limaçon of Pascal with $a = b$
$\boldsymbol{(x^2+y^2-2ax)^2 - b^2(x^2+y^2)}$



Limnescate of Bernoulli
$\boldsymbol{(x^2+y^2)^2 = 2a^2(x^2-y^2)}$

Hints: Try substitutions $r^2 = x^2 + y^2$ or $x = r\cos\theta$, $y = r\sin\theta$ or let $y = tx$ and try to obtain a parametric representation of the given curves.

Chapter 2

Differential Calculus

The history of mathematics presents the development of calculus as being accredited to Sir Isaac Newton (1642-1727) an English physicist, mathematician and Gottfried Wilhelm Leibnitz (1646-1716) a German physicist, mathematician. Portraits of these famous individuals are given in the figure 2-1. The introduction of calculus created an explosion in the development of the physical sciences and other areas of science as calculus provided a way of describing natural and physical laws in a mathematical format which is easily understood. The development of calculus also opened new areas of mathematics and science as individuals sought out new ways to apply the techniques of calculus.

Isaac Newton
(1642-1727)

Wilhelm Leibnitz
(1646-1716)

Figure 2-1. Joint developers of the calculus.

Calculus is **the study of things that change** and finding ways to represent these changes in a mathematical way. The symbol Δ will be used to represent change. For example, the notation Δy is to be read "The change in y".

Slope of Tangent Line to Curve

Consider a **continuous smooth**[1] **curve** $y = f(x)$, defined over a closed interval defined by the set of points $X = \{\, x \mid x \in [a, b]\,\}$. Here x is the **independent variable**, y

[1] A continuous smooth curve is an unbroken curve defined everywhere over the domain of definition of the function and is a curve which has no sharp edges. If P is a point on the curve and ℓ is the tangent line to the point P, then a smooth curve is said to have a continuously turning tangent line as P moves along the curve.

is the **dependent variable** and the function can be represented graphically as a curve defined by the set of points

$$\{ (x, y) \mid x \in X, \; y = f(x) \}$$

The **slope of the curve** at some given point P on the curve **is defined** to be the same as the **slope of the tangent line to the curve at the point** P.

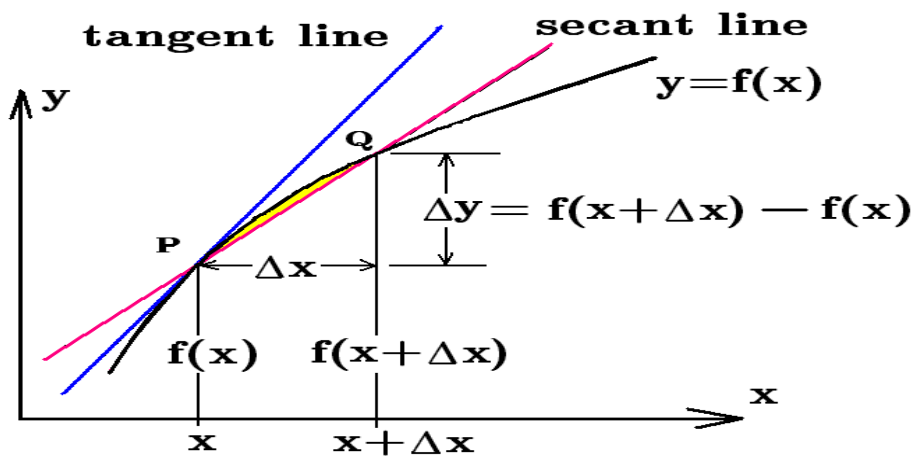


Figure 2-2. Secant line approaching tangent line as $Q \to P$

One can construct a tangent line to any point P on the curve as follows. On the curve $y = f(x)$ consider two neighboring points P and Q with coordinates $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$, as illustrated in the figure 2-2. In this figure Δx represents some small change in x and $\Delta y = f(x + \Delta x) - f(x)$ denotes the change in y in moving from point P to Q. In the figure 2-2, the near points P and Q on the curve define a straight line called **a secant line of the curve**. In the limit as the point Q approaches the point P the quantity Δx tends toward zero and the **secant line approaches the tangent line**. The quantity Δy also tends toward zero while the **slope of the secant line** m_s **approaches the slope of the tangent line** m_t **to the curve at the point** P.

The **slope of the secant line** m_s is given by $m_s = \dfrac{\Delta y}{\Delta x}$ and the **slope of the tangent line** m_t at the point P with coordinates $(x, f(x))$ is given by the derived function $f'(x)$ calculated from the limiting process

$$m_t = \frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{2.1}$$

if this limit exists.

The Derivative of $y = f(x)$

The secant line through the near points P and Q approaches the tangent line in the limit as Δx tends toward zero. The slope of the secant line $m_s = \dfrac{\Delta y}{\Delta x}$ represents the **average change of the height of the curve y with respect to changes in x over the interval Δx**. The derived function $f'(x) = \dfrac{dy}{dx}$ represents the **slope of the tangent line m_t at the point P and is obtained from equation (2.1) as a limiting process**. The derived function $f'(x)$ is called **the derivative of $y = f(x)$ with respect to x** and represents **the slope of the curve $y = f(x)$ at the point $(x, f(x))$.** It also represents the **instantaneous rate of change of $y = f(x)$ with respect to x at the point $(x, f(x))$ on the curve $y = f(x)$**. The derived function $f'(x)$ or derivative represents the slope m_t of the tangent line constructed through the point $(x, f(x))$ on the curve. The limit, defined by the equation (2.1), represents a process, called **differentiation**, for finding the derivative $\dfrac{dy}{dx} = f'(x)$. A function $y = f(x)$, where the differentiation process is successful, is called a **differentiable function**. The derived function $\dfrac{dy}{dx} = f'(x)$ obtained from the differentiation process is called **the derivative function associated with the given function** $y = f(x)$. By agreement, when referencing the derivative of an explicit function $y = f(x)$, it is understood to represent the ratio of changes of the dependent variable y, with respect to changes of the independent variable x, as these changes tend toward zero. There are alternative equivalent methods for calculating the derivative of a function $y = f(x)$. One alternative method is the following

$$\boldsymbol{\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\xi \to x} \frac{f(\xi) - f(x)}{\xi - x}} \tag{2.2}$$

if these limits exist. If $y = f(x)$ for $x \in X$, then the domain of definition of the derivative $f'(x)$ is the set X' defined by $X' = \{\, x \mid f'(x) \text{ exists } \}$. In general $X' \subseteq X$.

Example 2-1. Tangent line to curve

Given the parabola $y = f(x) = 16 - x^2$. Find the tangent lines which touch this curve at the points $(-2, 12)$, $(3, 7)$ and at a general point (x_0, y_0) on the curve.

Solution The derivative function $\dfrac{dy}{dx} = f'(x)$ associated with the parabolic function $y = f(x) = 16 - x^2$ represents the slope of the tangent line to the curve at the point $(x, f(x))$ on the curve. The derivative function is calculated using the limiting process defined by equation (2.1) or equation (2.2). One finds using the equation (2.1)

$$\begin{aligned} \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{16 - (x+h)^2 - (16 - x^2)}{h} \\ = \lim_{h \to 0} \frac{16 - (x^2 + 2xh + h^2) - (16 - x^2)}{h} &= \lim_{h \to 0} -(2x + h) = -2x \end{aligned}$$

or if one uses equation (2.2)

$$\frac{dy}{dx}=f'(x)=\lim_{\xi\to x}\frac{f(\xi)-f(x)}{\xi-x}=\lim_{\xi\to x}\frac{16-\xi^2-(16-x^2)}{\xi-x}=\lim_{\xi\to x}\frac{(x-\xi)(x+\xi)}{(\xi-x)}$$
$$=\lim_{\xi\to x}-(\xi+x)=-2x$$

Here the derivative function is $f'(x)=-2x$ and from the derivative function

the slope of the tangent line at $(-2,12)$ is $m_t=f'(-2)=-2(-2)=4$

the slope of the tangent line at $(3,7)$ is $m_t=f'(3)=-2(3)=-6$

Using the point-slope formula $y-y_0=m(x-x_0)$ for representing the equation of a line, one finds

tangent line through point $(-2,12)$ is $y-12=4(x+2)$

tangent line through point $(3,7)$ is $y-7=-6(x-3)$

Knowing that a function $y=f(x)$ has a derivative function $\frac{dy}{dx}=f'(x)$ which is defined and continuous for all values of the independent variable $x\in(a,b)$ implies that the given function $y=f(x)$ is a continuous function for $x\in(a,b)$. This is because the tangent line to a point P on the curve is a continuous turning tangent line as the point P moves along the curve. This is illustrated in the figure 2-3 where the tangent line to the curve is **continuously turning without any interruptions**, the slope moving continuously from a positive value, through zero to a negative value.

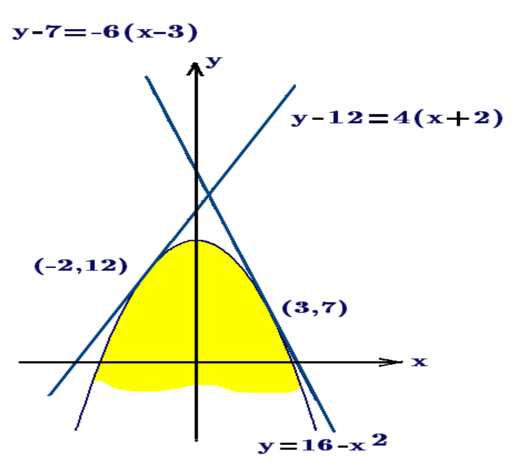


Figure 2-3.
Tangent lines to curve $y=16-x^2$

In general, at each point (x_0,y_0), where $y_0=f(x_0)=16-x_0^2$, the slope of the curve at that point is also the slope of the tangent line at that point and this slope is given by $f'(x_0)=-2x_0$. The equation of the tangent line to the curve $y=f(x)=16-x^2$ which passes through the point (x_0,y_0) is given by the point-slope formula

$$y-y_0=(-2x_0)(x-x_0),\quad \text{where } y_0=f(x_0)=16-x_0^2$$

and x_0 is some fixed abscissa value within the domain of definition of the function. The parabolic curve and tangent lines are illustrated in the figure 2-3.

Also note that for $x < 0$, the slope of the curve is positive and indicates that as x increases, y increases. For $x > 0$, the slope of the curve is negative and indicates that as x increases, y decreases. If the derivative function changes continuously from a positive value to a negative value, then it must pass through zero. Here the zero slope occurs where the function $y = f(x) = 16 - x^2$ has a maximum value. ■

Example 2-2. If $y = f(x) = \sin x$, then show $\frac{dy}{dx} = f'(x) = \frac{d}{dx}\sin x = \cos x$

Solution By definition $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$. Use the results from example 1-7 together with the trigonometric identity for the difference of two sine functions to obtain

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \cos(x + \frac{\Delta x}{2}) \lim_{\Delta x \to 0} \frac{\sin(\frac{\Delta x}{2})}{\frac{\Delta x}{2}} = \cos x$$

■

Example 2-3. If $y = g(x) = \cos x$, then show $\frac{dy}{dx} = g'(x) = \frac{d}{dx}\cos x = -\sin x$

Solution By definition $\frac{dy}{dx} = g'(x) = \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$. Use the results from example 1-7 together with the trigonometric identity for the difference of two cosine functions to obtain

$$\frac{dy}{dx} = g'(x) = \lim_{\Delta x \to 0} -\sin(x + \frac{\Delta x}{2}) \lim_{\Delta x \to 0} \frac{\sin(\frac{\Delta x}{2})}{\frac{\Delta x}{2}} = -\sin x$$

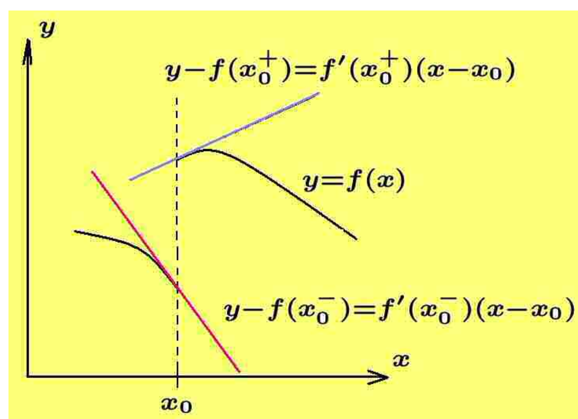
■

Right and Left-hand Derivatives

If a function $y = f(x)$ has a jump discontinuity at the point $x = x_0$, then one can define the **right-hand derivative of** $f(x)$ **at the point** $x = x_0$ as the following limit

$$f'(x_0^+) = \lim_{h \to 0^+} \frac{f(x_0^+ + h) - f(x_0^+)}{h}, \quad h > 0$$

if this limit exists.



The **left-hand derivative of** $f(x)$ **at the point** $x = x_0$ is defined as the limit

$$f'(x_0^-) = \lim_{h \to 0^-} \frac{f(x_0^- + h) - f(x_0^-)}{h}, \quad \text{where } h \text{ is restricted such that } h < 0$$

if this limit exists. If the left-hand derivative is different from the right-hand derivative, then there exists a tangent line through the point $(x_0^+, f(x_0^+))$ and a different tangent line through the point $(x_0^-, f(x_0^-))$. If the left-hand derivative equals the right-hand derivative then $f(x)$ is said to have a derivative at the point $x = x_0$.

Alternative Notations for the Derivative

Some of the notations used to represent the derivative of a function $y = f(x)$ are

$$\frac{dy}{dx} = \frac{d}{dx}y = f'(x) = y' = \frac{d}{dx}f(x) = \frac{df}{dx} = Df(x) = D_x f(x) \tag{2.3}$$

where $\frac{d}{dx}$, $D = \frac{d}{dx}$ and $D_x = \frac{d}{dx}$ are called **differentiation operators**. One can think of $\frac{d}{dx}$ as a derivative operator which operates upon a given function to produce the differences Δx and Δy and then evaluates the limit $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ to produce the derivative function.

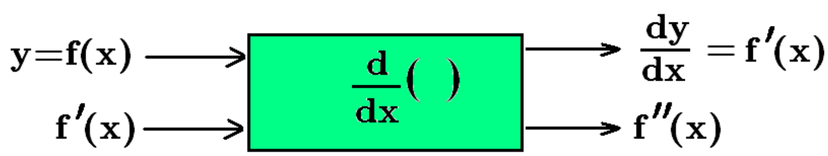


Figure 2-4. Differentiation performed by an operator box.

The figure 2-4 illustrates an operator box where functions that enter the operator box get operated upon using the differentiation process defined by equation (2.1) and the output from the box represents a derivative of the input function. Observe that a **derivative of a derivative function** is called a **second derivative function**. In general, a derivative, such as $\frac{dy}{dx} = f'(x)$, is a measure of the instantaneous rate of change of $y = f(x)$ with respect to a change in x. The notation $\frac{dy}{dx}$ for the derivative was introduced by Gottfried Leibnitz. The prime notation $f'(x)$ for the derivative of a function $f(x)$ was introduced by Joseph-Louis Lagrange[2]. If $y = y(t)$, Sir Isaac Newton used the dot notation $\dot{y}$, $\ddot{y}$, ... for representing the first, second and higher derivatives. The operator notation $D_x y$ was introduced by Leonhard Euler[3].

Higher Derivatives

If the derivative function $f'(x)$ is the input to the operator box illustrated in the figure 2-4, then the output function is denoted $f''(x)$ and represents a **derivative of a derivative called a second derivative**. Higher ordered derivatives are defined in a similar fashion with $\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^n y}{dx^n}$ which states that the derivative of the

[2] Joseph-Louis Lagrange (1736-1813) an Italian born French mathematician.
[3] Leonhard Euler (1707-1783) A Swiss mathematician.

$(n-1)st$ derivative is the nth derivative. The function $f'(x) = \frac{dy}{dx}$ is called a first derivative, $f''(x) = \frac{d^2y}{dx^2}$ is called a second derivative, $f'''(x) = \frac{d^3y}{dx^3}$ is called a third derivative,... $f^{(n)}(x) = \frac{d^ny}{dx^n}$ is called a n-th derivative. Other notations for higher ordered derivatives are as follows.

The first derivative of $y = f(x)$ is denoted

$$\frac{dy}{dx} = f'(x) \quad \text{or} \quad D_x y \quad \text{or} \quad Dy \quad \text{or} \quad y'$$

The second derivative of $y = f(x)$ is denoted

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x) \quad \text{or} \quad D_x^2 y \quad \text{or} \quad D^2 y \quad \text{or} \quad y''$$

The third derivative of $y = f(x)$ is denoted

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = f'''(x) \quad \text{or} \quad D_x^3 y \quad \text{or} \quad D^3 y \quad \text{or} \quad y'''$$

The n-th derivative of $y = f(x)$ is denoted

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = f^{(n)}(x) \quad \text{or} \quad D_x^n y \quad \text{or} \quad D^n y \quad \text{or} \quad y^{(n)}$$

Rules and Properties

The following sections cover fundamental material associated with the introduction of different kinds of functions and developing techniques to find the derivatives associated with these functions. The following list contains fundamental rules and properties associated with the differentiation of sums, products and quotients of functions. These fundamental properties should be memorized and recognized in applications. Note that most proofs of a differentiation property use one of the previous definitions of differentiation given above and so the student should memorize the definitions of a derivative as given by equations (2.2).

The derivative of a constant C is zero or $\frac{d}{dx}C = 0$. That is, if

$$y = f(x) = C = constant, \quad \text{then} \quad \frac{dy}{dx} = f'(x) = \frac{d}{dx}C = 0$$

Proof

Sketch the curve $y = f(x) = C = constant$ and observe that it has a zero slope everywhere. The derivative function represents the slope of the curve $y = C$ at the point with abscissa x and consequently $\frac{d}{dx}C = 0$ for all values of x since the slope is zero at every point on the curve and the height of the curve is not changing. Using the definition of a derivative one finds

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{C-C}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

The converse statement that if $f'(x) = 0$ for all values of x, then $y = f(x) = C$ is a constant also holds and will be proven later in this chapter.

The derivative of the function $\boldsymbol{y = f(x) = x}$, is $\boldsymbol{\frac{dy}{dx} = f'(x) = \frac{d}{dx}x = 1}$

Proof

Sketch the curve $y = f(x) = x$ and observe that it is a line which passes through the origin making an angle of $\pi/4$ with the x-axis. The slope of this line is given by $m = \tan\frac{\pi}{4} = 1$ for all values of x. Consequently, $f'(x) = \frac{d}{dx}x = 1$ for all values of x since the **derivative function represents the slope of the curve at the point** x. Using the definition of a derivative one finds

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}x = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = 1$$

for all values of x.

The derivative of the function $\boldsymbol{y = f(x) = x^n}$, where $\boldsymbol{n}$ is a nonzero integer, is given by $\boldsymbol{\frac{dy}{dx} = f'(x) = \frac{d}{dx}x^n = nx^{n-1}}$ or in words one can say the derivative of x to an integer power n equals the power n times x to the $(n-1)$st power.

Proof

Using the limiting process which defines a derivative one finds

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx}x^n = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

One can employ the binomial theorem to expand the numerator and obtain

$$\begin{aligned}\frac{dy}{dx} = f'(x) =& \frac{d}{dx}x^n = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + h^n - x^n}{h}\\ &= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + h^{n-1}\right]\\ &= nx^{n-1}\end{aligned}$$

Consequently, one can write $\boldsymbol{\frac{d}{dx}x^n = nx^{n-1}}$ where n is an integer.

Later it will be demonstrated that $\boldsymbol{\frac{d}{dx}x^r = rx^{r-1}}$ for all real numbers r which are different from zero.

The derivative of a constant times a function equals the constant times the derivative of the function or

$$\boldsymbol{\frac{d}{dx}\left[Cf(x)\right] = C\frac{d}{dx}f(x) = Cf'(x)}$$

Proof

Use the definition of a derivative applied to the function $g(x) = Cf(x)$ and show that

$$\begin{aligned}\frac{d}{dx}g(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} &= \lim_{h\to 0}\frac{Cf(x+h)-Cf(x)}{h}\\ &= \lim_{h\to 0} C\left(\frac{f(x+h)-f(x)}{h}\right)\end{aligned}$$

It is known that the limit of a constant times a function is the constant times the limit of the function and so one can write

$$\frac{d}{dx}g(x) = C\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = Cf'(x)$$

or

$$\frac{d}{dx}\left[Cf(x)\right] = C\frac{d}{dx}f(x) = Cf'(x)$$

The derivative of a sum is the sum of the derivatives or

$$\boldsymbol{\frac{d}{dx}\left[u(x)+v(x)\right] = \frac{d}{dx}u(x) + \frac{d}{dx}v(x) = \frac{du}{dx} + \frac{dv}{dx} = u'(x) + v'(x)} \tag{2.4}$$

This result can be extended to include n-functions

$$\boldsymbol{\frac{d}{dx}\left[u_1(x)+u_2(x)+\cdots+u_n(x)\right] = \frac{d}{dx}u_1(x) + \frac{d}{dx}u_2(x) + \cdots + \frac{d}{dx}u_n(x)}$$

Proof

If $y(x) = u(x) + v(x)$, then

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{y(x+h)-y(x)}{h}\\ &= \lim_{h\to 0}\frac{u(x+h)+v(x+h)-[u(x)+v(x)]}{h}\\ &= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}\right] + \lim_{h\to 0}\left[\frac{v(x+h)-v(x)}{h}\right]\end{aligned}$$

or

$$\boldsymbol{\frac{dy}{dx} = \frac{d}{dx}[u(x)+v(x)] = \frac{d}{dx}u(x) + \frac{d}{dx}v(x) = u'(x) + v'(x)}$$

This result follows from the limit property that the limit of a sum is the sum of the limits. The above proof can be extended to larger sums by breaking the larger sums into smaller groups of summing two functions.

Example 2-4. The above properties are combined into the following examples.

(a) If $y = F(x)$ is a function which is differentiable and C is a nonzero constant, then

$$\frac{d}{dx}\left[CF(x)\right] = C\frac{dF(x)}{dx} \qquad \frac{d}{dx}\left[5x^3\right] = 5\frac{d}{dx}x^3 = 5(3x^2) = 15x^2$$

$$\frac{d}{dx}\left[F(x)+C\right] = \frac{dF(x)}{dx} + \frac{d}{dx}C = \frac{dF(x)}{dx} \qquad \frac{d}{dx}\left[x^3+8\right] = \frac{d}{dx}x^3 + \frac{d}{dx}8 = 3x^2$$

since the derivative of a constant times a function equals the constant times the derivative of the function and the derivative of a sum is the sum of the derivatives.

(b) If $S = \{f_1(x), f_2(x), f_3(x), \dots, f_n(x), \dots\}$ is a set of functions, define the set of derivatives $\frac{dS}{dx} = \{\frac{df_1}{dx}, \frac{df_2}{dx}, \frac{df_3}{dx}, \dots, \frac{df_n}{dx}, \dots\}$. To find the derivatives of each of the functions in the set $S = \{1, x, x^2, x^3, x^4, x^5, \dots, x^{100}, \dots, x^m, \dots\}$, where m is a very large integer, one can use properties 1 , 2 and 3 above to write

$$\frac{dS}{dx} = \{0, 1, 2x, 3x^2, 4x^3, 5x^4, \dots, 100x^{99}, \dots, mx^{m-1}, \dots\}$$

(c) Consider the polynomial function $y = x^6 + 7x^4 + 32x^2 - 17x + 33$. To find the derivative of this function one can combine the properties 1, 2, 3, 4 to show

$$\frac{dy}{dx} = \frac{d}{dx}(x^6 + 7x^4 + 32x^2 - 17x + 33)$$

$$\frac{dy}{dx} = \frac{d}{dx}x^6 + 7\frac{d}{dx}x^4 + 32\frac{d}{dx}x^2 - 17\frac{d}{dx}x + \frac{d}{dx}(33)$$

$$\frac{dy}{dx} = 6x^5 + 7(4x^3) + 32(2x) - 17(1) + 0$$

$$\frac{dy}{dx} = 6x^5 + 28x^3 + 64x - 17$$

This result follows from use of the properties (i) the derivative of a sum is the sum of the derivatives (ii) the derivative of a constant times a function is that constant times the derivative of the function (iii) the derivative of x to a power is the power times x to the one less power and (iv) the derivative of a constant is zero.

(d) To find the derivative of a polynomial function

$$y = p_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n, \tag{2.5}$$

where $a_0, a_1, \dots, a_n$ are constants,with $a_0 \neq 0$, one can use the first four properties above to show that by differentiating each term one obtains the derivative function

$$\frac{dy}{dx} = a_0\left[nx^{n-1}\right] + a_1\left[(n-1)x^{n-2}\right] + a_2\left[(n-2)x^{n-3}\right] + \cdots a_{n-2}\left[2x\right] + a_{n-1}[1] + 0$$

(e) The polynomial function $p_n(x)$ of degree n given by equation (2.5) is a linear combination of terms involving x to a power. The first term $a_0 x^n$, with $a_0 \neq 0$, being the term containing the largest power of x. Make note of the higher derivatives associated with the function x^n. These derivatives are

$$\begin{aligned}
\frac{d}{dx}(x^n) =& nx^{n-1} \\
\frac{d^2}{dx^2}(x^n) =& n(n-1)x^{n-2} \\
\frac{d^3}{dx^3}(x^n) =& n(n-1)(n-2)x^{n-3} \\
\vdots \qquad & \qquad \vdots \\
\frac{d^n}{dx^n}(x^n) =& n(n-1)(n-2)\cdots(3)(2)(1)x^0 = n! \quad \text{Read } n\text{-factorial.} \\
\frac{d^{n+1}}{dx^{n+1}}(x^n) =& 0
\end{aligned}$$

This result demonstrates that the $(n+1)$st and higher derivatives of a polynomial of degree n will all be zero.

(f) One can readily verify the following derivatives

$$\begin{aligned}
\frac{d^3}{dx^3}(x^3) =& 3! = 3\cdot 2\cdot 1 = 6 \qquad & \frac{d^5}{x^5}(x^5) =& 5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120 \\
\frac{d^4}{dx^4}(x^3) =& 0 & \frac{d^6}{dx^6}(x^5) =& 0
\end{aligned}$$

■

The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function or

$$\begin{aligned}
\frac{d}{dx}\left[u(x)v(x)\right] =& u(x)\frac{dv}{dx} + v(x)\frac{du}{dx} = u(x)v'(x) + v(x)u'(x) \\
\text{or} \qquad \frac{d}{dx}\left[u(x)v(x)\right] =& u(x)v(x)\left(\frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)}\right)
\end{aligned} \tag{2.6}$$

Proof

Use the properties of limits along with the definition of a derivative to show that if $y(x) = u(x)v(x)$, then

$$\begin{aligned}
\frac{dy}{dx} =& \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} \\
=& \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\
=& \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x+h) + u(x)v(x+h) - u(x)v(x)}{h}
\end{aligned}$$

Where the term $u(x)v(x+h)$ has been added and subtracted to the numerator. Now rearrange terms and use the limit properties to write

$$\frac{dy}{dx} = \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}\right] v(x+h) + \lim_{h\to 0} u(x)\,\left[\frac{v(x+h)-v(x)}{h}\right]$$
$$\frac{dy}{dx} = \lim_{h\to 0} u(x) \lim_{h\to 0}\left[\frac{v(x+h)-v(x)}{h}\right] + \lim_{h\to 0} v(x+h) \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}\right]$$

or

$$\boldsymbol{\frac{dy}{dx} = \frac{d}{dx}[u(x)v(x)] = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx} = u(x)v'(x) + v(x)u'(x)}$$

The result given by equation (2.6) is known as **the product rule for differentiation**.

■

Example 2-5.

(a) To find the derivative of the function $y = (3x^2+2x+1)(8x+3)$ one should recognize the function is defined as a product of polynomial functions and consequently the derivative is given by

$$\begin{aligned}
\frac{dy}{dx} =& \frac{d}{dx}\left[(3x^2+2x+1)(8x+3)\right]\\
\frac{dy}{dx} =& (3x^2+2x+1)\frac{d}{dx}(8x+3) + (8x+3)\frac{d}{dx}(3x^2+2x+1)\\
\frac{dy}{dx} =& (3x^2+2x+1)(8) + (8x+3)(6x+2)\\
\frac{dy}{dx} =& 72x^2+50x+14
\end{aligned}$$

(b) The second derivative is by definition a derivative of the first derivative so that differentiating the result in part(a) gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\left(72x^2+50x+14\right) = 144x+50$$

Similarly, the third derivative is

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}(144x+50) = 144$$

and the fourth derivative and higher derivatives are all zero.

■

Example 2-6.

Consider the problem of differentiating the function $y = u(x)v(x)w(x)$ which is a product of three functions. To differentiate this function one can apply the product rule to the function $y = [u(x)v(x)]\cdot w(x)$ to obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left([u(x)v(x)]\cdot w(x)\right) = [u(x)v(x)]\frac{dw(x)}{dx} + w(x)\frac{d}{dx}[u(x)v(x)]$$

Applying the product rule to the last term one finds

$$\begin{aligned}
\frac{dy}{dx} =& \frac{d}{dx}\left[u(x)v(x)w(x)\right] = u(x)v(x)\frac{dw(x)}{dx} + u(x)\frac{dv(x)}{dx}w(x) + \frac{du(x)}{dx}v(x)w(x)\\
\frac{dy}{dx} =& \frac{d}{dx}\left[u(x)v(x)w(x)\right] = u(x)v(x)w'(x) + u(x)v'(x)w(x) + u'(x)v(x)w(x)
\end{aligned}$$

A generalization of the above procedure produces the generalized product rule for differentiating a product of n-functions

$$\begin{aligned}
\frac{d}{dx}\left[u_1(x)u_2(x)u_3(x)\cdots u_{n-1}(x)u_n(x)\right] =& u_1(x)u_2(x)u_3(x)\cdots u_{n-1}(x)\frac{du_n(x)}{dx}\\
&+u_1(x)u_2(x)u_3(x)\cdots\frac{du_{n-1}(x)}{dx}u_n(x)\\
&+\cdots\\
&+u_1(x)u_2(x)\frac{du_3(x)}{dx}\cdots u_{n-1}(x)u_n(x)\\
&+u_1(x)\frac{du_2(x)}{dx}u_3(x)\cdots u_{n-1}(x)u_n(x)\\
&+\frac{du_1(x)}{dx}u_2(x)u_3(x)\cdots u_{n-1}(x)u_n(x)
\end{aligned}$$

This result can also be expressed in the form

$$\frac{d}{dx}\left[u_1u_2u_3\cdots u_{n-1}u_n\right] = u_1u_2u_3\cdots u_n\left(\frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \frac{u_3'}{u_3} + \cdots + \frac{u_n'}{u_n}\right)$$

and is obtained by a repeated application of the original product rule for two functions.

■

The derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the denominator squared or

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}}{v^2(x)} = \frac{v(x)u'(x) - u(x)v'(x)}{v^2(x)} \qquad \textbf{(2.7)}$$

Proof

Let $y(x) = \dfrac{u(x)}{v(x)}$ and write

$$
\begin{aligned}
\frac{dy}{dx} = \lim_{h\to 0}\frac{y(x+h)-y(x)}{h} &= \lim_{h\to 0}\frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{v(x)u(x+h)-u(x)v(x)+u(x)v(x)-u(x)v(x+h)}{v(x+h)v(x)}}{h}\\
&= \lim_{h\to 0}\frac{v(x)\left[\dfrac{u(x+h)-u(x)}{h}\right] - u(x)\left[\dfrac{v(x+h)-v(x)}{h}\right]}{v(x+h)v(x)}\\
&= \frac{v(x)\lim\limits_{h\to 0}\left[\dfrac{u(x+h)-u(x)}{h}\right] - u(x)\lim\limits_{h\to 0}\left[\dfrac{v(x+h)-v(x)}{h}\right]}{\lim\limits_{h\to 0} v(x+h)v(x)}
\end{aligned}
$$

or

$$
\boldsymbol{\frac{dy}{dx} = \frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v(x)u'(x)-u(x)v'(x)}{v^2(x)},} \qquad \textbf{where } \boldsymbol{v^2(x) = [v(x)]^2}
$$

This result is known as **the quotient rule for differentiation.**

A special case of the above result is the differentiation formula

$$
\boldsymbol{\frac{d}{dx}\left(v(x)^{-1}\right) = \frac{d}{dx}\left[\frac{1}{v(x)}\right] = \frac{-1}{[v(x)]^2}\frac{dv}{dx} = \frac{-1}{[v(x)]^2}v'(x)} \tag{2.8}
$$

Example 2-7. If $y = \dfrac{3x^2+8}{x^3-x^2+x}$, then find $\dfrac{dy}{dx}$

Solution

Using the derivative of a quotient property one finds

$$
\begin{aligned}
\frac{dy}{dx} =& \frac{d}{dx}\left[\frac{3x^2+8}{x^3-x^2+x}\right] = \frac{(x^3-x^2+x)\frac{d}{dx}(3x^2+8) - (3x^2+8)\frac{d}{dx}(x^3-x^2+x)}{(x^3-x^2+x)^2}\\
=& \frac{(x^3-x^2+x)(6x)-(3x^2+8)(3x^2-2x+1)}{(x^3-x^2+x)^2} = \frac{-3x^4-21x^2+16x-8}{(x^3-x^2+x)^2}
\end{aligned}
$$

■

Differentiation of a Composite Function

If $y = y(u)$ is a function of u and $u = u(x)$ is a function of x, then the derivative of y with respect to x equals the derivative of y with respect to u times the derivative of u with respect to x or

$$
\boldsymbol{\frac{dy}{dx} = \frac{d}{dx}y(u) = \frac{dy}{du}\frac{du}{dx} = y'(u)u'(x)} \tag{2.9}
$$

This is known as **the composite function rule for differentiation** or **the chain rule for differentiation.** Note that the prime notation $'$ always denotes differentiation with respect to the argument of the function. For example $z'(\xi) = \dfrac{dz}{d\xi}$.

Proof

If $y = y(u)$ is a function of u and $u = u(x)$ is a function of x, then make note of the fact that if x changes to $x + \Delta x$, then u changes to $u + \Delta u$ and $\Delta u \to 0$ as $\Delta x \to 0$. Hence, if $\Delta u \neq 0$, one can use the identity

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

together with the limit theorem for products of functions, to obtain

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta u \to 0} \left(\frac{\Delta y}{\Delta u}\right) \cdot \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x}\right) = \frac{dy}{du} \cdot \frac{du}{dx} = y'(u)u'(x)$$

which is known as **the chain rule for differentiation.**

An alternative derivation of this rule makes use of the definition of a derivative given by equation (2.2). If $y = y(u)$ is a function of u and $u = u(x)$ is a function of x, then one can write

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} = \lim_{h \to 0} \frac{y(u(x+h)) - y(u(x))}{h} \tag{2.10}$$

In equation (2.10) make the substitutions $u = u(x)$ and $\xi = u(x+h)$ and write equation (2.10) in the form

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\xi \to u} \left[\frac{y(\xi) - y(u)}{(\xi - u)} \cdot \frac{(\xi - u)}{h}\right] \\
\frac{dy}{dx} &= \lim_{\xi \to u} \frac{y(\xi) - y(u)}{\xi - u} \cdot \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = y'(u)u'(x)
\end{aligned}$$

Here the chain rule is used to **differentiate a function of a function**. For example, if $y = f(g(x))$ is a function of a function, then make the substitution $u = g(x)$ and write $y = f(u)$, then by the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)u'(x) = f'(u)g'(x) \tag{2.11}$$

The derivative of a function $u = u(x)$ raised to a power n, n an integer, equals the power times the function to the one less power times the derivative of the function or

$$\boldsymbol{\frac{d}{dx}\left[u(x)^n\right] = n\,u(x)^{n-1}\,\frac{du}{dx} = n\,u(x)^{n-1}u'(x)} \tag{2.12}$$

This result is known as **the power rule for differentiation.**

Proof

This is a special case of the previous property. If $y = y(u) = u^n$ is a function of u and $u = u(x)$ is a function of x, then differentiation of these functions with respect to their independent variables gives the derivatives

$$\frac{dy}{du} = \frac{d}{du}u^n = nu^{n-1} \quad \text{and} \quad \frac{du}{dx} = \frac{d}{dx}u(x) = u'(x)$$

Using the chain rule for differentiation one finds

$$\frac{dy}{dx} = \frac{d}{dx}u(x)^n = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} = nu^{n-1}u'(x)$$

or

$$\boldsymbol{\frac{d}{dx}u(x)^n = nu(x)^{n-1}\frac{du}{dx} = nu(x)^{n-1}\,u'(x)}$$

The **general power rule for differentiation** is

$$\boldsymbol{\frac{d}{dx}u(x)^r = ru(x)^{r-1}\,\frac{du}{dx}} \tag{2.13}$$

where r is any real number. Here it is understood that for the derivative to exist, then $u(x) \neq 0$ and the function $u(x)^r$ is well defined everywhere. A proof of the general power rule is given later in this chapter.

Example 2-8. Find the derivative $\frac{dy}{dx}$ of the function $y = \sqrt[3]{x^2 + x}$

Solution Let $u = x^2 + x$ and write $y = u^{1/3}$. These functions have the derivatives

$$\frac{du}{dx} = \frac{d}{dx}(x^2 + x) = 2x + 1 \qquad \text{and} \qquad \frac{dy}{du} = \frac{d}{du}u^{1/3} = \frac{1}{3}u^{-2/3}$$

By the chain rule for differentiation

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{3u^{2/3}}(2x+1) = \frac{2x+1}{3(x^2+x)^{2/3}}$$

Using the general power rule one can write

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + x)^{1/3} = \frac{1}{3}(x^2+x)^{-2/3}(2x+1) = \frac{2x+1}{3(x^2+x)^{2/3}}$$

■

Example 2-9. Find the derivative $\frac{dy}{dx}$ of the function $y = \left(\frac{x^2-1}{x^4+1}\right)^3$

Solution Let $u = \frac{x^2-1}{x^4+1}$ and write $y = u^3$ so that by the chain rule for differentiation one has $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ where $\frac{dy}{du} = \frac{d}{du}u^3 = 3u^2$ and

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x^2-1}{x^4+1}\right) = \frac{(x^4+1)(2x)-(x^2-1)(4x^3)}{(x^4+1)^2} = \frac{-2x^5-4x^3+2x}{(x^4+1)^2}$$

This gives the final result

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2\frac{du}{dx} = 3\left(\frac{x^2-1}{x^4+1}\right)^2 \cdot \left(\frac{-2x^5-4x^3+2x}{(x^4+1)^2}\right)$$
$$\frac{dy}{dx} = \frac{6(x-4x^5+4x^7-x^9)}{(x^4+1)^4}$$

■

Differentials

If $y = f(x)$ is differentiable, then the limit $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$ exists. The quantity **Δx is called the increment given to x** and **$\Delta y = f(x+\Delta x) - f(x)$ is called the increment in $y = f(x)$ corresponding to the increment in x**. Since the derivative is determined by a limiting process, then one can define **$dx = \Delta x$ as the differential of x** and write

$$\Delta y = f(x+\Delta x) - f(x) = f'(x)\Delta x + \epsilon\Delta x = f'(x)\,dx + \epsilon\,dx \tag{2.14}$$

where $\epsilon \to 0$ as $\Delta x \to 0$. Define the quantity **$dy = f'(x)dx$ as the differential of y** which represents the **principal part of the change in** y **as** $\Delta x \to 0$. Note that the differential dy **does not equal** Δy because dy is only an **approximation** to **the actual change in** y. Using the above definitions one can write

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

Here $dx = \Delta x$, but dy is not Δy because $\Delta y = dy + \epsilon\,dx$, where $\epsilon \to 0$ as $\Delta x \to 0$.

Using the definition $dy = f'(x)\,dx$ one can verify the following differentials

$$df(x) = f'(x)\,dx \qquad d(uv) = u\,dv + v\,du$$
$$dC = 0 \quad \text{C is a constant} \qquad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$
$$d(Cu) = C\,du \qquad df = \frac{df}{du}\,du$$
$$d(u+v) = du + dv \qquad d(u^n) = nu^{n-1}\,du$$

Differentiation of Implicit Functions

A function defined by an equation of the form $F(x,y) = 0$, where one of the variables x or y is not explicitly solved for in terms of the other variable, then one says that y **is defined as an implicit function of** x. For example, the equation of the circle given by

$$F(x,y) = x^2 + y^2 - \rho^2 = 0, \qquad \rho \text{ is a constant}$$

is an example of an implicit function, where a dependent variable has not been defined explicitly in terms of an independent variable. In general, when given an implicit function $F(x,y) = 0$, there are times where it is possible to solve for one variable in terms of another and thereby convert the implicit form into an explicit form for representing the function. Note also that there are times where the implicit functions $F(x,y) = 0$ cannot be converted into an explicit form. Given an implicit function $F(x,y) = 0$, where it is not possible to solve for y in terms of x, it is still possible to calculate the derivative $\frac{dy}{dx}$ by **treating the function** $F(x,y) = 0$ **as a function** $F(x, y(x)) = 0$**, where it is to be understood, that theoretically the implicit function defines** y **as a function of** x. One can then differentiate every part of the implicit function with respect to x and then solve the resulting equation for the derivative term $\frac{dy}{dx}$.

Example 2-10. Given the implicit function $F(x,y) = x^3 + xy^2 + y^3 = 0$, find the derivative $\frac{dy}{dx}$.

Solution

Differentiate each term of the given implicit function with respect to x to obtain

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(xy^2) + \frac{d}{dx}(y^3) = \frac{d}{dx}0 \tag{2.15}$$

The derivative of the first term in equation (2.15) represents the derivative of x to a power. The second term in equation (2.15) represents the derivative of a product of two functions (the function x times the function $y^2(x)$). The third term in equation (2.15) represents the derivative of a function to a power (the function $y^3(x)$). Remember, that when dealing with implicit functions, it is understood that y is to be treated as a function of x. Calculate the derivatives in equation (2.15) using the product rule and general power rule and show there results

$$\begin{aligned}
3x^2 + x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) + 3y^2\frac{dy}{dx} =& 0\\
3x^2 + x\cdot 2y\frac{dy}{dx} + y^2(1) + 3y^2\frac{dy}{dx} =& 0\\
(3x^2 + y^2) + (2xy + 3y^2)\frac{dy}{dx} =& 0
\end{aligned}$$

Solving this last equation for the derivative term gives

$$\frac{dy}{dx} = \frac{-(3x^2 + y^2)}{2xy + 3y^2}$$

Make note that once the derivative is solved for, then **the form for representing the derivative can be changed by using some algebra along with the given original implicit form** $y^3 = -x^3 - xy^2$. For example, one can write

$$\frac{dy}{dx} = \frac{-(3x^2 + y^2)}{2xy + 3y^2} = \frac{-3x^2y - y^3}{2xy^2 + 3y^3} = \frac{-3x^2y - (-xy^2 - x^3)}{2xy^2 + 3(-xy^2 - x^3)} = \frac{3xy - x^2 - y^2}{3x^2 + y^2}$$

An alternative method to solve the above problem is to **use differentials** and find the differential of each term to obtain

$$3x^2\,dx + x\cdot 2y\,dy + dx\cdot y^2 + 3y^2\,dy = 0$$

Divide each term by dx and combine like terms to obtain

$$(2xy + 3y^2)\frac{dy}{dx} = -(3x^2 + y^2)$$

and solving for $\frac{dy}{dx}$ obtain the same result as above.

■

Example 2-11.

Find the equation of the tangent line to the circle $x^2 + 2x + y^2 - 6y - 15 = 0$ which passes through the point $(2, 7)$.

Solution

The given equation is an implicit equation defining the circle. By completing the square on the x and y terms one can convert this equation to the form

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = 15 + 10 \qquad \Longrightarrow \qquad (x+1)^2 + (y-3)^2 = 25$$

which represents a circle centered at the point $(-1, 3)$ with radius 5.

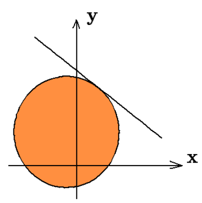


Figure 2-5.
Circle centered at $(-1,3)$ with radius 5.

This circle is illustrated in the figure 2-5. One can verify that the point $(2,7)$ is on the circle by substituting the values $x=2$ and $y=7$ into the given equation to show that these values do indeed satisfy the equation. Use implicit differentiation and show

$$2x+2+2y\frac{dy}{dx}-6\frac{dy}{dx}=0 \text{ or } (2y-6)\frac{dy}{dx}=-(2x+2)$$

and so the derivative is given by $\dfrac{dy}{dx}=-\dfrac{(x+1)}{y-3}$

This derivative represents the slope of the circle at a point (x_0,y_0) on the circle which is the same as the slope of the tangent line to the point (x_0,y_0) on the circle. Therefore, the slope of the tangent line to the circle at the point $(2,7)$ is obtained by evaluating the derivative at this point. The notation used to denote a derivative $\dfrac{dy}{dx}$ being evaluated at a point (x_0,y_0) is $\dfrac{dy}{dx}\Big|_{(x_0,y_0)}$. For example, one can say the slope of the tangent line to the circle at the point $(2,7)$ is given by

$$m_t=\frac{dy}{dx}\Big|_{(2,7)}=\frac{-(x+1)}{(y-3)}\Big|_{(2,7)}=\frac{-(2+1)}{(7-3)}=-\frac{3}{4}$$

The equation of the tangent line to the circle which passed through the point $(2,7)$ is obtained from the point-slope formula $y-y_0=m_t(x-x_0)$ for the equation of a line. One finds the equation of the tangent line which passes through the point $(2,7)$ on the circle is given by

$$y-7=-(3/4)(x-2)$$

■

Example 2-12.

(a) Consider two lines ℓ_1 and ℓ_2 which intersect to form supplementary angles α and β as illustrated in the figure 2-6. Let α equal the counterclockwise angle from line ℓ_1 to line ℓ_2. One could define either angle α or β as the angle of intersection between the two lines. To avoid confusion as to which angle to use, **define the point of intersection of the two lines as a point of rotation.**

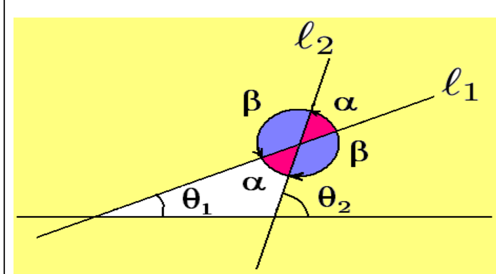


Figure 2-6.
Intersection of two lines.

One can think of line ℓ_1 as being rotated about this point to coincide with the line ℓ_2 or line ℓ_2 as being rotated to coincide with line ℓ_1. **The smaller angle of rotation, either counterclockwise or clockwise, is defined as the angle of intersection between the two lines.**

Assume the lines ℓ_1 and ℓ_2 have slopes $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$ which are well defined. We know the exterior angle of a triangle must equal the sum of the two opposite interior angles so one can write $\alpha = \theta_2 - \theta_1$ and consequently,

$$\tan\alpha = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2} = \frac{m_2 - m_1}{1 + m_1 m_2} \tag{2.16}$$

Here α denotes the counterclockwise angle from line ℓ_1 to line ℓ_2. If **the lines are perpendicular**, then they are said to **intersect orthogonally**. In this case the formula given by equation (2.16) becomes meaningless because when the lines intersect orthogonally then the slopes satisfy $m_1 m_2 = -1$.

(b) If two curves C_1 and C_2 intersect at a point P, the **angle of intersection of the two curves** is defined as the **angle of intersection of the tangent lines to the curves** C_1 and C_2 at the intersection point P. Two curves are said to intersect orthogonally when their intersection is such that the tangent lines at the point of intersection form right angles.

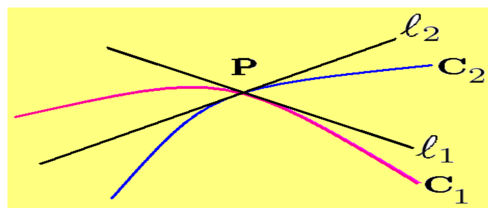


Figure 2-7.

Intersection of two curves.

(c) Find the angle of intersection between the circles

$$x^2 + 2x + y^2 - 4y = 0 \qquad \text{and} \qquad x^2 - 4x + y^2 - 6y + 8 = 0$$

Solution First find the points where the two circles intersect. Eliminating the terms x^2 and y^2 by subtracting the equations of the circle shows that the two circles must intersect at points which lie on the line $y = 4 - 3x$. Substitute this value for y into either of the equations for the circle and eliminate y to obtain a quadratic equation in x and show the points of intersection are $(0,4)$ and $(1,1)$. As a check, show that these values satisfy both the given equations.

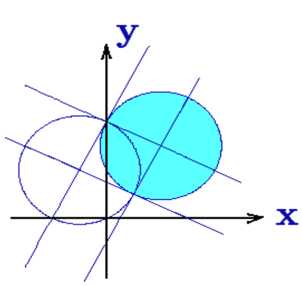


Figure 2-8.

Intersection of two circles.

To find the slopes of the tangent lines at these two points of intersection, use implicit differentiation to differentiate the given equations for the circles. These differentiations produce the following equations.

$$\begin{aligned} x^2+2x+y^2-4y =&0 \\ \frac{d}{dx}(x^2+2x+y^2-4y) =&\frac{d}{dx}(0) \\ 2x+2+2y\frac{dy}{dx}-4\frac{dy}{dx} =&0 \\ \frac{dy}{dx} =& \frac{-(2x+2)}{(2y-4)} \end{aligned} \qquad \begin{aligned} x^2-4x+y^2-6y+8 =&0 \\ \frac{d}{dx}(x^2-4x+y^2-6y+8) =&\frac{d}{dx}(0) \\ 2x-4+2y\frac{dy}{dx}-6\frac{dy}{dx} =&0 \\ \frac{dy}{dx} =& \frac{-(2x-4)}{(2y-6)} \end{aligned}$$

The slopes of the tangent lines at the point $(0,4)$ are given by

$$\text{For the first circle} \qquad m_1 = \frac{dy}{dx}\Big|_{(0,4)} = \frac{-(2x+2)}{(2y-4)}\Big|_{(0,4)} = \frac{-1}{2}$$

$$\text{and for the second circle} \qquad m_2 = \frac{dy}{dx}\Big|_{(0,4)} = \frac{-(2x-4)}{(2y-6)}\Big|_{(0,4)} = 2$$

This gives the equations of the tangent lines to the point $(0,4)$ as $y-4=(-1/2)x$ and $y-4=2x$. Note that the product of the slopes gives $m_1m_2=-1$ indicating the curves intersect orthogonally.

Similarly, the slopes of the tangent lines at the point $(1,1)$ are given by

$$m_1 = \frac{dy}{dx}\Big|_{(1,1)} = \frac{-(2x+2)}{2y-4)}\Big|_{(1,1)} = 2$$

$$\text{and} \quad m_2 = \frac{dy}{dx}\Big|_{(1,1)} = \frac{-(2x-4)}{(2y-6)}\Big|_{1,1)} = \frac{-1}{2}$$

This gives the equations of the tangent lines to the point $(1,1)$ as $y-1=2(x-1)$ and $y-1=(-1/2)(x-1)$. The product of the slopes gives $m_1m_2=-1$ indicating the curves intersect orthogonally. The situation is illustrated in the figure 2-8.

■

Importance of Tangent Line and Derivative Function $f'(x)$

Given a curve C defined by the set of points $\{\,(x,y)\mid\; y=f(x),\; a\le x\le b\,\}$ where $y=f(x)$ is a differentiable function for $a<x<b$. The following is a short list of things that can be said about the curve C, the function $f(x)$ defining the curve and the derivative function $f'(x)$ associated with $f(x)$.

1. If P is a point on the curve C, having coordinates $(x_0, f(x_0))$, **the slope of the curve at the point P is given by** $f'(x_0)$, where $f'(x)$ is the derivative function associated with the function rule $y=f(x)$ defining the ordinate of the curve.

2. The **tangent line to the curve** C **at the point** P is given by the **point-slope formula**

$$y - f(x_0) = f'(x_0)(x - x_0) \quad \text{the point } (x_0, f(x_0)) \text{ is a fixed point on the curve.}$$

3. The **curve** C **is called a smooth curve over the interval** $a \le x \le b$ **if it has a continuously turning tangent line as the point** P **moves from** $(a, f(a)$ **to** $(b, f(b))$**.**

4. If the continuously turning tangent line suddenly changes at a point $(x_1, f(x_1))$, then the derivative function $f'(x)$ is said to have a **jump discontinuity at the point** $x = x_1$. See figure 2-9(a).

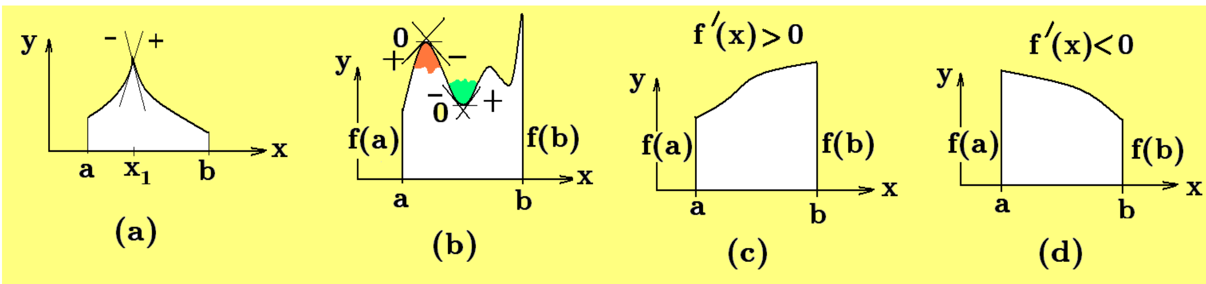


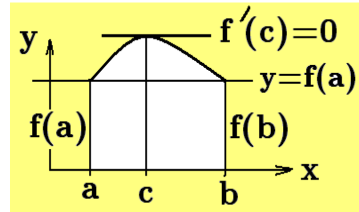
Figure 2-9. Analysis of the derivative function $f'(x)$.

5. If **as** x **increases, the tangent line to the curve continuously changes from a positive slope to a zero slope followed by a negative slope, the curve** C **is said to have a local maximum or relative maximum at the point where the slope is zero.** If this local maximum occurs at the point $(x_1, f(x_1))$,**then** $f(x_1) \ge f(x)$ **for all points** x **near** x_1**.** Similarly, if **as** x **increases the tangent line continuously changes from a negative slope to a zero slope followed by a positive slope, the curve** C **is said to have a local minimum or relative minimum at the point where the slope is zero.** If the local minimum occurs at the point $(x_2, f(x_2))$, **then** $f(x_2) \le f(x)$ **for all points** x **near** x_2**.** See figure 2-9(b)

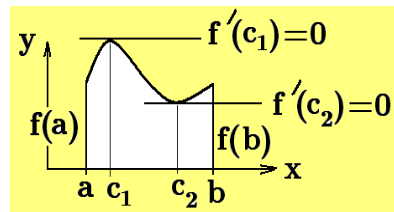
6. If $f'(x) > 0$ **for all values of** x **as** x **moves from** a **to** b**, the continuously turning tangent line always has a positive slope which indicates that the function** $y = f(x)$ **is an increasing function of** x **over the interval** (a, b)**. Functions with this property are called monotone increasing functions.** See figure 2-9(c).

7. If $f'(x) < 0$ **for all values of** x **as** x **moves from** a **to** b**, the continuously turning tangent line always has a negative slope which indicates that the function** $y = f(x)$ **is a decreasing function of** x **over the interval** (a, b)**. Functions with this property are called monotone decreasing functions.** See figure 2-9(d)

Rolle's Theorem[4]



If $y = f(x)$ is a function satisfying (i) it is continuous for all $x \in [a, b]$ (ii) it is differentiable for all $x \in (a, b)$ and (iii) $f(a) = f(b)$, then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.



This result is known as **Rolle's theorem**. If $y = f(x)$ is a constant, the theorem is true so assume $y = f(x)$ is different from a constant. If the slope $f'(x)$ is always positive or always negative for $a \le x \le b$, then $f(x)$ would be either continuously increasing or continuously decreasing between the endpoints $x = a$ and $x = b$ and so it would be impossible for $y = f(x)$ to have the same value at both endpoints. This implies that in order for $f(a) = f(b)$ the derivative function $f'(x)$ must change sign as x moves from a to b. If the derivative function changes sign it must pass through zero and so one can say there exists at least one number $x = c$ where $f'(c) = 0$.

The Mean-Value Theorem

If $y = f(x)$ is a continuous function for $x \in [a, b]$ and is differentiable so that $f'(x)$ exists for $x \in (a, b)$, then there exists at least one number $x = c \in (a, b)$ such that the slope m_t of the tangent line at $(c, f(c))$ is the same as the slope m_s of the secant line passing through the points $(a, f(a))$ and $(b, f(b))$ or

$$\boldsymbol{m_t = f'(c) = \frac{f(b) - f(a)}{b - a} = m_s \qquad a < c < b}$$

This result is known as the mean-value theorem and its implications are illustrated in the figure 2-10.

Proof

A sketch showing the secant line and tangent line having the same slope is given in the figure 2-10. In this figure note the secant line passing through the points $(a, f(a))$ and $(b, f(b))$ and verify that the equation of this secant line is given by the point-slope formula

$$y - f(a) = \left[\frac{f(b) - f(a)}{b - a}\right](x - a)$$

[4] Michel Rolle (1652-1719) A French mathematician. His name is pronounced "Roll".

Also construct the vertical line $x = \xi$, where $a < \xi < b$. This line intersects the curve $y = f(x)$ at the point P with coordinates $(\xi, f(\xi))$ and it intersects the secant line at point Q with coordinates $(\xi, f(a) + \left[\frac{f(b)-f(a)}{b-a}\right](\xi - a))$. Denote the distance from Q to P as $h(\xi)$ and verify that

$$h(\xi) = f(\xi) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](\xi - a) \tag{2.17}$$

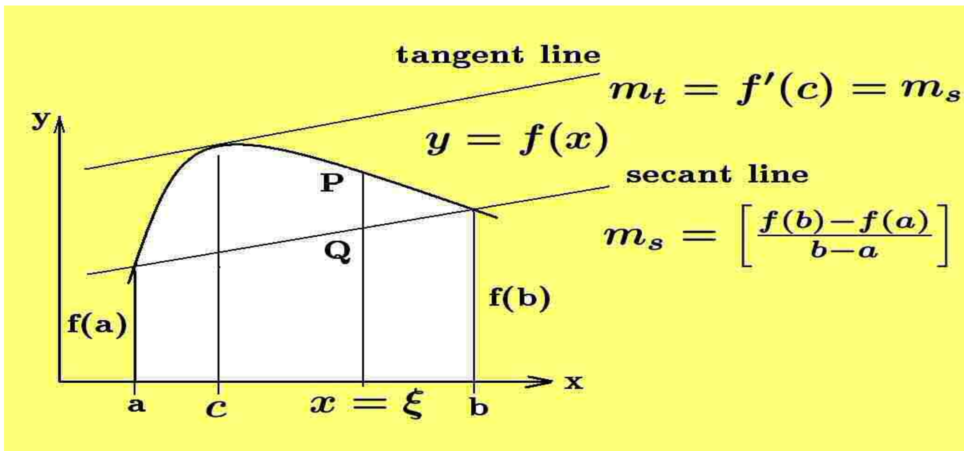


Figure 2-10. Construction of secant line to curve $y = f(x)$

Note that $h(\xi)$ varies with ξ and satisfies $h(a) = h(b) = 0$. The function $h(\xi)$ satisfies all the conditions of Rolle's theorem so one can say there exists at least one point $x = c$ where $h'(c) = 0$. Differentiate the equation (2.17) with respect to ξ and show

$$h'(\xi) = \frac{dh}{d\xi} = f'(\xi) - \frac{f(b) - f(a)}{b - a} \tag{2.18}$$

If there is a value $\xi = c$ such that $h'(c) = 0$, then equation (2.18) reduces to

$$m_t = f'(c) = \frac{f(b) - f(a)}{b - a} = m_s \tag{2.19}$$

which shows that there must exist a point $x = c$ such that the slope of the tangent line at the point $(c, f(c))$ is the same as the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$ as illustrated in the figure 2-10.

The mean-value theorem can be expressed in a slightly different form if in equation (2.19) one makes the substitution $b - a = h$, so that $b = a + h$. This produces the form

$$f(a + h) = f(a) + hf'(c) \qquad \text{where } a < c < a + h \tag{2.20}$$

Let β denote a real number between 0 and 1 and express the number c in the form $c = a + \beta h$, then another form for the mean-value theorem is

$$f(a+h) = f(a) + hf'(a+\beta h), \qquad 0 < \beta < 1 \tag{2.21}$$

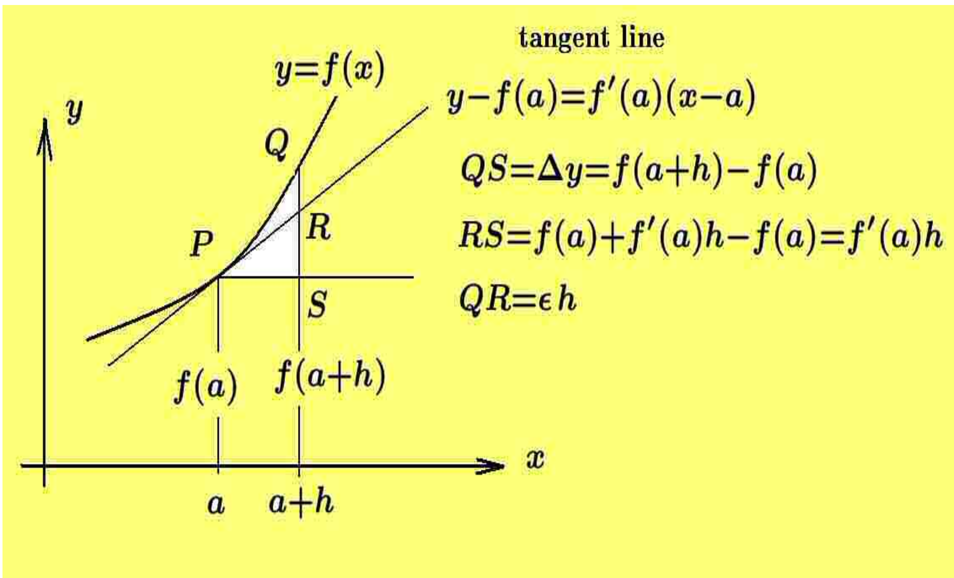


Figure 2-11. Another form for the mean value theorem.

A physical interpretation of the mean-value theorem, which will prove to be useful in later sections, is obtained from an examination of the figure 2-11.

In this figure let $QR = \epsilon h$ where $\epsilon \to 0$ as $h \to 0$, then one can write

$$\begin{aligned} QS =& QR + RS \\ \text{or} \qquad \Delta y = f(a+h) - f(a) =& \epsilon h + f'(a) h \\ \text{or by the mean-value theorem} \qquad \frac{f(a+h) - f(a)}{h} =& f'(a) + \epsilon = f'(a + \beta h) \end{aligned} \tag{2.22}$$

where $\epsilon \to 0$ as $h \to 0$. This result was used earlier in equation (2.14).

In summary, if $\dfrac{d}{dx} G(x) = G'(x) = g(x)$, then one form for the mean-value theorem is

$$\Delta G = G(a+h) - G(a) = g(a)h + \epsilon h \quad \text{or} \quad \Delta G = G(a+h) - G(a) = G'(a)h + \epsilon h \tag{2.23}$$

where $\epsilon \to 0$ as $h \to 0$.

Cauchy's Generalized Mean-Value Theorem

Let $f(x)$ and $g(x)$ denote two functions which are continuous on the interval $[a,b]$. Assume the derivatives $f'(x)$ and $g'(x)$ exist and **do not vanish simultaneously for all** $x \in [a,b]$ **and that** $g(b) \neq g(a)$**.** Construct the function

$$y(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)] \tag{2.24}$$

and note that $y(a) = y(b) = f(a)g(b) - f(b)g(a)$ and so all the conditions exist such that Rolle's theorem can be applied to this function. The derivative of the function given by equation (2.24) is

$$y'(x) = f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)]$$

and Rolle's theorem states that there must exist a value $x = c$ satisfying $a < c < b$ such that

$$y'(c) = f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] = 0 \tag{2.25}$$

By hypothesis the quantity $g(b) - g(a) \neq 0$ and $g'(c) \neq 0$, for if $g'(c) = 0$, then equation (2.25) would require that $f'(c) = 0$, which contradicts our assumption that the derivatives $f'(x)$ and $g'(x)$ cannot be zero simultaneously. Rearranging terms in equation (2.25) gives **Cauchy's generalized mean-value theorem** that $f(x)$ and $g(x)$ must satisfy

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \qquad a < c < b \tag{2.26}$$

Note the special case $g(x) = x$ reduces equation (2.26) to the form of equation (2.19).

Derivative of the Logarithm Function

Assume $b > 0$ is constant and $y = y(x) = \log_b x$. Use the definition of a derivative and write

$$\frac{dy}{dx} = y'(x) = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

$$\frac{dy}{dx} = y'(x) = \lim_{\Delta x \to 0} \frac{\log_b(x + \Delta x) - \log_b(x)}{\Delta x}$$ and use the properties of logarithms to write

$$\frac{dy}{dx} = y'(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \log_b \left(\frac{x + \Delta x}{x} \right)$$
$$\frac{dy}{dx} = y'(x) = \lim_{\Delta x \to 0} \frac{1}{x} \left(\frac{x}{\Delta x} \right) \log_b \left(1 + \frac{\Delta x}{x} \right)$$
$$\frac{dy}{dx} = y'(x) = \lim_{\Delta x \to 0} \frac{1}{x} \log_b \left(1 + \frac{\Delta x}{x} \right)^{x/\Delta x} \tag{2.27}$$

In equation (2.27) make the substitution $h = \dfrac{\Delta x}{x}$ and make note of the fact that $h \to 0$ as $\Delta x \to 0$ to obtain

$$\frac{dy}{dx} = y'(x) = \frac{1}{x} \lim_{h \to 0} \log_b (1 + h)^{1/h} \tag{2.28}$$

Recall from chapter 1 that $\lim_{h \to 0} (1+h)^{1/h} = e$ and use this result to simplify the equation (2.28) to the form

$$\boldsymbol{\frac{dy}{dx} = y'(x) = \frac{d}{dx} \log_b x = \frac{1}{x} (\log_b e)} \tag{2.29}$$

Observe that in the special case $b = e$ one can use the result $\log_e e = \ln e = 1$ to simplify the equation (2.29) to the following result.

$$\textbf{If} \qquad \boldsymbol{y = \ln x, \quad x > 0,} \ \textbf{then} \ \ \boldsymbol{\frac{dy}{dx} = \frac{d}{dx} \ln x = \frac{1}{x}, \qquad x \neq 0} \tag{2.30}$$

If $y = \log_b u$, where $u = u(x) > 0$, the chain rule for differentiation can be employed to obtain the results

$$\boldsymbol{\frac{d}{dx} \log_b u = \frac{d}{du} \log_b u \cdot \frac{du}{dx}} \quad \textbf{or} \quad \boldsymbol{\frac{d}{dx} \log_b u = (\log_b e) \cdot \frac{1}{u} \cdot \frac{du}{dx}} \tag{2.31}$$

and in the special case $y = \ln u,\ u = u(x) > 0$, then

$$\boldsymbol{\frac{d}{dx} \ln u = \frac{d}{du} \ln u \cdot \frac{du}{dx}} \quad \textbf{or} \quad \boldsymbol{\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}} \tag{2.32}$$

The more general situation is that for

$$\boldsymbol{y = \ln |x|,} \quad \textbf{then} \quad \boldsymbol{\frac{dy}{dx} = \frac{d}{dx} \ln |x| = \frac{1}{x}, \quad x \neq 0} \tag{2.33}$$

and if

$$\boldsymbol{y = \ln |u|,} \quad \textbf{then} \qquad \boldsymbol{\frac{dy}{dx} = \frac{d}{dx} \ln |u| = \frac{1}{u(x)} \frac{du}{dx}, \quad u(x) \neq 0} \tag{2.34}$$

In differential notation one can write

$$\boldsymbol{d \ln |u| = \frac{du}{u}} \tag{2.35}$$

Example 2-13. Find the derivatives of the following functions

$$(a) \quad y = \ln|\cos x|, \qquad (b) \quad y = \log_{10}|x|, \qquad (c) \quad y = \log_b |u(x)|$$

Solution

(a) $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln|\cos x| = \dfrac{1}{\cos x}\dfrac{d}{dx}\cos x = -\dfrac{\sin x}{\cos x} = -\tan x$

(b) $\dfrac{dy}{dx} = \dfrac{d}{dx}\log_{10}|x| = (\log_{10} e)\dfrac{1}{x}, \quad x \neq 0$

(c) $\dfrac{dy}{dx} = \dfrac{d}{dx}\log_b|u(x)| = (\log_b e)\dfrac{1}{u(x)}\dfrac{du}{dx}, \quad u(x) \neq 0$

■

Derivative of the Exponential Function

Let $y = y(x) = b^x$, with $b > 0$ denote a general exponential function. Knowing how to differentiate the logarithm function can be used to find a derivative formula for the exponential function. Recall that

$$\boldsymbol{y = b^x \quad \text{if and only if} \quad x = \log_b y} \tag{2.36}$$

Make use of the chain rule for differentiation and differentiate both sides of the equation $x = \log_b y$ with respect to x to obtain

$$\begin{aligned} \frac{d}{dx}x =& \frac{d}{dx}\log_b y \\ \frac{d}{dx}x =& \frac{d}{dy}\log_b y \cdot \frac{dy}{dx} \\ 1 =& \frac{1}{y}\log_b e \cdot \frac{dy}{dx} \end{aligned} \tag{2.37}$$

This result can be expressed in the alternative form

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{\log_b e}\, y \quad \text{or} \quad \frac{d}{dx}(b^x) = \frac{1}{\log_b e}\, b^x} \tag{2.38}$$

Using the identity[5] $\log_b e = \dfrac{1}{\ln b}$ the equation (2.38) can be expressed in the alternative form

$$\boldsymbol{\frac{d}{dx}(b^x) = (\ln b)\cdot b^x} \tag{2.39}$$

In the special case $b = e$ there results $\log_b e = \ln e = 1$, so that the equations (2.38) and (2.39) simplify to the result

$$\boldsymbol{\frac{d}{dx}e^x = e^x} \tag{2.40}$$

---

[5] Use the change of base relation $\log_b a = \frac{\log_b x}{\log_a x}$ in the special case $a = e$ and $x = b$.

Note the exponential function $y = e^x$ is the only function equal to its own derivative. Often times the exponential function $y = e^x$ is expressed using the notation $y = \exp(x)$. This is usually done whenever the exponent x is replaced by some expression difficult to typeset as an exponent. Also note that the functions $y = e^x$ and $y = \ln x$ are inverse functions having the property that

$$e^{\ln x} = x \quad \text{for } x > 0 \qquad \text{and} \qquad \ln(e^x) = x \quad \text{for all values of } x$$

If $u = u(x)$, then a generalization of the above results is obtained using the chain rule for differentiation. These generalizations are

$$\begin{aligned} &\frac{d}{dx}(b^u) = \frac{d}{du}(b^u) \cdot \frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}(b^u) = (\ln b) \cdot b^u \cdot \frac{du}{dx} \\ \text{and} \quad &\frac{d}{dx}(e^u) = \frac{d}{du}(e^u) \cdot \frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx} \end{aligned} \tag{2.41}$$

Because the exponential function $y = e^u$ is easy to differentiate, many differentiation problems are converted to this form. For example, writing $y = b^x = e^{x \ln b}$, then

$$\frac{dy}{dx} = \frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{x \ln b}\right) = e^{x \ln b} \frac{d}{dx}[x \ln b]$$

which simplifies to the result given by equation (2.39).

In differential notation, one can write

$$\begin{aligned} d\, e^u =& e^u\, du \\ d\, a^u =& a^u \ \ln a\, du \\ d\left(\frac{a^u}{\ln a}\right) =& a^u\, du, \quad 0 < a < 1 \text{ or } a > 1 \end{aligned} \tag{2.42}$$

Example 2-14. The differentiation formula $\frac{d}{dx} x^n = n x^{n-1}$ was derived for n an integer. Show that for $x > 0$ and r any real number one finds that $\frac{d}{dx} x^r = r x^{r-1}$

Solution Use the exponential function and write $y = x^r$ as $y = e^{r \ln x}$, then

$$\frac{dy}{dx} = \frac{d}{dx} e^{r \ln x} = e^{r \ln x} \frac{d}{dx}(r \ln x) = x^r \frac{r}{x} = r x^{r-1}$$

■

Example 2-15. If $y = |\sin x|$, find $\frac{dy}{dx}$

Solution Use the exponential function and write $y = |\sin x| = e^{\ln|\sin x|}$, then

$$\begin{aligned}\frac{dy}{dx} = \frac{d}{dx}e^{\ln|\sin x|} =& e^{\ln|\sin x|}\frac{d}{dx}\ln|\sin x| \\ =& |\sin x|\frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{|\sin x|}{\sin x}\cos x = \begin{cases} \cos x & \text{if} \quad \sin x > 0 \\ -\cos x & \text{if} \quad \sin x < 0 \end{cases}\end{aligned}$$

■

Example 2-16. If $y = x^{\cos x}$ with $x > 0$, find $\frac{dy}{dx}$

Solution Write $y = x^{\cos x}$ as $y = e^{(\cos x)\ln x}$, then

$$\begin{aligned}\frac{dy}{dx} = \frac{d}{dx}e^{(\cos x)\ln x} =& e^{(\cos x)\ln x}\frac{d}{dx}[(\cos x)\ln x] \\ =& x^{\cos x}\left[\cos x \cdot \frac{1}{x} + \ln x \cdot (-\sin x)\right] \\ =& x^{\cos x}\left[\frac{\cos x}{x} - (\ln x)(\sin x)\right]\end{aligned}$$

■

Example 2-17. The general power rule for differentiation is expressed

$$\boldsymbol{\frac{d}{dx}u(x)^r = ru(x)^{r-1}\frac{du}{dx}} \tag{2.43}$$

where r can be any real number. This is sometimes written as

$$\boldsymbol{\begin{aligned}\frac{d}{dx}u(x)^r =& \frac{d}{dx}e^{r\ln u(x)} \\ =& e^{r\ln u(x)}\frac{d}{dx}[r\ln u(x)] \\ =& u(x)^r \cdot r\frac{1}{u(x)}\frac{du}{dx} \\ =& ru(x)^{r-1}\frac{du}{dx}\end{aligned}} \tag{2.44}$$

which is valid whenever $u(x) \neq 0$ with $\ln|u(x)|$ and $u(x)^r$ well defined.

■

Example 2-18. The exponential function can be used to differentiate the general power function $y = y(x) = u(x)^{v(x)}$, where $u = u(x) > 0$ and $u(x)^{v(x)}$ is well defined. One can write $y = u(x)^{v(x)} = e^{v(x)\ln u(x)}$ and by differentiation obtain

$$\begin{aligned}\frac{dy}{dx} =& \frac{d}{dx} e^{v(x)\ln u(x)} \\ =& e^{v(x)\ln u(x)} \cdot \frac{d}{dx}\left[v(x)\ln u(x)\right] \\ =& u(x)^{v(x)} \cdot \left[v(x)\frac{1}{u(x)}\frac{du}{dx} + \frac{dv}{dx}\cdot \ln u(x)\right]\end{aligned}$$

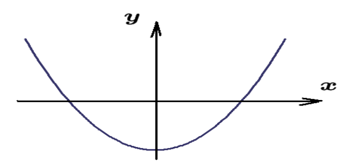
■

Derivative and Continuity

If a function $y = f(x)$ is such that both the function $f(x)$ and its derivative $f'(x)$ are continuous functions for all values of x over some interval $[a, b]$, then the function $y = f(x)$ is called **a smooth function and its graph is called a smooth curve**. A smooth function is characterized by **an unbroken curve with a continuously turning tangent** .

Example 2-19.

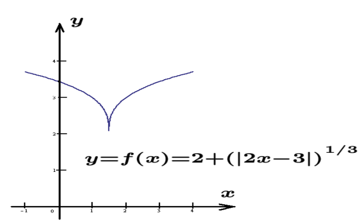
(a) The function $y = f(x) = x^2 - 4$ has the derivative $\frac{dy}{dx} = f'(x) = 2x$ which is everywhere continuous and so the graph is called a smooth curve.



$y=f(x)=x^2-4$

(b) The function

$$y = f(x) = 2 + \exp\left(\frac{1}{3}\ln \mid 2x - 3 \mid\right) = 2 + (|2x-3|)^{1/3}$$



has the derivative

$$\frac{dy}{dx} = e^{\frac{1}{3}\ln|2x-3|} \cdot \frac{2}{3(2x-3)}$$

which has a discontinuity in its derivative at the point $x = 3/2$ and so the curve is not a smooth curve.

■

Maxima and Minima

Examine the curve $y = f(x)$ illustrated in the figure 2-12 which is defined and continuous for all values of x satisfying $a \le x \le b$. Start at the point $x = a$ and move along the x-axis to the point b examining the heights of the curve $y = f(x)$ as you move left to right.

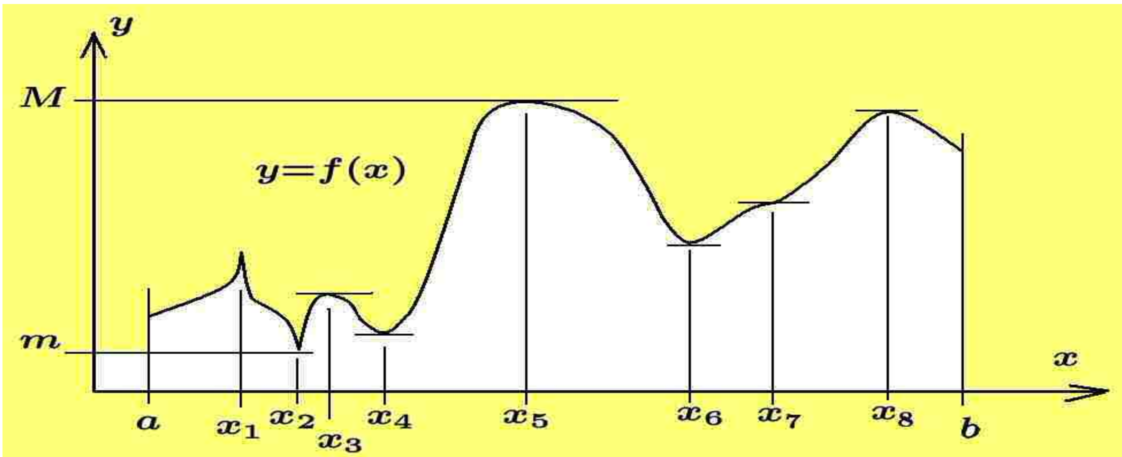


Figure 2-12. Curve $y = f(x)$ with horizontal line indicating critical points.

A local maximum or relative maximum value for $f(x)$ is said to occur at those points where in moving from left to right the height of the curve increases, then stops and begins to decrease. A local minimum or relative minimum value of $f(x)$ is said to occur at those points where in moving from left to right the height of the curve decreases, then stops and begins to increase. In figure 2-12 the points x_1, x_3, x_5, x_8 are where the function $f(x)$ has local maximum values. The points x_2, x_4, x_6 are where $f(x)$ has local minimum values. The end points where $x = a$ and $x = b$ are always tested separately for the existence of a local maximum or minimum value.

Definition: (Absolute maximum) *A function is said to have an absolute maximum M or global maximum M at a point $(x_0, f(x_0))$ if $f(x_0) \geq f(x)$ for all $x \in D$, where D is the domain of definition of the function and $M = f(x_0)$.*

Definition: (Absolute minimum) *A function is said to have an absolute minimum m or global minimum m at a point $(x_0, f(x_0))$ if $f(x_0) \leq f(x)$ for all $x \in D$, where D is the domain of definition of the function and $m = f(x_0)$.*

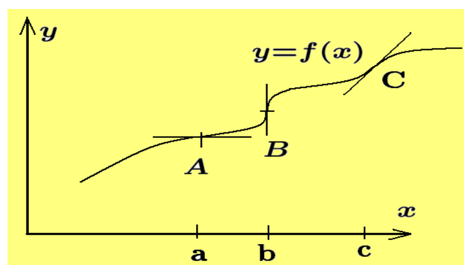
For $x \in D$ one can write $m \leq f(x) \leq M$ where m and M are referred to as extreme values of the function $y = f(x)$. In the figure 2-12 the point where $x = x_5$ gives $M = f(x_5)$ and the point where $x = x_2$ gives $m = f(x_2)$.**Note that for functions defined on a closed interval, the end points $x = a$ and $x = b$ must be tested separately for a maximum or minimum value.**

Definition: (Relative maximum) *A function is said to have a relative maximum or local maximum at a point* $(x_0, f(x_0)$ *if* $f(x_0) \geq f(x)$ *for all* x *in some open interval containing the point* x_0*.*

Definition: (Relative minimum) *A function is said to have a relative minimum or local minimum at a point* $(x_0, f(x_0))$ *if* $f(x_0) \leq f(x)$ *for all values of* x *in some open interval containing the point* x_0*.*

Concavity of Curve

If the graph of a function $y = f(x)$ is such that $f(x)$ lies above all of its tangents on some interval, then the curve $y = f(x)$ is called **concave upward** on the interval. In this case one will have throughout the arc of the curve $f''(x) > 0$ which indicates that as x moves from left to right, then $f'(x)$ is increasing. If the graph of the function $y = f(x)$ is such that $f(x)$ always lies below all of its tangents on some interval, then the curve $y = f(x)$ is said to be **concave downward** on the interval. In this case one will have throughout the arc of the curve $f''(x) < 0$, which indicates that as x moves from left to right, then $f'(x)$ is decreasing. Related to the second derivative are points known as points of inflection.



Definition: (Point of inflection)
Assume $y = f(x)$ *is a continuous function which has a first derivative* $f'(x)$ *and a second derivative* $f''(x)$ *defined in the domain of definition of the function. A point* $(x_0, f(x_0))$ *is called an* **inflection point** *if the concavity of the curve changes at that point. The second derivative* $f''(x_0)$ *may or may not equal zero at an inflection point. One can state that a point* $(x_0, f(x_0))$ *is an inflection point associated with the curve* $y = f(x)$ *if there exists a small neighborhood of the point* x_0 *such that*

$$(i) \text{ for } x < x_0, \text{ one finds } f''(x) > 0 \text{ and for } x > x_0, \text{ one finds } f''(x) < 0$$

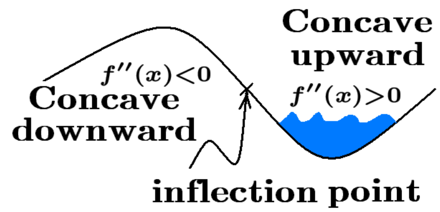
$$\text{or} \quad (ii) \text{ for } x < x_0, \text{ one finds } f''(x) < 0 \text{ and for } x > x_0, \text{ one find } f''(x) > 0$$

No local minimum value or maximum value occurs at an inflection point.

A **horizontal inflection point** is characterized by a tangent line parallel with the x-axis with $f'(a) = 0$ and $f''(a) = 0$ as illustrated by point A. A **vertical inflection point** is characterized by a tangent line parallel with the y-axis with $f'(b) = \pm\infty$ as illustrated by the point B. The point C illustrates a point of inflection where

$f'(c) \neq 0$ and $f''(c) = 0$ and the concavity changes from concave up to concave down as x increases across the inflection point.

If the curve is continuous with continuous derivatives, those points where the concavity changes, from upward to downward or from downward to upward, are inflection points with a zero second derivative.



Sections of the curve which are concave upward will hold water, while those sections that are concave downward will not hold water.

Comments on Local Maxima and Minima

Examine the figure 2-12 and make note of the following.

(1) **The words extrema (plural) or extremum (singular) are often used when referring to the maximum and minimum values associated with a given function $y = f(x)$.**

(2) At a local maximum or local minimum value the **tangent line to the curve is parallel to one of the coordinate axes.**

(3) A local maximum or local minimum value is associated with those points x where $f'(x) = 0$. The roots of the equation $f'(x) = 0$ are called **critical points**. Critical points must then be tested to see if they correspond to a local maximum, local minimum or neither, such as the point x_7 in figure 2-12.

(4) Continuous curves which have abrupt changes in their derivative at a single point are said to have **cusps at these points**. For example, the points where $x = x_1$ and $x = x_2$ in the figure 2-12 are called cusps. At these cusps one finds that either $f'(x) = \pm\infty$ or $f'(x)$ has a jump discontinuity. These points must be tested separately to determine if they correspond to local maximum or minimum values for $y = f(x)$.

(5) **The end points of the interval of definition $x = a$ and $x = b$ must be tested separately to determine if a local maximum or minimum value exists.**

(6) The conditions $f'(x) = 0$ or $f'(x) = \pm\infty$ at a point x_0 **are not sufficient conditions for an extremum value for the function** $y = f(x)$ as these conditions may produce an inflection point or an asymptotic line and so additional tests for local maximum and minimum values are needed.

(7) If the function $y = f(x)$ is continuous, then between two equal values for the function, where $f'(x)$ is not zero everywhere, at least one maximum or one minimum value must exist. One can also say that between two maximum values

there is at least one minimum value or between two minimum values there is at least one maximum value.

(8) In the neighborhood of a local maximum value, as x increases the function increases, then stops changing and starts to decrease. Similarly, in the neighborhood of a local minimum value, as x increases the function decreases, then stops changing and starts to increase. In terms of a particle moving along the curve, one can say that the particle change becomes **stationary** at a local maximum or minimum value of the function. The terminology of finding **stationary values of a function** is often used when referring to maximum and minimum problems.

First Derivative Test

The **first derivative test** for extreme values of a function tests the slope of the curve at near points on either side of a critical point. That is, to test a given function $y = f(x)$ for maximum and minimum values, one first calculates the derivative function $f'(x)$ and then solves the equation $f'(x) = 0$ to find the critical points. If x_0 is a root of the equation $f'(x) = 0$, then $f'(x_0) = 0$ and then one must examine how $f'(x)$ changes as x moves from left to right across the point x_0.

Slope Changes in Neighborhood of Critical Point

If the slope $f'(x)$ changes from

(i) $+$ to 0 to $-$, then a local maximum occurs at the critical point.

(ii) $-$ to 0 to $+$, then a local minimum occurs at the critical point.

(iii) $+$ to 0 to $+$, then a point of inflection is said to exist at the critical point.

(iv) $-$ to 0 to $-$, then a point of inflection is said to exist at the critical point.

Given a curve $y = f(x)$ for $x \in [a, b]$, the values of $f(x)$ at the end points where $x = a$ and $x = b$ must be tested separately to determine if they represents relative or absolute extreme values for the function. Also points where the slope of the curve changes abruptly, such as the point where $x = x_1$ in figure 2-12, must also be tested separately for local extreme values of the function.

Second Derivative Test

The **second derivative test** for extreme values of a function $y = f(x)$ assumes that the second derivative $f''(x)$ is continuous in the neighborhood of a critical point. One can then say in the neighborhood of a local minimum value the curve will be concave upward and in the neighborhood of a local maximum value the concavity of the curve will be downward. This gives the following second derivative test for local maximum and minimum values.

(*i*) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has relative minimum value at x_0.

(*ii*) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum value at x_0.

If $f''(x_0) = 0$, then the second derivative test fails and one must use the first derivative test. The second derivative test is often used because it is convenient. The second derivative test is not as general as the first derivative test. If the second derivative test fails, then resort back to the more general first derivative test.

Example 2-20.

Find the maximum and minimum values of the function $y = f(x) = x^3 - 3x$

Solution

The derivative of the given function is $\dfrac{dy}{dx} = \dfrac{d}{dx}x^3 - 3\dfrac{d}{dx}x = 3x^2 - 3 = f'(x)$. Setting $f'(x) = 0$ one finds

$$3(x^2-1) = 3(x-1)(x+1) = 0 \quad \text{with roots} \quad x = 1 \quad \text{and} \quad x = -1$$

being the critical points.

First derivative test

Selecting the points $x = -3/2$, $x = -1$ and $x = -1/2$ one finds that

$$f'(-3/2) = 3x^2 - 3\Big|_{x=-3/2} = \frac{15}{4}, \quad f'(-1) = 0, \quad f'(-1/2) = 3x^2 - 3\Big|_{x=-1/2} = -\frac{9}{4}$$

and so the slope of the curve changes from + to 0 to − indicating a local maximum value for the function.

Selecting the points $x = 1/2$, $x = 1$ and $x = 3/2$ one finds

$$f'(1/2) = 3x^2 - 3\Big|_{x=1/2} = -\frac{9}{4}, \quad f'(1) = 0, \quad f'(3/2) = 3x^2 - 3\Big|_{3/2} = \frac{15}{4}$$

and so the slope of the curve changes from − to 0 to + indicating a local minimum value for the function.

Second derivative test

The second derivative of the given function is $\dfrac{d^2y}{dx^2} = f''(x) = 6x$. The first and second derivatives evaluated at the critical points gives

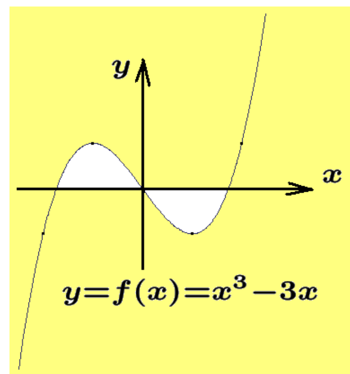
(i) at $x = -1$ one finds $f'(-1) = 0$ and $f''(-1) = 6(-1) = -6 < 0$ indicating the curve is concave downward. Therefore, the critical point $x = -1$ corresponds to a local maximum.

(ii) at $x = 1$ one finds $f'(1) = 0$ and $f''(1) = 6(1) = 6 > 0$ indicating the curve is concave upward. Therefore, the critical point $x = 1$ corresponds to a local minimum value.

Sketching the curve

The local minimum value at $x = 1$ is $f(1) = (1)^3 - 3(1) = -2$ and the local maximum value at $x = -1$ is $f(-1) = (-1)^3 - 3(-1) = 2$. Consequently, the curve passes through the points $(1, -2)$ being concave upward and it passes through the point $(-1, 2)$ being concave downward.

Select random points in the neighborhood of these points for additional information about the curve. Select the points where $x = -2$, $x = 0$ and $x = 2$ and show the points $(-2, -2)$, $(0, 0)$ and $(2, 2)$ lie on the curve. Plotting these points and connecting them with a smooth curve gives the following sketch.



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Example 2-21. Snell's Law

Refraction is the process where a light or sound wave changes direction when passing from one isotropic[6] medium to another. Examine the figure 2-13 illustrating a ray of light moving from point P in air to a point Q in water. The point P is a height h above the air-water interface and the point Q is at a depth d below the air-water interface. The normal to the air-water interface is used to define angles of incidence and refraction.

In figure 2-13 the symbol i denotes the angle of incidence and r denotes the angle of refraction of the ray of light moving from point P in air to a point Q in water and ℓ is the x-distance between the points P and Q. Fermat's law[7] states that light will travel from point P in air to point Q in water along a path POQ which

[6] Having the same physical properties in all directions.

[7] Pierre de Fermat (1601-1665) A French lawyer famous for his developing many results in number theory and calculus.

minimizes the time travel. If c_1 denotes the speed of light in air and c_2 denotes the speed of light in water one can find from tables the approximate values

$$c_1 \approx 2.99\,(10)^{10} \text{ cm/sec} \qquad\qquad c_2 \approx 2.26\,(10)^{10} \text{ cm/sec}$$

Find the relation between the angles i and r such that Fermat's law is satisfied.

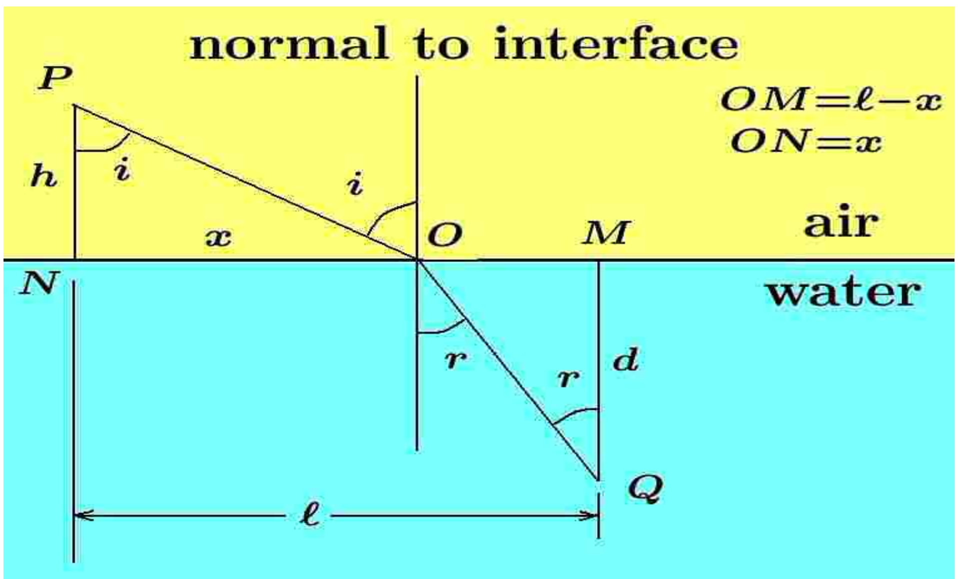


Figure 2-13. Light ray moving from point P in air to point Q in water.

Solution

Use the formula $\quad Distance = (Velocity)\,(Time) \quad$ to obtain the following values. The time of travel for light in air to move from point P to O is

$$T_{air} = \frac{\overline{PO}}{c_1} = \frac{\sqrt{x^2+h^2}}{c_1}$$

The time of travel for light in water to move from point O to Q is

$$T_{water} = \frac{\overline{OQ}}{c_2} = \frac{\sqrt{(\ell-x)^2+d^2}}{c_2}$$

The total time to travel from point P to Q is therefore

$$T = T(x) = T_{air} + T_{water} = \frac{\sqrt{x^2+h^2}}{c_1} + \frac{\sqrt{(\ell-x)^2+d^2}}{c_2}$$

Calculating the derivative one finds

$$\frac{dT}{dx} = \frac{1}{c_1}\frac{x}{\sqrt{x^2+h^2}} - \frac{1}{c_2}\frac{(\ell - x)}{\sqrt{(\ell-x)^2+d^2}}$$

If the time T has an extreme value, then $\frac{dT}{dx} = 0$ and x is required to satisfy the equation

$$\frac{1}{c_1}\frac{x}{\sqrt{x^2+h^2}} = \frac{1}{c_2}\frac{(\ell - x)}{\sqrt{(\ell-x)^2+d^2}}$$

and from this equation one can theoretically solve for the value of x which makes $T = T(x)$ have a critical value. This result can be expressed in a slightly different form. Examine the geometry in the figure 2-13 and verify that

$$\sin i = \frac{x}{\sqrt{x^2+h^2}} \qquad \text{and} \qquad \sin r = \frac{(\ell - x)}{\sqrt{(\ell-x)^2+d^2}}$$

so the condition for an extreme value can be written in the form

$$\boldsymbol{\frac{\sin i}{c_1} = \frac{\sin r}{c_2}} \tag{2.45}$$

This result is known as Snell's law.[8] Show that the second derivative simplifies to

$$\frac{d^2T}{dx^2} = \frac{1}{c_1}\frac{h^2}{(h^2+x^2)^{3/2}} + \frac{1}{c_2}\frac{d^2}{((\ell-x)^2+d^2)^{3/2}} > 0$$

By the second derivative test the critical point corresponds to a minimum value for $T = T(x)$

■

Example 2-22. Consider the function $y = f(x) = \dfrac{x^2-x+1}{x^2+x+1}$ and ask the question "Is this function defined for all values of x?" If the denominator is not zero, then one can answer yes to this question. If $x^2+x+1 = 0$, then $x = \frac{-1\pm\sqrt{1-4}}{2} = \frac{1}{2}(-1 \pm i\sqrt{3})$ which is a complex number and so for real values of x the denominator is never zero. One can then say the domain of definition for the function is $D = R$. To determine the range for the function, rewrite the function in the form $x^2(1-y) - x(1+y) + (1-y) = 0$

[8] This law was discovered by Willebrord Snell (1591-1526) a Dutch astronomer. Let c denote the speed of light in vacuum and c_m the speed of light in medium m. The ratio $n_m = c/c_m$ is called the absolute index of refraction and the more general form of Snell's law is $n_1 \sin i = n_2 \sin r$.

In order that x be a real quantity it is necessary for

$$\begin{aligned}(1+y)^2 >& 4(1-y)^2\\ 1+2y+y^2 >& 4(1-2y+y^2)\\ -3y^2+10y-3 >& 0\\ -(3y-1)(y-3) >& 0\end{aligned}$$

This requires that $\frac{1}{3} < y < 3$ which determines the range for the function.

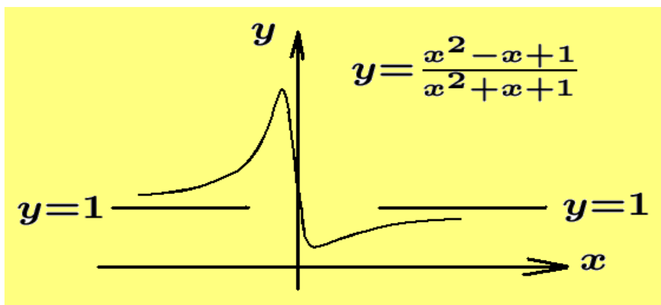


Figure 2-14. Sketch of $y = \dfrac{x^2-x+1}{x^2+x+1}$

As x increases without bound one can write

$$\lim_{x\to\infty} \frac{x^2-x+1}{x^2+x+1} = \lim_{x\to\infty} \frac{1-1/x+1/x^2}{1+1/x+1/x^2} = 1$$

so that $y=1$ is an asymptotic line.

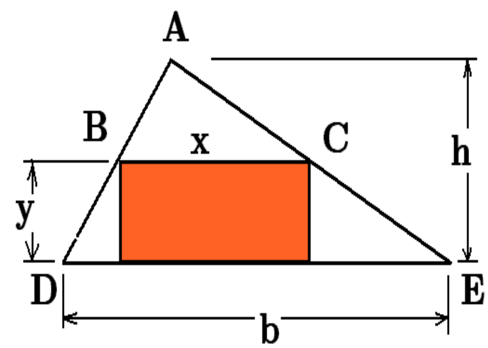
Differentiating the given function one finds

$$\frac{dy}{dx} = f'(x) = \frac{(x^2+x+1)(2x-1)-(x^2-x+1)(2x+1)}{(x^2+x+1)^2} = \frac{2(x^2-1)}{(x^2+x+1)^2}$$

The slope of the curve $f'(x)$ is zero when $x=1$ or $x=-1$. These are the critical points to be tested. One finds that at $x=1$ the height of the curve is $y=f(1)=1/3$ and when $x=-1$, the height of the curve is $y=f(-1)=3$. A sketch of the function is given in the figure 2-14. By the first derivative test for $x<-1, f'(x)>0$ and for $x>-1, f'(x)<0$ so that $x=-1$ corresponds to an absolute maximum value. It is similarly demonstrated that the point $x=1$ corresponds to an absolute minimum value.

■

Example 2-23.
Find the largest rectangle that can be inscribed in a given triangle, where the base of the rectangle lies on the base of the triangle. Let b denote the base of the triangle and let h denote the height of the triangle.



Solution

Let x denote the base of the rectangle and y the height of the rectangle, then the area of the rectangle to be maximized is given by $A = xy$. This expresses the area as a function of two variables. If y can be related to x, then the area can be expressed in terms of a single variable and the area can be differentiated. In this way one can apply the previous max-min methods for analyzing this problem. To begin, observe that the triangles ABC and ADE are similar triangles so one can write

$$\frac{h-y}{h} = \frac{x}{b} \quad \text{or} \quad x = b\frac{h-y}{h} \quad \text{or} \qquad y = h - \frac{h}{b}x$$

This gives a relationship between the values of x and y. Note that as y varies from $y = 0$ to the value $y = h$, the area $A = xy$ will vary from 0 to a maximum value and then back to 0. The area of the rectangle can now be expressed as either a function of x or as a function of y. For example, if $A = xy$, then one can write either

$$\begin{aligned} A =& x\left(h - \frac{h}{b}x\right) = hx - \frac{h}{b}x^2, \qquad 0 \le x \le b \\ \text{or} \qquad A =& b\left(\frac{h-y}{h}\right)y = \frac{b}{h}(hy - y^2), \qquad 0 \le y \le h \end{aligned}$$

These representation for the area can be differentiated to determine maximum and minimum values for the area A. Differentiating with respect to x one finds

$$\frac{dA}{dx} = h - 2\frac{h}{b}x$$

and a critical value occurs when $\frac{dA}{dx} = 0$ or $h - 2\frac{h}{b}x = 0$, which requires $x = b/2$ with $y = h/2$. Alternatively, if one differentiates with respect to y one finds

$$\frac{dA}{dy} = \frac{b}{h}(h - 2y)$$

and a critical value occurs when $\frac{dA}{dy} = 0$, or when $y = h/2$ which then gives $x = b/2$. In both cases one find the maximum area as $A = hb/4$. Note that there is only one critical point as y varies from 0 to h, since the area is zero at the end points where $y=0$ and $y=h$, the Rolle's theorem implies there must be a maximum value somewhere between. That is, if the area is a continuous function of y and A increases as y increases from 0, then the only way for A to return to zero is for it to reach a maximum value, stop and then return to zero.

■

Logarithmic Differentiation

Whenever one is confronted with functions which are represented by complicated products and quotients such as

$$y = f(x) = \frac{x^2\sqrt{3+x^2}}{(x+4)^{1/3}}$$

or functions of the form $y=f(x)=u(x)^{v(x)}$, where $u=u(x)$ and $v=v(x)$ are complicated functions, then it is recommended that you take logarithms before starting the differentiation process. For example, to differentiate the function $y=f(x)=u(x)^{v(x)}$, first take logarithms to obtain

$$\ln y = \ln\left[u(x)^{v(x)}\right] \quad \text{which simplifies to} \quad \ln y = v(x)\ln u(x)$$

The right-hand side of the resulting equation is a product function which can then be differentiated. Differentiating both sides of the resulting equation, one finds

$$\begin{aligned}\frac{d}{dx}\ln y =& \frac{d}{dx}\left[v(x)\ln u(x)\right] = v(x)\frac{d}{dx}\ln u(x) + \ln u(x)\,\frac{d}{dx}v(x)\\ \frac{1}{y}\cdot\frac{dy}{dx} =& v(x)\frac{1}{u(x)}\frac{du(x)}{dx} + \ln u(x)\cdot\frac{dv(x)}{dx}\end{aligned}$$

Solve this equation for the derivative term to obtain

$$\frac{dy}{dx} = y\cdot\left[v(x)\frac{1}{u(x)}\frac{du(x)}{dx} + \ln u(x)\cdot\frac{dv(x)}{dx}\right] \tag{2.46}$$

where y can be replaced by $u(x)^{v(x)}$.

Differentiation of Inverse Functions

Assume that $y = f(x)$ is a single-valued function of x in an interval (a, b) and the derivative function $\frac{dy}{dx} = f'(x)$ exists and is different from zero in this interval. If the inverse function $x = f^{-1}(y)$ exists, then it has the derivative

$$\boldsymbol{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}}$$

Proof

By hypothesis the function $y = f(x)$ is differentiable and the derivative is nonzero in an interval (a, b) and so one can use implicit differentiation and differentiate both sides of $y = f(x)$ with respect to y to obtain $\dfrac{d}{dy}y = \dfrac{d}{dy}f(x)$ which by the chain rule becomes

$$\frac{d}{dy}y = \frac{d}{dx}f(x) \cdot \frac{dx}{dy} \quad \text{or} \quad 1 = f'(x) \cdot \frac{dx}{dy}$$

Consequently, if $f'(x) \neq 0$, then one can write

$$\frac{dx}{dy} = \frac{1}{f'(x)} = \frac{1}{\frac{dy}{dx}}$$

An alternative way to view this result is as follows. If $y = f(x)$, then one can interchange x and y and write

$$\boldsymbol{x = f(y) \quad \textbf{and solving for } y \textbf{ obtain} \quad y = f^{-1}(x)}$$

Observe that by employing the chain rule there results

$$\frac{dx}{dy} = f'(y) \qquad \text{and} \qquad \frac{d}{dy}y = \frac{d}{dy}f^{-1}(x) = \frac{d}{dx}f^{-1}(x) \cdot \frac{dx}{dy}$$

This last equation reduces to

$$1 = \frac{d}{dx}f^{-1}(x) \cdot \frac{dx}{dy} \qquad \text{or} \qquad \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))} = \frac{d}{dx}f^{-1}(x) = f^{-1\prime}(x)$$

which gives the result

$$\boldsymbol{\frac{d}{dx}f^{-1}(x) = f^{-1\prime}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}} \tag{2.47}$$

provided that the denominator is different from zero.

Example 2-24. If $y = f(x) = \dfrac{x+1}{x}$, show that $\dfrac{dy}{dx} = f'(x) = \dfrac{-1}{x^2}$. Solving for x one finds $x = f^{-1}(y) = \dfrac{1}{y-1}$ with derivative $\dfrac{dx}{dy} = \dfrac{d}{dy}f^{-1}(y) = \dfrac{-1}{(y-1)^2}$. Note that

$$\frac{dx}{dy} = \frac{d}{dy}f^{-1}(y) = \frac{-1}{(y-1)^2} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{-1}{x^2}} = -x^2 = \frac{-1}{(y-1)^2}$$

Approached from a different point of view one finds that by interchanging x and y in the given function gives $x = f(y) = \dfrac{y+1}{y}$ and solving for y gives the inverse function $y = f^{-1}(x) = \dfrac{1}{x-1}$. This function has the derivative

$$\frac{d}{dx}f^{-1}(x) = f^{-1\prime}(x) = \frac{d}{dx}(x-1)^{-1} = \frac{-1}{(x-1)^2}$$

Using the equation (2.47) one can write

$$\frac{d}{dx}f^{-1}(x) = f^{-1\prime}(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\dfrac{-1}{\left(f^{-1}(x)\right)^2}} = -\left(f^{-1}(x)\right)^2 = \frac{-1}{(x-1)^2}$$

■

Example 2-25. If $y = f(x) = e^x$, then interchanging x and y gives $x = f(y) = e^y$ and solving for y one obtains $y = f^{-1}(x) = \ln x$. Here the functions e^x and $\ln x$ are inverse functions of one another. Differentiate $y = f^{-1}(x) = \ln x$ to obtain

$$\frac{dy}{dx} = \frac{d}{dx}f^{-1}(x) = f^{-1\prime}(x) = \frac{d}{dx}\ln x = \frac{1}{x}$$

Here $f'(x) = e^x$ and $f'(f^{-1}(x)) = e^{\ln x} = x$ and

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}\ln x = \frac{1}{x} = f^{-1\prime}(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{x}$$

■

Differentiation of Parametric Equations

If $x = x(t)$ and $y = y(t)$ are a given set of parametric equations which define y as a function of x by eliminating the parameter t and the functions $x(t)$ and $y(t)$ are continuous and differentiable, then by the chain rule one can write

$$\boldsymbol{\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}} \qquad \textbf{or} \qquad \boldsymbol{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}} \quad \textbf{provided } \boldsymbol{x'(t) \neq 0.}$$

Differentiation of the Trigonometric Functions

To derive the derivatives associated with the trigonometric functions one can utilize the trigonometric identities

$$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$$
$$\cos(A+B) - \cos(A-B) = -2\sin A \sin B$$
$$\sin A \cos B - \cos A \sin B = \sin(A-B)$$

as well as the limit relation $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ previously derived in the example 1-6.

Example 2-26. Find the derivative of $y = \sin x$ and then generalize this result to differentiate $y = \sin u(x)$ where $u = u(x)$ is an arbitrary function of x.

Solution

Using the definition of a derivative, if $y = \sin x$, then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$
$$\frac{dy}{dx} = \lim_{h\to 0} \frac{2\sin(\frac{h}{2})\cos(x+\frac{h}{2})}{h} = \lim_{h\to 0} \frac{\sin(\frac{h}{2})}{\frac{h}{2}} \lim_{h\to 0} \cos(x+\frac{h}{2}) = \cos x$$

Therefore, the derivative of the sine function is the cosine function and one can write

$$\boldsymbol{\frac{d}{dx}\sin x = \cos x} \tag{2.48}$$

Using the chain rule for differentiation this result can be generalized. If $y = \sin u$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ or $\frac{d}{dx}\sin u = \frac{d}{du}\sin u \frac{du}{dx}$ or

$$\boldsymbol{\frac{d}{dx}\sin u = \cos u \frac{du}{dx}} \tag{2.49}$$

■

Example 2-27. Some examples involving differentiation of the sine function are the following.

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)\cdot 2x \qquad \frac{d}{dx}\sin(e^{3x}) = \cos(e^{3x})\cdot 3e^{3x}$$
$$\frac{d}{dx}\sin(\alpha x+\beta) = \cos(\alpha x+\beta)\cdot\alpha \qquad \frac{d}{dx}\sin(\sqrt{x^2+1}) = \cos(\sqrt{x^2+1})\cdot\frac{x}{\sqrt{x^2+1}}$$

■

Example 2-28. Find the derivative of $y = \cos x$ and then generalize this result to differentiate $y = \cos u(x)$ where $u = u(x)$ is an arbitrary function of x.

Solution

Using the definition of a derivative, if $y = \cos x$, then

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
\frac{dy}{dx} &= \lim_{h\to 0} \frac{-2\sin(\frac{h}{2})\sin(x+\frac{h}{2})}{h} = -\lim_{h\to 0} \frac{\sin(\frac{h}{2})}{\frac{h}{2}} \cdot \lim_{h\to 0} \sin(x+\frac{h}{2}) \\
\frac{dy}{dx} &= -\sin x
\end{aligned}$$

One finds that the derivative of the cosine function is the negative of the sine function giving

$$\boldsymbol{\frac{d}{dx}\cos x = -\sin x} \tag{2.50}$$

This result can be generalized using the chain rule for differentiation to obtain the result $\frac{d}{dx}\cos u = \frac{d}{du}\cos u \frac{du}{dx}$ or

$$\boldsymbol{\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}} \tag{2.51}$$

■

Example 2-29. Some examples involving the derivative of the cosine function are the following.

$$\frac{d}{dx}\cos(x^2) = -\sin(x^2)\cdot 2x \qquad \frac{d}{dx}\cos(e^{\alpha x}) = -\sin(e^{\alpha x})\cdot e^{\alpha x}\alpha$$

$$\frac{d}{dx}\cos(\alpha x+\beta) = -\sin(\alpha x+\beta)\cdot\alpha \qquad \frac{d}{dx}\cos(\sqrt{x^3+1}) = -\sin(\sqrt{x^3+1})\cdot\frac{3x^2}{2\sqrt{x^3+1}}$$

■

Example 2-30. Find the derivative of $y = \tan x$ and then generalize this result to differentiate $y = \tan u(x)$ where $u = u(x)$ is an arbitrary function of x.

Solution

Until you get to a point where you memorize all the rules for differentiating a function and learn how to combine all these results you are restricted to using the definition of a derivative.

If $y = \tan x$, then

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{y(x+h)-y(x)}{h} = \lim_{h\to 0} \frac{\tan(x+h)-\tan x}{h}\\
\frac{dy}{dx} &= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right] = \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos x\cos(x+h)}\\
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{1}{\cos x\cos(x+h)}\\
\frac{dy}{dx} &= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}
$$

If you know the derivatives of $\sin x$ and $\cos x$ you can derive the derivative of the $\tan x$ by using the quotient rule for differentiation and write

$$
\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \frac{d}{dx}\sin x - \sin x \cdot \frac{d}{dx}\cos x}{\cos^2 x}\\
\frac{d}{dx}\tan x &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}
$$

One finds

$$
\boldsymbol{\frac{d}{dx}\tan x = \sec^2 x} \tag{2.52}
$$

The chain rule can be utilized to show $\frac{d}{dx}\tan u = \frac{d}{du}\tan u\,\frac{du}{dx}$ or

$$
\boldsymbol{\frac{d}{dx}\tan u = \sec^2 u\,\frac{du}{dx}} \tag{2.53}
$$

■

Example 2-31. Some examples involving the derivative of the tangent function are the following.

$$
\frac{d}{dx}\tan(x^2) = \sec^2(x^2)\cdot 2x \qquad \frac{d}{dx}\tan(e^{\alpha x}) = \sec^2(e^{\alpha x})\cdot e^{\alpha x}\alpha
$$

$$
\frac{d}{dx}\tan(\alpha x+\beta) = \sec^2(\alpha x+\beta)\cdot\alpha \qquad \frac{d}{dx}\tan(\sqrt{x^2+x}) = \sec^2(\sqrt{x^2+x})\cdot\frac{1+2x}{2\sqrt{x^2+x}}
$$

■

Example 2-32. Find the derivative of $y = \cot x$ and then generalize this result to differentiate $y = \cot u(x)$ where $u = u(x)$ is an arbitrary function of x.

Solution

Use the trigonometric identity $y = \cot x = \dfrac{\cos x}{\sin x}$ and write

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = \lim_{h\to 0} \frac{\cot(x+h) - \cot x}{h} \\
&= \lim_{h\to 0} \frac{\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}}{h} = \lim_{h\to 0} \frac{\cos(x+h)\sin x - \cos x \sin(x+h)}{h \sin x \sin(x+h)} \\
&= \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{-1}{\sin x \sin(x+h)} \\
\frac{d}{dx}\cot x &= \frac{-1}{\sin^2 x} = -\csc^2 x
\end{aligned}$$

so that

$$\boldsymbol{\frac{d}{dx}\cot x = -\csc^2 x} \tag{2.54}$$

Using the chain rule for differentiation one finds

$$\boldsymbol{\frac{d}{dx}\cot u(x) = -\csc^2 u(x)\,\frac{du}{dx}} \tag{2.55}$$

Example 2-33. Find the derivative of $y = \sec x$ and then generalize this result to differentiate $y = \sec u(x)$ where $u = u(x)$ is an arbitrary function of x.

Solution

Use the trigonometric identity $y = \sec x = \dfrac{1}{\cos x}$ and write

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\sec(x+h) - \sec x}{h} = \lim_{h\to 0} \frac{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}}{h} \\
&= \lim_{h\to 0} \frac{\cos x - \cos(x+h)}{h \cos x \cos(x+h)} = \lim_{h\to 0} \frac{\sin(\frac{h}{2})}{\frac{h}{2}} \cdot \lim_{h\to 0} \frac{\sin(x+\frac{h}{2})}{\cos x \cos(x+h)} \\
&= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x
\end{aligned}$$

so that

$$\boldsymbol{\frac{d}{dx}\sec x = \sec x \tan x} \tag{2.56}$$

Using the chain rule for differentiation one finds

$$\boldsymbol{\frac{d}{dx}\sec u(x) = \sec u(x) \tan u(x)\,\frac{du}{dx}} \tag{2.57}$$

Example 2-34. Find the derivative of $y = \csc x$ and then generalize this result to differentiate $y = \csc u(x)$ where $u = u(x)$ is an arbitrary function of x.

Solution

Use the trigonometric identity $y = \csc x = \dfrac{1}{\sin x}$ and write

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\csc(x+h) - \csc x}{h} = \lim_{h\to 0} \frac{\frac{1}{\sin(x+h)} - \frac{1}{\sin x}}{h} \\
&= \lim_{h\to 0} \frac{\sin x - \sin(x+h)}{h \sin x \sin(x+h)} = \lim_{h\to 0} \frac{\sin(\frac{h}{2})}{\frac{h}{2}} \cdot \lim_{h\to 0} \frac{-\cos(x+\frac{h}{2})}{\sin x \sin(x+h)} \\
&= -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x}\frac{\cos x}{\sin x} = -\csc x \cot x
\end{aligned}$$

so that

$$\boldsymbol{\frac{d}{dx} \csc x = -\csc x \cot x} \tag{2.58}$$

Using the chain rule, show that

$$\boldsymbol{\frac{d}{dx} \csc u(x) = -\csc u(x) \cot u(x) \frac{du}{dx}} \tag{2.59}$$

Example 2-35. Some curves are easily expressed in terms of a parameter. For example, examine the figure 2-15 which illustrates a circle with radius a which rolls without slipping along the x-axis. On this circle there is attached a fixed arm of length $0P = r$, which rotates with the circle. At the end of the arm is a point P which sweeps out a curve as the circle rolls without slipping. This arm initially lies on the y-axis and the coordinates of the point P in this initial position is $(0, -(r-a))$. As the circle rolls along the x-axis without slipping, the point P has coordinates (x, y). From the geometry of the problem the coordinates of point P in terms of the parameter θ are given by

$$x = a\theta - x_0 = a\theta - r\sin\theta \qquad\qquad y = a + y_0 = a - r\cos\theta \tag{2.60}$$

The term $a\theta$ in the parametric equations (2.60) represents arc length as the circle rolls and the terms $r\sin\theta$ and $r\cos\theta$ represent projections of the arm onto the x and y axes respectively.

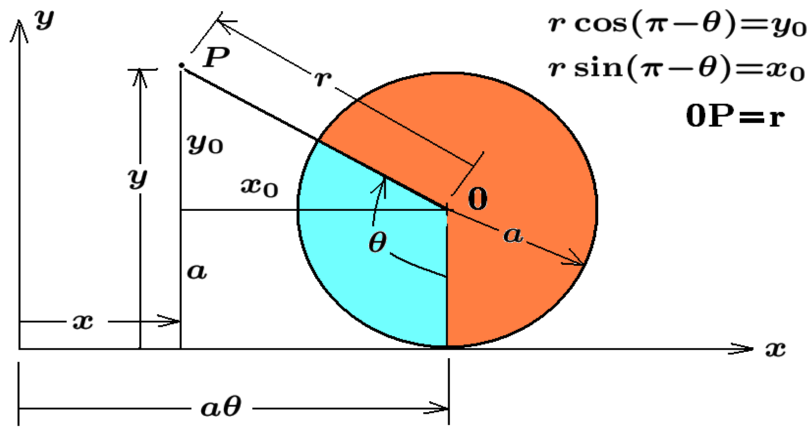


Figure 2-15. Circle rolling without slipping.

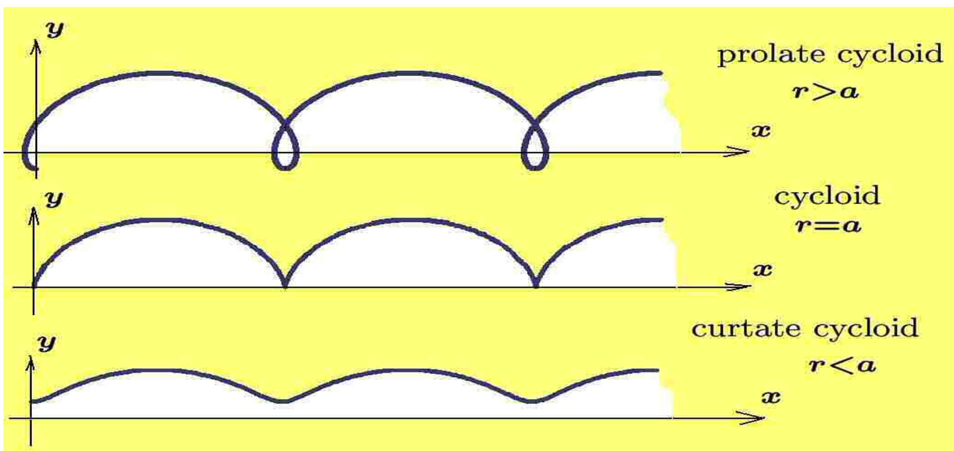


Figure 2-16. Cycloid curves for $r = \frac{3}{2}a$, $r = a$ and $r = \frac{a}{2}$

The curve that the point P sweeps out as the circle rolls without slipping has different names depending upon whether $r > a$, $r = a$ or $r < a$, where a is the radius of the circle. These curves are called

a prolate cycloid if $r > a$

a cycloid if $r = a$

a curtate cycloid if $r < a$

These curves are illustrated in the figure 2-16.

To construct a tangent line to some point (x_0, y_0) on one of the cycloids, one must be able to find the slope of the curve at this point. Using chain rule differentiation one finds $\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta}$ or

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}, \qquad \text{where} \quad \frac{dx}{d\theta} = a - r\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = r\sin\theta$$

The point (x_0, y_0) on the cycloid corresponds to some value θ_0 of the parameter. The slope of the tangent line at this point is given by

$$m_t = \frac{dy}{dx} = \frac{r\sin\theta}{a - r\cos\theta}\Big|_{\theta=\theta_0}$$

and the equation of the tangent line at this point is $y - y_0 = m_t(x - x_0)$.

■

Simple Harmonic Motion

If the motion of a particle or center of mass of a body can be described by either of the equations

$$y = y(t) = A\cos(\omega t - \phi_0) \qquad \text{or} \qquad y = y(t) = A\sin(\omega t - \phi_0) \tag{2.61}$$

where A, ω and ϕ_0 are constants, then the particle or body is said to undergo a **simple harmonic motion**. This motion is **periodic** with least period $T = 2\pi/|\omega|$. The **amplitude of the motion** is $|A|$ and the quantity ϕ_0 is called a **phase constant or phase angle**.

Note 1: By changing the phase constant, one of the equations (2.61) can be transformed into the other. For example,

$$A\sin(\omega t - \phi_0) = A\cos\left[(\omega t - \phi_0) - \pi/2\right] = A\cos(\omega t - \theta_0), \qquad \theta_0 = \phi_0 + \pi/2$$

and similarly

$$A\cos(\omega t - \phi_0) = A\sin\left[(\omega t - \phi_0) - \pi/2\right] = A\sin(\omega t - \theta_0), \qquad \theta_0 = \phi_0 + \pi/2$$

Note 2: Particles having the equation of motion

$$y = y(t) = \alpha\sin\omega t + \beta\cos\omega t$$

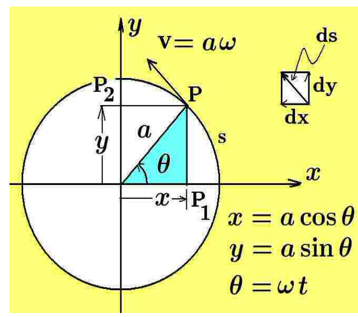
where α, β and ω are constants, can be written in the form of either equation in (2.61) by multiplying both the numerator and denominator by $\sqrt{\alpha^2+\beta^2}$ to obtain

$$y = y(t) = \sqrt{\alpha^2+\beta^2}\left[\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\sin\omega t + \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\cos\omega t\right] \tag{2.62}$$

The substitutions $A = \sqrt{\alpha^2+\beta^2}$, $\sin\phi_0 = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$, $\cos\phi_0 = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$ reduces equation (2.62) to the form

$$y = y(t) = A\left(\sin\phi_0 \sin\omega t + \cos\phi_0\cos\omega t\right) = A\cos(\omega t - \phi_0)$$

Example 2-36.



Consider a particle P moving around a circle of radius a with constant angular velocity ω. The points P_1 and P_2 are the projections of P onto the x and y axes. The distance of these points from the origin are described by the x and y-positions of the particle and are given by $x = x(t) = a\cos\omega t$ and $y = y(t) = a\sin\omega t$. Here both P_1 and P_2 exhibit a simple harmonic motion about the origin as the particle P moves counterclockwise about the circle. This simple harmonic motion has a time period $2\pi/\omega$ and amplitude a.

The derivatives $\dfrac{dx}{dt} = x'(t) = -a\omega\sin\omega t$ and $\dfrac{dy}{dt} = y'(t) = a\omega\cos\omega t$ represent the velocities of the points P_1 and P_2. These velocities can be used to determine the velocity of the particle P on the circle. Velocity is the change in distance with respect to time. If $s = a\theta$ is the distance traveled by the particle along the circle, then $v = \dfrac{ds}{dt} = a\dfrac{d\theta}{dt} = a\omega$ is the velocity of the particle. This same result can be obtained from the following analysis. The quantity $dx = -a\omega\sin\omega t\,dt$ represents a small change of P in x-direction and the quantity $dy = a\omega\cos\omega t\,dt$ represents a small change of P in the y-direction. One can define an element of arc length squared given by $ds^2 = dx^2 + dy^2$. This result can be represented in the form

$$ds = \sqrt{(dx)^2+(dy)^2} \implies \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{2.63}$$

and when the derivatives $x'(t)$ and $y'(t)$ are substituted into equation (2.63) there results $v = \dfrac{ds}{dt} = a\omega$. The second derivatives of $x = x(t)$ and $y = y(t)$ are found to be

$$\frac{d^2x}{dt^2} = \ddot{x} = -a\omega^2\cos\omega t \qquad \text{and} \qquad \frac{d^2y}{dt^2} = \ddot{y} = -a\omega^2\sin\omega t \tag{2.64}$$

which can be written in the form

$$\ddot{x} = -\omega^2 x \qquad \text{and} \qquad \ddot{y} = -\omega^2 y \tag{2.65}$$

This shows that one of the characteristics of simple harmonic motion is that the magnitude of the acceleration of either the point P_1 or P_2 is **always proportional to the displacement from the origin** and the **direction of the acceleration is always opposite to that of the displacement**.

■

L´Hôpital's Rule

One form of L´Hôpital's[9] rule, used to evaluate the indeterminate form $\frac{0}{0}$, is the following. If $f(x)$ and $g(x)$ are both differentiable functions and satisfy the properties

$$\lim_{x\to x_0} f(x) = 0, \qquad \lim_{x\to x_0} g(x) = 0, \qquad \lim_{x\to x_0} g'(x) \neq 0,$$

then one can write

$$\boldsymbol{\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}} \tag{2.66}$$

provided the limit on the right-hand side exists.

The proof of the above statement is obtained by using the definition of the derivative and properties of the limiting process. One can write

$$\begin{aligned}\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \frac{f'(x_0)}{g'(x_0)} &= \frac{\displaystyle\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}}{\displaystyle\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}} \\ &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{x\to x_0} \frac{f(x)}{g(x)}\end{aligned}$$

because $f(x_0) = g(x_0) = 0$ by hypothesis.

The L´Hôpital's rule can also be used to evaluate the indeterminate form $\frac{\infty}{\infty}$. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then one can write

$$\boldsymbol{\lim_{x\to \infty} \frac{f(x)}{g(x)} = \lim_{x\to \infty} \frac{f'(x)}{g'(x)}} \tag{2.67}$$

To show this is true make the substitution $x = 1/t$ so that as $x \to \infty$, then $t \to 0$ and write

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{f(1/t)}{g(1/t)}, \qquad t > 0 \tag{2.68}$$

[9] Guillaume François Antoine Marquis L´Hôpital (1661-1704) French mathematician who wrote the first calculus book. L´Hôpital's name is sometimes translated as L'Hospital with the s silent.

and then apply L´Hôpital's rule to the right-hand side of equation (2.68) to obtain

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{t\to 0}\frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)}=\lim_{t\to 0}\frac{f'(1/t)}{g'(1/t)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

Still another form of L´Hôpital's rule is that if x_0 is a finite real number and

$$\lim_{x\to x_0} f(x)=\infty \qquad \text{and} \qquad \lim_{x\to x_0} g(x)=\infty$$

then one can write

$$\boldsymbol{\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}} \tag{2.69}$$

This result can be established using the result from equation (2.67). Make the substitution $x=x_0+1/t$ so that as $x\to x_0$, then $t\to\infty$ and write

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{t\to\infty}\frac{f(x_0+1/t)}{g(x_0+1/t)}=\lim_{t\to\infty}\frac{F(t)}{G(t)} \tag{2.70}$$

Applying L´Hôpital's rule from equation (2.67) one finds

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{t\to\infty}\frac{F(t)}{G(t)}=\lim_{t\to\infty}\frac{F'(t)}{G'(t)}=\lim_{t\to\infty}\frac{f'(x_0+1/t)(-1/t^2)}{g'(x_0+1/t)(-1/t^2)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

Note that sometimes L´Hôpital's rule must be applied multiple times. That is, if $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ is an indeterminate form, then apply L´Hôpital's rule again and write

$$\boldsymbol{\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=\lim_{x\to x_0}\frac{f''(x)}{g''(x)}}$$

Example 2-37. Find $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Solution Use L´Hôpital's rule multiple times and write

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{-(-\sin x)}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

■

Make note of the fact that the functions that are used in equations (2.66), (2.67), and (2.69) can themselves be derivatives.

One final note about L´Hôpital's rule. There may occur limits[10] where a repeated application of L´Hôpital's rule puts you into an infinite loop and in such cases alternative methods for determining the limits must be employed.

[10] For example, L´Hôpital's rule applied to $\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x}$ produces an infinite loop.

Example 2-38.

(a) Evaluate the limit $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

Solution Using the L´Hôpital's rule one finds $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{x\to 0} \dfrac{\cos x}{1} = 1$

(b) Evaluate the limit $\lim\limits_{x\to \infty} \dfrac{\ln x}{x}$

Solution By L´Hôpital's rule $\lim\limits_{x\to \infty} \dfrac{\ln x}{x} = \lim\limits_{x\to \infty} \dfrac{1/x}{1} = 0$

(c) Evaluate the limit $\lim\limits_{x\to 0} \dfrac{\ln(\sin x)}{\ln(\tan x)}$

Solution By L´Hôpital's rule $\lim\limits_{x\to 0} \dfrac{\ln(\sin x)}{\ln(\tan x)} = \lim\limits_{x\to 0} \dfrac{\dfrac{1}{\sin x}\cdot \cos x}{\dfrac{1}{\tan x}\cdot \sec^2 x} = \lim\limits_{x\to 0} \cos^2 x = 1$

■

Example 2-39. Use L´Hôpital's rule to show $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{n}\right)^n = e$

Solution Write $\left(1+\dfrac{1}{n}\right)^n = e^{n\ln(1+\frac{1}{n})}$, then by L´Hôpital's rule

one can show that $\lim\limits_{x\to\infty} \dfrac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = 1$

■

Differentiation of Inverse Trigonometric Functions

Examine the inverse trigonometric functions illustrated in the figures 1-18 and 1-19 presented in chapter 1. Observe that these functions are multi-valued functions and because of this their derivatives depend upon which branch of the function you are dealing with. In the example 1-5 a branch was assigned to each inverse trigonometric function. You are not restricted to use these branches all the time. In using mathematics to solve applied problems it is customary to select the branch of the inverse trigonometric function which is applicable to the kind of problem you are solving. In this section derivations for the derivatives of the inverse trigonometric functions will be given for all possible branches that you might want to deal with.

By definition $y = \sin^{-1} u$ is equivalent to $\sin y = u$ and the branch where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ has been selected. Hence, to find the derivative of $y = \sin^{-1} u$ one can differentiate instead the equivalent relationship $\sin y = u$. Differentiating with respect to x one finds

$$\frac{d}{dx}\sin y = \frac{du}{dx} \quad \text{or} \quad \cos y \frac{dy}{dx} = \frac{du}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\cos y}\frac{du}{dx}$$

Consequently,

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1} u = \frac{1}{\cos y}\frac{du}{dx}$$

Observe that $u = \sin y$ is related to $\cos y$, since $\sin^2 y + \cos^2 y = 1$ so that one can write

$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - u^2} \tag{2.71}$$

The sign selected for the square root function depends upon where y is located. If $y = \sin^{-1} u$ is restricted to the first and fourth quadrant, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, then $\cos y$ is positive and so the plus sign is selected for the square root. However, if $y = \sin^{-1} u$ is restricted to the second or third quadrant, where $\frac{\pi}{2} < \sin^{-1} u < \frac{3\pi}{2}$, then the function $\cos y$ is negative and so the minus sign is selected for the square root function. This gives the following differentiation formula for the function $y = \sin^{-1} u = \arcsin u$

$$\boldsymbol{\frac{d}{dx}\arcsin u = \frac{d}{dx}\sin^{-1} u = \begin{cases} \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, & |u| < 1, \quad -\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2} \\ \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}, & |u| < 1, \quad \frac{\pi}{2} < \sin^{-1} u < \frac{3\pi}{2} \end{cases}} \tag{2.72}$$

To differentiate the function $y = \arccos u = \cos^{-1} u$ write the equivalent statement $\cos y = u$ and then differentiate both sides of this equivalent equation with respect to x to obtain

$$\frac{d}{dx}\cos y = -\sin y \frac{dy}{dx} = \frac{du}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{-1}{\sin y}\frac{du}{dx}$$

Here $u = \cos y$ and $\sin y$ are related by

$$\sin y = \pm\sqrt{1 - \cos^2 u} = \pm\sqrt{1 - u^2}$$

where the sign assigned to the square root function depends upon where y lies. If $y = \cos^{-1} u$ lies in the first or second quadrant, then $\sin y$ is positive and so the plus sign is selected. If $y = \cos^{-1} u$ is the third or fourth quadrant, then $\sin y$ is negative and so the minus sign is selected. One can then show that

$$\boldsymbol{\frac{d}{dx}\arccos u = \frac{d}{dx}\cos^{-1} u = \begin{cases} \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}, & |u| < 1, \quad 0 < \cos^{-1} u < \pi \\ \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, & |u| < 1, \quad \pi < \cos^{-1} u < 2\pi \end{cases}} \tag{2.73}$$

In a similar fashion, if $y = \arctan u = \tan^{-1} u$, then write $\tan y = u$ and differentiate both sides of this equation with respect to x and show

$$\frac{d}{dx}\tan y = \sec^2 y \frac{dy}{dx} = \frac{du}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\sec^2 y}\frac{du}{dx}$$

Use the identity $1+\tan^2 y = \sec^2 y = 1+u^2$ and show

$$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx} \tag{2.74}$$

This result holds independent of which quadrant the angle $y = \tan^{-1} u$ lies in.

In a similar fashion one can derive the derivative formulas for the inverse functions $\cot^{-1} u$, $\sec^{-1} u$ and $\csc^{-1} u$. One finds

$$\frac{d}{dx}\cot^{-1} u = \frac{-1}{1+u^2}\frac{du}{dx} \tag{2.75}$$

a result which holds independent of which quadrant the angle $y = \cot^{-1} u$ lies in. The derivatives for the inverse secant and cosecant functions are found to be

$$\frac{d}{dx}\sec^{-1} u = \begin{cases} \dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}, & 0 < \sec^{-1} u < \dfrac{\pi}{2} \quad \text{or} \quad \pi < \sec^{-1} u < \frac{\pi}{2} \\ \dfrac{-1}{u\sqrt{u^2-1}}\dfrac{du}{dx}, & \dfrac{\pi}{2} < \sec^{-1} u < \pi \quad \text{or} \quad \frac{3\pi}{2} < \sec^{-1} u < 2\pi \end{cases} \tag{2.76}$$

$$\frac{d}{dx}\csc^{-1} u = \begin{cases} \dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}, & \dfrac{\pi}{2} < \csc^{-1} u < \pi \quad \text{or} \quad \frac{3\pi}{2} < \csc^{-1} u < 2\pi \\ \dfrac{-1}{u\sqrt{u^2-1}}\dfrac{du}{dx}, & 0 < \csc^{-1} u < \dfrac{\pi}{2} \quad \text{or} \quad \pi < \csc^{-1} u < \frac{3\pi}{2} \end{cases} \tag{2.77}$$

Hyperbolic Functions and their Derivatives

The hyperbolic functions were introduced around the year 1760 by the mathematicians Vincenzo Riccati[11] and Johan Heinrich Lambert.[12] These functions were previously defined in terms of the exponential functions e^x and e^{-x}. Hyperbolic functions occur in many areas of physics, engineering and related sciences. Recall that these functions are defined

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} = \frac{e^{2x}-1}{2e^x} && \text{hyperbolic sine function} \\ \cosh x &= \frac{e^x + e^{-x}}{2} = \frac{e^{2x}+1}{2e^x} && \text{hyperbolic cosine function} \\ \tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1} && \text{hyperbolic tangent function} \end{aligned} \tag{2.78}$$

[11] Vincenzo Riccati (1707-1775) An Italian mathematician.
[12] Johan Heinrich Lambert (1728-1777) A French mathematician.

Analogous to the definition of the trigonometric functions the $\coth x$, $\operatorname{sech} x$ and $\operatorname{csch} x$ are defined.

$$
\begin{aligned}
\coth x =& \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1} && \text{hyperbolic cotangent function}\\
\operatorname{sech} x =& \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x}+1} && \text{hyperbolic secant function}\\
\operatorname{csch} x =& \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} = \frac{2e^x}{e^{2x}-1} && \text{hyperbolic cosecant function}
\end{aligned}
\tag{2.79}
$$

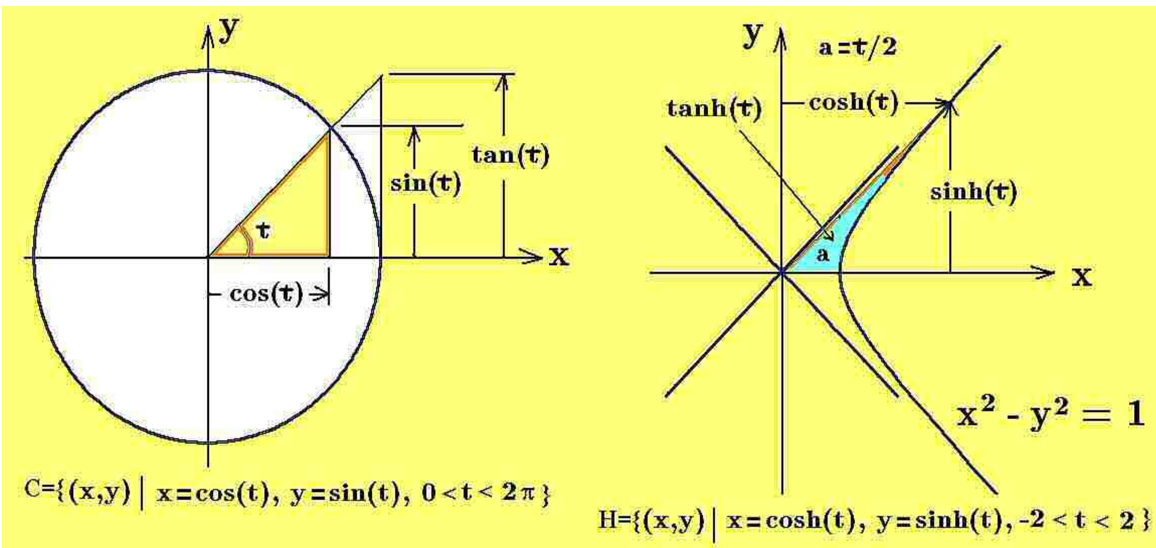


Figure 2-17. Circular functions and hyperbolic functions.

The set of points $C = \{\,(x,y) \mid x = \cos(t),\ y = \sin(t),\ 0 < t < 2\pi\,\}$ defines a circle of unit radius centered at the origin as illustrated in the figure 2-17. The parameter t has the physical significance of representing an angle of rotation. This representation for the circle gives rise to the terminology of calling trigonometric functions circular functions. In a similar fashion, the set of points

$$H = \{\,(x,y) \mid x = \cosh(t),\ y = \sinh(t),\ t \in R\,\}$$

defines the right-half of the equilateral hyperbola defined by $x^2 - y^2 = 1$ as illustrated in the figure 2-17. If the point $(x,y) = (\cosh(t), \sinh(t))$ on the hyperbola is connected to the origin by a straight line, then an area between the line, the x-axis and the hyperbola is formed having an area a. The physical significance of the parameter t in this representation is that $t = 2a$. This representation gives rise to the terminology of calling the functions in equations (2.78) and (2.79) the hyperbolic functions. Graphs of the hyperbolic functions, defined by the equations (2.79), are illustrated in the figures 2-18 and 2-19.

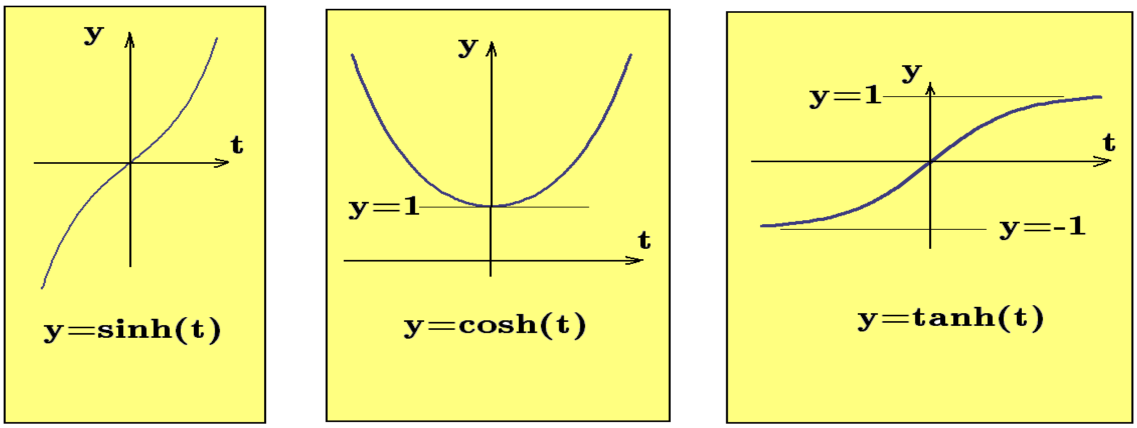


Figure 2-18. Hyperbolic functions $\sinh(t)$, $\cosh(t)$, $\tanh(t)$.

An examination of the hyperbolic functions illustrated in the figures 2-18 and 2-19 show that

$$\begin{aligned} \sinh(-x) &= -\sinh(x) & \qquad \operatorname{csch}(-x) &= -\operatorname{csch}(x) \\ \cosh(-x) &= \cosh(x) & \qquad \operatorname{sech}(-x) &= \operatorname{sech}(x) \\ \tanh(-x) &= -\tanh(x) & \qquad \coth(-x) &= -\coth(x) \end{aligned} \tag{2.80}$$

which shows that the functions $\cosh(x)$ and $\operatorname{sech}(x)$ are even function of x symmetric about the y-axis and the functions $\sinh(x)$, $\tanh(x)$, $\operatorname{csch}(x)$ and $\coth(x)$ are odd functions of x being symmetric about the origin.

Approximations

For large values of $|x|$, with $x > 0$

$$\cosh x \approx \sinh x \approx \frac{1}{2}e^{x}$$

$$\tanh x \approx \coth x \approx 1$$

$$\operatorname{sech} x \approx \operatorname{csch} x \approx 2e^{-x} \approx 0$$

For large values of $|x|$, with $x < 0$

$$\cosh x \approx -\sinh x \approx \frac{1}{2}e^{-x}$$

$$\tanh x \approx \coth x \approx -1$$

$$\operatorname{sech} x \approx -\operatorname{csch} x \approx 2e^{x} \approx 0$$

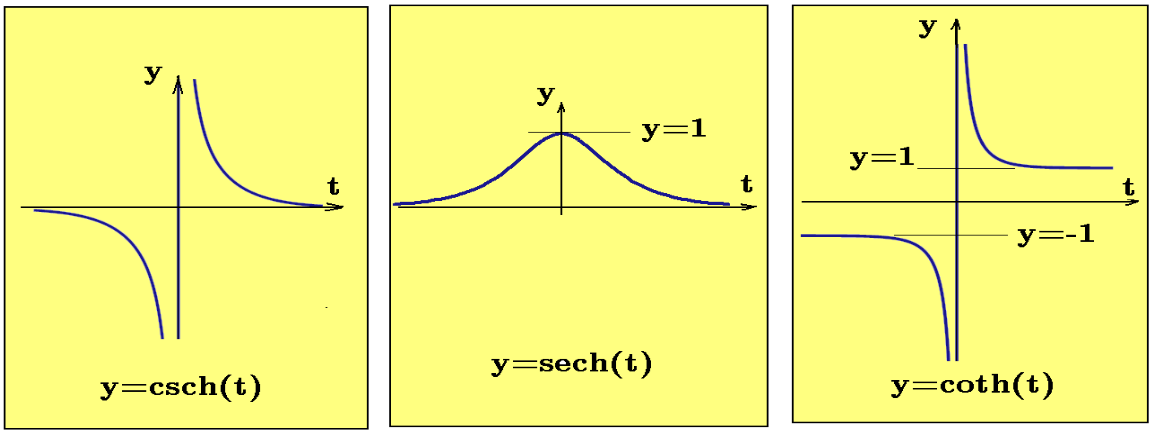


Figure 2-19. Hyperbolic functions $\operatorname{csch}(t)$, $\operatorname{sech}(t)$, $\coth(t)$

Hyperbolic Identities

One can readily show that the hyperbolic functions satisfy many properties similar to the trigonometric identities. For example, one can use algebra to verify that

$$\cosh x + \sinh x = e^{x} \qquad \text{and} \qquad \cosh x - \sinh x = e^{-x} \tag{2.81}$$

and

$$\cosh^2 x - \sinh^2 x = 1, \qquad 1 - \operatorname{sech}^2 x = \tanh^2 x, \qquad 1 + \operatorname{csch}^2 x = \coth^2 x \tag{2.82}$$

Algebra can also be used to prove the addition formula

$$\begin{aligned} \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\ \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \\ \tanh(x+y) &= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \end{aligned} \tag{2.83}$$

Example 2-39. Show that $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

Solution

Use the law of exponents and show that

$$\sinh(x+y) = \frac{e^{x+y} - e^{-x-y}}{2} = \frac{e^{x}e^{y} - e^{-x}e^{-y}}{2}$$

Now use the results from the equations (2.81) to show

$$\sinh(x+y) = \frac{1}{2}\left[(\cosh x + \sinh x)(\cosh y + \sinh y) - (\cosh x - \sinh x)(\cosh y - \sinh y)\right]$$

which when expanded simplifies to the desired result.

■

Replacing y by $-y$ in the equations (2.83) produces the difference expansions

$$\begin{aligned}
\sinh(x-y) &= \sinh x \cosh y - \cosh x \sinh y \\
\cosh(x-y) &= \cosh x \cosh y - \sinh x \sinh y \\
\tanh(x-y) &= \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}
\end{aligned} \tag{2.84}$$

Substituting $y = x$ in the equations (2.83) produces the results

$$\begin{aligned}
\sinh(2x) &= 2\sinh x \cosh x \\
\cosh(2x) &= \cosh^2 x + \sinh^2 y = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x \\
\tanh(2x) &= \frac{2\tanh x}{1 + \tanh^2 x}
\end{aligned} \tag{2.85}$$

It is left for the exercises to verify the additional relations

$$\begin{aligned}
\sinh x + \sinh y &= 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right) \\
\cosh x + \cosh y &= 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right) \\
\tanh x + \tanh y &= \frac{\sinh(x+y)}{\cosh x \cosh y}
\end{aligned} \tag{2.86}$$

$$\begin{aligned}
\sinh x - \sinh y &= 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right) \\
\cosh x - \cosh y &= 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right) \\
\tanh x - \tanh y &= \frac{\sinh(x-y)}{\cosh x \cosh y}
\end{aligned} \tag{2.87}$$

$$\begin{aligned}
\sinh\left(\frac{x}{2}\right) &= \sqrt{\frac{1}{2}(\cosh x - 1)} \\
\cosh\left(\frac{x}{2}\right) &= \sqrt{\frac{1}{2}(\cosh x + 1)} \\
\tanh\left(\frac{x}{2}\right) &= \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}
\end{aligned} \tag{2.88}$$

Euler's Formula

Sometime around the year 1790 the mathematician Leonhard Euler[13] discovered the following relation

$$\boldsymbol{e^{ix} = \cos x + i \sin x} \tag{2.89}$$

where i is an imaginary unit with the property $i^2 = -1$. This formula is known as **Euler's formula** and is one of the most important formulas in all of mathematics. The Euler formula can be employed to make a connection between the trigonometric functions and the hyperbolic functions.

In order to prove the Euler formula given by equation (2.89) the following result is needed.

If a function $f(x)$ has a derivative $f'(x)$ which is everywhere zero within an interval, then the function $f(x)$ must be a constant for all values of x within the interval.

The above result can be proven using the mean-value theorem considered earlier. If $f'(c) = 0$ for all values c in an interval and $x_1 \neq x_2$ are arbitrary points within the interval, then the mean-value theorem requires that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

This result implies that $f(x_1) = f(x_2)$ for all values $x_1 \neq x_2$ in the interval and hence $f(x)$ must be a constant throughout the interval.

To prove the Euler formula examine the function

$$F(x) = (\cos x - i \sin x)e^{ix}, \qquad i^2 = -1 \tag{2.90}$$

where i is an imaginary unit, which is treated as a constant. Differentiate this product and show

$$\begin{aligned} \frac{d}{dx}F(x) = F'(x) =& (\cos x - i \sin x)e^{ix} i + (-\sin x - i \cos x)e^{ix} \\ F'(x) =& \left[i \cos x + \sin x - \sin x - i \cos x\right] e^{ix} = 0 \end{aligned} \tag{2.91}$$

Since $F'(x) = 0$ for all values of x, then one can conclude that $F(x)$ must equal a constant for all values of x. Substituting the value $x = 0$ into equation (2.90) gives

$$F(0) = (\cos 0 - i \sin 0)e^{i0} = 1$$

[13] Leonhard Euler (1707-1783) A famous Swiss mathematician.

so that the constant value is 1 and consequently

$$1 = (\cos x - i\sin x)e^{ix} \tag{2.92}$$

Multiply both sides of equation (2.92) by $(\cos x + i\sin x)$ and show

$$\begin{aligned}
\cos x + i\sin x =& (\cos x + i\sin x)(\cos x - i\sin x)e^{ix}\\
\cos x + i\sin x =& (\cos^2 x + \sin^2 x)e^{ix}\\
\cos x + i\sin x =& e^{ix}
\end{aligned}$$

which is the Euler formula given by equation (2.89).

Note that if

$$\boldsymbol{e^{ix} = \cos x + i\sin x} \tag{2.93}$$

$$\textbf{then} \qquad \boldsymbol{e^{-ix} = \cos x - i\sin x} \tag{2.94}$$

since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$. Adding and subtracting the above equations produces the results

$$\boldsymbol{\sin x = \frac{e^{ix} - e^{-ix}}{2i}} \qquad \textbf{and} \qquad \boldsymbol{\cos x = \frac{e^{ix} + e^{-ix}}{2}} \tag{2.95}$$

Examine the equations (2.95) and then examine the definitions of the hyperbolic sine and hyperbolic cosine functions to obtain the immediate result that

$$\boldsymbol{i\sin x = \sinh(ix)} \qquad \textbf{and} \qquad \boldsymbol{\cos x = \cosh(ix)} \tag{2.96}$$

which states that complex values of the hyperbolic sine and cosine functions give relations involving the trigonometric functions sine and cosine. Replacing x by ix in the equations (2.96) produces the results

$$\boldsymbol{\sinh x = -i\sin(ix)} \qquad \textbf{and} \qquad \boldsymbol{\cosh x = \cos(ix)} \tag{2.97}$$

The results from equations (2.96) and (2.97) together with the definition of the hyperbolic functions gives the additional relations

$$\boldsymbol{\tanh x = -i\tan(ix)} \qquad \textbf{and} \qquad \boldsymbol{\tanh(ix) = i\tan x} \tag{2.98}$$

Properties of these complex functions are examined in more detail in a course dealing with complex variables and their applications.

Example 2-40. Using the above definitions one can show that an alternative form of de Moivre's theorem is

$$(\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx$$

■

Derivatives of the Hyperbolic Functions

For $u = u(x)$ an arbitrary function of x, the derivatives of the exponential functions

$$\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx} \qquad \text{and} \qquad \frac{d}{dx}e^{-u} = -e^{-u} \cdot \frac{du}{dx} \tag{2.99}$$

can be used to calculate the derivatives of the hyperbolic functions, since they are all defined in terms of exponential functions.

Example 2-41. Find the derivatives of the functions $\sinh u$ and $\cosh u$ where $u = u(x)$ is a function of x.

Solution

Use the definitions of the hyperbolic sine and cosine functions and write

$$\begin{aligned}
\frac{d}{dx}\sinh u =& \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) = \frac{e^u + e^{-u}}{2} \cdot \frac{du}{dx} = \cosh u \cdot \frac{du}{dx} \\
\frac{d}{dx}\cosh u =& \frac{d}{dx}\left(\frac{e^u + e^{-u}}{2}\right) = \frac{e^u - e^{-u}}{2} \cdot \frac{du}{dx} = \sinh u \cdot \frac{du}{dx}
\end{aligned}$$

■

Following the above example, the derivatives of all the hyperbolic functions can be calculated. One can verify that the following results are obtained.

$$\begin{aligned}
\frac{d}{dx}\sinh u &= \cosh u \cdot \frac{du}{dx} & \frac{d}{dx}\operatorname{csch} u &= -\operatorname{csch} u \coth u \cdot \frac{du}{dx} \\
\frac{d}{dx}\cosh u &= \sinh u \cdot \frac{du}{dx} & \frac{d}{dx}\operatorname{sech} u &= -\operatorname{sech} u \tanh u \cdot \frac{du}{dx} \\
\frac{d}{dx}\tanh u &= \operatorname{sech}^2 u \cdot \frac{du}{dx} & \frac{d}{dx}\coth u &= -\operatorname{csch}^2 u \cdot \frac{du}{dx}
\end{aligned} \tag{2.100}$$

Inverse Hyperbolic Functions and their Derivatives

One can define the inverse hyperbolic functions[14] in a manner analogous to how the inverse trigonometric functions were defined. For example,

[14] Sometimes the inverse hyperbolic functions are represented using the notations, arcsinh, arccosh, arctanh, arccoth, arcsech, arccsch

$$
\begin{aligned}
y = \sinh^{-1} x \quad &\text{if and only if} \quad \sinh y = x \\
y = \cosh^{-1} x \quad &\text{if and only if} \quad \cosh y = x \\
y = \tanh^{-1} x \quad &\text{if and only if} \quad \tanh y = x \\
y = \coth^{-1} x \quad &\text{if and only if} \quad \coth y = x \\
y = \operatorname{sech}^{-1} x \quad &\text{if and only if} \quad \operatorname{sech} y = x \\
y = \operatorname{csch}^{-1} x \quad &\text{if and only if} \quad \operatorname{csch} y = x
\end{aligned}
\tag{2.101}
$$

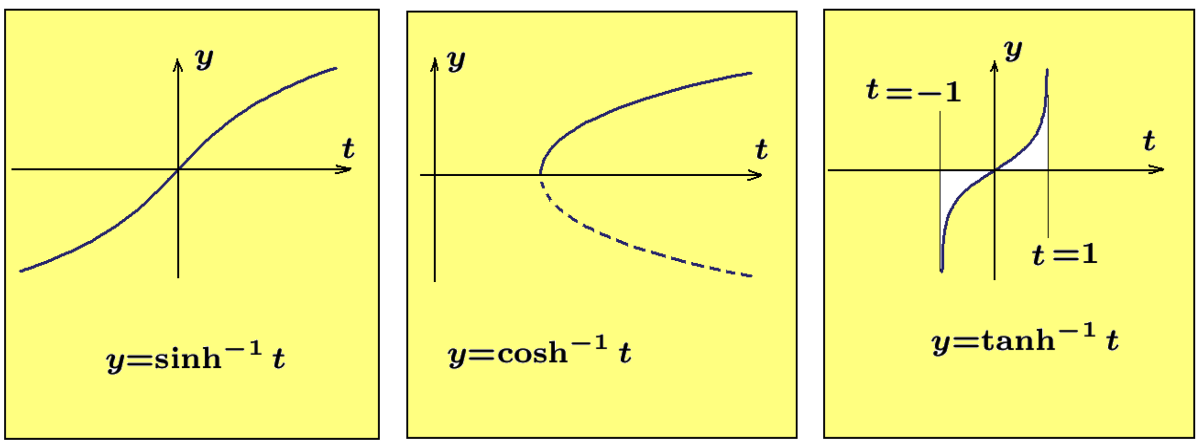


Figure 2-20. Inverse Hyperbolic functions $\sinh^{-1}(t)$, $\cosh^{-1}(t)$, $\tanh^{-1}(t)$.

Graphs of the inverse hyperbolic functions can be obtained from the graphs of the hyperbolic functions by interchanging x and y on the graphs and axes and then re-orienting the graph. The sketches given in the figures 2-20 and 2-21 illustrate the inverse hyperbolic functions.

Examine the figures 2-20 and 2-21 and note the functions $\cosh^{-1} t$ and $\operatorname{sech}^{-1} t$ are multi-valued functions. The other inverse functions are single-valued. The branches where $\cosh^{-1} t$ and $\operatorname{sech}^{-1} t$ are positive are selected as the principal branches. If you want the negative values of these functions, then use the functions $-\cosh^{-1} t$ and $-\operatorname{sech}^{-1} t$.

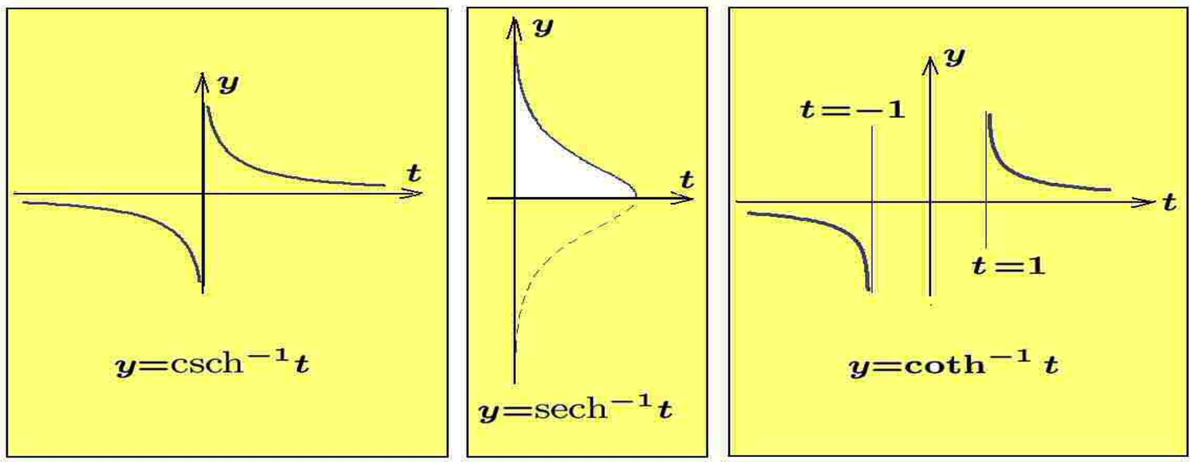


Figure 2-21. Inverse Hyperbolic functions $\operatorname{csch}^{-1}(t)$, $\operatorname{sech}^{-1}(t)$, $\coth^{-1}(t)$

The hyperbolic functions are defined in terms of exponential functions. The inverse hyperbolic functions can be expressed in terms of logarithm functions, which is the inverse function associated with the exponential functions.

Example 2-42. Express $y = \sinh^{-1} x$ in terms of logarithms.

Solution

If $y = \sinh^{-1} x$, then $\sinh y = x$ or

$$\sinh y = \frac{e^y - e^{-y}}{2} = x \qquad \text{or} \qquad e^y - e^{-y} = 2x$$

This last equation can be converted to a quadratic equation in the unknown e^y. This is accomplished by a multiplication of the equation $e^y - e^{-y} = 2x$ by e^y to obtain

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Solving this quadratic equation for the unknown e^y one finds

$$e^y = x \pm \sqrt{x^2 + 1} \tag{2.102}$$

Observe that e^y is always positive for real values of y and $\sqrt{x^2+1} > x$. Hence, in order that e^y remain positive, one must select the positive square root in equation (2.102). Then solving the equation (2.102) for y one obtains the result

$$y = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \qquad -\infty < x < \infty$$

∎

Example 2-43. Express $y = \tanh^{-1} x$ in terms of logarithms.

Solution

If $y = \tanh^{-1} x$, then $\tanh y = x$, which implies $-1 < x < 1$. Use the definition of the hyperbolic tangent to write

$$\tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = x \qquad \text{or} \qquad e^y - e^{-y} = xe^y + xe^{-y}$$

Multiply the last equation through by e^y and then solve for e^{2y} to obtain

$$(1-x)e^{2y} = (1+x) \qquad \text{giving} \qquad e^y = \sqrt{\frac{1+x}{1-x}}$$

Solving for y gives

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad -1 < x < 1$$

■

Example 2-44. Find the derivative of $y = \cosh^{-1} u$ where $u = u(x)$ is a function of x.

Solution One can write $\cosh y = u$ and then differentiate both side with respect to x and obtain

$$\sinh y \frac{dy}{dx} = \frac{du}{dx}$$

and then solve for $\frac{dy}{dx}$ to obtain $\frac{dy}{dx} = \frac{1}{\sinh y}\frac{du}{dx} = \frac{1}{\pm\sqrt{\cosh^2 y - 1}}\frac{du}{dx} = \pm\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$ where one uses the $+$ sign in the principal value region where $y > 0$.

■

The previous examples demonstrate how one can establish the representations

$$\begin{aligned}
\sinh^{-1} x &= \ln\left(x + \sqrt{x^2+1}\right), \quad -\infty < x < \infty \\
\cosh^{-1} x &= \ln(x + \sqrt{x^2-1}), \quad x \ge 1 \\
\tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad -1 < x < 1 \\
\coth^{-1} x &= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \quad x > 1 \text{ or } x < -1 \\
\operatorname{sech}^{-1} x &= \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right), \quad 0 < x < 1 \\
\operatorname{csch}^{-1} x &= \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right), \quad x \ne 0
\end{aligned} \tag{2.103}$$

Relations between Inverse Hyperbolic Functions

In the previous equations (2.103) replace x by $\frac{1}{x}$ and show

$$\begin{aligned}\sinh^{-1}\left(\frac{1}{x}\right) &= \operatorname{csch}^{-1}x\\ \cosh^{-1}\left(\frac{1}{x}\right) &= \operatorname{sech}^{-1}x\\ \tanh^{-1}\left(\frac{1}{x}\right) &= \coth^{-1}x\end{aligned} \tag{2.104}$$

Example 2-45.

(a) Examine the logarithm of the product $(x+\sqrt{x^2+1})(-x+\sqrt{x^2+1})=1$ and observe that $\ln(x+\sqrt{x^2+1})=-\ln(-x+\sqrt{x^2+1})$. This result can be used to show

$$\sinh^{-1}(-x)=\ln(-x+\sqrt{x^2+1})=-\sinh^{-1}x$$

(b) If $\tanh^{-1}x=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, then

$$\tanh^{-1}(-x)=\frac{1}{2}\ln\left(\frac{1-x}{1+x}\right)=-\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)=-\tanh^{-1}x$$

(c) If $y=\operatorname{sech}^{-1}x$, with $y>0$ and $0<x<1$, then $x=\operatorname{sech}y$. By definition $\operatorname{sech}y=\dfrac{1}{\cosh y}$, so that one can write

$$\frac{1}{x}=\cosh y \qquad \text{or} \qquad y=\cosh^{-1}\left(\frac{1}{x}\right)=\operatorname{sech}^{-1}x$$

are equivalent statements.

■

Using the techniques illustrated in the previous example one can verify the following identities

$$\begin{aligned}\boldsymbol{\sinh^{-1}(-x)} &= \boldsymbol{-\sinh^{-1}x}\\ \boldsymbol{\tanh^{-1}(-x)} &= \boldsymbol{-\tanh^{-1}x}\\ \boldsymbol{\coth^{-1}(-x)} &= \boldsymbol{-\coth^{-1}x}\\ \boldsymbol{\operatorname{csch}^{-1}(-x)} &= \boldsymbol{-\operatorname{csch}^{-1}x}\end{aligned} \tag{2.105}$$

Derivatives of the Inverse Hyperbolic Functions

To obtain the derivatives of the inverse hyperbolic functions one can differentiate the functions given by the equations (2.103). For example, if

$$y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

then
$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}\sinh^{-1} x =& \frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right)\\
=& \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\left(x+(x^2+1)^{1/2}\right)\\
=& \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)\\
=& \frac{x+\sqrt{x^2+1}}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}}\\
\frac{d}{dx}\sinh^{-1} x =& \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

One can use the chain rule for differentiation to generalize this result and obtain

$$\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$

where $u = u(x)$ is a function of x. In a similar fashion all the inverse hyperbolic functions can be differentiated and one can verify that

$$\begin{aligned}
&\boldsymbol{\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx},} \qquad && \boldsymbol{-\infty < u < \infty}\\
&\boldsymbol{\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx},} \qquad && \boldsymbol{u > 1}\\
&\boldsymbol{\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx},} \qquad && \boldsymbol{-1 < u < 1}
\end{aligned} \tag{2.106}$$

Example 2-46. Find the derivative of $y = \operatorname{sech}^{-1} x$ with $y > 0$.

Solution

If $\quad y = \operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$

then
$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}\operatorname{sech}^{-1} x =& \frac{d}{dx}\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \qquad 0 < x < 1\\
=& \frac{1}{\dfrac{1+\sqrt{1-x^2}}{x}}\cdot\left[\frac{\frac{-x^2}{\sqrt{1-x^2}} - 1 - \sqrt{1-x^2}}{x^2}\right]\\
=& \frac{-1}{x\sqrt{1-x^2}}, \qquad 0 < x < 1
\end{aligned}$$

The chain rule for differentiation can be used to generalize this result to

$$\frac{d}{dx}\,\mathrm{sech}^{-1}u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}, \qquad 0<u<1$$

If the lower half of the hyperbolic secant curve is used, then the sign of the above result must be changed.

■

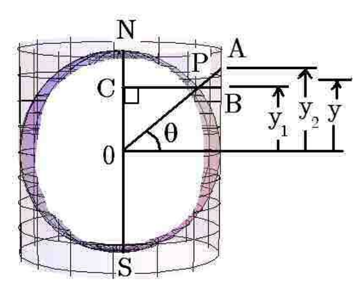
Differentiation of the logarithmic functions which define the inverse hyperbolic functions, one obtains the results

$$\begin{aligned}
&\frac{d}{dx}\coth^{-1}u = \frac{-1}{u^2-1}\frac{du}{dx}, \qquad u^2>1\\
&\frac{d}{dx}\,\mathrm{sech}^{-1}u = \begin{cases} \dfrac{-1}{u\sqrt{1+u^2}}\dfrac{du}{dx}, & \mathrm{sech}^{-1}u>0,\ 0<u<1\\[2ex] \dfrac{1}{u\sqrt{1+u^2}}\dfrac{du}{dx}, & \mathrm{sech}^{-1}u<0,\ 0<u<1 \end{cases}\\
&\frac{d}{dx}\,\mathrm{csch}^{-1}u = \begin{cases} \dfrac{-1}{u\sqrt{1+u^2}}\dfrac{du}{dx}, & u>0\\[2ex] \dfrac{1}{u\sqrt{1+u^2}}\dfrac{du}{dx}, & u<0 \end{cases}
\end{aligned} \tag{2.107}$$

Some additional relations involving the inverse hyperbolic functions are the following.

$$\begin{aligned}
\sinh^{-1}x &= \tanh^{-1}\frac{x}{\sqrt{x^2+1}} & \sinh^{-1}x &= -\,i\,\sin^{-1}(ix)\\
\sinh^{-1}x &= \pm\,\cosh^{-1}\sqrt{x^2+1} & \cosh^{-1}x &= \pm\,i\,\cos^{-1}x\\
\tanh^{-1}x &= \sinh^{-1}\frac{x}{\sqrt{1-x^2}}, \quad |x|<1 & \tanh^{-1}x &= -\,i\,\tan^{-1}(ix)
\end{aligned}$$

Example 2-47. As an exercise study **Mercator projections** and **conformal mappings** and show projections of point P using line from 0 to A gives y_2 latitude which distorts map shape and distances and projections of point P using the line C to B also distorts shapes and distances. Show the correct conformal projection is y between y_1 and y_2 such that $\frac{dy}{d\theta}=\sec\theta$ and



$$y = ln[\sec\theta+\tan\theta] = \tanh^{-1}[\sin\theta]$$

■

Table of Derivatives	
Function $f(x)$	Derivative $\frac{df}{dx}$
$y = x^m$	$\frac{dy}{dx} = mx^{m-1}$
$y = a^x$	$\frac{dy}{dx} = a^x \ln a$
$y = e^x$	$\frac{dy}{dx} = e^x$
$y = \sin x$	$\frac{dy}{dx} = \cos x$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$
$y = \tan x$	$\frac{dy}{dx} = \sec^2 x$
$y = \cot x$	$\frac{dy}{dx} = -\csc^2 x$
$y = \sec x$	$\frac{dy}{dx} = \sec x \tan x$
$y = \csc x$	$\frac{dy}{dx} = -\csc x \cot x$
$y = \sin^{-1} x$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$
$y = \cos^{-1} x$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$
$y = \tan^{-1} x$	$\frac{dy}{dx} = \frac{1}{1+x^2}$
$y = \cot^{-1} x$	$\frac{dy}{dx} = \frac{-1}{1+x^2}$
$y = \sec^{-1} x$	$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$
$y = \csc^{-1} x$	$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$
$y = \sinh x$	$\frac{dy}{dx} = \cosh x$
$y = \cosh x$	$\frac{dy}{dx} = \sinh x$
$y = \tanh x$	$\frac{dy}{dx} = \mathrm{sech}^2 x$
$y = \coth x$	$\frac{dy}{dx} = -\mathrm{csch}^2 x$
$y = \mathrm{sech}\, x$	$\frac{dy}{dx} = -\mathrm{sech}\, x \tanh x$
$y = \mathrm{csch}\, x$	$\frac{dy}{dx} = -\mathrm{csch}\, x \coth x$
$y = \sinh^{-1} x = \ln(x + \sqrt{1+x^2})$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$
$y = \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$	$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$
$y = \tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$	$\frac{dy}{dx} = \frac{1}{1-x^2}$
$y = \coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$	$\frac{dy}{dx} = \frac{-1}{x^2-1}$
$y = \mathrm{sech}^{-1} x = \cosh^{-1}(\frac{1}{x})$	$\frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}}$
$y = \csc^{-1} x = \sinh^{-1}(\frac{1}{x})$	$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2+1}}$

Table of Differentials

$$d(c\,u) = c\,du$$

$$d(u+v) = du + dv$$

$$d(u+v+w) = du + dv + dw$$

$$d(u\,v) = u\,dv + v\,du$$

$$d(u\,v\,w) = u\,v\,dw + u\,dv\,w + du\,v\,w$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

$$d\,(u^n) = nu^{n-1}du$$

$$d\,(u^v) = v\,u^{v-1}\,du + u^n\,(\ln u)\,dv$$

$$d\,(u^u) = u^u\,(1+\ln u)\,du$$

$$d\,(e^u) = e^u\,du$$

$$d\,(b^u) = b^u\,(\ln b)\,du$$

$$d(\ln u) = \frac{1}{u}\,du$$

$$d(\log_b u) = \frac{1}{u}\,(\log_b e)\,du$$

$$d\,\sin u = \cos u\,du$$

$$d\,\cos u = -\sin u\,du$$

$$d\,\tan u = \sec^2 u\,du$$

$$d\,\cot u = -\csc^2 u\,du$$

$$d\,\sec u = \sec u \tan u\,du$$

$$d\,\csc u = -\csc u \cot u\,du$$

$$d\,\sin^{-1} u = (1-u^2)^{-1/2}\,du$$

$$d\,\cos^{-1} u = -(1-u^2)^{-1/2}\,du$$

$$d\,\tan^{-1} u = (1+u^2)^{-1}\,du$$

$$d\,\cot^{-1} u = -(1+u^2)^{-1}\,du$$

$$d\,\sec^{-1} u = \frac{1}{u}(u^2-1)^{1/2}\,du$$

$$d\,\csc^{-1} u = -\frac{1}{u}(u^2-1)^{-1/2}\,du$$

All angles in first quadrant.

$$d\,\sinh u = \cosh u\,du$$

$$d\,\cosh u = \sinh u\,du$$

$$d\,\tanh u = \text{sech}^2 u\,du$$

$$d\,\coth u = -\,\text{csch}^2 u\,du$$

$$d\,\text{sech}\, u = -\,\text{sech}\, u \tanh u\,du$$

$$d\,\text{csch}\, u = -\,\text{csch}\, u \coth u\,du$$

$$d\,\sinh^{-1} u = (u^2+1)^{-1/2}\,du$$

$$d\,\cosh^{-1} u = (u^2-1)^{-1/2}\,du$$

$$d\,\tanh^{-1} u = (1-u^2)^{-1}\,du$$

$$d\,\coth^{-1} u = -(u^2-1)^{-1}\,du$$

$$d\,\text{sech}^{-1} u = -\frac{1}{u}(1-u^2)^{-1/2}\,du$$

$$d\,\text{csch}^{-1} u = -\frac{1}{u}(u^2+1)^{-1/2}\,du$$

$$d(xy) = x\,dy + y\,dx$$

$$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$$

$$d\left(\frac{x^2+y^2}{2}\right) = x\,dx + y\,dy$$

$$d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$$

$$d[\tan^{-1}\left(\frac{y}{x}\right)] = \frac{x\,dy - y\,dx}{x^2+y^2}$$

Partial Derivatives

If $u = u(x_1, x_2, x_3, \dots, x_n)$ is a function of several independent variables, the differentiation with respect to one of the variables is done by treating all the other variables as constants. The notations $\boldsymbol{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \dots, \frac{\partial}{\partial x_n}}$ are used to denote these differentiations. The partial derivative symbol $\frac{\partial}{\partial x_i}$ indicates all variables different from x_i are being held constant. For example, if $u = u(x, y)$ is a function of two real variables x and y, then the partial derivatives of u with respect to x and y are defined

$$\boldsymbol{\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}, \qquad \frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y},}$$

provided these limits exist. The partial derivative operator $\frac{\partial}{\partial x}$ is used to indicate a differentiation with respect to x holding all other variables constant during the differentiation processes. So treat the partial derivative operator just like an ordinary derivative, except all other variables are held constant during the differentiation with respect to x. Similarly, the partial differential operator $\frac{\partial}{\partial y}$ is just like an ordinary derivative with respect to y while holding all other variables constant during the differentiation with respect to y.

Example 2-48. If $u = u(x, y) = x^3y^2 - \sin y + \cos x$, then find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

Solution

Treating y as a constant one finds

$$\begin{aligned}
\frac{\partial u}{\partial x} =& \frac{\partial}{\partial x}(x^3y^2 - \sin y + \cos x) \\
=& \frac{\partial (x^3y^2)}{\partial x} - \frac{\partial (\sin y)}{\partial x} + \frac{\partial (\cos x)}{\partial x} \\
\frac{\partial u}{\partial x} =& y^2 \frac{\partial (x^3)}{\partial x} - 0 + \frac{\partial (\cos x)}{\partial x} \quad \text{If } y \text{ constant, then } \sin y \text{ is constant.} \\
=& y^2(3x^2) - \sin x
\end{aligned}$$

In a similar fashion, if x is held constant, then

$$\begin{aligned}
\frac{\partial u}{\partial y} =& \frac{\partial}{\partial y}(x^3y^2 - \sin y + \cos x) \\
\frac{\partial u}{\partial y} =& \frac{\partial (x^3y^2)}{\partial y} - \frac{\partial (\sin y)}{\partial y} + \frac{\partial (\cos x)}{\partial y} \\
=& x^3 \frac{\partial (y^2)}{\partial y} - \frac{\partial (\sin y)}{\partial y} + 0 \quad \text{If } x \text{ is constant, } \cos x \text{ is constant.} \\
\frac{\partial u}{\partial y} =& x^3(2y) - \cos y
\end{aligned}$$

■

Higher partial derivatives are defined as a derivative of a lower ordered derivative. For example, The second partial derivatives of $u = u(x, y)$ are defined

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

The second derivatives $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$, $\dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$ are called **mixed partial derivatives.** If both the function $u = u(x, y)$ and its first ordered partial derivatives are continuous functions, then the mixed partial derivatives are equal to one another, in which case it doesn't matter as to the order of the differentiation and consequently $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$.

Total Differential

If $u = u(x, y)$ is a continuous function of two variables, then as x and y change, the change in u is written

$$\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$$

Add and subtract the term $u(x, y + \Delta y)$ to the change in u and write

$$\Delta u = [u(x + \Delta x, y + \Delta y] - u(x, y + \Delta y)] + [u(x, y + \Delta y) - u(x, y)]$$

which can also be expressed in the form

$$\Delta u = \left[\frac{u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)}{\Delta x}\right] \Delta x + \left[\frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}\right] \Delta y \tag{2.108}$$

Now use the mean-value theorem on the terms in brackets to show

$$\begin{aligned} \left[\frac{u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)}{\Delta x}\right] &= \frac{\partial u}{\partial x} + \epsilon_1 \\ \left[\frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}\right] &= \frac{\partial u}{\partial y} + \epsilon_2 \end{aligned} \tag{2.109}$$

where ϵ_1 approaches zero as $\Delta x \to 0$ and ϵ_2 approaches zero as $\Delta y \to 0$. One can then express the change in u as

$$\Delta u = \frac{\partial u}{\partial x}\,\Delta x + \frac{\partial u}{\partial y}\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y \tag{2.110}$$

Define the differentials $dx = \Delta x$ and $dy = \Delta y$ and write the change in u as

$$\Delta u = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \epsilon_1\,dx + \epsilon_2\,dy \tag{2.111}$$

Define the **total differential of** u as

$$\boldsymbol{du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy} \tag{2.112}$$

and note the total differential du differs from Δu by an infinitesimal of higher order than dx or dy because ϵ_1 and ϵ_2 approach zero as $\Delta x \to 0$ and $\Delta y \to 0$. The **total differential** du, given by equation (2.112) is sometimes called the **principal part in the change in** u.

Notation

Partial derivatives are sometimes expressed using a **subscript notation**. Some examples of this notation are the following.

$$\begin{array}{ccc}
 & & \dfrac{\partial^3 u}{\partial x^3} = u_{xxx} \\
 & \dfrac{\partial^2 u}{\partial x^2} = u_{xx} & \dfrac{\partial^3 u}{\partial x^2 \partial y} = u_{xxy} \\
\dfrac{\partial u}{\partial x} = u_x & \dfrac{\partial^2 u}{\partial x\, \partial y} = u_{xy} & \\
\dfrac{\partial u}{\partial y} = u_y & & \dfrac{\partial^3 u}{\partial x \partial y^2} = u_{xyy} \\
 & \dfrac{\partial^2 u}{\partial y^2} = u_{yy} & \\
 & & \dfrac{\partial^3 u}{\partial y^3} = u_{yyy}
\end{array}$$

In general, if $f = f(x, y)$ is a function of x and y and $m = i + j$ is an integer, then $\dfrac{\partial^m f}{\partial x^i \partial y^j}$ is the representation of a mixed partial derivative of f.

Differential Operator

If $u = u(x, y)$ is a function of two variables, then the differential of u is defined

$$\boldsymbol{du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy} \tag{2.113}$$

and if the variables $x = x(t)$ and $y = y(t)$ are functions of t, then u becomes a function of t with derivative

$$\boldsymbol{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}} \tag{2.114}$$

This is obtained by dividing both sides of equation (2.113) by dt. One can think of equation (2.114) as defining the differential operator

$$\boldsymbol{\frac{d[\ ]}{dt} = \frac{\partial[\ ]}{\partial x}\frac{dx}{dt} + \frac{\partial[\ ]}{\partial y}\frac{dy}{dt}} \tag{2.115}$$

where the quantity to be substituted inside the brackets can be any function of x and y where both x and y are functions of another variable t.

By definition a second derivative is the derivative of a first derivative and so one can write

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\frac{du}{dt} = \frac{d}{dt}\left(\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{\partial u}{\partial x}\frac{dx}{dt}\right) + \frac{d}{dt}\left(\frac{\partial u}{\partial y}\frac{dy}{dt}\right) \tag{2.116}$$

since the derivative of a sum is the sum of derivatives. The quantities inside the parentheses represents a product of functions which can be differentiated using the product rule for differentiation. Applying the product rule one obtains

$$\begin{aligned}\frac{d^2u}{dt^2} =&\frac{\partial u}{\partial x}\frac{d}{dt}\frac{dx}{dt} + \frac{dx}{dt}\frac{d}{dt}\left[\frac{\partial u}{\partial x}\right] + \frac{\partial u}{\partial y}\frac{d}{dt}\frac{dy}{dt} + \frac{dy}{dt}\frac{d}{dt}\left[\frac{\partial u}{\partial y}\right]\\ \frac{d^2u}{dt} =&\frac{\partial u}{\partial x}\frac{d^2x}{dt^2} + \frac{dx}{dt}\frac{d}{dt}\left[\frac{\partial u}{\partial x}\right] + \frac{\partial u}{\partial y}\frac{d^2y}{dt^2} + \frac{dy}{dt}\frac{d}{dt}\left[\frac{\partial u}{\partial y}\right]\end{aligned} \tag{2.117}$$

The equation (2.115) tells us how to differentiate the terms inside the brackets. Here both $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are some functions of x and y and so using the equation (2.115) one finds

$$\begin{aligned}\frac{d^2u}{dt^2} =&\frac{\partial u}{\partial x}\frac{d^2x}{dt^2} + \frac{dx}{dt}\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{dy}{dt}\right]\\ &+\frac{\partial u}{\partial y}\frac{d^2y}{dt^2} + \frac{dy}{dt}\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{dy}{dt}\right]\\ \frac{d^2u}{dt^2} =&\frac{\partial u}{\partial x}\frac{d^2x}{dt^2} + \frac{dx}{dt}\left[\frac{\partial^2 u}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{dy}{dt}\right]\\ &+\frac{\partial u}{\partial y}\frac{d^2y}{dt^2} + \frac{dy}{dt}\left[\frac{\partial^2 u}{\partial y\,\partial x}\frac{dx}{dt} + \frac{\partial^2 u}{\partial y^2}\frac{dy}{dt}\right]\end{aligned} \tag{2.118}$$

Functions of more than two variables are treated in a similar fashion.

Maxima and Minima for Functions of Two Variables

Given that $f = f(x,y)$ and its partial derivatives f_x and f_y are all continuous and well defined in some domain D of the x,y-plane. For $R > 0$, the set of points $N = \{\ (x,y)\ \ |\ (x-x_0)^2 + (y-y_0)^2 \le R^2\ \}$ is called a neighborhood of the fixed point (x_0, y_0), where it is assumed that the point (x_0, y_0) and the neighborhood N are in the domain D. The function $f = f(x,y)$ can be thought of as defining a surface S over the domain D with the set of points $S = \{\ (x,y,f)\ |\ x,y \in D,\ \text{and}\ f = f(x,y)\ \}$ defining the surface. The function f is said to have

a relative or local minimum value at (x_0, y_0) *if* $f(x,y) \ge f(x_0,y_0)$ for $(x,y) \in N$.

a relative or local maximum value at (x_0, y_0) *if* $f(x,y) \le f(x_0,y_0)$ for $(x,y) \in N$.

Determining relative maximum and minimum values of a function of two variables can be examined by reducing the problem to a one dimensional problem. Note that the planes $x = x_0 = a\ constant$ and $y = y_0 = a\ constant$ cut the surface $f = f(x, y)$ in one-dimensional curves. One can examine these one-dimensional curves for local maximum and minimum values. For example, consider the curves defined by

$$C_x = \{\ (x, y_0, f) \mid f = f(x, y_0)\ \} \qquad\qquad C_y = \{\ (x_0, y, f) \mid f = f(x_0, y)\}$$

These curves have tangent lines with the slope of the tangent line to the curve C_x given by $\frac{\partial f}{\partial x}\Big|_{y=y_0} = f_x(x, y_0)$ and the slope of the tangent line to the curve C_y given by $\frac{\partial f}{\partial y}\Big|_{x=x_0} = f_y(x_0, y)$. At a local maximum or minimum value these slopes must be zero. Consequently, one can say that a necessary condition for the point (x_0, y_0) to corresponds to a local maximum or minimum value for f is that the conditions

$$\boldsymbol{\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = f_x(x_0, y_0) = 0} \qquad \textbf{and} \qquad \boldsymbol{\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} = f_y(x_0, y_0) = 0} \quad \textbf{simultaneously.}$$

These are **necessary conditions for an extreme value** but they are not sufficient conditions. The problem of determining a sufficient condition for an extreme value will be considered in a later chapter and it will be shown that

If the function $f = f(x, y)$ and its derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ exist and are continuous at the point (x_0, y_0), then for $f = f(x, y)$ to have an extreme value at the point (x_0, y_0) the conditions

$$\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = f_x(x_0, y_0) = 0 \qquad \textit{and} \qquad \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} = f_y(x_0, y_0) = 0$$

together with the condition $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 > 0$ that must be satisfied. One can then say

$$f(x_0, y_0) \textit{ is a relative maximum value if } f_{xx}(x_0, y_0) < 0$$
$$f(x_0, y_0) \textit{ is a relative minimum value if } f_{xx}(x_0, y_0) > 0$$

Implicit Differentiation

If $F(x, y, \dots, z)$ is a continuous function of $n-$variables with continuous partial derivatives, then the total differential of F is given by

$$\boldsymbol{dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy + \cdots + \frac{\partial F}{\partial z}\, dz} \tag{2.119}$$

In two dimensions, if $F(x,y)=0$ is an implicit function defining y as a function of x, then by taking the total differential one obtains

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

and solving for $\frac{dy}{dx}$ the derivative is calculated as

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y} = \frac{F_x}{F_y},$$

provided that $\frac{\partial F}{\partial y} = F_y \neq 0$.

In three dimensions, if $F(x,y,z)=0$, is an implicit function of three variables which defines z as a function of x and y, then one can write the total differential as

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dx = 0 \tag{2.120}$$

Solving for dz in the equation (2.120) one finds

$$dz = \left(-\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z}\right)\,dx + \left(-\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z}\right)\,dy = \left(\frac{-F_x}{F_z}\right)\,dx + \left(\frac{-F_y}{F_z}\right)\,dy \tag{2.121}$$

Note that if the implicit function $F(x,y,z)=0$ defines z as a function of independent variables x and y, then one can write $z=z(x,y)$ and the total differential of z would be given by

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \tag{2.122}$$

Comparing the equations (2.122) and (2.121), there results the relations

$$\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} = \frac{-F_x}{F_z} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z} = \frac{-F_y}{F_z} \tag{2.123}$$

provided that $\frac{\partial F}{\partial z} \neq 0$.

Given an implicit equation $F(x,y,z)=0$, one could assume one of the following.

(a) x and y are independent variables.

(b) x and z are independent variables.

(c) y and z are independent variables.

The derivatives in these various cases all give results similar to equations (2.123) and (2.122) derived above.

Exercises

▶ **2-1.**

(a) Sketch the curve $y = x^2$

(b) Find the equation of the tangent line to this curve which passes through the point $(2, 4)$.

(c) Find the equation of the tangent line to this curve which passes through the point $(-2, 4)$

(d) Find the equation of the tangent line to this curve which passes through the point $(0, 0)$

▶ **2-2.** Find the derivatives of the following functions.

$(a)\ y = 3x^5 + 2x^2 - x + 4$ $\quad(d)\ y = \dfrac{1}{ax^2+bx+c}$ $\quad(g)\ y = \sqrt{x^2+1}$

$(b)\ y = 4x^3 - x^2 + x + 1$ $\quad(e)\ y = \dfrac{1}{3x^2-x+1}$ $\quad(h)\ y = (2x+1)^3(3x^2-x)^2$

$(c)\ y = ax^2 + bx + c$ $\quad(f)\ y = \dfrac{1}{x^2+x}$ $\quad(i)\ y = \left(\dfrac{x-1}{x+1}\right)^2$

▶ **2-3.** Find the derivatives of the given functions.

$(a)\ y = \dfrac{1}{x^3}$ $\quad(d)\ y = \dfrac{1}{\sqrt{x}}$ $\quad(g)\ y = 3 + 4t + 5t^2$ $\quad(j)\ y = x\ln x$

$(b)\ y = \sqrt[3]{x}$ $\quad(e)\ y = \sqrt[3]{t^2}$ $\quad(h)\ y = \dfrac{2x^2+x}{x+1}$ $\quad(k)\ y = e^x \ln x$

$(c)\ y = x^{3/2}$ $\quad(f)\ y = \dfrac{1}{t} + t$ $\quad(i)\ y = \sqrt{x} + \dfrac{1}{\sqrt{2t}}$ $\quad(l)\ y = \sqrt[n]{a+x}$

▶ **2-4.** Find the derivatives of the given functions.

$(a)\ y = \dfrac{3}{x} + \dfrac{4}{x^2}$ $\quad(d)\ y = (x^2+4)^3$ $\quad(g)\ y = e^{3x}$ $\quad(j)\ y = \tan^{-1}(\tanh 3x)$

$(b)\ y = \dfrac{\sqrt{x}}{1+x}$ $\quad(e)\ y = \dfrac{x+1}{\sqrt{x}}$ $\quad(h)\ y = \ln(3x)$ $\quad(k)\ y = \sin^{-1}\dfrac{1}{\sqrt{1+x^2}}$

$(c)\ y = \sqrt{x}(1+x)$ $\quad(f)\ y = x\sqrt{x^3-x^2}$ $\quad(i)\ y = e^{\sin x}$ $\quad(l)\ y = \tan^{-1}(\ln x)$

▶ **2-5.** Find the derivatives of the following functions.

$(a)\ A = \pi r^2$ $\quad(d)\ y = \sin(3\theta)$ $\quad(g)\ y = \sin^2(3\theta)$ $\quad(j)\ y = \sin(a + bx^n)$

$(b)\ V = \dfrac{4}{3}\pi r^3$ $\quad(e)\ y = \cos(3\theta)$ $\quad(h)\ y = \cos^2(3\theta)$ $\quad(k)\ y = \cos^{-1}(ax - bx^2)$

$(c)\ S = 4\pi r^2$ $\quad(f)\ y = \tan(3\theta)$ $\quad(i)\ y = \tan^2(3\theta)$ $\quad(l)\ y = x^x + x^{1/x}$

▶ **2-6.** Find the derivatives of the following functions.

$(a)\ y = 3^x$ $(b)\ y = \frac{1}{a}\tan^{-1}\frac{x}{a}$ $(c)\ y = e^{3x}$

$(d)\ y = e^{3x^2}$ $(e)\ y = \sin(e^x)$ $(f)\ y = \cos(\tan x)$

$(g)\ y = \sqrt[3]{1+\sin t}$ $(h)\ y = \sqrt{t^2 - 3t}$ $(i)\ y = e^{\sqrt{t}}$

$(j)\ y = \sqrt{\frac{1-x^2}{1+x^2}}$ $(k)\ y = \sec(\ln(\sqrt{a+bx+cx^2}))$ $(l)\ y = e^{x^x}$

▶ **2-7.** Find the derivative $\frac{dy}{dx} = y'(x)$ if $y = y(x)$ is defined by the equation

$(a)\ x^2 + y^2 = 24$ $(b)\ x^3 - y^2 = 1$ $(c)\ xy + x^3y^3 = 1$

$(d)\ x^3 + y^3 = 6x$ $(e)\ x^3 + y^3 = 6y$ $(f)\ x^3 + y^3 = 6xy$

$(g)\ x^2 - y^2 = 1$ $(h)\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(i)\ x^2 = 4py$

$(j)\ xy = (x+y)^3$ $(k)\ y = x\ln\left(\frac{y}{1+x}\right)$ $(l)\ y = x^y$

▶ **2-8.** Find the derivative $\frac{dy}{dx}$ associated with the given functions.

$(a)\ y = \sin^{-1}(3x)$ $(b)\ y = \cos^{-1}(3x)$ $(c)\ y = \tan^{-1}(3x)$

$(d)\ y = \sin^{-1}(x^2)$ $(e)\ y = \cos^{-1}(x^2)$ $(f)\ y = \tan^{-1}(x^2)$

$(g)\ y = (1+x)\sin^{-1}x$ $(h)\ y = (1+x)\cos^{-1}x$ $(i)\ y = (1+x)\tan^{-1}x$

$(j)\ y = (\cos 3x)^x$ $(k)\ y = \left(1+\frac{1}{x}\right)^x$ $(l)\ xy = x^2y^3 + x + 3$

▶ **2-9.** Differentiate the given functions.

$(a)\ y = \ln(3x)$ $(b)\ y = \ln(ax)$ $(c)\ y = \ln\left(\frac{x}{x+1}\right)$

$(d)\ y = (3+x)^x,\ x > 0$ $(e)\ (3+x)^{x^2}$ $(f)\ (3+x)^{x^3}$

$(g)\ y = \ln(ax+b)$ $(h)\ y = e^{ax+b}$ $(i)\ y = x^x$

$(j)\ y = \frac{1}{\sqrt{x^2+x}}$ $(k)\ y = x + \frac{1}{x^2}$ $(l)\ y = x\sqrt{\sin^3(4x)}$

▶ **2-10.** Find the derivative of the given functions.

$(a)\ y = \sin^{-1}(ax^2)$ $(b)\ y = \cos^{-1}(ax^2)$ $(c)\ y = \tan^{-1}(ax^2)$

$(d)\ y = \sinh(3x)$ $(e)\ y = \cosh(3x)$ $(f)\ y = \tanh(3x)$

$(g)\ y = \sinh^{-1}(3x)$ $(h)\ y = \cosh^{-1}(3x)$ $(i)\ y = \tanh^{-1}(3x)$

$(j)\ y = \frac{a+bx+c^2}{x+1}$ $(k)\ y = \ln(\cosh 3x)$ $(l)\ y = \sin^{-1}x^3$

▶ **2-11.** Show the derivative of a function $f(x)$ at a fixed point x_0 can be written $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$ Hint: Make a substitution.

▶ **2-12.** Use the quotient rule to differentiate the functions

$$(a)\ y = \cot x = \frac{\cos x}{\sin x}, \quad (b)\ y = \sec x = \frac{1}{\cos x}, \quad (c)\ y = \csc x = \frac{1}{\sin x}$$

▶ **2-13.** Find the first and second derivatives of the following functions.

$$(a)\quad y = \frac{1}{t}e^{-3t} \qquad (c)\quad y = te^{-3t} \qquad (e)\quad y = \frac{x}{(x-a)(x-b)}$$

$$(b)\quad y = \frac{1}{x}\sqrt{1-x^2} \qquad (d)\quad y = x\sqrt{4+3\sin x} \qquad (f)\quad y = \frac{1}{x^2+a^2}$$

▶ **2-14.** Find the first derivative $\frac{dy}{dx}$ and second derivative $\frac{d^2y}{dx^2}$ associated with the given parametric curve.

$$(a)\ x = a\cos t, \quad y = b\sin t \qquad (d)\ x = at, \quad y = bt^2$$

$$(b)\ x = 4\cos t, \quad y = 4\sin t \qquad (e)\ x = a\cosh t, \quad y = b\sinh t$$

$$(c)\ x = 3t^2, \quad y = 2t \qquad (f)\ x = \sin(3t+4), \quad y = \cos(5t+2)$$

▶ **2-15.** Differentiate the given functions.

$$(a)\ y = x\sin(4x^2) \qquad (d)\ y = \cos(4x^2) \qquad (g)\ y = \ln(3x+4)\sin(x^2)$$

$$(b)\ y = x^2e^{-3x} \qquad (e)\ y = x^2\ln(3x),\ x > 0 \qquad (h)\ y = \ln(x^2+x)\cos(x^2)$$

$$(c)\ y = xe^{-x} \qquad (f)\ y = x^2\tan(3x) \qquad (i)\ y = \tan x \sec x$$

▶ **2-16.** Differentiate the given functions.

$$(a)\ y = \frac{\sin x}{x} \qquad (d)\ y = \frac{x}{\sqrt{x+1}} \qquad (g)\ y = x^2\ln(x^2)$$

$$(b)\ y = \frac{\cos x}{x} \qquad (e)\ y = \frac{x}{(1+x^2)^{3/2}} \qquad (h)\ y = \sin(x^2)\ln(x^3)$$

$$(c)\ y = \frac{\sqrt{x+1}}{x} \qquad (f)\ y = \frac{(1+x^2)^{3/2}}{x} \qquad (i)\ y = \sin(x^2)\cos(x^2)$$

▶ **2-17.** Find the first and second derivatives of the given functions.

$$(a)\ y = x + \frac{1}{x} \qquad (d)\ y = x^2 + 2x + 3 + \frac{4}{x} \qquad (g)\ y = x\sin x$$

$$(b)\ y = \sin^2(3x) \qquad (e)\ y = \cos^2(3x) \qquad (h)\ y = x^2\cos x$$

$$(c)\ y = \frac{x}{x+1} \qquad (f)\ y = \tan(2x) \qquad (i)\ y = x\cos(x^2)$$

▶ **2-18.** Find the tangent line to the given curve at the specified point.

$$(a)\ y = x + \frac{1}{x},\ \text{at } (1,2) \qquad (b)\ y = \sin x,\ \text{at } (\pi/4, 1/\sqrt{2}) \qquad (c)\ y = x^3,\ \text{at } (2,8)$$

▶ **2-19.** Show that Rolle's theorem can be applied to the given functions. Find all values $x = c$ such that Rolle's theorem is satisfied.

$$(a)\ f(x) = 2 + \sin(2\pi x),\ x \in [0,1] \qquad (b)\ f(x) = x + \frac{1}{x},\ x \in [1/2, 2]$$

▶ **2-20.** Sketch the given curves and where appropriate describe the domain of the function, symmetry properties, x and $y-$intercepts, asymptotes, relative maximum and minimum points, points of inflection and how the concavity changes.

$$(a) \quad y = x + \frac{1}{x} \qquad (c) \quad y = x^4 - 6x^2 \qquad (e) \quad y = \sin x + \cos x$$
$$(b) \quad y = \frac{x}{x+1} \qquad (d) \quad y = (x-1)^2(x-4) \qquad (f) \quad y = x^4 + 12x^3 + 1$$

▶ **2-21.** If $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

(a) Show that $f''(x) = \lim_{h\to 0} \dfrac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$

(b) Substitute $f(x) = x^3$ into the result from part (a) and show both sides of the equation give the same result.

(c) Substitute $f(x) = \cos x$ into the result from part (a) and show both sides of the equation give the same result. Hint: Use L´Hôpital's rule with respect to the variable h.

▶ **2-22.** Find the local maximum and minimum values associated with the given curves.

$$(a)\ y = x^2 - 4x + 3 \qquad (d)\ y = -5 - 48x + x^3 \qquad (g)\ y = \sin(2\pi x),\ \text{all x}$$
$$(b)\ y = \frac{x^2 - x + 1}{x^2 + 1} \qquad (e)\ y = \sin x,\ \text{all x} \qquad (h)\ y = \cos(2\pi x),\ \text{all x}$$
$$(c)\ y = \frac{2}{x^2 + 4} \qquad (f)\ y = \cos x,\ \text{all x} \qquad (i)\ y = \frac{3}{5 - 4\cos x}, \quad x \in [-16, 16]$$

▶ **2-23.** Find the absolute maximum and absolute minimum value of the given functions over the domain D.

$$(a)\ y = f(x) = x^2 + \frac{2}{x}, \quad D = \{\ x \ \mid\ x \in [1/2, 2]\ \}$$
$$(b)\ y = f(x) = \frac{x}{x+1}, \quad D = \{\ x \ \mid\ x \in [1, 2]\ \}$$
$$(c)\ y = f(x) = \sin x + \cos x, \quad D = \{\ x \ \mid\ x \in [0, 2\pi]\ \}$$
$$(d)\ y = f(x) = \frac{x}{1 + x^2}, \quad D = \{\ x \ \mid\ x \in [-2, 2]\ \}$$

▶ **2-24.** Show that the given functions satisfy the conditions of the mean-value theorem. Find all values $x = c$ such that the mean-value theorem is satisfied.

$$(a) \quad f(x) = -4 + (x-2)^2,\ x \in [2, 6] \qquad (b) \quad f(x) = 4 - x^2,\ x \in [0, 2]$$

▶ **2-25.** A wire of length $\ell = 4 + \pi$ is to be cut into two parts. One part is bent into the shape of a square and the other part is bent into the shape of a circle. Determine how to cut the wire so that the area of the square plus the area of the circle has a minimum value?

▶ **2-26.** A wire of length $\ell = 9 + 4\sqrt{3}$ is to be cut into two parts. One part is bent into the shape of a square and the other part is bent into the shape of an equilateral triangle. Show how the wire is to be cut if the area of the square plus the area of the triangle is to have a minimum value?

▶ **2-27.** A wire of length $\ell = 9 + \sqrt{3}$ is to be cut into two parts. One part is bent into the shape of an equilateral triangle and the other part is bent into the shape of a circle. Show how the wire is to be cut if the area of the triangle plus the area of the circle is to have a minimum value?

▶ **2-28.** Find the critical values and determine if the critical values correspond to a maximum value, minimum value or neither.

$$(a)\ y = (x-1)(x-2)^2 \qquad (c)\ y = f(x),\ \text{where } f'(x) = x(x-1)^2(x-3)^3$$
$$(b)\ y = \frac{x^2 - 7x + 10}{x - 10} \qquad (d)\ y = f(x),\ \text{where } f'(x) = x^2(x-1)^2(x-3)$$

▶ **2-29.** Evaluate the given limits.

$$(a)\ \lim_{x\to 1} \frac{x^n - 1}{x - 1} \qquad (d)\ \lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} \qquad (g)\ \lim_{x\to 0} \frac{a^x - 1}{b^x - 1}$$
$$(b)\ \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x \qquad (e)\ \lim_{x\to 1} \frac{\ln x}{x^2 - 1} \qquad (h)\ \lim_{x\to 1} \frac{\sin \pi x}{x - 1}$$
$$(c)\ \lim_{x\to 0} \frac{ax^2 + bx}{bx^2 + ax} \qquad (f)\ \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} \qquad (i)\ \lim_{x\to 0} \frac{x - \sin x}{x^2}$$

▶ **2-30.** Determine where the graph of the given functions are (a) increasing and (b) decreasing. Sketch the graph.

$$(a)\ y = 8 - 10x + x^2 \qquad (b)\ y = 3x^2 - 5x + 2 \qquad (c)\ y = (x^2 - 1)^2$$

▶ **2-31.** Verify the Leibnitz differentiation rule for the nth derivative of a product of two functions, for the cases $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

$$D^n[u(x)v(x)] = D^n[uv] = \sum_{i=0}^{n} \binom{n}{i} \left(D^{n-i}u\right)\left(D^i v\right)$$
$$D^n[uv] = \binom{n}{0}\left(D^n u\right)v + \binom{n}{1}\left(D^{n-1}u\right)Dv + \binom{n}{2}\left(D^{n-2}u\right)D^2 v + \cdots + \binom{n}{n} u D^n v$$

where $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$ are the binomial coefficients. The general case can be proven using mathematical induction.

▸ **2-32.** Use Euler's formula and show

(a) $e^{i\,(\theta+2n\pi)}=e^{i\,\theta}$ where n is an integer.

(b) The polar form of the complex number $x+i\,y=re^{i\,\theta}$

(c) Show de Moivre's theorem can be expressed $(re^{i\,\theta})^n=r^n\,e^{i\,n\theta}$

▸ **2-33.** Find the partial derivatives $\dfrac{\partial u}{\partial x},\ \dfrac{\partial u}{\partial y},\ \dfrac{\partial^2 u}{\partial x^2},\ \dfrac{\partial^2 u}{\partial x\,\partial y},\ \dfrac{\partial^2 u}{\partial y^2}$

$(a)\ u=x^2y+xy^3 \qquad (c)\ u=\sqrt{x^2+y^2} \qquad (e)\ u=xye^{xy}$

$(b)\ u=(x^2+y^2)^3 \qquad (d)\ u=\sqrt{x^2-y^2} \qquad (f)\ u=xe^y+ye^x$

▸ **2-34.**

(a) Show $\cosh^{-1}x=\ln(x+\sqrt{x^2-1}),\quad x\ge 1$

(b) Show $\operatorname{sech}^{-1}x=\ln\left(\dfrac{1}{x}+\sqrt{\dfrac{1}{x^2}-1}\right),\quad 0<x<1$

(c) Show $\sinh^{-1}(-x)=-\sinh^{-1}x$

(d) Show $\dfrac{d}{dx}\cosh^{-1}x=\dfrac{1}{\sqrt{x^2-1}},\quad x>1$

▸ **2-35.** Define the operators $D=\dfrac{d}{dx},\ D^2=\dfrac{d^2}{dx^2},\ldots,D^n=\dfrac{d^n}{dx^n}$, with $D^nf(x)=\dfrac{d^nf(x)}{dx^n}$ representing the nth derivative of $f(x)$. Find a formula for the indicated derivatives.

$(a)\ D^n(e^{\alpha x}) \qquad (d)\ D^m(x^n),\quad m<n \qquad (g)\ D^n(\sin^3 x) \qquad (j)\ D^n(\cos(ax+b))$

$(b)\ D^n(a^x) \qquad (e)\ D^n(\sin x) \qquad (h)\ D^n(\dfrac{1}{x^2-a^2}) \qquad (k)\ D^n(\ln(x+a))$

$(c)\ D^n(\ln x),\quad x>0 \qquad (f)\ D^n(\cos x) \qquad (i)\ D^n(\sin(ax+b)) \qquad (l)\ D^n(\dfrac{1}{x+a})$

▸ **2-36.** Find the first and second derivatives $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ if $x^2y+y^2x=1$

▸ **2-37.** Find the partial derivatives $\dfrac{\partial\phi}{\partial x},\ \dfrac{\partial\phi}{\partial y},\ \dfrac{\partial^2\phi}{\partial x^2},\ \dfrac{\partial^2\phi}{\partial x\,\partial y},\ \dfrac{\partial^2\phi}{\partial y^2}$

$(a)\ \phi=x^3+yx^2-3axy \qquad (d)\ \phi=ax+\dfrac{b}{y}+cxy$

$(b)\ \phi=x^2+y^2+xy \qquad (e)\ \phi=ax+by+cxy+dx^2y$

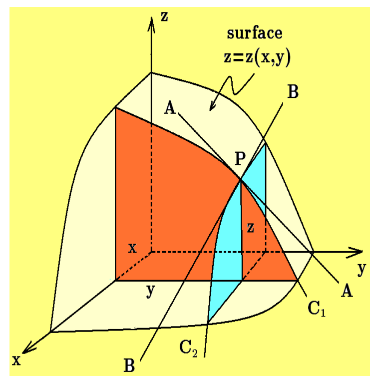
$(c)\ \phi=\sin(xy) \qquad (f)\ \phi=3\dfrac{x^2}{y}+4\dfrac{y^2}{x}$

▸ **2-38.** Sketch the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ and then find the tangent lines to this ellipse at the following points $(a)\ (0,-3)\quad (b)\ (\sqrt{3},-3/2)\quad (c)\ (2,0)\quad (d)\quad (\sqrt{3},3/2)\quad (e)\ (0,3)$

▶ **2-39.** The area of a circle is given by $A = \pi r^2$. If the radius $r = r(t)$ changes with time, then how does the area change with time?

▶ **2-40.** If s describes the displacement of a particle from some fixed point, as measured along a straight line, and $s = s(t)$ is a function of time t, then the velocity v of the particle is given by the change in the displacement with respect to time or $v = v(t) = \frac{ds}{dt}$. The acceleration a of the particle is defined as the rate of change of the velocity with respect to time and so one can write $a = a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$. If $s = s(t) = \frac{t^4}{4} - 2t^3 + \frac{11t^2}{2} - 6t$, find the velocity and acceleration as a function of time. Find where s increases and decreases.

▶ **2-41.**



Let $z = z(x, y)$ denote a function representing a surface in three-dimensional space. Let P denote a point on this surface with coordinates $(x_0, y_0, z(x_0, y_0))$.

(a) If $\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} = \lim_{\Delta x \to 0} \frac{z(x_0 + \Delta x, y_0) - z(x_0, y_0)}{\Delta x}$ and

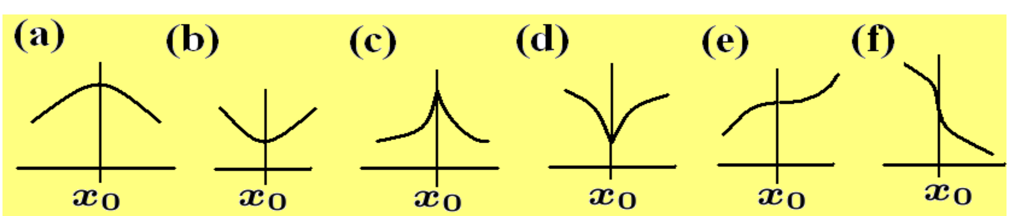
$$\left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} = \lim_{\Delta y \to 0} \frac{z(x_0, y_0 + \Delta y) - z(x_0, y_0)}{\Delta y}$$

are the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ evaluated at the point P, then what is the geometric interpretation of these partial derivatives?

(b) Let the planes $x = x_0 =$ a constant, and $y = y_0 =$ a constant, intersect the surface $z = z(x, y)$ in curves C_1 and C_2 as illustrated. Find the equations of the tangent lines $\overline{AA}$ and $\overline{BB}$ to the curves C_1 and C_2 at their point of intersection P.

▶ **2-42.** Derive the absolute value rule $\frac{d}{dx} \mid u \mid = \frac{u}{\mid u \mid} \frac{du}{dx}$, where $u = u(x)$ is a function of x and test this rule using the function $u = u(x) = x$. Hint: $|u| = \sqrt{u^2}$

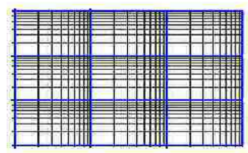
▶ **2-43.** Determine the sign of the slope to the left and right of the given critical point.



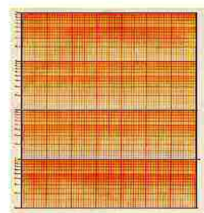
► **2-44.**

(a) Semi-log graph paper has two perpendicular axes with a logarithmic scale on one axis and an ordinary scale on the other axis. Show curves of the form $y = \alpha\beta^x$, $\alpha > 0$, $\beta > 0$ are straight lines when plotted on semi-log graph paper.

(b) Log-Log graph paper has two perpendicular axes with a logarithmic scale on both axes. Show curves of the form $y = \alpha x^\beta$, $\alpha > 0$ are straight lines when plotted on log-log graph paper.



3 x 3 cycle log-log paper



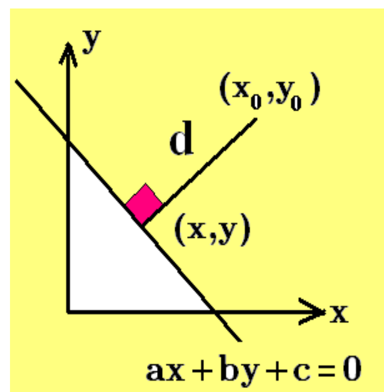
4-cycle semi-log paper

► **2-45.**

Let s denote the distance between a fixed point (x_0, y_0) and an arbitrary point (x, y) lying on the line $ax + by + c = 0$

(a) Show that the quantity s^2 is a minimum when the line through the points (x_0, y_0) and (x, y) is perpendicular to the line $ax + by + c = 0$.

(b) Show the minimum distance d from the point (x_0, y_0) to the line $ax + by + c = 0$ is given by $d = \dfrac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$



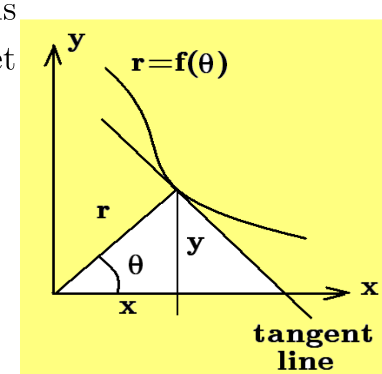
► **2-46.**

If $r = f(\theta)$ is the polar equation of a curve, then this curve can be represented in cartesian coordinates as a set of parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Show the slope of the curve can be represented

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$



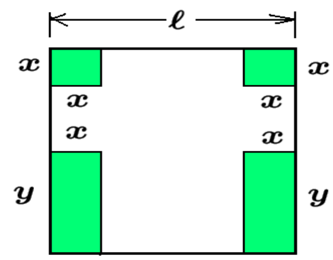
▶ **2-47.** The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. If the radius $r = r(t)$ changes with time, then how does the volume change with time?

▶ **2-48.** A pool is constructed 15 meters long, 8 meters wide and 4 meters deep. When completed, water is pumped into the pool at the rate of 2 cubic meters per minute.

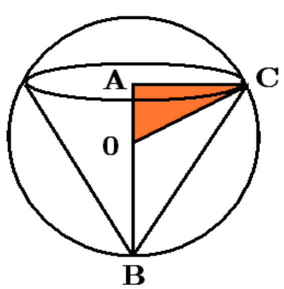
(i) At what rate is the water level rising?
(ii) How long does it take to fill the pool?

▶ **2-49.**

A box having a lid is to be constructed from a square piece of cardboard having sides of length ℓ. The box is to be constructed by cutting squares with sides x from two corners and then cutting rectangles with sides of length x and y from the opposite corners as illustrated in the figure. The sides are folded up and the lid folded over with the sides to be taped. Find the dimensions of the box having the largest volume.



▶ **2-50.**



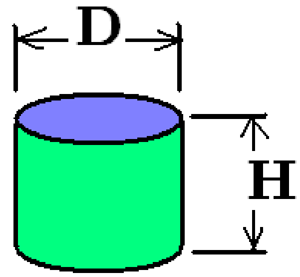
Determine the right circular cone of maximum volume that can be inscribed inside a given sphere having a radius R. The situation is illustrated in the figure where

$AC = r =$ base radius of cone. $0B = R =$ radius of sphere.

$AB = h =$ altitude of cone. $0C = R =$ radius of sphere.

▶ **2-51.** Sketch the function $y = \dfrac{x}{2} + \sin x$ and determine where the maximum and minimum values are.

▶ **2-52.** A cylindrical can is to be made such that it has a fixed volume V. Show that the can using the least amount of material has its height equal to its diameter.



▶ **2-53.** Consider a rectangle inscribed inside a circle of radius R. Find the dimensions of the rectangle with maximum perimeter.

▶ **2-54.** Find the first and second derivatives if

$(a)\quad y = \cos(x^3 + x) \qquad (c)\quad y = \dfrac{x}{\sqrt{x^2+1}} \qquad (e)\quad y = \tan(3x)$

$(b)\quad y = x^2 + \dfrac{1}{x^3} \qquad (d)\quad y = \sin(x^2 + x) \qquad (f)\quad y = e^{3x} + \cosh(2x)$

▶ **2-55.** Find the first and second derivatives $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ if

$(a)\quad x = \sin(2t), \quad y = \cos(2t) \qquad (c)\quad y = t^2\sin(2t)e^{3t} \qquad (e)\quad y = \sin u, \quad u = x^2 + x$

$(b)\quad x = t^2, \quad y = t^3 \qquad (d)\quad y = u^3, \quad u = x^2 + x \qquad (f)\quad y = \sinh(3x^2) + \cosh(3x)$

▶ **2-56.** Sketch the curve $y = \frac{9}{x^2+3}$

(a) Find regions where the slope is positive.
(b) Find regions where the slope is negative.
(c) Find where the slope is zero.
(d) Find regions where the curve is concave up.
(e) Find regions where the curve is concave down.

▶ **2-57.** Find the first and second derivatives of the given functions.

$(a)\quad y = \sin^{-1}(3x) \qquad (c)\quad y = \tan^{-1}(\sqrt{x}) \qquad (e)\quad y = \sec^{-1}(3x^2)$

$(b)\quad y = \cos^{-1}(1 - x^2) \qquad (d)\quad y = \cot^{-1}(x^2 + x) \qquad (f)\quad y = \csc^{-1}(\dfrac{x}{3})$

▶ **2-58.** Find the derivatives of the given functions.

$(a)\quad y = \ln(x + \sqrt{1+x^2}) \qquad (c)\quad y = \alpha x^2 + \beta x \ln|\alpha x + \beta| \qquad (e)\quad y = \sin(3x)\cos(2x)$

$(b)\quad y = \sin^2(e^x) \qquad (d)\quad y = \cos x\, e^{\sin x} \qquad (f)\quad y = \dfrac{1}{x}\tan x$

▶ **2-59.** Use derivative information to sketch the curve over the domain specified.

(a) $y = -1 + 3x^2 - x^3$ for $-1 \le x \le 4$
(b) $y = 1 + (x-1)^3(x-5)$ for $-1 \le y \le 6$

▶ **2-60.** Determine the following limits

$(a)\quad \lim_{x\to\alpha} \dfrac{x^m - \alpha^m}{x^n - \alpha^n} \qquad (b)\quad \lim_{x\to 0}(1 + ax)^{1/x} \qquad (c)\quad \lim_{x\to 0} \dfrac{\sin mx}{\sin nx}$

▶ **2-61.** Given the function $F(x,y,z) = ax^2 + by^2 + cxyz - d = 0$, where a, b, c, d are nonzero constants. Assume x and y are independent variables and find the following partial derivatives. $(a)\quad \dfrac{\partial z}{\partial x} \qquad (b)\quad \dfrac{\partial z}{\partial y}$

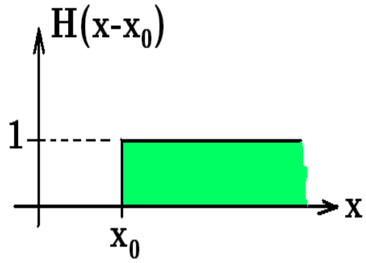
► **2-62.** Given the function $F(x,y,z) = ax^2 + by^2 + cxyz - d = 0$, where a,b,c,d are nonzero constants. Assume x and z are independent variables and find the following partial derivatives. (a) $\dfrac{\partial y}{\partial x}$ (b) $\dfrac{\partial y}{\partial z}$

► **2-63.** Given the function $F(x,y,z) = ax^2 + by^2 + cxyz - d = 0$, where a,b,c,d are nonzero constants. Assume y and z are independent variables and find the following partial derivatives. (a) $\dfrac{\partial x}{\partial y}$ (b) $\dfrac{\partial x}{\partial z}$

► **2-64.**

The Heaviside[15] step function $H(\xi)$ is defined

$$H(\xi) = \begin{cases} 0, & \xi < 0 \\ 1, & \xi > 0 \end{cases}$$



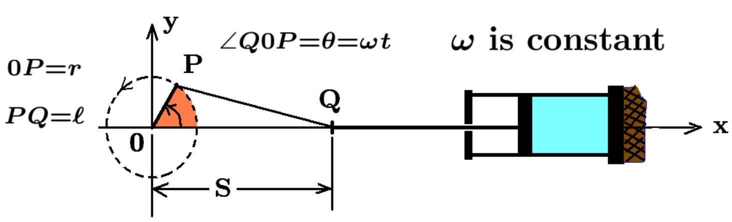
The figure illustrates the step function $H(x - x_0)$

(i) Sketch the functions $y_1(x) = H(x - x_0)$ and $y_2(x) = H(x - (x_0 + \epsilon))$

(ii) Sketch the function $y(x) = y_1(x) - y_2(x)$

(ii) Define the Dirac Delta function $\delta(x - x_0) = \lim\limits_{\epsilon \to 0} \dfrac{y_1(x) - y_2(x)}{\epsilon}$ and give a physical interpretation as to the meaning of this function.

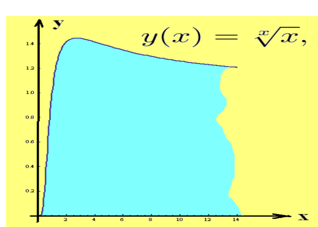
► **2-65.**



The crank arm $0P$, of length r (cm), revolves with constant angular velocity ω (radians/sec). The connecting rod PQ, of length ℓ (cm), moves the point Q back and forth driving a piston. Show that point Q has the velocity (cm/sec), given by

$$\frac{dS}{dt} = V = -\omega r \sin \omega t - \frac{r^2 \omega \sin \omega t \cos \omega t}{\sqrt{\ell^2 - r^2 \sin^2 \omega t}} \qquad \text{Hint: Use law of cosines}$$

► **2-66.** Find the global maximum of the function $y(x) = \sqrt[x]{x},\ 0 < x < 10$ illustrated.



15 Oliver Heaviside (1850-1925) An English engineer and mathematician.

Chapter 3

Integral Calculus

The integral calculus is closely related to the differential calculus presented in the previous chapter. One of the fundamental uses for the integral calculus is the construction of methods for finding areas, arc lengths, surface areas and volumes associated with plane curves and solid figures. Many of the applications of the differential and integral calculus are also to be found in selected areas of engineering, physics, business, chemistry and the health sciences. These application areas require additional background material and so investigation into these applied areas are presented in a later chapter after certain fundamental concepts are developed. Various concepts related to the integral calculus requires some preliminary background material concerning summations.

Summations

The mathematical symbol $\sum$ (Greek letter sigma) is used to denote a **summation of terms**. If $f = f(x)$ is a function whose domain contains all the integers and m is an integer, then the notation

$$\boldsymbol{\sum_{j=1}^{m} f(j) = f(1) + f(2) + f(3) + \cdots + f(m)} \tag{3.1}$$

is used to denote the summation of the terms $f(j)$ as j varies from 1 to m. Here $j = 1$ is called the **starting index** for the sum and the m above the sigma sign is used to denote the **ending index** for the sum. The quantity j is called the **dummy summation index** because the letter j does not occur in the answer and j can be replaced by some other index if one desires to do so.

The following are some examples illustrating how the summation notation is employed.

(a) If m, n are integers satisfying $1 < m < n$, then a summation from 1 to n of the $f(j)$ terms can be broken up and written as a sum of m terms followed by a summation of $(n - m)$ terms by writing

$$\sum_{j=1}^{n} f(j) = \sum_{j=1}^{m} f(j) + \sum_{j=m+1}^{n} f(j) \tag{3.2}$$

(b) The summation index can be **shifted** to represent summations in different forms. For example, the representation $S = \displaystyle\sum_{j=1}^{n} (j-1)^2$ can be modified by making the substitution $k = j-1$ and noting that when $j=1$, then $k=0$ and when $j=n$, then $k=n-1$, so that the sum S can also be expressed $S = \sum_{k=0}^{n-1} k^2$.

As another example, the sum $\sum_{j=m+1}^{n} f(j)$ can be expressed in the form $\sum_{k=1}^{n-m} f(m+k)$. This is called **shifting the summation index**. This result is obtained by making the substitution $j = m+k$ and then finding the summation range for the index k. For example, when $j = m+1$, then $k=1$ and when $j=n$, then $k = n-m$ giving the above result.

(c) If c_1, c_2 are constants and $f(x)$ and $g(x)$ are functions, then one can write

$$\boldsymbol{\sum_{k=1}^{n} (c_1 f(k) + c_2 g(k)) = c_1 \sum_{k=1}^{n} f(k) + c_2 \sum_{k=1}^{n} g(k)} \tag{3.3}$$

where the constant terms can be placed in front of the summation signs.

(d) If $f(x) = c = constant$, for all values of x, then

$$\begin{aligned} \sum_{j=m}^{n} f(j) =& f(m) + f(m+1) + f(m+2) + \cdots + f(n) \\ =& \underbrace{c + c + c + \cdots + c}_{(n-m+1) \text{ values of } c} \\ =& c(n-m+1) \end{aligned} \tag{3.4}$$

The special sum $\sum_{j=1}^{n} 1 = \underbrace{1+1+1+\cdots+1}_{n \text{ ones}} = n$ occurs quite often.

(e) The notation $\sum_{j=1}^{\infty} f(j)$ is used to denote the **limiting process** $\lim_{n\to\infty} \sum_{j=1}^{n} f(j)$ if this limit exists. These summations are sometimes referred to as **infinite sums.**

(f) Summations can be combined. For example,

$$\sum_{j=1}^{m-2} f(j) + f(m-1) + f(m) = \sum_{j=1}^{m} f(j) \tag{3.5}$$

Special Sums

If $f(x) = a + xd$, then the sum $S = \sum_{j=0}^{n-1} f(j) = \sum_{j=0}^{n-1} (a + jd)$ or

$$\boldsymbol{S = \sum_{j=0}^{n-1} f(j) = a + (a+d) + (a+2d) + (a+3d) + \cdots + (a+(n-1)d)} \tag{3.6}$$

is known as an **arithmetic series** with a called the **first term**, d called **the common difference between successive terms**, $\ell = a + (n-1)d$ is the **last term** and n is the **number of terms**. Reverse the order of the terms in equation (3.6) and write

$$\boldsymbol{S = (a + (n-1)d) + (a + (n-2)d) + \cdots + (a + d) + a} \tag{3.7}$$

and then add the equations (3.6) and (3.7) on a term by term basis to show

$$2S = n[a + a + (n-1)d] = n(a + \ell) \tag{3.8}$$

Solving equation (3.8) for S one finds the sum of an arithmetic series is given by

$$\boldsymbol{S = \sum_{j=0}^{n-1}(a + jd) = \frac{n}{2}(a + \ell) = n\frac{a + \ell}{2}} \tag{3.9}$$

which says **the sum of an arithmetic series is given by the number of terms multiplied by the average of the first and last terms of the sum.**

If $f(x) = ar^x$, with a and r nonzero constants, the sum $S = \sum_{j=0}^{n-1} f(j) = \sum_{j=0}^{n-1} ar^j$ or

$$\boldsymbol{S = \sum_{j=0}^{n-1} f(j) = \sum_{j=0}^{n-1} ar^j = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}} \tag{3.10}$$

is known as a **geometric series**, where a is the **first term of the sum**, r is **the common ratio of successive terms** and n is **the number of terms in the summation**. Multiply equation (3.10) by r to obtain

$$\boldsymbol{rS = \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n} \tag{3.11}$$

and then subtract equation (3.11) from equation (3.10) to show

$$\boldsymbol{(1-r)S = a - ar^n \qquad \text{or} \qquad S = \frac{a - ar^n}{1 - r}} \tag{3.12}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ and in this special case one can write

$$\boldsymbol{S_\infty = \lim_{n\to\infty} \sum_{j=0}^{n-1} ar^j = \lim_{n\to\infty} \frac{a - ar^n}{1 - r} = \frac{a}{1 - r}, \qquad |r| < 1} \tag{3.13}$$

Archimedes of Syracuse, (287-212 BCE), used infinite summation processes to find the areas under plane curves and to find the volume of solids. In addition to the arithmetic and geometric series, Archimedes knew the following special sums

$$\begin{aligned}
\sum_{j=1}^{n} 1 =& \underbrace{1+1+1+\cdots+1}_{n\ terms} = n \\
\sum_{j=1}^{n} j =& 1+2+3+\cdots+n = \frac{1}{2}n(n+1) \\
\sum_{j=1}^{n} j^2 =& 1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}(2n^3+3n^2+n) \\
\sum_{j=1}^{n} j^3 =& 1^3+2^3+3^3+\cdots+n^3 = \frac{1}{4}(n^4+2n^3+n^2)
\end{aligned} \tag{3.14}$$

Modern day mathematicians now know how to generalize these results to obtain sums of the form

$$S_n = \sum_{j=1}^{n} j^p = 1^p + 2^p + 3^p + \cdots + n^p \tag{3.15}$$

where p is any positive integer. They have found that the sum S_n of the series given by equation (3.15) must be a polynomial of degree $p+1$ of the form

$$S_n = a_0 n^{p+1} + a_1 n^p + a_2 n^{p-1} + \cdots + a_p n \tag{3.16}$$

where n is the number of terms in the series and $a_0, a_1, a_2, \ldots, a_p$ are constants to be determined. A more general representation for the sum (3.15) can be found on page 349.

Example 3-1. Sum the series $\sum_{j=1}^{n} j^4 = 1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4$

Solution Let S_n denote the sum of the series and make use of the fact that S_n must be a polynomial of degree 5 having the form

$$S_n = a_0 n^5 + a_1 n^4 + a_2 n^3 + a_3 n^2 + a_4 n \tag{3.17}$$

where a_0, a_1, a_2, a_3, a_4 are constants to be determined and n is the number of terms to be summed. The sums S_1, S_2, S_3, S_4, S_5 give the five conditions

$$S_1 = \sum_{j=1}^{1} j^4 = 1^4 = 1$$
$$S_2 = \sum_{j=1}^{2} j^4 = 1^4 + 2^4 = 1 + 16 = 17$$
$$S_3 = \sum_{j=1}^{3} j^4 = 1^4 + 2^4 + 3^4 = 1 + 16 + 81 = 98$$
$$S_4 = \sum_{j=1}^{4} j^4 = 1^4 + 2^4 + 3^4 + 4^4 = 1 + 16 + 81 + 256 = 354$$
$$S_5 = \sum_{j=1}^{5} j^4 = 1^4 + 2^4 + 3^4 + 4^4 + 5^4 = 1 + 16 + 81 + 256 + 625 = 979$$

to determine the constants a_0, a_1, a_2, a_3, a_4. That is, if S_n has the form given by equation (3.17), then

$$\begin{aligned}
S_1 = a_0 + a_1 + a_2 + a_3 + a_4 &= 1 \\
S_2 = a_0(2)^5 + a_1(2)^4 + a_2(2)^3 + a_3(2)^2 + a_4(2) &= 17 \\
S_3 = a_0(3)^5 + a_1(3)^4 + a_2(3)^3 + a_3(3)^2 + a_4(3) &= 98 \\
S_4 = a_0(4)^5 + a_1(4)^4 + a_2(4)^3 + a_3(4)^2 + a_4(4) &= 354 \\
S_5 = a_0(5)^5 + a_1(5)^4 + a_2(5)^3 + a_3(5)^2 + a_4(5) &= 979
\end{aligned} \tag{3.18}$$

The equations (3.18) represent 5-equations in 5-unknowns which can be solved using algebra. After a lot of work one finds the solutions

$$a_0 = \frac{1}{5} = \frac{6}{30}, \quad a_1 = \frac{1}{2} = \frac{15}{30}, \quad a_2 = \frac{1}{3} = \frac{10}{30}, \quad a_3 = 0, \quad a_4 = \frac{-1}{30}$$

This gives the result $S_n = \sum_{j=1}^{n} j^4 = 1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{1}{30}(6n^5 + 15n^4 + 10n^3 - n)$

■

Integration

The mathematical process which represents the **inverse of differentiation** is known as **integration**. In the differential calculus the differential operator $\frac{d}{dx}$, which performed differentiation, was employed as a shorthand notation for the limiting process required for differentiation. Define the **integral symbol** $\int (\ )\, dx$ as an operator that performs **the inverse of differentiation** which is called integration.

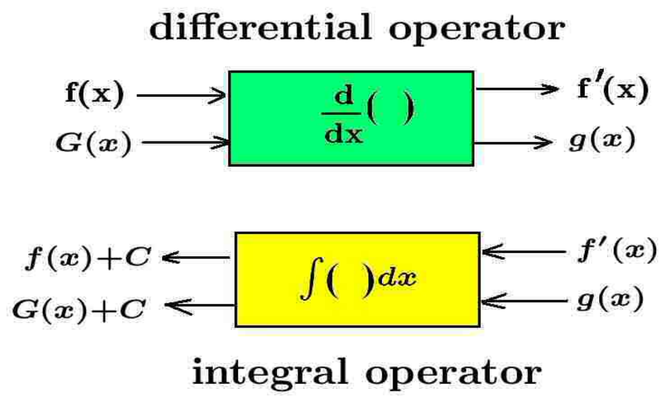


Figure 3-1. Differential and integral operators.

Examine the operator boxes illustrated in the figure 3-1 where one box represents a differential operator and the other box represents an integral operator. If $f(x)$ is an input to the differential operator box, then the output is denoted $\frac{d}{dx}f(x) = f'(x)$. Suppose it is required to undo what has just been done. To **reverse the differentiation process**, insert the derivative function into the integral operator box. The output from the integral operator box is called **an indefinite integral** and is written

$$\int f'(x)\,dx = f(x) + C \tag{3.19}$$

and the equation (3.19) is sometimes read as **"The indefinite integral of** $f'(x)\,dx$ **is equal to** $f(x)+C$**"**. Here $f'(x)$ is called the **integrand**, $f(x)$ is called **a particular integral** and $f(x) + C$ is called **the general integral** of the indefinite integral of $f'(x)\,dx$ and C is called the **constant of integration**. Recall that two functions $f(x)$ and $f(x) + C$, C constant, both have the same derivative $f'(x)$, this is because the derivative of a sum is the sum of the derivatives and the derivative of a constant is zero. It is customary when performing an indefinite integral to always **add a constant of integration** in order to get the more general result.

Examine the notation for the inputs and outputs associated with the operator boxes illustrated in the figure 3-1. One can state that if $\frac{d}{dx}G(x) = g(x)$ then by definition one can express the indefinite integral in any of the forms

$$\int \frac{dG(x)}{dx}\,dx = G(x) + C \quad \text{or} \quad \int g(x)\,dx = G(x) + C, \quad \text{or} \quad \int dG(x) = G(x) + C \tag{3.20}$$

because $G(x) + C$ is the more general function which has the derivative $g(x) = \frac{dG(x)}{dx}$.

The symbol $\int$ is called an **integral sign** and is sometimes replaced by the words, **"The function whose differential is"**. The symbol x used in the indefinite integral given by equation (3.20) is called a **dummy variable of integration**. It can be replaced by some other symbol. For example,

$$\text{if} \quad \frac{d}{d\xi}G(\xi) = g(\xi) \qquad \text{then} \qquad \int g(\xi)\,d\xi = G(\xi) + C \tag{3.21}$$

where C is called a constant of integration.

Example 3-2.

The following integrals occur quite often and should be memorized.

$$\begin{array}{rlll}
\text{If} \quad \dfrac{d}{dx}x = 1, & \text{then} & \displaystyle\int 1\,dx = x + C & \text{or} \displaystyle\int dx = x + C \\
\text{If} \quad \dfrac{d}{dx}x^2 = 2x, & \text{then} & \displaystyle\int 2x\,dx = x^2 + C & \text{or} \displaystyle\int d(x^2) = x^2 + C \\
\text{If} \quad \dfrac{d}{dx}x^3 = 3x^2, & \text{then} & \displaystyle\int 3x^2\,dx = x^3 + C & \text{or} \displaystyle\int d(x^3) = x^3 + C \\
\text{If} \quad \dfrac{d}{dx}x^n = nx^{n-1}, & \text{then} & \displaystyle\int nx^{n-1}\,dx = x^n + C & \text{or} \displaystyle\int d(x^n) = x^n + C \\
\text{If} \quad \dfrac{d}{du}\left(\dfrac{u^{m+1}}{m+1}\right) = u^m, & \text{then} & \displaystyle\int u^m\,du = \frac{u^{m+1}}{m+1} + C & \text{or} \displaystyle\int d\left(\frac{u^{m+1}}{m+1}\right) = \frac{u^{m+1}}{m+1} + C \\
\text{If} \quad \dfrac{d}{dt}\sin t = \cos t, & \text{then} & \displaystyle\int \cos t\,dt = \sin t + C & \text{or} \displaystyle\int d(\sin t) = \sin t + C \\
\text{If} \quad \dfrac{d}{dt}\cos t = -\sin t, & \text{then} & \displaystyle\int \sin t\,dt = -\cos t + C & \text{or} -\displaystyle\int d(\cos t) = -\cos t + C
\end{array}$$

■

Properties of the Integral Operator

$$\textbf{If} \quad \int \boldsymbol{f(x)\,dx = F(x) + C}, \quad \textbf{then} \quad \boldsymbol{\frac{d}{dx}F(x) = f(x)}$$

That is, to check that the integration performed is accurate, observe that one must have the derivative of the particular integral $F(x)$ always equal to the integrand function $f(x)$.

If $\int \boldsymbol{f(x)\,dx = F(x) + C}$, **then**

$$\int \boldsymbol{\alpha f(x)\,dx = \alpha \int f(x)\,dx = \alpha\,[F(x) + C] = \alpha F(x) + K}$$

for all constants $\boldsymbol{\alpha}$. Here $K = \alpha C$ is just some new constant of integration. This property is read, "The integral of a constant times a function equals the constant times the integral of the function."

If $\displaystyle\int f(x)\,dx = F(x) + C$ **and** $\displaystyle\int g(x)\,dx = G(x) + C$**, then**

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx = F(x) + G(x) + C$$

This property states that the integral of a sum is the sum of the integrals. The constants C in each of the above integrals are **not the same constants.** The symbol C represents an arbitrary constant and all $C's$ are not the same. That is, the sum of arbitrary constants is still an arbitrary constant. For example, examine the statement that the integral of a sum is the sum of the integrals. If for $i = 1, 2, \ldots, m$ you know $\displaystyle\int f_i(x)\,dx = F_i(x) + C_i$, where each C_i is an arbitrary constant, then one could add a constant of integration to each integral and write

$$\begin{aligned}\int [f_1(x) + f_2(x) + \cdots + f_m(x)]\,dx =& \int f_1(x)\,dx + \int f_2(x)\,dx + \cdots + \int f_m(x)\,dx \\ =&[F_1(x) + C_1] + [F_2(x) + C_2] + \cdots + [F_m(x) + C_m] \\ =&F_1(x) + F_2(x) + \cdots + F_m(x) + C\end{aligned}$$

All the arbitrary constants of integration can be combined to form just one arbitrary constant of integration.

Notation

There are different notations for representing an integral. For example, **if** $\dfrac{d}{dx}F(x) = f(x)$**, then** $dF(x) = f(x)\,dx$ **and** $\int dF(x) = \int f(x)\,dx = F(x) + C$ **or**

$$\int f(x)\,dx = \int \frac{d}{dx}F(x)\,dx = \int dF(x) = F(x) + C \tag{3.22}$$

Examine equation (3.22) and observe $\displaystyle\int dF(x) = F(x) + C$. One can think of the differential operator d and the integral operator $\displaystyle\int$ as being inverse operators of each other where the product of operators $\displaystyle\int d$ produces unity. These operators are commutative so that $\displaystyle d\int$ also produces unity. For example,

$$d\left[\int f(x)\,dx\right] = d[F(x) + C] = dF(x) + dC = f(x)\,dx$$

Some additional examples of such integrals are the following.

$$\textbf{If } d(uv) = u\,dv + v\,du, \textbf{ then } \int \left(u\frac{dv}{dx} + v\frac{du}{dx}\right)\,dx = \int d(uv) = uv + C$$

$$\textbf{If } d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}, \textbf{ then } \int \left(\frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}\right)\,dx = \int d\left(\frac{u}{v}\right) = \frac{u}{v} + C$$

$$\textbf{In general, if } dw = \frac{dw}{dx}\,dx, \textbf{ then } \int \frac{dw}{dx}\,dx = \int dw = w + C$$

Integration of derivatives

If $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$, or $\frac{d}{dx}\left(f'(x)\right) = f''(x)$, then multiplying both sides of this equation by dx and integrating both sides of the equation gives

$$\begin{aligned}\int \frac{d}{dx}\left(\frac{dy}{dx}\right)\,dx &= \int \frac{d^2y}{dx^2}\,dx \\ \int d\left(\frac{dy}{dx}\right) &= \int \frac{d^2y}{dx^2}\,dx\end{aligned} \qquad \text{or} \qquad \begin{aligned}\int \frac{d}{dx}\left(f'(x)\right)\,dx &= \int f''(x)\,dx \\ \int d\left(f'(x)\right) &= \int f''(x)\,dx\end{aligned}$$

Since $\int dw = w + C$, one finds

$$\int \frac{d^2y}{dx^2}\,dx = \int d\left(\frac{dy}{dx}\right) = \frac{dy}{dx} + C \quad \text{or} \quad \int f''(x)\,dx = \int d(f'(x)) = f'(x) + C \tag{3.23}$$

In a similar fashion one can demonstrate that in general

$$\int \frac{d^{n+1}y}{dx^{n+1}}\,dx = \frac{d^ny}{dx^n} + C \quad \text{or} \quad \int f^{(n+1)}(x)\,dx = f^{(n)}(x) + C \tag{3.24}$$

for $n = 1, 2, 3, \ldots$.

Polynomials

Use the result $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ obtained from example 3-2 to evaluate the integral of a polynomial function

$$p_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are constants. Also use the result that the integral of a sum is the sum of the integrals and the integral of a constant times a function is that constant times the integral of a function. One can then integrate the given polynomial function to obtain

$$\begin{aligned}\int p_n(x)\,dx = &\int \left(a_0x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n\right)\,dx \\ =&a_0\int x^n\,dx + a_1\int x^{n-1}\,dx + \cdots + a_{n-2}\int x^2\,dx + a_{n-1}\int x\,dx + a_n\int dx \\ =&a_0\frac{x^{n+1}}{n+1} + a_1\frac{x^n}{n} + \cdots + a_{n-2}\frac{x^3}{3} + a_{n-1}\frac{x^2}{2} + a_nx + C\end{aligned}$$

Example 3-3. Recall that if functions are scaled, then the chain rule for differentiation is used to find the derivative of the scaled function. If you know $\frac{d}{dx}F(x) = f(x)$, then you know $\frac{d}{du}F(u) = f(u)$, no matter what u is, so long as it is different from zero and well behaved. Say for example that you are required to differentiate the function $y = F(ax)$, where a is a constant different from zero, then you would use the chain rule for differentiation. Make the substitution $u = ax$ and write $y = F(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}F(u) \cdot \frac{du}{dx} = f(u)\,a = f(ax)\,a$$

In a similar fashion **integrals must be adjusted when a scaling occurs**. If you know $\int f(x)\,dx = F(x) + C$, then you know $\int f(u)\,du = F(u) + C$, no matter what u is, so long as it is different from zero and well behaved. Consequently, to evaluate the integral $\int f(ax)\,dx$ you would make the substitution $u = ax$ with $du = a\,dx$ and then multiply and divide the given integral by the required scale factor and write the integral in the form

$$\int f(ax)\,dx = \frac{1}{a}\int f(ax)\,a\,dx = \frac{1}{a}\int f(u)\,du = \frac{1}{a}F(u) + C = \frac{1}{a}F(ax) + C$$

As another example, if you know $\int x^2\,dx = \frac{x^3}{3} + C$, then you know $\int u^2\,du = \frac{u^3}{3} + C$ since x is a dummy variable of integration and can be replaced by some other symbol. To find the integral given by $I = \int (3x+7)^2\,dx$ you would make a substitution $u = 3x+7$ with $du = 3\,dx$ and then perform the necessary scaling to write

$$I = \frac{1}{3}\int (3x+7)^2\,3\,dx = \frac{1}{3}\int u^2\,du = \frac{1}{3}\frac{u^3}{3} + C = \frac{1}{3}\frac{(3x+7)^3}{3} + C = \frac{1}{9}(3x+7)^3 + C$$

■

Example 3-4. If $\int \cos u\,du = \sin u + C$, then to find $\int \cos(ax)\,dx$ one can scale the integral by letting $u = ax$ with $du = a\,dx$ to obtain

$$\frac{1}{a}\int \cos u\,du = \frac{1}{a}\sin u + C = \frac{1}{a}\sin(ax) + C$$

■

General Considerations

If you plot the functions $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, y, $\int y(x)\,dx$, $\int \left[\int y(x)\,dx\right] dx$ you will find that differentiation is **a roughening process** and integration is **a smoothing process**.

If you are given a function, say $y = y(x) = x^3 e^{5x}$, then you can use the rules for differentiation of a product of two functions to obtain

$$\frac{dy}{dx} = y'(x) = x^3(5e^{5x}) + (3x^2)e^{5x} = (5x^3 + 3x^2)\, e^{5x}$$

One topic in integral calculus develops ways that enable one to reverse the steps used in differentiation and work backwards to obtain the original function which was differentiated plus a constant of integration representing the more general function $y_g = y_g(x) = x^3 e^{5x} + C$. In the study of integral calculus one develops **integration methods** whereby the integral

$$\int (5x^3 + 3x^2)\, e^{5x}\, dx = x^3 e^{5x} + C$$

can be obtained. This result can also be expressed in the form

$$\int \frac{d}{dx}(x^3 e^{5x})\, dx = x^3 e^{5x} + C$$

and illustrates the basic relation between differentiation and integration, that if you know a derivative $\frac{dF(x)}{dx} = f(x)$, then you can immediately write down the integral

$$\int \frac{dF(x)}{dx}\, dx = \int F'(x)\, dx = \int f(x)\, dx = F(x) + C \tag{3.25}$$

Many integrals can be simplified by making a change of variable within the integral. For example, if it is required to evaluate an integral $I = \int f(x)\, dx$, then sometimes one can find a change of variables $x = g(u)$ with $dx = g'(u)\, du$ which changes the integration to $I = \int f(g(u))g'(u)\, du$ which may or may not be an easier integral to evaluate. In the sections that follow we will investigate various methods which will aid in evaluating difficult integrals.

Another thing to look for in performing integrations, is that an integration might produce two results which **appear to be different**. For example, student A might perform an integration and get the result

$$\int f(x)\, dx = F(x) + C$$

and student B might perform **the same integration** and get the result

$$\int f(x)\, dx = G(x) + C$$

If both students results are correct, then (i) the constants of integration C need not be the same constants and (ii) there must exist some relationship between the functions $F(x)$ and $G(x)$ because they have the same derivative of $f(x)$.

In the differential calculus, if one finds two functions $F(x)$ and $G(x)$ having derivatives $F'(x)$ and $G'(x)$ which are equal and satisfy $F'(x) = G'(x)$, over an interval (a, b), then one can say that the functions $F(x)$ and $G(x)$ differ by a constant and one can write $\int F'(x)\,dx = \int G'(x)\,dx$ or $F(x) = G(x) + c$.

Example 3-5. Consider the functions $F(x) = \cos^2 x$ and $G(x) = -\sin^2 x$, these functions have the derivatives $F'(x) = \dfrac{dF}{dx} = -2\cos x \sin x$ and $G'(x) = -2\sin x \cos x$ which are equal. Consequently one can state that

$$F(x) = G(x) + c \qquad \text{or} \qquad \cos^2 x = -\sin^2 x + c \tag{3.26}$$

for all values of x. Substituting $x = 0$ into equation (3.26) one finds $1 = c$ and consequently comes up with the trigonometric identity $\cos^2 x + \sin^2 x = 1$.

This result can also be illustrated using integration. Consider the evaluation of the integral

$$\int 2\sin x \cos x\,dx$$

Student A makes the substitution $u = \sin x$ with $du = \cos x\,dx$ and obtains the solution

$$\int 2\sin x \cos x\,dx = 2\int u\,du = 2\frac{u^2}{2} + C_1 = \sin^2 x + C_1$$

Student B makes the substitution $v = \cos x$ with $dv = -\sin x$ and obtains the solution

$$\int 2\sin x \cos x\,dx = -2\int v\,dv = -2\frac{v^2}{2} + C_2 = -\cos^2 x + C_2$$

The two integrals appear to be different, but because of the trigonometric identity $\cos^2 x + \sin^2 x = 1$, the results are really the same as one result is expressed in an alternative form of the other and the results differ by some constant.

■

Table of Integrals

If you know a differentiation formula, then you immediately obtain an integration formula. That is, if

$$\boldsymbol{\frac{d}{du}F(u) = f(u), \qquad \text{then} \qquad \int f(u)\,du = F(u) + C} \tag{3.27}$$

Going back and examining all the derivatives that have been calculated one can reverse the process and create a **table of derivatives and integrals** such as the Tables I and II on the following pages.

Table I	Derivatives and Integrals	
Function $f(u)$	**Derivative**	**Integral**
$y = u^p$	$\frac{dy}{du} = pu^{p-1}$	$\int u^p\,du = \frac{u^{p+1}}{p+1} + C,\ p \neq -1$
$y = \ln u$	$\frac{dy}{du} = \frac{1}{u}$	$\int \frac{du}{u} = \ln \mid u \mid + C$
$y = a^u$	$\frac{dy}{du} = a^u \ln a$	$\int a^u\,du = \frac{a^u}{\ln a} + C$
$y = e^u$	$\frac{dy}{du} = e^u$	$\int e^u\,du = e^u + C$
$y = \sin u$	$\frac{dy}{du} = \cos u$	$\int \cos u\,du = \sin u + C$
$y = \cos u$	$\frac{dy}{du} = -\sin u$	$\int \sin u\,du = -\cos u + C$
$y = \tan u$	$\frac{dy}{du} = \sec^2 u$	$\int \sec^2 u\,du = \tan u + C$
$y = \cot u$	$\frac{dy}{du} = -\csc^2 u$	$\int \csc^2 u\,du = -\cot u + C$
$y = \sec u$	$\frac{dy}{du} = \sec u \tan u$	$\int \sec u \tan u\,du = \sec u + C$
$y = \csc u$	$\frac{dy}{du} = -\csc u \cot u$	$\int \csc u \cot u\,du = \csc u + C$
$y = \sin^{-1} u$	$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$	$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$
$y = \cos^{-1} u$	$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$	$\int \frac{du}{\sqrt{1-u^2}} = -\cos^{-1} u + C$
$y = \tan^{-1} u$	$\frac{dy}{du} = \frac{1}{1+u^2}$	$\int \frac{du}{1+u^2} = \tan^{-1} u + C$
$y = \cot^{-1} u$	$\frac{dy}{du} = \frac{-1}{1+u^2}$	$\int \frac{du}{1+u^2} = -\cot^{-1} u + C$
$y = \sec^{-1} u$	$\frac{dy}{du} = \frac{1}{u\sqrt{u^2-1}}$	$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C$
$y = \csc^{-1} u$	$\frac{dy}{du} = \frac{-1}{u\sqrt{u^2-1}}$	$\int \frac{du}{u\sqrt{u^2-1}} = -\csc^{-1} u + C$
$y = \sinh u$	$\frac{dy}{du} = \cosh u$	$\int \cosh u\,du = \sinh u + C$
$y = \cosh u$	$\frac{dy}{du} = \sinh u$	$\int \sinh u\,du = \cosh u + C$
$y = \tanh u$	$\frac{dy}{du} = \operatorname{sech}^2 u$	$\int \operatorname{sech}^2 u\,du = \tanh u + C$

Table II	Derivatives and Integrals	
Function $f(u)$	**Derivative**	**Integral**
$y = \coth u$	$\dfrac{dy}{du} = -\operatorname{csch}^2 u$	$\int \operatorname{csch}^2 u\, du = -\coth u + C$
$y = \operatorname{sech} u$	$\dfrac{dy}{du} = -\operatorname{sech} u \tanh u$	$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$
$y = \operatorname{csch} u$	$\dfrac{dy}{du} = -\operatorname{csch} u \coth u$	$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$
$y = \sinh^{-1} u$ $y = \ln(u + \sqrt{1+u^2})$	$\dfrac{dy}{du} = \dfrac{1}{\sqrt{1+u^2}}$	$\int \dfrac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + C$
$y = \cosh^{-1} u$ $y = \ln(u + \sqrt{u^2-1})$	$\dfrac{dy}{du} = \dfrac{1}{\sqrt{u^2-1}}$	$\int \dfrac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$
$y = \tanh^{-1} u$ $y = \dfrac{1}{2}\ln\left(\dfrac{1+u}{1-u}\right)$	$\dfrac{dy}{du} = \dfrac{1}{1-u^2}$	$\int \dfrac{du}{1-u^2} = \tanh^{-1} u + C$
$y = \coth^{-1} u$ $y = \dfrac{1}{2}\ln\left(\dfrac{u+1}{u-1}\right)$	$\dfrac{dy}{du} = \dfrac{-1}{u^2-1}$	$\int \dfrac{du}{u^2-1} = -\coth^{-1} u + C$
$y = \operatorname{sech}^{-1} u$ $y = \cosh^{-1} \dfrac{1}{u}$	$\dfrac{dy}{du} = \dfrac{-1}{u\sqrt{1-u^2}}$	$\int \dfrac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1} u + C$
$y = \operatorname{csch}^{-1} u$ $y = \sinh^{-1} \dfrac{1}{u}$	$\dfrac{dy}{du} = \dfrac{-1}{u\sqrt{u^2+1}}$	$\int \dfrac{du}{u\sqrt{u^2+1}} = -\operatorname{csch}^{-1} u + C$

Example 3-6. In the above tables of derivative and integrals the symbol u is a dummy variable of integration. If $u = u(x)$ is a function of x, then to use an integration formula from the above table there may be occasions where it is necessary to **scale the integral** to be evaluated in order that it **agree exactly** with the form given in the above tables.

(a) To evaluate the integral $I_a = \int (5x^2+7)^5\, x\, dx$ one can make the substitution $u = 5x^2 + 7$ and make sure that **the correct differential** $du = 10x\, dx$ is used in the integral formula. This may or may not require that scaling by a constant be performed. Observe that the given integral needs a constant factor of 10 to have the correct du to go along with the u specified. Consequently, one can multiply and divide by 10 in order to change the form of the given integral. This gives

$$I_a = \frac{1}{10}\int (5x^2+7)^5\,(10x\,dx) = \frac{1}{10}\int u^5\,du = \frac{1}{10}\frac{u^6}{6} + C = \frac{1}{60}(5x^2+7)^6 + C$$

(b) In a similar fashion the integral $I_b = \displaystyle\int e^{3x^2}\,x\,dx$ is evaluated. If one makes the substitution $u = 3x^2$, then $du = 6x\,dx$ is the required form necessary to use the above table. This again requires that some type of scaling be performed. One can write

$$I_b = \frac{1}{6}\int e^{3x^2}(6x\,dx) = \frac{1}{6}\int e^u\,du = \frac{1}{6}e^u + C = \frac{1}{6}e^{3x^2} + C$$

(c) To evaluate the integral $I_c = \displaystyle\int \sin(x^4)\,x^3\,dx$ make the substitution $u = x^4$ with $du = 4x^3\,dx$ and then scale the given integral by writing

$$I_c = \frac{1}{4}\int \sin(x^4)(4x^3\,dx) = \frac{1}{4}\int \sin u\,du = \frac{-1}{4}\cos u + C = \frac{-1}{4}\cos(x^4) + C$$

(d) To evaluate the integral $I_d = \displaystyle\int \sin\beta x\,dx$ let $u = \beta x$ with $du = \beta\,dx$ and scale the integral by writing

$$I_d = \frac{1}{\beta}\int \sin\beta x\,\beta dx = \frac{1}{\beta}\int \sin u\,du = -\frac{1}{\beta}\cos\beta x + C$$

(e) Each of the above integrals has been scaled and placed into the form

$$\alpha\int f(g(x))g'(x)\,dx$$

where α is some scaling constant. These type of integrals occur quite frequently and when you recognize them it is customary to make the substitution $u = g(x)$ with $du = g'(x)\,dx$ and simplify the integral to the form

$$\alpha\int f(u)\,du$$

Always perform scaling if necessary to get the correct form for du.

■

Trigonometric Substitutions

The integration tables given above can **be expanded by developing other types of integrals**. The appendix C gives an extended table of integrals representing just a sampling of the thousands of integrals that have been constructed since calculus was created.

Always examine the integrand of an integral and try to learn some of the algebraic and trigonometric forms that can be converted to integrals of a simpler type.

All of the trigonometric identities that you have learned are available and can be thought of as possible aids for evaluating integrals where the integrand involves trigonometric functions.

One type of integrand to look for is the **powers of the trigonometric functions**. Recall the de Moivre[1] theorem that states

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx \tag{3.28}$$

where i is an imaginary unit satisfying $i^2 = -1$. Let $\cos x + i \sin x = y$ and then multiply both sides of this equation by $\cos x - i \sin x$ to obtain

$$(\cos x - i \sin x)(\cos x + i \sin x) = y\,(\cos x - i \sin x)$$

and then expand the left-hand side to show that if

$$\cos x + i \sin x = y \qquad \text{then} \qquad \cos x - i \sin x = \frac{1}{y} \tag{3.29}$$

An addition and subtraction of the equations (3.29) produces the relations

$$2\cos x = y + \frac{1}{y} \qquad \text{and} \qquad 2i \sin x = y - \frac{1}{y} \tag{3.30}$$

Apply de Moivre's theorem to the quantities y and $1/y$ from equation (3.29) to show

$$\cos nx + i \sin nx = y^n \qquad \text{and} \qquad \cos nx - i \sin nx = \frac{1}{y^n} \tag{3.31}$$

Adding and subtracting the equations (3.31) gives the relations

$$2\cos nx = y^n + \frac{1}{y^n} \qquad \text{and} \qquad 2i \sin nx = y^n - \frac{1}{y^n} \tag{3.32}$$

where n is an integer. The above relations can now be employed to calculate **trigonometric identities for powers of** $\sin x$ **and** $\cos x$. Recall the powers of the imaginary unit i are represented $i^2 = -1$, $i^3 = -i$, $i^4 = -i^2 = 1, i^5 = i$, etc, so that the mth power of either $2i \sin x$ or $2\cos x$ can be calculated by employing the binomial expansion to expand the terms $\left(y - \frac{1}{y}\right)^m$ or $\left(y + \frac{1}{y}\right)^m$ and then using the relations from equations (3.32) to simplify the results.

[1] Abraham de Moivre (1667-1754) a French mathematician.

Using the results from the equations (3.30) one can verify the following algebraic operations

$$
\begin{aligned}
2^2 i^2 \sin^2 x =& \left(y - \frac{1}{y}\right)^2 = \left(y^2 + \frac{1}{y^2}\right) - 2 = 2\cos 2x - 2 \\
\text{or} \qquad \sin^2 x =& \frac{1}{2}(1 - \cos 2x)
\end{aligned}
\tag{3.33}
$$

In a similar fashion show that

$$
\begin{aligned}
2^3 i^3 \sin^3 x =& \left(y - \frac{1}{y}\right)^3 = y^3 - 3y + \frac{3}{y} - \frac{1}{y^3} \\
8(-i)\sin^3 x =& \left(y^3 - \frac{1}{y^3}\right) - 3\left(y - \frac{1}{y}\right) = 2i\sin 3x - 3(2i\sin x) \\
\text{or} \qquad \sin^3 x =& \frac{3}{4}\sin x - \frac{1}{4}\sin 3x
\end{aligned}
\tag{3.34}
$$

To calculate the fourth power of $\sin x$ write

$$
\begin{aligned}
2^4 i^4 \sin^4 x =& \left(y - \frac{1}{y}\right)^4 = y^4 - 4y^3 + 6 - \frac{4}{y^2} + \frac{1}{y^4} \\
16\sin^4 x =& \left(y^4 + \frac{1}{y^4}\right) - 4\left(y^2 + \frac{1}{y^2}\right) + 6 = 2\cos 4x - 4(2\cos 2x) + 6 \\
\text{or} \qquad \sin^4 x =& \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x
\end{aligned}
\tag{3.35}
$$

In summary, the use of de Moivre's theorem together with some algebra produced the trigonometric identities

$$
\begin{aligned}
\sin^2 x =& \frac{1}{2}(1 - \cos 2x) \\
\sin^3 x =& \frac{3}{4}\sin x - \frac{1}{4}\sin 3x \\
\sin^4 x =& \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x
\end{aligned}
\tag{3.36}
$$

In a similar fashion one can use the results from equation (3.30) and establish the following identities

$$
\begin{aligned}
2^2\cos^2 x = \left(y + \frac{1}{y}\right)^2 \quad &\Longrightarrow \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \\
2^3\cos^3 x = \left(y + \frac{1}{y}\right)^3 \quad &\Longrightarrow \quad \cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x \\
2^4\cos^4 x = \left(y + \frac{1}{y}\right)^4 \quad &\Longrightarrow \quad \cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x
\end{aligned}
\tag{3.37}
$$

Verifying the above results is left as an exercise. The calculation of representations for higher powers of $\sin x$ and $\cos x$ are obtained using an expansion similar to the above examples.

Example 3-7. Evaluate the integrals $\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$

Solution Using the trigonometric identities for $\sin^2 x$ and $\cos^2 x$ from equations (3.36) and (3.37) one can write

$$\begin{aligned}\int \sin^2 x\,dx =&\frac{1}{2}\int (1-\cos 2x)\,dx\\ \int \sin^2 x\,dx =&\frac{1}{2}\int dx - \frac{1}{4}\int \cos 2x\,2dx\\ \int \sin^2 x\,dx =&\frac{1}{2}x - \frac{1}{4}\sin 2x + C\end{aligned}$$

$$\begin{aligned}\int \cos^2 x\,dx =&\frac{1}{2}\int (1+\cos 2x)\,dx\\ \int \cos^2 x\,dx =&\frac{1}{2}\int dx + \frac{1}{4}\int \cos 2x\,2dx\\ \int \cos^2 x\,dx =&\frac{1}{2}x + \frac{1}{4}\sin 2x + C\end{aligned}$$

where C represents an arbitrary constant of integration.

■

Example 3-8. Evaluate the integrals $\int \sin^3 x\,dx$ and $\int \cos^3 x\,dx$

Solution Using the trigonometric identities for $\sin^3 x$ and $\cos^3 x$ from equations (3.36) and (3.37) one can write

$$\begin{aligned}\int \sin^3 x\,dx =& \int \left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right)\,dx\\ \int \sin^3 x\,dx =&\frac{3}{4}\int \sin x\,dx - \frac{1}{12}\int \sin 3x\,3dx\\ \int \sin^3 x\,dx =& -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C\end{aligned}$$

$$\begin{aligned}\int \cos^3 x\,dx =& \int (\frac{3}{4}\cos x + \frac{1}{4}\cos 3x)\,dx\\ \int \cos^3 x\,dx =&\frac{3}{4}\int \cos x + \frac{1}{12}\int \cos 3x\,3dx\\ \int \cos^3 x\,dx =&\frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C\end{aligned}$$

■

Example 3-9. Using the substitutions for $\sin^4 x$ and $\cos^4 x$ from the equation (3.36) and (3.37) one can verify the integrals

$$\begin{aligned}\int \sin^4 x\,dx =&\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C\\ \int \cos^4 x\,dx =&\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C\end{aligned}$$

■

Trigonometric substitution is just one of many methods which can be applied to aid in the evaluation of an integral where the integrand contains trigonometric functions.

Example 3-10. Alternative methods for the integration of odd powers of $\sin x$ and $\cos x$ involve using the trigonometric identity $\sin^2 x + \cos^2 x = 1$ as illustrated below.

(a) Integral of odd power of $\sin x$

$$\int \sin^{2n+1} x\, dx = \int (\sin^2 x)^n \, \sin x\, dx = \int (1-\cos^2 x)^n \, \sin x\, dx$$

Make the substitution $\xi = \cos x$ with $d\xi = -\sin x\, dx$ and express the above integral in the form

$$\int \sin^{2n+1} x\, dx = -\int (1-\xi^2)^n \, d\xi, \qquad \xi = \cos x$$

The quantity $(1-\xi^2)^n$ can be expanded by the binomial expansion. This creates a sum of integrals, each of the form $\int \xi^m \, d\xi = \frac{\xi^{m+1}}{m+1}$ where m is some constant integer.

(b) Integral of odd power of $\cos x$

$$\int \cos^{2n+1} x\, dx = \int (\cos^2 x)^n \, \cos x\, dx = \int (1-\sin^2 x)^n \, \cos x\, dx$$

Make the substitution $\xi = \sin x$ with $d\xi = \cos x\, dx$ and express the above integral in the form

$$\int \cos^{2n+1} x\, dx = \int (\cos^2 x)^n \, \cos x\, dx = \int (1-\xi^2)^n \, d\xi, \qquad \xi = \sin x$$

Expand the quantity $(1-\xi^2)^n$ using the binomial theorem and then like the previous example integrate each term of the expansion. Note that each term is again an integral of the form $\int \xi^m \, d\xi = \frac{\xi^{m+1}}{m+1}$.

■

Products of Sines and Cosines

To evaluate integrals which are products of the sine and cosine functions such as

$$\int \sin mx \, \sin nx \, dx, \qquad \int \sin mx \, \cos nx \, dx, \qquad \int \cos mx \, \cos nx \, dx$$

one can use the **addition and subtraction formulas from trigonometry**

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
\sin(A-B) &= \sin A \cos B - \cos A \sin B \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
\cos(A-B) &= \cos A \cos B + \sin A \sin B
\end{aligned} \tag{3.38}$$

to obtain the product relations

$$\begin{aligned}\sin mx\,\sin nx =&\frac{1}{2}\left[\cos(m-n)x-\cos(m+n)x\right]\\ \sin mx\,\cos nx =&\frac{1}{2}\left[\sin(m-n)x+\sin(m+n)x\right]\\ \cos mx\,\cos nx =&\frac{1}{2}\left[\cos(m-n)x+\cos(m+n)x\right]\end{aligned}\tag{3.39}$$

which, with proper scaling, reduce the above integrals to forms involving simple integration of sine and cosine functions.

Example 3-11. Evaluate the integral $I=\int \sin 5x\,\sin 3x\,dx$

Solution Using the above trigonometric substitution one can write

$$I=\int \frac{1}{2}\left[\cos 2x-\cos 8x\right]\,dx=\frac{1}{4}\int \cos 2x\,2dx-\frac{1}{16}\int \cos 8x\,8dx$$

to obtain, after proper scaling of the integrals,

$$I=\int \sin 5x\,\sin 3x\,dx=\frac{1}{4}\sin 2x-\frac{1}{16}\sin 8x+C$$

■

Special Trigonometric Integrals

Examining the previous tables of derivatives and integrals one finds that integrals of the trigonometric functions $\tan x$, $\cot x$, $\sec x$ and $\csc x$ are missing. Let us examine the integration of each of these functions.

Integrals of the form $\int \boldsymbol{\tan u\,du}$

To evaluate this integral express it in the form $\int \frac{dw}{w}$ as this is a form which can be found in the previous tables. Note that

$$\int \tan u\,du=\int \frac{\sin u}{\cos u}\,du=-\int \frac{d(\cos u)}{\cos u}=-\ln|\cos u|+C$$

An alternative approach is to write

$$\int \tan u\,du=\int \frac{\sec u\tan u}{\sec u}\,du=\int \frac{d(\sec u)}{\sec u}=\ln|\sec u|+C$$

Therefore one can write

$$\boldsymbol{\int \tan u\,du=-\ln|\cos u|+C=\ln|\sec u|+C}$$

The reason that there are two equivalent representations for the integral of the tangent function is because of a trigonometric identity and properties of the logarithm function. Note that $\cos u \sec u = 1$ and taking logarithms gives

$$\begin{aligned} \ln(\cos u \sec u) &= \ln 1 \\ \ln|\cos u| + \ln|\sec u| &= 0 \\ \text{or} \qquad \ln|\sec u| &= -\ln|\cos u| \end{aligned}$$

Integrals of the form $\displaystyle\int \boldsymbol{\cot u\, du}$

The integral of the cotangent function is treated much the same way as the integral of the tangent function. One can write

$$\int \cot u\, du = \int \frac{\cos u}{\sin u}\, du = \int \frac{d(\sin u)}{\sin u} = \ln|\sin u| + C$$

One can then show that

$$\boldsymbol{\int \cot u\, du = \ln|\sin u| + C = -\int \frac{-\csc u\, \cot u}{\csc u}\, du = -\ln|\csc u| + C}$$

From this result can you determine a relationship between $\ln|\sin u|$ and $-\ln|\csc u|$?

Integrals of the form $\displaystyle\int \boldsymbol{\sec u\, du}$

The integral of the secant function can be expressed in the form $\int \frac{dw}{w}$ by writing

$$\int \sec u\, du = \int \sec u \, \frac{\sec u + \tan u}{\sec u + \tan u}\, du = \int \frac{\sec u \tan u + \sec^2 u}{\sec u + \tan u}\, du$$

so that

$$\boldsymbol{\int \sec u\, du = \int \frac{d(\sec u + \tan u)}{\sec u + \tan u} = \ln|\sec u + \tan u| + C}$$

Integrals of the form $\displaystyle\int \boldsymbol{\csc u\, du}$

In a similar fashion one can verify that

$$\boldsymbol{\int \csc u\, du = -\int \frac{d(\csc u + \cot u)}{\csc u + \cot u} = -\ln|\csc u + \cot u| + C}$$

Method of Partial Fractions

The **method of partial fractions** is used to integrate rational functions $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions and the degree of $P(x)$ is **less than** the degree of $Q(x)$. If a rational function $\dfrac{R(x)}{Q(x)}$ is such that the degree of $R(x)$ is **greater than** the degree of $Q(x)$, then one must use long division and write the rational function in the form

$$\frac{R(x)}{Q(x)} = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n + \frac{P(x)}{Q(x)}$$

where now $P(x)$ is a remainder term with the degree of $P(x)$ **less than** the degree of $Q(x)$ and our object is to integrate each term of the above representation.

Example 3-12. The function $y = \dfrac{x^3}{(x-1)(x-2)}$ is a rational function with degree of numerator greater than the degree of the denominator. One can use long division and write

$$y = \frac{x^3}{x^2 - 3x + 2} = x + 3 + \frac{7x - 6}{(x-1)(x-2)}$$

■

Recall from the study of algebra that when one sums fractions it is customary to get a common denominator and then sum the numerators. In developing integration techniques for rational functions the algebra mentioned above is reversed. It has been found that to integrate a rational function $f(x) = \frac{P(x)}{Q(x)}$, where the degree of $P(x)$ is less than the degree of $Q(x)$, it is easier to first factor the numerator and denominator terms and then split the fraction into the sum of fractions with simpler denominators. The function $f(x)$ is then said to have been converted into its **simplest fractional component form** and these resulting fractions are called the **partial fractions associated with the given rational function**. The following cases are considered.

Case 1 ***The denominator $Q(x)$ has only* first degree factors*, none of which are repeated. For example, $Q(x)$ has the form***

$$\boldsymbol{Q(x) = (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)}$$

where $x_0 \neq x_1 \neq x_2 \neq \cdots \neq x_n$. One can then write

$$\boldsymbol{f(x) = \frac{P(x)}{Q(x)} = \frac{A_0}{x - x_0} + \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \cdots + \frac{A_n}{x - x_n}}$$

where $A_0, A_1, A_2, \ldots, A_n$ are constants to be determined.

Example 3-13. Evaluate the integral $I = \displaystyle\int \frac{11x - 43}{x^2 - 6x + 5}\,dx$

Solution

Here the integrand $f(x) = \dfrac{11x - 43}{x^2 - 6x + 5}$ is a rational function with the degree of the numerator less than the degree of the denominator. Observe that the denominator has linear factors and so one can write

$$f(x) = \frac{11x - 43}{x^2 - 6x + 5} = \frac{11x - 43}{(x-1)(x-5)} = \frac{A_1}{x - 1} + \frac{A_2}{x - 5} \tag{3.40}$$

where A_1, A_2 are constants to be determined. Multiply both sides of equation (3.40) by the factor $(x - 1)$ and show

$$\frac{11x - 43}{x - 5} = A_1 + \frac{A_2(x - 1)}{x - 5} \tag{3.41}$$

Evaluate equation (3.41) using the value $x = 1$ to show $A_1 = 8$. Next multiply equation (3.40) on both sides by the other factor $(x-5)$ and show

$$\frac{11x-43}{x-1} = \frac{A_1(x-5)}{x-1} + A_2 \tag{3.42}$$

Evaluate the equation (3.42) using the value $x = 5$ to show $A_2 = 3$. One can then write

$$I = \int \frac{11x-43}{x^2-6x+5}\,dx = \int \left[\frac{8}{x-1} + \frac{3}{x-5}\right]\,dx = 8\int \frac{dx}{x-1} + 3\int \frac{dx}{x-5}$$

Both integrals on the right-hand side of this equation are of the form $\displaystyle\int \frac{du}{u}$ and consequently one finds

$$I = 8\ln|x-1| + 3\ln|x-5| + C$$

where C is a constant of integration. Observe that C is an arbitrary constant and so one can replace C by $\ln K$, to make the algebra easier, where $K > 0$ is also an arbitrary constant. This is done so that all the terms in the solution will be logarithm terms and therefore can be combined. This results in the solution being expressed in the form

$$I = \ln\left|K(x-1)^8(x-5)^3\right|$$

■

Case 2 ***The denominator $Q(x)$ has only*** **first degree factors*****, but some of these factors may be*** **repeated factors*****. For example, the denominator $Q(x)$ might have a form such as***

$$\boldsymbol{Q(x) = (x-x_0)^k(x-x_1)^\ell \cdots (x-x_n)^m}$$

where $k, \ell, \ldots, m$ are integers. Here the denominator has repeated factors of orders $k, \ell, \cdots, m$. In this case one can write the rational function in the form

$$\begin{aligned}
\boldsymbol{f(x) = \frac{P(x)}{Q(x)}} =& \boldsymbol{\frac{A_1}{x-x_0} + \frac{A_2}{(x-x_0)^2} + \cdots + \frac{A_k}{(x-x_0)^k}}\\
&+ \boldsymbol{\frac{B_1}{x-x_1} + \frac{B_2}{(x-x_1)^2} + \cdots + \frac{B_\ell}{(x-x_1)^\ell}}\\
&+ \qquad \cdots\\
&+ \boldsymbol{\frac{C_1}{x-x_n} + \frac{C_2}{(x-x_n)^2} + \cdots + \frac{C_m}{(x-x_n)^m}}
\end{aligned}$$

where $A_1, \ldots, A_k$, $B_1, \ldots, B_\ell, \ldots, C_1, \ldots, C_m$ are constants to be determined.

Example 3-14. Evaluate the integral $I = \int \frac{8x^4 - 132x^3 + 673x^2 - 1183x + 560}{(x-2)^3(x-8)(x-9)}\,dx$

Solution

Using the method of partial fractions the integrand can be expressed in the simpler form

$$\frac{8x^4 - 132x^3 + 673x^2 - 1183x + 560}{(x-2)^3(x-8)(x-9)} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3} + \frac{B_1}{x-8} + \frac{C_1}{x-9} \tag{3.43}$$

where A_1, A_2, A_3, B_1, C_1 are constants to be determined. The constants B_1 and C_1 are found as in the previous example. One can verify that

$$\begin{aligned} C_1 =& \frac{8x^4 - 132x^3 + 673x^2 - 1183x + 560}{(x-2)^3(x-8)}\bigg|_{x=9} = 2 \\ \text{and} \qquad B_1 =& \frac{8x^4 - 132x^3 + 673x^2 - 1183x + 560}{(x-2)^3(x-9)}\bigg|_{x=8} = 3 \end{aligned}$$

One can then write

$$\frac{8x^4 - 132x^3 + 673x^2 - 1183x + 560}{(x-2)^3(x-8)(x-9)} - \frac{2}{x-9} - \frac{3}{x-8} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3}$$

which simplifies to

$$\frac{3x^2 - 8x + 3}{(x-2)^3} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3} \tag{3.44}$$

Multiply both sides of equation (3.44) by $(x-2)^3$ to obtain

$$3x^2 - 8x + 3 = A_1(x-2)^2 + A_2(x-2) + A_3 \tag{3.45}$$

Differentiate equation (3.45) and show

$$6x - 8 = 2A_1(x-2) + A_2 \tag{3.46}$$

Differentiate equation (3.46) and show

$$6 = 2A_1 \tag{3.47}$$

giving $A_1 = 3$. Evaluate equations (3.45) and (3.46) at $x = 2$ to show $A_3 = -1$ and $A_2 = 4$. The given integral can now be represented in the form

$$I = 3\int \frac{dx}{x-2} + 4\int \frac{dx}{(x-2)^2} - \int \frac{dx}{(x-2)^3} + 3\int \frac{dx}{x-8} + 2\int \frac{dx}{x-9}$$

where each term can be integrated to obtain

$$I = 3\ln|x-2| - \frac{4}{x-2} + \frac{1}{2}\frac{1}{(x-2)^2} + 3\ln|x-8| + 2\ln|x-9| + C$$

or

$$I = \ln\left|(x-2)^3(x-8)^3(x-9)^2\right| + \frac{1}{2}\frac{1}{(x-2)^2} - \frac{4}{x-2} + C$$

■

Case 3 ***The denominator*** $\boldsymbol{Q(x)}$ ***has one or more*** **quadratic factors** ***of the form*** $\boldsymbol{ax^2+bx+c}$ ***none of which are repeated. In this case, for each quadratic factor there corresponds a partial fraction of the form***

$$\boldsymbol{\frac{A_0x+B_0}{ax^2+bx+c}}$$

where $\boldsymbol{A_0}$ ***and*** $\boldsymbol{B_0}$ ***are constants to be determined.***

Example 3-15. Evaluate the integral $I = \displaystyle\int \frac{11x^2+18x+43}{(x-1)(x^2+2x+5)}\,dx$

Solution

Use partial fractions and express the integrand in the form

$$\frac{11x^2+18x+43}{(x-1)(x^2+2x+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+5}$$

where A, B, C are constants to be determined. As in the previous example, the constant A is given by

$$A = \frac{11x^2+18x+43}{(x^2+2x+5)}\bigg|_{x=1} = 9$$

One can then write

$$\frac{11x^2+18x+43}{(x-1)(x^2+2x+5)} - \frac{9}{x-1} = \frac{Bx+C}{x^2+2x+5}$$

Simplify the left-hand side of this last equation and show

$$\frac{2x+2}{x^2+2x+5} = \frac{Bx+C}{x^2+2x+5}$$

giving $B=2$ and $C=2$. Here partial fractions were use to convert the given integral to the form

$$I = \int \frac{8}{x-1}\,dx + \int \frac{2x+2}{x^2+2x+5}\,dx$$

which can be easily integrated to obtain $I = 9\ln|x-1| + \ln|x^2+2x+5| + \ln K$. This result can be further simplified and one finds $I = \ln\left|K(x-1)^9(x^2+2x+5)\right|$ where K is an arbitrary constant.

■

Case 4 ***The denominator $Q(x)$ has one or more*** **quadratic factors*, some of which are*** **repeated quadratic factors*. In this case, for each repeated quadratic factor $(ax^2+bx+c)^k$ there corresponds a sum of partial fractions of the form***

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

where $A_1, B_1, \ldots, A_k, B_k$ are constants to be determined.

Before giving an example of this last property let us investigate the use of partial fractions to evaluate **special integrals** which arise during the application of case 4 above. These special integrals will then be summarized in a table for later reference.

Integrals of the form $\displaystyle\int \frac{dx}{\beta^2 - x^2}$, where $x < \beta$.

Use partial fractions and write

$$\begin{aligned}\int \frac{dx}{\beta^2-x^2} =& \frac{1}{2\beta}\int\left(\frac{1}{\beta+x} + \frac{1}{\beta-x}\right)dx \\ =& \frac{1}{2\beta}\left[\ln|\beta+x| - \ln|\beta-x|\right] + C\end{aligned}$$

$$\text{so that} \quad \int \frac{dx}{\beta^2-x^2} = \frac{1}{2\beta}\ln\left|\frac{\beta+x}{\beta-x}\right| + C = -\frac{1}{\beta}\tanh^{-1}\left(\frac{x}{\beta}\right) + C, \quad x<\beta$$

See for example the previous result of equation (2.103) with x replaced by x/β.

Integrals of the form $\displaystyle\int \frac{dx}{x^2 - \beta^2}$, where $x > \beta$.

Use partial fractions and show

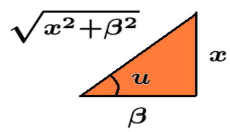
$$\begin{aligned}\int \frac{dx}{x^2-\beta^2} =& \frac{1}{2\beta}\int\left(\frac{1}{x-\beta} - \frac{1}{x+\beta}\right)dx \\ =& \frac{1}{2\beta}\left[\ln|x-\beta| - \ln|x+\beta|\right] + C\end{aligned}$$

This can be simplified to one of the forms

$$\int \frac{dx}{x^2-\beta^2} = \frac{1}{2\beta}\ln\left|\frac{x-\beta}{x+\beta}\right| + C = -\frac{1}{\beta}\coth^{-1}\left(\frac{x}{\beta}\right) + C = -\frac{1}{\beta}\tanh^{-1}\left(\frac{\beta}{x}\right) + C, \quad x>\beta$$

Here the previous results from equation (2.103) have been used to produce the alternative form above.

Integrals of the form $\displaystyle\int \frac{dx}{x^2+\beta^2}$



$\cos u = \frac{\beta}{\sqrt{x^2+\beta^2}}$

$\sec u = \frac{\sqrt{x^2+\beta^2}}{\beta}$

Make the substitution $x = \beta \tan u$ with $dx = \beta \sec^2 u\, du$ to obtain

$$\int \frac{dx}{x^2+\beta^2} = \int \frac{\beta \sec^2 u\, du}{\beta^2(\tan^2 u + 1)} = \frac{1}{\beta}\int du = \frac{1}{\beta}u + C$$

where $u = \tan^{-1}\left(\frac{x}{\beta}\right)$. Therefore one can write

$$\int \frac{dx}{x^2+\beta^2} = \frac{1}{\beta}\tan^{-1}\left(\frac{x}{\beta}\right) + C$$

or by constructing a right triangle representing the substitution, one can write the equivalent forms

$$\int \frac{dx}{x^2+\beta^2} = \frac{1}{\beta}\cos^{-1}\frac{\beta}{\sqrt{x^2+\beta^2}} + C = \frac{1}{\beta}\sec^{-1}\frac{\sqrt{x^2+\beta^2}}{\beta} + C$$

Integrals of the form $\displaystyle\int \frac{dx}{(x^2+\beta^2)^2}$

Make the trigonometric substitution $x = \beta\tan\theta$ with $dx = \beta\sec^2\theta\, d\theta$ and show

$$\begin{aligned}\int \frac{dx}{(x^2+\beta^2)^2} =& \int \frac{\beta \sec^2\theta\, d\theta}{\beta^4(\tan^2\theta+1)^2} = \frac{1}{\beta^3}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \frac{1}{\beta^3}\int \cos^2\theta\, d\theta\\ =&\frac{1}{2\beta^3}\int (1+\cos 2\theta)\, d\theta = \frac{1}{2\beta^3}\left[\theta + \frac{1}{2}\sin 2\theta\right] = \frac{1}{2\beta^3}\left[\theta + \sin\theta\cos\theta\right]\end{aligned}$$

Using back substitution representing θ in terms of x one finds

$$\int \frac{dx}{(x^2+\beta^2)^2} = \frac{1}{2\beta^3}\left[\tan^{-1}\left(\frac{x}{\beta}\right) + \frac{\beta x}{x^2+\beta^2}\right] + C$$

where C is a general constant of integration added to make the result more general.

Integrals of the form $\displaystyle\int \frac{dx}{ax^2+bx+c}$

Integrals having the form $I = \int \frac{dx}{Q(x)}$, where $Q(x) = ax^2+bx+c$ is a quadratic factor, can be evaluated if one first performs a completing the square operation on the quadratic term. One finds that either

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right] \qquad \text{where } 4ac-b^2>0$$

$$\text{or} \quad ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right] \qquad \text{where } b^2-4ac>0$$

Case 1 If $4ac - b^2 > 0$, make the substitution $\beta^2 = \dfrac{4ac - b^2}{4a^2}$ so that

$$\int \frac{dx}{ax^2 + bx + c} = \int \frac{dx}{a\left[\left(x + \frac{b}{2a}\right)^2 + \beta^2\right]}$$

and then make the additional substitution $X = x + \frac{b}{2a}$ with $dX = dx$. One then obtains

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a}\int \frac{dX}{X^2 + \beta^2} = \frac{1}{a}\cdot\frac{1}{\beta}\tan^{-1}\left(\frac{X}{\beta}\right) + C$$

Back substitution and simplifying gives the result

$$\int \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}}\tan^{-1}\left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right) + C, \qquad 4ac - b^2 > 0$$

Case 2 If $b^2 - 4ac > 0$, make the substitution $\beta^2 = \dfrac{b^2 - 4ac}{4a^2}$ and write

$$\int \frac{dx}{ax^2 + bx + c} = \int \frac{dx}{a\left[\left(x + \frac{b}{2a}\right)^2 - \beta^2\right]}$$

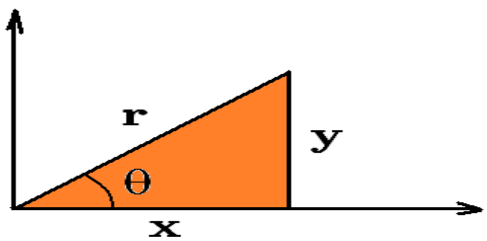
and then make the additional substitution $X = x + \frac{b}{2a}$ with $dX = dx$. This produces the simplified form

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a}\int \frac{dX}{X^2 - \beta^2} = \frac{1}{a}\cdot\frac{1}{\beta}\ln\left|\frac{X - \beta}{X + \beta}\right| + C$$

Back substitution and simplifying then gives the final result

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{\sqrt{b^2 - 4ac}}\ln\left|\frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}\right| + C, \qquad b^2 - 4ac > 0$$

Sums and Differences of Squares



Use the Pythagorean theorem and the definitions of the trigonometric functions as follows. In the right triangle illustrated one finds

$$x^2 + y^2 = r^2 \tag{3.48}$$

Divide each term of equation (3.48) by r^2 and write

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \qquad \Longrightarrow \qquad \cos^2\theta + \sin^2\theta = 1 \tag{3.49}$$

Divide each term of equation (3.48) by x^2 and write

$$\mathbf{1 + \frac{y^2}{x^2} = \frac{r^2}{x^2} \qquad \Longrightarrow \qquad 1 + \tan^2\theta = \sec^2\theta} \tag{3.50}$$

Divide each term of equation (3.48) by y^2 and write

$$\mathbf{\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2} \qquad \Longrightarrow \qquad \cot^2\theta + 1 = \csc^2\theta} \tag{3.51}$$

The above identities are known as the **Pythagorean identities** and can be used when one recognizes **sums and differences of squared quantities** in the integrand of an integral. Sometimes the integrand is simplified by using one of these identities.

Integrals of the form $\displaystyle\int \frac{dx}{\sqrt{\beta^2 - x^2}}$

Make the substitution $x = \beta\sin\theta$ with $dx = \beta\cos\theta\,d\theta$ to obtain

$$\int \frac{dx}{\sqrt{\beta^2 - x^2}} = \int \frac{\beta\cos\theta}{\sqrt{\beta^2 - \beta^2\sin^2\theta}}\,d\theta = \int \frac{\cos\theta}{\sqrt{1 - \sin^2\theta}}\,d\theta = \int d\theta = \theta + C = \sin^{-1}\frac{x}{\beta} + C$$

This gives the general result

$$\int \frac{du}{\sqrt{\beta^2 - u^2}} = \sin^{-1}\frac{u}{\beta} + C \tag{3.52}$$

Integrals of the form $\displaystyle\int \frac{dx}{\sqrt{x^2 + \beta^2}}$

Let $x = \beta\tan u$ with $dx = \beta\sec^2 u\,du$ and then form a right triangle with one angle u and appropriate sides of x and β. One can then show

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2 + \beta^2}} &= \int \frac{\beta\sec^2 u\,du}{\beta\sqrt{\tan^2 u + 1}} = \int \sec u\,du \\
&= \ln|\sec u + \tan u| + C_1 \\
&= \ln\left|\frac{x}{\beta} + \frac{\sqrt{x^2 + \beta^2}}{\beta}\right| + C_1 = \ln|x + \sqrt{x^2 + \beta^2}| + C
\end{aligned}$$

where $C = C_1 - \ln\beta$ is just some new constant. In general one can write

$$\int \frac{du}{\sqrt{u^2 + \beta^2}} = \ln|u + \sqrt{u^2 + \beta^2}| + C$$

Integrals of the form $\displaystyle\int \frac{dx}{\sqrt{x^2 - \beta^2}}$

Let $x = \beta\sec u$ with $dx = \beta\sec u\tan u\,du$ and form a right triangle with one angle u and appropriate sides x and β. One can then show

$$\int \frac{dx}{\sqrt{x^2-\beta^2}} = \int \frac{\beta \sec u \tan u\, du}{\beta\sqrt{\sec^2 u - 1}} = \int \sec u\, du = \ln|\sec u + \tan u| + C_1$$
$$= \ln|\frac{x}{\beta} + \frac{\sqrt{x^2-\beta^2}}{\beta}| + C_1 = \ln|x + \sqrt{x^2-\beta^2}| + C$$

where $C = C_1 - \ln\beta$ is some new constant of integration. In general, one can write

$$\boldsymbol{\int \frac{du}{\sqrt{u^2-\beta^2}} = \ln|u + \sqrt{u^2-\beta^2}| + C}$$

Example 3-16. Evaluate the integral $I = \displaystyle\int \frac{du}{u^4 + 18x^2 + 81}$

Solution Recognize the denominator is the square of (u^2+9) and write $I = \displaystyle\int \frac{du}{(u^2+9)^2}$

This is an integral of the form $\displaystyle\int \frac{dx}{(x^2+\beta^2)^2}$ previously investigated, so that one can write

$$I = \frac{1}{2\beta^3}\left[\tan^{-1}\left(\frac{x}{\beta}\right) + \frac{\beta x}{x^2+\beta^2}\right] + C$$

where $\beta = 3$.

■

Example 3-17. The Pythagorean identities can be employed when one recognizes the integrand has sums or differences of squared quantities. Sometimes it is necessary to **complete the square on quadratic terms** in order to obtain a sum or difference of squared terms. For example, to evaluate the integral

$$I = \int \frac{dx}{36x^2 + 48x + 41}$$

one can write

$$I = \frac{1}{36}\int \frac{dx}{x^2 + \frac{48}{36}x + \frac{41}{36}} = \frac{1}{36}\int \frac{dx}{x^2 + \frac{4}{3}x + \left(\frac{2}{3}\right)^2 + \frac{41}{36} - \left(\frac{2}{3}\right)^2} = \frac{1}{36}\int \frac{dx}{(x+\frac{2}{3})^2 + \frac{25}{36}}$$

One can now make the substitution $u = x + 2/3$ with $du = dx$ to obtain

$$I = \frac{1}{36}\int \frac{du}{u^2+\beta^2}, \quad \text{where} \quad \beta = 5/6$$

One then finds

$$I = \frac{1}{36}\frac{1}{\beta}\tan^{-1}\frac{u}{\beta} + C = \frac{1}{36}\frac{1}{5/6}\tan^{-1}\left(\frac{x+2/3}{5/6}\right) + C = \frac{1}{30}\tan^{-1}\left(\frac{6x+4}{5}\right) + C$$

■

Example 3-18. Evaluate the integral $I = \displaystyle\int \frac{4x^5 - 15x^4 + 13x^3 - 4x^2 + 13x + 89}{(x-3)^2(x^2+1)^2}\, dx$

Solution The denominator of the integrand has a repeated linear factor and a repeated quadratic factor and so by the properties of partial fractions one can write

$$\frac{4x^5 - 15x^4 + 13x^3 - 4x^2 + 13x + 89}{(x-3)^2(x^2+1)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} \tag{3.53}$$

where A, B, C, D, E, F are constants to be determined. Multiply both sides of equation(3.53) by $(x-3)^2$ and show

$$\frac{4x^5 - 15x^4 + 13x^3 - 4x^2 + 13x + 89}{(x^2+1)^2} = A(x-3) + B + (x-3)^2 \left[\frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}\right] \tag{3.54}$$

Evaluate equation (3.54) using the value $x = 3$ to find

$$B = \frac{4x^5 - 15x^4 + 13x^3 - 4x^2 + 13x + 89}{(x^2+1)^2}\bigg|_{x=3} = 2$$

The equation (3.53) can therefore be written as

$$\frac{4x^5 - 15x^4 + 13x^3 - 4x^2 + 13x + 89}{(x-3)^2(x^2+1)^2} - \frac{2}{(x-3)^2} = \frac{A}{x-3} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} \tag{3.55}$$

The left-hand side of equation (3.55) simplifies to the form

$$\frac{4x^4 - 5x^3 - 3x^2 - 14x - 29}{(x-3)(x^2+1)^2} = \frac{A}{x-3} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} \tag{3.56}$$

Multiply both sides of equation (3.56) by the factor $(x-3)$ and show

$$\frac{4x^4 - 5x^3 - 3x^2 - 14x - 29}{(x^2+1)^2} = A + (x-3)\left[\frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}\right] \tag{3.57}$$

Evaluating equation (3.57) using the value $x = 3$ gives the result

$$A = \frac{4x^4 - 5x^3 - 3x^2 - 14x - 29}{(x^2+1)^2}\bigg|_{x=3} = 1$$

Consequently, the equation (3.56) can be written in the form

$$\frac{4x^4 - 5x^3 - 3x^2 - 14x - 29}{(x-3)(x^2+1)^2} - \frac{1}{x-3} = \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} \tag{3.58}$$

The left-hand side of equation (3.58) simplifies to give the result

$$\frac{3x^3 + 4x^2 + 8x + 10}{(x^2+1)^2} = \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} \tag{3.59}$$

Multiply equation (3.59) on both sides by the factor $(x^2+1)^2$ and then expand the right-hand side of the equation to obtain

$$\begin{aligned} 3x^3+4x^2+8x+10 =&(Cx+D)(x^2+1)+(Ex+F) \\ \text{or} \qquad 3x^3+4x^2+8x+10 =&Cx^3+Dx^2+(E+C)x+(F+D) \end{aligned} \tag{3.60}$$

Comparing the left and right-hand sides of equation (3.60) one finds

$$C=3, \qquad D=4, \qquad E+C=8, \qquad F+D=10$$

From the last two equations one finds $E=5$ and $F=6$. All this algebra reduces the integrand of the given integral to a summation of simpler terms where each term can be easily integrated using a table of integrals if necessary.[2] One finds

$$I=\int \left(\frac{1}{x-3}+\frac{2}{(x-3)^2}+\frac{3x+4}{x^2+1}+\frac{5x+6}{(x^2+1)^2} \right) dx \tag{3.61}$$

The first integral in equation (3.61) is

$$\int \frac{dx}{x-3}=\ln|x-3| \tag{3.62}$$

and the second integral in equation (3.61) is

$$2\int \frac{dx}{(x-3)^2}=\frac{-2}{(x-3)} \tag{3.63}$$

The third integral in equation (3.61) needs to be scaled to get part of it in the form $\dfrac{du}{u}$ which can then be integrated. One can write

$$3\int \frac{(x+4/3)}{x^2+1}\,dx=\frac{3}{2}\int \frac{(2x+8/3)}{x^2+1}\,dx=\frac{3}{2}\int \frac{2x\,dx}{x^2+1}+4\int \frac{dx}{x^2+1}$$

Evaluating these integrals gives the result

$$\int \frac{3x+4}{x^2+1}\,dx=\frac{3}{2}\ln(x^2+1)+4\tan^{-1}x \tag{3.64}$$

The last integral in equation (3.61) can be scaled and written

$$\int \frac{5x+6}{(x^2+1)^2}\,dx=5\int \frac{(x+6/5)}{(x^2+1)^2}\,dx=\frac{5}{2}\int \frac{2x+12/5}{(x^2+1)^2}\,dx=\frac{5}{2}\int \frac{2x\,dx}{(x^2+1)^2}+6\int \frac{dx}{(x^2+1)^2}$$

[2] A table of integrals is given in the appendix C.

and then integrated to obtain

$$\int \frac{5x+6}{(x^2+1)^2}\,dx = \frac{5}{2}\left[\frac{-1}{x^2+1}\right] + 3\left[\tan^{-1}x + \frac{x}{x^2+1}\right] = \frac{6x-5}{2(x^2+1)} + 3\tan^{-1}x \tag{3.65}$$

Combining the above results gives

$$I = \ln|x-3| - \frac{2}{x-3} + \frac{3}{2}\ln(x^2+1) + 4\tan^{-1}x + \frac{6x-5}{2(x^2+1)} + 3\tan^{-1}x + C \tag{3.66}$$

where C is a constant of integration. To check that what has been done is correct one should note that the final result should satisfy $\frac{dI}{dx} = f(x)$, where $f(x)$ is the integrand of the original integral. This check is left as an exercise.

■

Note that in the case the denominator has a single linear factor $(x-a)$, then one can write $\frac{f(x)}{(x-a)g(x)} = \frac{A}{x-a} + \frac{\alpha(x)}{\beta(x)}$ where A is a constant which can be determined from the relation $\frac{f(x)}{g(x)} = A + (x-a)\frac{\alpha(x)}{\beta(x)}$ evaluated at $x=a$.

Example 3-19. Find the partial fraction expansion for representing a function having the form

$$f(x) = \frac{ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g}{(x-1)(x-2)(x-3)^3(x^2+x+1)(x^2+3x+1)^4}$$

where a, b, c, d, e, f, g are known constants.

Solution Here the denominator has the unrepeated linear factors $(x-1)$ and $(x-2)$. The linear factor $(x-3)$ is repeated three times. The quadratic factor (x^2+x+1) is unrepeated and the quadratic factor (x^2+3x+1) is repeated four times. Using the properties of partial fractions, represented by the previous cases 1 through 4, the form for the partial fraction representation of the given function is

$$\begin{aligned} f(x) =& \frac{A_0}{x-1} + \frac{B_0}{x-2} + \frac{C_0}{x-3} + \frac{D_0}{(x-3)^2} + \frac{E_0}{(x-3)^3} + \frac{F_0x+G_0}{x^2+x+1} \\ &+ \frac{A_1x+B_1}{x^2+3x+1} + \frac{A_2x+B_2}{(x^2+3x+1)^2} + \frac{A_3x+B_3}{(x^2+3x+1)^3} + \frac{A_4x+B_4}{(x^2+3x+1)^4} \end{aligned}$$

where $A_0, B_0, \ldots, A_4, B_4$ are constants to be determined.

■

The following table III is a summary of previous results.

Table III Summary of Integrals
$\displaystyle\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
$\displaystyle\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
$\displaystyle\int \sin^3 u\,du = -\frac{3}{4}\cos u + \frac{1}{12}\cos 3u + C$
$\displaystyle\int \cos^3 u\,du = \frac{3}{4}\sin u + \frac{1}{12}\sin 3u + C$
$\displaystyle\int \sin^4 u\,du = \frac{3}{8} - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u + C$
$\displaystyle\int \cos^4 u\,du = \frac{3}{8}u + \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u + C$
$\displaystyle\int \tan u\,du = -\ln \mid \cos u \mid + C$
$\displaystyle\int \cot u\,du = \ln \mid \sin u \mid + C$
$\displaystyle\int \sec u\,du = \ln \mid \sec u + \tan u \mid + C$
$\displaystyle\int \csc u\,du = \ln \mid \csc u + \cot u \mid + C$
$\displaystyle\int \frac{du}{\beta^2 - u^2} = \frac{1}{2\beta}\ln\left\lvert\frac{\beta+u}{\beta-u}\right\rvert + C$
$\displaystyle\int \frac{du}{u^2 - \beta^2} = \frac{1}{2\beta}\ln\left\lvert\frac{u-\beta}{u+\beta}\right\rvert + C$
$\displaystyle\int \frac{du}{u^2 + \beta^2} = \frac{1}{\beta}\tan^{-1}\left(\frac{u}{\beta}\right) + C$
$\displaystyle\int \frac{du}{(u^2 + \beta^2)^2} = \frac{1}{2\beta^3}\left[\tan^{-1}\left(\frac{u}{\beta}\right) + \frac{\beta u}{u^2+\beta^2}\right] + C$
$\displaystyle\int \frac{du}{au^2 + bu + c} = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2au+b}{\sqrt{4ac-b^2}}\right) + C, \qquad 4ac - b^2 > 0$
$\displaystyle\int \frac{du}{au^2 + bu + c} = \frac{1}{\sqrt{b^2-4ac}}\ln\left\lvert\frac{2au+b-\sqrt{b^2-4ac}}{2au+b+\sqrt{b^2-4ac}}\right\rvert + C, \qquad b^2 - 4ac > 0$
$\displaystyle\int \frac{du}{\sqrt{\beta^2 - u^2}} = \sin^{-1}\left(\frac{u}{\beta}\right) + C$
$\displaystyle\int \frac{du}{\sqrt{\beta^2 + u^2}} = \ln\left\lvert u + \sqrt{u^2+\beta^2}\right\rvert + C$
$\displaystyle\int \frac{du}{\sqrt{u^2 - \beta^2}} = \ln\left\lvert u + \sqrt{u^2-\beta^2}\right\rvert + C$

Integration by parts

If $d(UV) = U\,dV + V\,dU$ then one can write $U\,dV = d(UV) - V\,dU$ and so by integrating both sides of this last equation one obtains the result

$$\begin{aligned}\int \boldsymbol{U\,dV} &= \int \boldsymbol{d(UV)} - \int \boldsymbol{V\,dU}\\ \int \boldsymbol{U\,dV} &= \boldsymbol{UV} - \int \boldsymbol{V\,dU}\end{aligned} \tag{3.67}$$

The equation (3.67) is known as the **integration by parts formula**. Another form for the integration by parts formula is

$$\int \boldsymbol{U(x)V'(x)\,dx} = \boldsymbol{U(x)V(x)} - \int \boldsymbol{V(x)U'(x)\,dx} \tag{3.68}$$

When using integration by parts try to select $U = U(x)$ such that $V(x)U'(x)\,dx$ is easy to integrate. If this is not possible, then alternative methods of integration have to be investigated. Integration by parts is a powerful method for evaluating many types of integrals. Sometimes it is necessary to apply the method of integration by parts multiple times before a result is obtained.

Example 3-20. Evaluate the integral $I = \int \arctan x\,dx$

Solution

For the given example, let $U = \arctan x$ and $dV = dx$ then one can calculate

$$dU = d(\arctan x) = \frac{dx}{1+x^2} \qquad \text{and} \qquad \int dV = \int dx \quad \text{or} \quad V = x \tag{3.69}$$

Substituting the results from the equations (3.69) into the integration by parts formula (3.67) one finds

$$\int \arctan x\,dx = x\,\arctan x - \int \frac{x\,dx}{1+x^2}$$

In order to evaluate the last integral, use the integration formula $\int \frac{dU}{U} = \ln U + C$ and recognize that if $U = 1 + x^2$, then it is necessary that $dU = 2x\,dx$ and so a scaling must be performed on the last integral. Perform the necessary scaling and express the integration by parts formula in the form

$$\begin{aligned}\int \arctan x\,dx =& x\,\arctan x - \frac{1}{2}\int \frac{2x\,dx}{1+x^2}\\ =& x\,\arctan x - \frac{1}{2}\ln(1+x^2) + C\end{aligned}$$

where C is a constant of integration.

Note when using integration by parts and you perform an integration to find V, it is not necessary to add the constant of integration for if V is replaced by $V+C$ in equation (3.67) one would obtain

$$\int U\,dV = U\,(V+C) - \int (V+C)\,dU = U\,V + C\,U - \int V\,dU - C\,U$$

and the constant would disappear. **You can always add a general constant of integration after performing the last integral.** This is usually done to make the final result more general.

■

The integration by parts formula can be written in different ways. Using the rule for differentiation of a product, write

$$\frac{d}{dx}(UV) = U\frac{dV}{dx} + V\frac{dU}{dx} \quad \text{and consequently} \quad UV = \int U\frac{dV}{dx}\,dx + \int V\frac{dU}{dx}\,dx$$

or

$$\int \boldsymbol{U\frac{dV}{dx}\,dx = UV - \int V\frac{dU}{dx}\,dx} \tag{3.70}$$

which is the form for integration by parts previously presented. In equation (3.70) make the substitution $\frac{dV}{dx} = W(x)$ with $V(x) = \int W(x)\,dx$, then equation (3.70) takes on the form

$$\int \boldsymbol{U(x)W(x)\,dx = U(x)\left[\int W(x)\,dx\right] - \int \frac{dU}{dx}\left[\int W(x)\,dx\right]dx} \tag{3.71}$$

and interchanging the functions $U(x)$ and $W(x)$ gives the alternative result

$$\int \boldsymbol{U(x)W(x)\,dx = W(x)\left[\int U(x)\,dx\right] - \int \frac{dW}{dx}\left[\int U(x)\,dx\right]dx} \tag{3.72}$$

The above two integration by parts formulas tells us that to integrate a product of two functions one can select either of the equations (3.71) or (3.72) to aid in the evaluation of the integral. One usually selects from the above two formulas that formula which produces an easy to obtain result, if this is at all possible.

Example 3-21. Evaluate the integral $\int x^2 \sin nx\,dx$

Solution The integration by parts formula may be repeated many times to evaluate an integral. For the given integral one can employ integration by parts, with $U = x^2$ and $dV = \sin nx\,dx$ to obtain

$$\int x^2 \sin nx\, dx = x^2 \left(\frac{-\cos nx}{n}\right) - \int 2x \left(\frac{-\cos nx}{n}\right) dx$$

Use scaling and apply integration by parts on the last integral, with $U = 2x$ and $dV = -\frac{\cos nx}{n}\, dx$, to obtain

$$\int 2x \left(\frac{-\cos nx}{n}\right) dx = 2x \left(\frac{-\sin nx}{n^2}\right) - \int 2 \left(\frac{-\sin nx}{n^2}\right) dx$$

The last integral can be scaled and integrated. One finds that the last integral becomes

$$\int 2 \left(\frac{-\sin nx}{n^2}\right) dx = \frac{2}{n^3} \cos nx$$

Back substitution gives the results

$$\int x^2 \sin nx\, dx = x^2 \left(\frac{-\cos nx}{n}\right) - 2x \left(\frac{-\sin nx}{n^2}\right) + \frac{2}{n^3} \cos nx + C$$

where C is a general constant of integration which can be added at the end of any indefinite integral.

■

Reduction Formula

The use of the integration by parts formula $\int U\, dV = UV - \int V\, dU$ to evaluate an integral gives a representation of a first integral in terms of a second integral. Sometimes, when an integration by parts is performed on the second integral, one finds that it can be reduced to a form of the first integral. When this happens one can usually obtain a general formula, known as **a reduction formula**, for evaluating the first and sometimes the second integral.

Example 3-22. Evaluate the integral $I_m = \int \sin^m x\, dx$ where m is a positive integer.

Solution Write the integral as $I_m = \int \sin^{m-1} x \sin x\, dx$ and use integration by parts with

$$U = \sin^{m-1} x \qquad dV = \sin x\, dx$$
$$dU = (m-1) \sin^{m-2} x \cos x\, dx \qquad V = -\cos x$$

to obtain

$$I_m = -\sin^{m-1} x \cos x + (m-1) \int \sin^{m-2} x \cos^2 x\, dx$$
$$I_m = -\sin^{m-1} x \cos x + (m-1) \int \sin^{m-2} x\, (1 - \sin^2 x)\, dx$$
$$I_m = -\sin^{m-1} x \cos x + (m-1) \left[I_{m-2} - I_m\right]$$

and using algebra one can solve for I_m to obtain

$$I_m = \frac{-1}{m}\sin^{m-1} x\,\cos x + \frac{(m-1)}{m} I_{m-2}$$

or

$$\boldsymbol{\int \sin^m x\, dx = \frac{-1}{m}\sin^{m-1} x\,\cos x + \frac{(m-1)}{m}\int \sin^{m-2} x\, dx} \qquad \textbf{(3.73)}$$

This result is known as a reduction formula where the integral of a higher power of $\sin x$ is expressed in terms of an integral involving a lower power of $\sin x$.

Substitute $m = 2$ into the reduction formula (3.73) and show

$$\int \sin^2 x\, dx = \frac{-1}{2}\sin x\,\cos x + \frac{1}{2}\int dx = \frac{x}{2} - \frac{1}{2}\sin x\,\cos x + C \qquad (3.74)$$

where a general constant of integration has been added to the final result.

Substitute $m = 3$ into the reduction formula (3.73) gives

$$\int \sin^3 x\, dx = \frac{-1}{3}\sin^2 x\,\cos x + \frac{2}{3}\int \sin x\, dx = \frac{-1}{3}\sin^2 x\,\cos x - \frac{2}{3}\cos x + C \qquad (3.75)$$

where C is some general constant of integration that has been added to obtain a more general result. It is left as an exercise to show that the results given by equations (3.74) and (3.75) are alternative forms of the results obtained in the examples 3-5 and 3-6.

■

Example 3-23. Using integration by parts on the integral $J_m = \int \cos^m x\, dx$ one can verify the reduction formula

$$J_m = \frac{1}{m}\cos^{m-1} x\,\sin x + \frac{(m-1)}{m} J_{m-2}$$

or

$$\boldsymbol{\int \cos^m x\, dx = \frac{1}{m}\cos^{m-1} x\,\sin x + \frac{(m-1)}{m}\int \cos^{m-2} x\, dx}$$

■

Example 3-24. For m and n integers and held constant during the integration process, evaluate the integrals

$$S_m = \int x^m \sin nx\, dx \qquad \text{and} \qquad C_m = \int x^m \cos nx\, dx$$

Solution Use integration by parts on the S_m integral with $U = x^m$ and $dV = \sin nx\, dx$. One finds $dU = mx^{m-1}\, dx$ and $V = \dfrac{-\cos nx}{n}$ so that

$$S_m = -x^m \frac{\cos nx}{n} + \frac{m}{n}\int x^{m-1}\cos nx\, dx \quad \text{or} \quad S_m = -x^m \frac{\cos nx}{n} + \frac{m}{n} C_{m-1} \qquad (3.76)$$

An integration by parts applied to the C_m integral with $U = x^m$ and $dV = \cos nx\, dx$ produces $dU = mx^{m-1}\, dx$ and $V = \frac{1}{n}\sin nx$. The C_m integral then can be represented

$$C_m = x^m \frac{\sin nx}{n} - \frac{m}{n}\int x^{m-1}\sin nx\, dx \quad \text{or} \quad C_m = x^m\frac{\sin nx}{n} - \frac{m}{n}S_{m-1} \tag{3.77}$$

In the equations (3.76) and (3.77) replace m by $m-1$ everywhere and use the resulting equations to show

$$\begin{aligned} S_m =& -x^m\frac{\cos nx}{n} + \frac{m}{n}\left[x^{m-1}\frac{\sin nx}{n} - \frac{m-1}{n}S_{m-2}\right] \\ C_m =& x^m\frac{\sin nx}{n} - \frac{m}{n}\left[-x^{m-1}\frac{\cos nx}{n} + \frac{m-1}{n}C_{m-2}\right] \end{aligned}$$

which simplify to the reduction formulas

$$\boldsymbol{S_m = -x^m\frac{\cos nx}{n} + mx^{m-1}\frac{\sin nx}{n^2} - \frac{m(m-1)}{n^2}S_{m-2}} \tag{3.78}$$

$$\boldsymbol{C_m = x^m\frac{\sin nx}{n} + mx^{m-1}\frac{\cos nx}{n^2} - \frac{m(m-1)}{n^2}C_{m-2}} \tag{3.79}$$

These reduction formula can be used as follows. First show that

$$S_0 = \int \sin nx\, dx = -\frac{\cos nx}{n} \quad \text{and} \quad C_0 = \int \cos nx\, dx = \frac{\sin nx}{n}$$

and then use integration by parts to show

$$\begin{aligned} S_1 =& \int x\sin nx\, dx, & C_1 =& \int x\cos nx\, dx \\ S_1 =& -x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}, & C_1 =& x\frac{\sin nx}{n} + \frac{\cos nx}{n^2} \end{aligned}$$

where the general constants of integration have been omitted. Knowing S_0, S_1, C_0, C_1 the reduction equations (3.78) and (3.79) can be used to calculate

$$S_2,\ C_2,\ S_3,\ C_3,\ S_4,\ C_4, \ldots$$

■

The Definite Integral

Consider the problem of finding the area bounded by a given curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis. The area to be determined is illustrated in the figure 3-2.

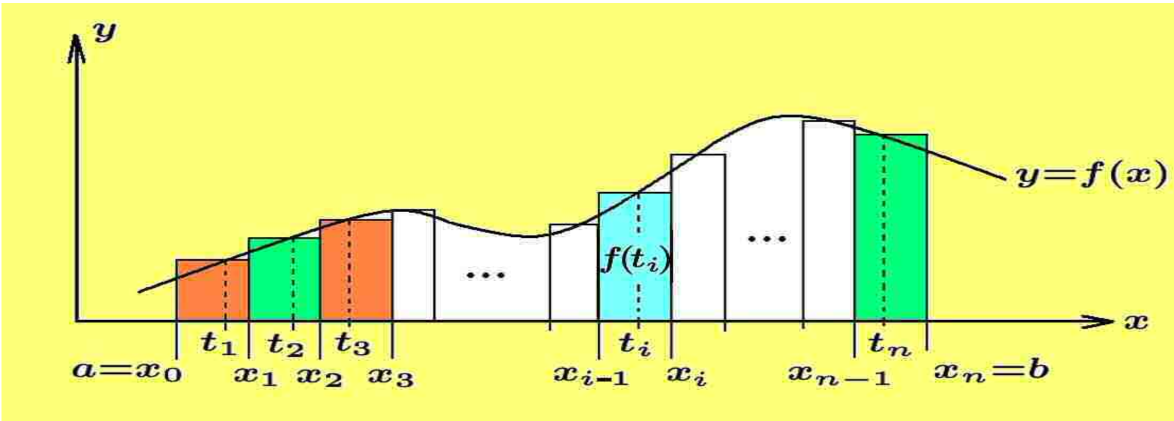


Figure 3-2. Area under curve and partitioning the interval $[a,b]$ into n-parts.

The curve $y=f(x)$ is assumed to be such that $y>0$ and continuous for all $x\in[a,b]$. To find an approximation to the area desired, construct a series of rectangles as follows.

(1) Divide the interval $[a,b]$ into n-parts by defining a step size $\Delta x=\dfrac{b-a}{n}$ and then define the points

$$\begin{aligned} x_0 =&a\\ x_1 =&a+\Delta x = x_0+\Delta x\\ x_2 =&a+2\Delta x = x_1+\Delta x\\ \vdots\quad &\quad\vdots\\ x_i =&a+i\Delta x = x_{i-1}+\Delta x\\ \vdots\quad &\quad\vdots\\ x_n =&a+n\Delta x = a+n\frac{(b-a)}{n} = b = x_{n-1}+\Delta x \end{aligned} \tag{3.80}$$

This is called partitioning the interval (a,b) into n-parts.

(2) Select arbitrary points t_i, within each Δx interval, such that $x_{i-1}\le t_i\le x_i$ for all values of i ranging from $i=1$ to $i=n$. Then for all values of i ranging from 1 to n construct rectangles of height $f(t_i)$ with the bottom corners of the rectangle touching the x-axis at the points x_{i-1} and x_i as illustrated in the figure 3-2.

(3) The area of the ith rectangle is denoted A_i=(height)(base), where the height of the rectangle is $f(t_i)$ and its base is $\Delta x_i = x_i - x_{i-1}$. The sum of all the rectangles is given by $S_n^t = \displaystyle\sum_{i=1}^n A_i = \sum_{i=1}^n f(t_i)\,\Delta x_i$ which is called the **Riemann[3] sum** for the

[3] Georg Friedrich Bernhard Riemann (1826-1866) A German mathematician.

function $y = f(x)$. The resulting sum is determined by the partition constructed. This Riemannian sum represents an approximation to the area under the curve. This approximation gets better as each Δx_i gets smaller.

Define the limit of the Riemann sum

$$\lim_{n\to\infty} S_n^t = \lim_{n\to\infty} \sum_{i=1}^{n} A_i = \lim_{\substack{\Delta x\to 0\\ n\to\infty}} \sum_{i=1}^{n} f(t_i)\Delta x_i = \int_a^b f(x)\,dx, \qquad x_{i-1} \le t_i \le x_i \quad (3.81)$$

where the quantity on the right-hand side of equation (3.81) is called the **definite integral** from a to b of $f(x)\,dx$ and the quantity on the left-hand side of equation (3.81) is the limit of the sum of rectangles as Δx tends toward zero. The notation for the definite integral from a to b of $f(x)\,dx$ has the physical interpretation illustrated in the figure 3-3.

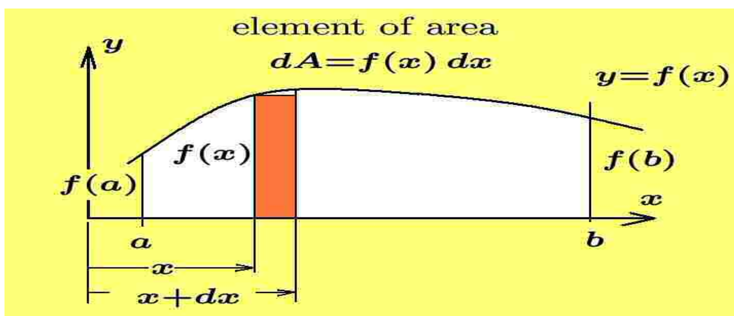


Figure 3-3. Mnemonic device for determining area under curve.

The quantity $f(x)\,dx = dA$ is to represent an **element of area** which is a rectangle positioned a distance x from the origin, having height $f(x)$ and base dx. The integral sign $\int$ is an elongated S to remind you that rectangles are being summed and the lower limit a and upper limit b on the integral sign is to remind you that the summation of rectangles is a limiting process taking place between the limits $x = a$ and $x = b$.

Observe that if $F(x)$ is a differentiable function which **is continuous over the interval** $[a, b]$ **and** $F(x)$ **is selected as a particular integral of** $f(x)$, then one can write $\frac{dF(x)}{dx} = F'(x) = f(x)$, or one can write the indefinite integral

$$\int f(x)\,dx = \int \frac{dF(x)}{dx}\,dx = \int dF(x) = F(x) + C$$

It will now be demonstrated that **any function** $F(x)$ **which is a particular integral of** $f(x)$ can be used to evaluate the definite integral $\int_a^b f(x)\,dx$. To accomplish this graph the function $y = F(x)$, satisfying $F'(x) = f(x)$, between the values $x = a$ and $x = b$ and then partition the interval $[a, b]$ into n-parts in the same way as for the original function $y = f(x)$. Apply **the mean-value theorem for derivatives to the points** $(x_{i-1}, F(x_{i-1}))$ **and** $(x_i, F(x_i))$ **associated with the** i**th** Δx **interval of the curve** $y = F(x)$. In using the mean-value theorem make special note that the function $F(x)$ is related to $f(x)$ by way of differentiation so that $F'(x) = f(x)$ is the slope of the curve $y = F(x)$ at the point x. One can then calculate the slope of the secant line through the points $(x_{i-1}, F(x_{i-1}))$ and $(x_i, F(x_i))$ as

$$m_s = \frac{\Delta F}{\Delta x} = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

The mean-value theorem says there must exist a point c_i satisfying $x_{i-1} \le c_i \le x_i$, such that the slope of the tangent line to the curve $y = F(x)$, at the point $x = c_i$, is the same as the slope of the secant line. By our choice of $F(x)$, the slope of the tangent line to $F(x)$ at the point $x = c_i$ is given by $f(c_i)$ since $F'(x) = f(x)$. Therefore, one can write

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = f(c_i) \quad \text{or} \quad F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i, \qquad \Delta x_i = x_i - x_{i-1} \tag{3.82}$$

This mean-value relationship can be applied to each Δx **interval for all values of** i **ranging from** 1 **to** n**.**

Make note of the fact the points t_i, $i = 1, \ldots, n$, **used to evaluate the definite integral in equation (3.81) were not specified.** They were arbitrary points satisfying $x_{i-1} \le t_i \le x_i$ for each value of the index i ranging from 1 to n. Note the values c_i, $i = 1, \ldots, n$ which satisfy the mean-value equation (3.82) are special values. Suppose one selects for equation (3.81) the values $t_i = c_i$ as i ranges from 1 to n. In this **special case** the summation given by equation (3.81) becomes

$$\begin{aligned} S_n^c = \sum_{i=1}^n f(c_i)\Delta x_i =& [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + \cdots + [F(x_n) - F(x_{n-1})] \\ \sum_{i=1}^n f(c_i)\Delta x_i =& F(x_n) - F(x_0) = F(b) - F(a) \end{aligned} \tag{3.83}$$

Observe that the summation of terms on the right-hand side of equation (3.83) is a telescoping sum and can be written as

$$
\begin{aligned}
&[F(x_1)-F(x_0)]+[F(x_2)-F(x_1)]\\
&\qquad+[F(x_3)-F(x_2)]+[F(x_4)-F(x_3)]\\
&\qquad\quad\vdots\\
&\qquad+[F(x_{n-1})-F(x_{n-2})]+[F(x_n)-F(x_{n-1})]=F(x_n)-F(x_0)=F(b)-F(a)
\end{aligned}
$$

where as i ranges from 1 to $n-1$, for each term $F(x_i)$ there is a $-F(x_i)$ and so these terms always add to zero and what is left is just the last term minus the first term. **This result still holds as** $\Delta x \to 0$ and so one can state that the area bounded by the curve $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$ is given by the definite integral

$$\boldsymbol{\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)} \tag{3.84}$$

where $F(x)$ is any particular integral of $f(x)$.

Observe that the constant of integration associated with the indefinite integral can be omitted when dealing with definite integrals. If this constant were used, then equation (3.84) would become $[F(b) + C] - [F(a) + C] = F(b) - F(a)$ which is the same result as given in equation (3.84). Also note that the name **"definite integral"** indicates that the integral has a **definite value of** $F(b) - F(a)$, which does not contain the symbol x or the constant C. The symbol x is called **a dummy variable of integration** in the definite integral and can be replaced by some other symbol.

The above result is a special case of **the fundamental theorem of integral calculus**.

Fundamental theorem of integral calculus

Let $f(x) > 0$ denote a continuous function over the interval $a \le x \le b$. Partition the interval (a, b) into n subintervals $[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$ which may or may not be of equal length. Select an arbitrary point $t_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \dots, n$ and construct the rectangles of height $f(t_i)$ and base $\Delta x_i = x_i - x_{i-1}$ with $A_i = f(t_i)\Delta x_i$ the area of the ith rectangle. A special case of the above situation is illustrated in the figure 3-2.

Let $A_n = \sum_{i=1}^n A_i = \sum_{i=1}^n f(t_i)\Delta x_i$ denote the Riemannian sum which equals the sum of the areas of these rectangles and let $F(x)$ denote any function which is an integral of $f(x)$ with the property $F(x) = \int f(x)\,dx$ or $\dfrac{dF(x)}{dx} = f(x)$. Then the fundamental theorem of integral calculus can be expressed

$$\lim_{n\to\infty} A_n = \lim_{\substack{n\to\infty\\ \Delta x\to 0}} \sum_{k=1}^n f(t_k)\Delta x = \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Since $F'(x) = f(x)$, the fundamental theorem of integral calculus is sometimes expressed in the form $\displaystyle\int_a^b \boldsymbol{F'(x)\,dx = F(x)}\Big|_{x=a}^{x=b} \boldsymbol{= F(b) - F(a)}$

Note that the limiting summation of rectangles that represents the definite integral of $f(x)$ from $x = a$ to $x = b$ **has the same value no matter how the points t_i are selected inside the interval** $[x_{i-1}, x_i]$**, for** $i = 1, 2, \ldots, n$**.** This is because of **the continuity of $f(x)$ over the interval** $[a, b]$. Recall that $f(x)$ is a continuous function if for every ϵ_1 there exists a positive number δ_1 such that

$$|f(t_i) - f(c_i)| < \epsilon_1 \qquad \text{whenever} \qquad |t_i - c_i| < \delta_1 \tag{3.85}$$

Suppose it is required that the condition given by (3.85) be satisfied for each value $i = 1, 2, \ldots, n$. If $\epsilon_1 = \dfrac{\epsilon}{(b-a)}$, with ϵ as small as desired, and n is selected large enough such that $\Delta x = \dfrac{b-a}{n} < \delta_1$, then one can compare the two summations

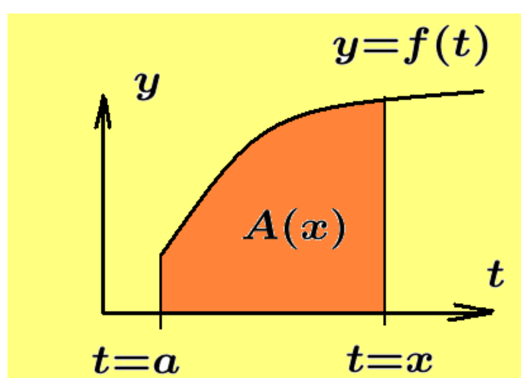
$$S_n^t = \sum_{i=1}^n f(t_i)\Delta x_i \qquad \text{and} \qquad S_n^c = \sum_{i=1}^n f(c_i)\Delta x_i$$

One finds the absolute value of the difference of these sums satisfies

$$|S_n^t - S_n^c| = \left|\sum_{i=1}^n \left[f(t_i)\Delta x_i - f(c_i)\Delta x_i\right]\right| \le \sum_{i=1}^n |f(t_i) - f(c_i)|\,\Delta x_i$$

For n large enough such that each $\Delta x_i = \dfrac{b-a}{n} < \delta_1$ for all values of the index i and $|f(t_i) - f(c_i)| < \epsilon_1$ for all values of the index i, then $\left|S_n^t - S_n^c\right| \le n\epsilon_1\Delta x = n\dfrac{\epsilon}{b-a}\dfrac{b-a}{n} = \epsilon$ This states that the difference between the two sums S_n^t and S_n^c can be made as small as desired for n large enough and in the limit these sums are the same. A similar type of argument can be made for an arbitrary, unequally spaced, partitioning of the interval $[a, b]$.

Properties of the Definite Integral



1. If $F'(x) = \dfrac{dF(x)}{dx} = f(x)$, then the definite integral

$$\boldsymbol{A(x) = F(x) - F(a) = \int_a^x f(t)\,dt}$$

represents the **area under the curve $\boldsymbol{y = f(t)}$ between the limits $\boldsymbol{t = a}$ and $\boldsymbol{t = x}$**. Whenever the upper or lower limit of integration involves the symbol x, then the dummy variable of integration in the definite integral is usually replaced by a different symbol.

2. Differentiate the above result to show

$$\frac{dA(x)}{dx} = \frac{dF(x)}{dx} = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x) \tag{3.86}$$

This shows that to differentiate a definite integral with respect to x, where the upper limit of integration is x and the lower limit of integration is a constant, one obtains the integrand evaluated at the upper limit x.

3. If the direction of integration is changed, then the sign of the integral changes

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \tag{3.87}$$

4. The interval of integration $[a,b]$ can be broken up into smaller subintervals, say, $[a,\xi_1],[\xi_1,\xi_2],[\xi_2,b]$ and the integral written

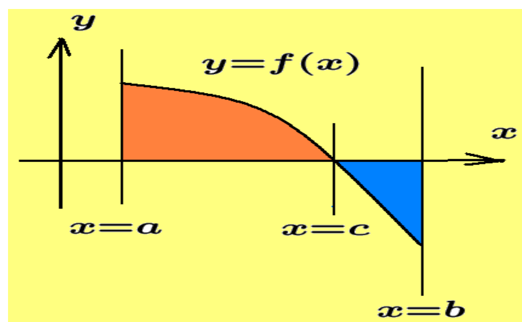
$$\int_a^b f(x)\,dx = \int_a^{\xi_1} f(x)\,dx + \int_{\xi_1}^{\xi_2} f(x)\,dx + \int_{\xi_2}^b f(x)\,dx \tag{3.88}$$

5. Assume the curve $y=f(x)$ crosses the x-axis at some point $x=c$ between the lines $x=a$ and $x=b$, such that $f(x)$ is positive for $a \le x \le c$ and $f(x)$ is negative for $c \le x \le b$, then

$$\int_a^c f(x)\,dx \text{ represents a } \textbf{positive area}$$

and the integral

$$\int_c^b f(x)\,dx \text{ represents a } \textbf{negative area}.$$



The definite integral $\int_a^b f(x)\,dx$ represents the summation of the **signed areas** above and below the x-axis. The integral $\int_a^b |f(x)|\,dx$ represents a summation of positive areas.

6. The summation $\sum_{i=1}^n f(t_i)$ represents the sum of the heights associated with the rectangles constructed in the figure 3-2, and the sum $\bar{y} = \frac{1}{n}\sum_{i=1}^n f(t_i)$ represents the **average height** of these rectangles. Using the equation (3.81) show that in the limit as $\Delta x \to 0$ and using $\Delta x_i = \frac{b-a}{n}$, this average height can be represented

$$\bar{y} = \lim_{\substack{\Delta x \to 0 \\ n\to\infty}} \frac{1}{n}\sum_{i=1}^n f(t_i) = \lim_{\substack{\Delta x \to 0 \\ n\to\infty}} \frac{1}{b-a}\sum_{i=1}^n f(t_i)\Delta x_i = \frac{1}{b-a}\int_a^b f(x)\,dx$$

This states that the average value for the height of the curve $y = f(x)$ between the limits $x = a$ and $x = b$ is given by

$$\textbf{Average height of curve} = \bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{3.89}$$

7. The integral of a constant times a function equals the constant times the integral of the function or

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

8. Change of variables in a definite integral

Given an integral of the form $I = \int_a^b f(g(x))\,g'(x)\,dx$ it is customary to make the substitution $u = g(x)$ with $du = g'(x)\,dx$. When this is done in a definite integral, then the limits of integration must also be changed. Thus, when $x = a$, then $u = g(a)$ and when $x = b$, then $u = g(b)$ so that if $f(u)$ is well defined on the interval $[g(a), g(b)]$, then the given integral can be reduced to the form

$$I = \int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

9. Area between curves

Let $y = f(x)$ and $y = g(x)$ denote two curves which are continuous on the interval $[a, b]$ and assume that $f(x) \le g(x)$ for all $x \in [a, b]$, then one can state that

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \quad \text{or} \quad \int_a^b g(x)\,dx \ge \int_a^b f(x)\,dx$$

Sketch the curves $y = f(x)$, $y = g(x)$, the lines $x = a$ and $x = b$ and sketch in a rectangular element of area dA representative of all the rectangular elements being summed to find the area bounded by the curves and the lines $x = a$ and $x = b$. Note that the element of area is given by

$$dA = (\text{y of upper curve} - \text{y of lower curve})\,dx$$

and these elements of area must be summed between the lines $x = a$ and $x = b$. The situation is illustrated in the figure 3-4(a). The area between the curves is obtained by a summation of the rectangular elements to obtain

$$\text{Area} = \int_a^b dA = \int_a^b (g(x) - f(x))\, dx$$

There may be times when the given curves dictate that a horizontal element of area between the curves be used to calculate the area between the curves.

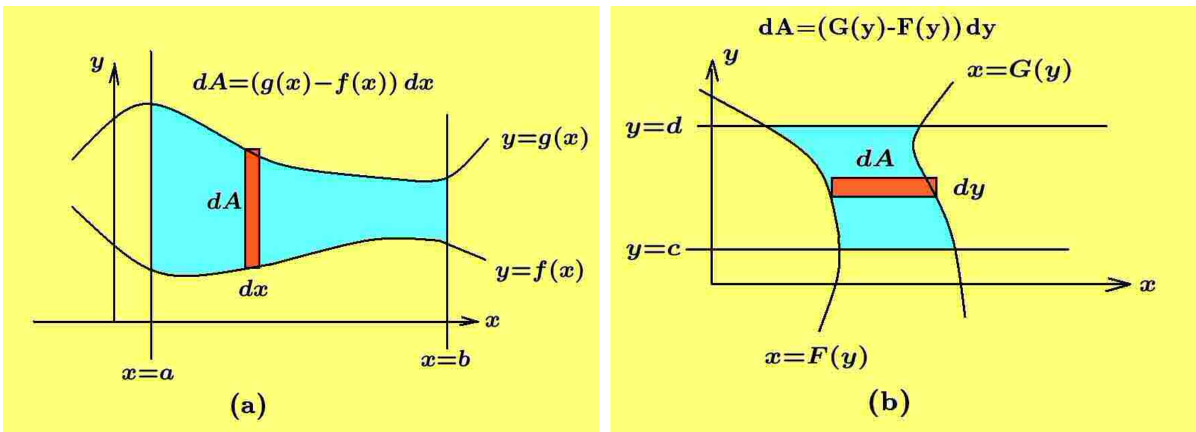


Figure 3-4. Area between curves.

For example, if $x = F(y)$ and $x = G(y)$ are two curves where a vertical element of area is not appropriate, then try using a horizontal element of area with the element of area dA given by

$$dA = (\text{x of right-hand curve} - \text{x of left-hand curve})\, dy$$

and then sum these elements of area between the lines $y = c$ and $y = d$ to obtain

$$\text{Area} = \int_c^d dA = \int_c^d (G(y) - F(y))\, dy$$

The situation is illustrated in the figure 3-4(b).

If the curves are intersecting curves between the limits of integration and situations arise where the upper curve switches and becomes a lower curve, then the integral representing the area **must be broken up into integrals over sections** otherwise one obtains a **summation of "signed"areas**.

Example 3-25. Consider the definite integral $I = \int_0^{\pi/6} \sin 2x \cos^4 2x\, dx$
Make the change of variable $u = \cos 2x$ with $du = -2\sin 2x\, dx$. The new limits of integration are found by substituting $x = 0$ and $x = \pi/6$ into the equation $u = \cos 2x$ to obtain $u_a = \cos 2x \Big|_{x=0} = 1$ and $u_b = \cos 2x \Big|_{x=\pi/6} = \frac{1}{2}$. Here $du = -2\sin 2x\, dx$ and so the given integral must be scaled. The scaled integral can then be written

$$I = \frac{-1}{2}\int_0^{\pi/6} (\cos 2x)^4 (-2\sin 2x\, dx)$$

where now the substitutions for u, du and new limits on integrations can be performed to obtain

$$I = \frac{-1}{2}\int_1^{1/2} u^4\, du = \frac{1}{2}\int_{1/2}^{1} u^4\, du = \frac{1}{2}\frac{u^5}{5}\Big|_{1/2}^{1} = \frac{1}{10}\left(1 - \frac{1}{32}\right) = \frac{31}{320}$$

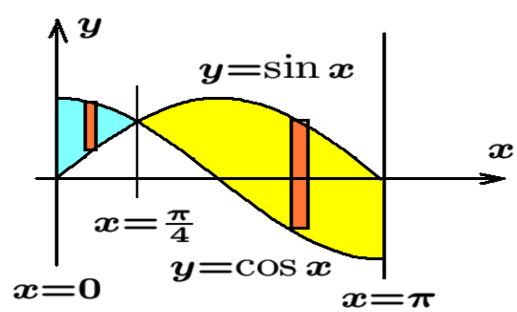
■

Example 3-26. Find the area between the curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le \pi$.

Solution

Sketch the given curves over the domain specified and show the curves intersect where $x = \pi/4$. The given integral can then be broken up into two parts and one can write

$$A_1 = \int_0^{\pi/4} [\cos x - \sin x]\, dx = \sin x + \cos x \Big|_0^{\pi/4} = \sqrt{2} - 1$$
$$A_2 = \int_{\pi/4}^{\pi} [\sin x - \cos x]\, dx = -\cos x - \sin x \Big|_{\pi/4}^{\pi} = 1 + \sqrt{2}$$



The total area is then $A_1 + A_2 = 2\sqrt{2}$
A summation of the signed areas is given by

$$\int_0^{\pi} [\cos x - \sin x]\, dx = (\sqrt{2} - 1) - (1 + \sqrt{2}) = -2 = A_1 - A_2$$

■

Example 3-27. Find the area of the triangle bounded by the x-axes, the line $y = \frac{h}{b_1}x$ and the line $y = h - \frac{h}{b_2}(x - b_1)$, where $b = b_1 + b_2$.

Solution

Get into the habit of

(*i*) Sketching the curve $y = f(x)$ to be integrated.

(*ii*) Sketching in an element of area $dA = f(x)\,dx$ or $dA = y\,dx$

(*iii*) Labeling the height and base of the rectangular element of area.

(*iv*) Sketching the lines $x = a$ and $x = b$ for the limits of integration.

Sketching the above lines one obtains the figure 3-5 illustrated below.

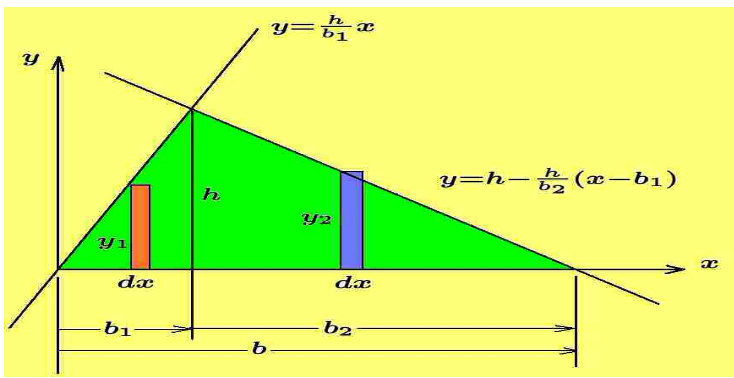


Figure 3-5. Triangle defined by x-axis, $y = \frac{h}{b_1}x$ and $y = h - \frac{h}{b_2}(x - b_1)$

The big triangle is built up of two smaller right triangles and an element of area has been constructed inside each of the smaller right triangles. The area of the left smaller right triangle is given by

$$A_1 = \int_0^{b_1} y_1\,dx = \int_0^{b_1} \frac{h}{b_1}x\,dx = \frac{h}{b_1}\int_0^{b_1} x\,dx = \frac{h}{b_1}\frac{x^2}{2}\Big|_{x=0}^{x=b_1} = \frac{h}{b_1}\frac{b_1^2}{2} = \frac{1}{2}hb_1$$

where the element of rectangular area $y_1\,dx$ is summed from $x = 0$ to $x = b_1$. This result says the area of a right triangle is one-half the base times the height. The area of the other right triangle is given by

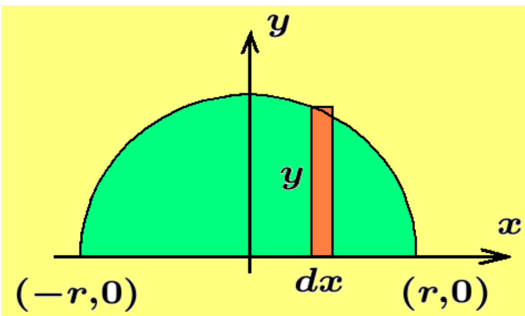
$$\begin{aligned} A_2 = &\int_{b_1}^{b} y_2\,dx = \int_{b_1}^{b}\left[h - \frac{h}{b_2}(x - b_1)\right]\,dx = \int_{b_1}^{b} h\,dx - \int_{b_1}^{b}\frac{h}{b_2}(x - b_1)\,dx \\ =&h\int_{b_1}^{b} dx - \frac{h}{b_2}\int_{b_1}^{b}(x - b_1)\,dx = h\,x\Big|_{x=b_1}^{x=b} - \frac{h}{b_2}\frac{(x - b_1)^2}{2}\Big|_{x=b_1}^{x=b} \\ =&h(b - b_1) - \frac{h}{b_2}\frac{(b - b_1)^2}{2} = hb_2 - \frac{h}{b_2}\frac{b_2^2}{2} = \frac{1}{2}hb_2 \end{aligned}$$

where the rectangular element of area $y_2\,dx$ was summed from $x=b_1$ to $x=b=b_1+b_2$. Adding the areas A_1 and A_2 gives the total area A where

$$A=A_1+A_2=\frac{1}{2}hb_1+\frac{1}{2}hb_2=\frac{1}{2}h(b_1+b_2)=\frac{1}{2}hb$$

That is, the area of a general triangle is one-half the base times the height. ∎

Example 3-28.



The curve

$$\{\,(x,y)\mid\ x=r\cos\theta,\ y=r\sin\theta,\ 0\le\theta\le\pi\,\}$$

is the set of points (x,y) defined by the parametric equations $x=r\cos\theta$ and $y=r\sin\theta$ as θ varies from 0 to π. Sketch this curve and show it represents the upper half of a circle with radius r centered at the origin as illustrated in the figure. An element of rectangular area $dA=y\,dx$ is constructed at a general point $(x,0)$, where the height of the rectangle is y and the base of the rectangle is dx. The area between the semi-circle and the x-axis is given by $A=\int_{-r}^{r}y\,dx$ which says the elements of area are to be summed between the limits $x=-r$ and $x=r$. Make the substitutions $y=r\sin\theta$ and $dx=-r\sin\theta\,d\theta$ and note that when $x=-r$, then $\theta=\pi$ and when $x=r$, then $\theta=0$. This gives the integral for the area as

$$A=\int_{-r}^{r}y\,dx=\int_{\pi}^{0}r\sin\theta(-r\sin\theta\,d\theta)=r^2\int_0^{\pi}\sin^2\theta\,d\theta$$

Make the trigonometric substitution $\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$ and then perform the integrations, after appropriate scaling, by using the previous table of integrals to show

$$A=\frac{r^2}{2}\int_0^{\pi}(1-\cos 2\theta)\,d\theta=\frac{r^2}{2}\left[\int_0^{\pi}d\theta-\frac{1}{2}\int_0^{\pi}\cos 2\theta(2d\theta)\right]$$
$$A=\frac{r^2}{2}\theta\Big|_0^{\pi}-\frac{1}{2}\sin 2\theta\Big|_0^{\pi}=\frac{\pi r^2}{2}$$

This shows the area of the semi-circle is $\pi r^2/2$ and so the area of the full circle is πr^2.

As an alternative, one can construct an element of area in the shape of a rectangle which is parallel to the x-axis as illustrated in the figure below. Due to symmetry this element of area is represented $dA = 2x\,dy$ and summing these elements of area in the y-direction from 0 to r gives the total area as

$$A = \int_0^r dA = \int_0^r 2x\,dy$$

Substituting in the values $x = r\cos\theta$ and $dy = r\cos\theta\,d\theta$ and noting that $y = 0$, corresponds to $\theta = 0$ and the value $y = r$, corresponds to $\theta = \pi/2$, one obtains the representation

$$A = \int_0^{\pi/2} 2(r\cos\theta)(r\cos\theta\,d\theta) = 2r^2 \int_0^{\pi/2} \cos^2\theta\,d\theta$$

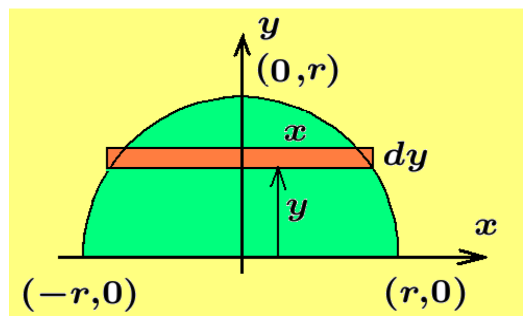
Using the trigonometric identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ the above integral for the area becomes

$$A = 2r^2 \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta = r^2\left[\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta\,(2\,d\theta)\right]$$

where the integral of $\cos 2\theta$ has been appropriately scaled. Performing the integrations one finds

$$A = r^2\left[\theta\Big|_0^{\pi/2} + \frac{1}{2}\sin 2\theta\Big|_0^{\pi/2}\right]$$
$$A = \frac{\pi r^2}{2}$$

which is the same as our previous answer.



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Solids of Revolution

Examine the shaded areas in each of the figures 3-6(a),(b),(c) and (d). These areas are going to be rotated about some axis to create a solid of revolution. The solid of revolution created depends upon what line is selected for the axis of rotation.

Figure 3-6(a)

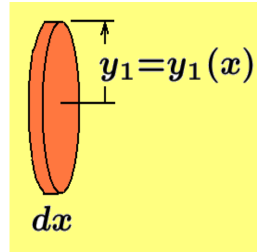
Examine rotation of element of area about lines $x = 0$, $y = 0$, $x = x_0$ and $y = y_0$.

Consider a general curve $y_1 = y_1(x)$ for $a \le x \le b$ such as the curve illustrated in the figure 3-6(a). To find the area bounded by the curve, the $x-$axis, and the lines $x = a$ and $x = b$ one would construct an element of area $dA = y_1(x)\,dx$ and then sum these elements from a to b to obtain the area

$$A = \int_a^b y_1(x)\,dx \tag{3.90}$$

If this area is rotated about the $x-$axis a solid of revolution is created. To find the volume of this solid the element of area is rotated about the $x-$axis to create a volume element **in the shape of a disk with thickness** dx. The radius of the disk is $y_1(x)$ and the volume element is given by

$$dV = \pi y_1^2(x)\, dx$$



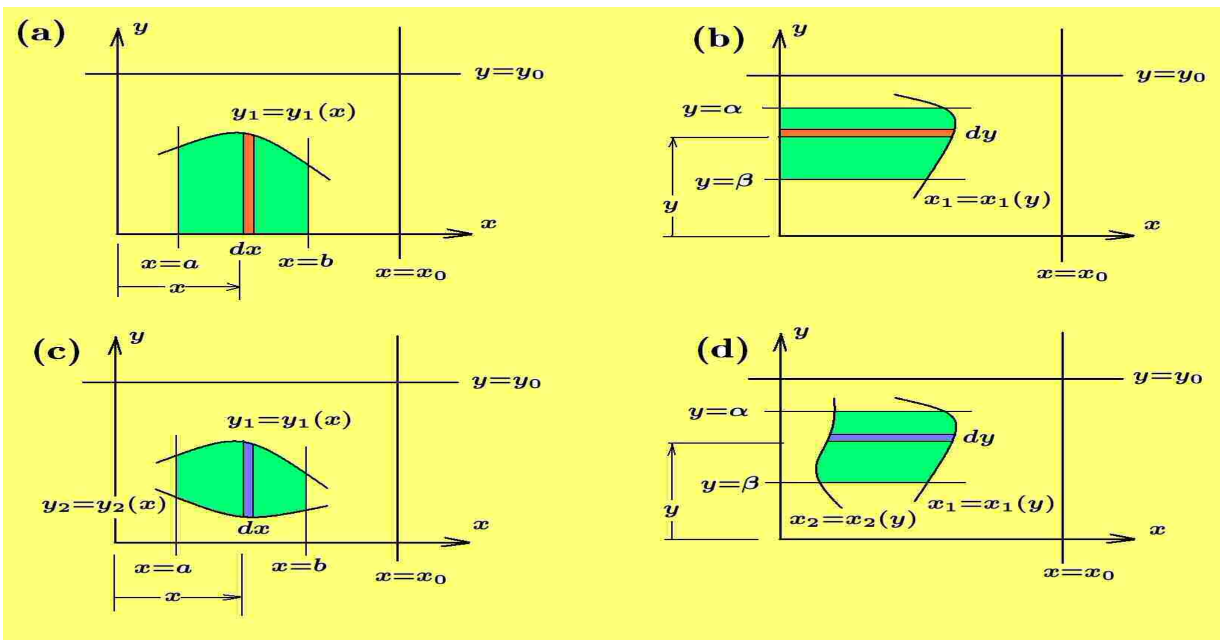


Figure 3-6.
Area element to be rotated about an axis to create volume element.

A summation of these volume elements from a to b gives the volume of the solid as

$$V = \pi \int_a^b y_1^2(x)\, dx \tag{3.91}$$

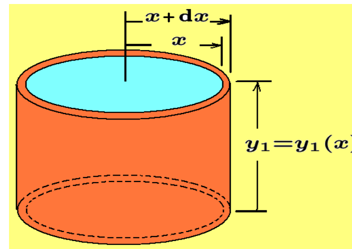
If the shaded area of figure 3-6(a) is rotated about the $y-$axis one can create a **cylindrical shell volume element** with inner radius x, outer radius $x + dx$ and height $y_1(x)$. The cylindrical shell volume element is given by

$$dV = (\text{Volume of outer cylinder} - \text{Volume of inner cylinder})(\text{height})$$

$$dV = \pi\left[(x+dx)^2 - x^2\right] y_1(x) = \pi\left[2x\,dx + (dx)^2\right] y_1(x)$$

The term $\pi\,(dx)^2\,y_1(x)$ is an infinitesimal of second order and can be neglected so that the **volume of the cylindrical shell element** is given by

$$dV = 2\pi x\, y_1(x)\, dx \tag{3.92}$$

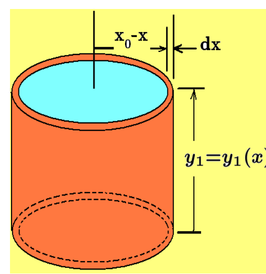


A summation of these cylindrical shell volume elements gives the total volume

$$V = 2\pi \int_a^b x\, y_1(x)\, dx \tag{3.93}$$

If the element of area $dA = y_1\,dx$ is rotated about the line $x = x_0$ one obtains the volume element in the shape of a **cylindrical shell** with the volume element given by

$$dV = 2\pi(x_0 - x) y_1(x)\, dx$$



and the volume of the solid of revolution is obtained from the integral

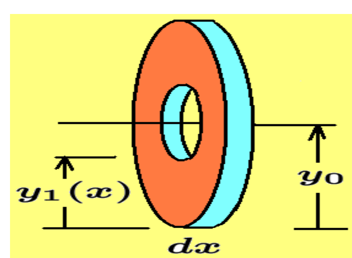
$$V = 2\pi \int_a^b (x_0 - x) y_1(x)\, dx$$

which represents a summation of these volume elements to generate the volume of revolution.

If the element of area is rotated about the line $y = y_0$ one obtains a volume element in the shape of **a washer** with the volume element represented

$$dV = [\text{Area outer circle} - \text{Area of inner circle}]\,(\text{Thickness})$$

$$dV = \pi\left[y_0^2 - (y_0 - y_1(x))^2\right] dx$$



The total volume is then given by a summation of these volume elements

$$V = \pi \int_a^b [y_0^2 - (y_0 - y_1(x))^2]\, dx$$

Figure 3-6(b)

Examine rotation of element of area about lines $x = 0$, $y = 0$, $x = x_0$ and $y = y_0$. Examine the figure 3-6(b) and show that to determined the area bounded by the curve $x_1 = x_1(y)$, the $y-$axis and the lines $y = \alpha$, $y = \beta$ is obtained by a summation of the area element $dA = x_1(y)\, dy$ from α to β. The total area is given by

$$A = \int_\alpha^\beta x_1(y)\, dy$$

If the area is **rotated about a line, then a solid of revolution is created.** To find the volume of the solid one can **rotate the element of area about the line to create an element of volume** which can then be summed. Consider the element of area illustrated as being rotated about the axes (i) the $x-$axis, (ii) the $y-$axis, (iii) the line $x = x_0$ and (iv) the line $y = y_0$ to obtain respectively elements of volumes in the shapes of (i)**a cylindrical shell element**, (ii) **a disk element**, (iii) **a washer element** and (iv) another **cylindrical shell element**. Show these volume elements are given by

$$\begin{array}{llll} (i) & dV = 2\pi y\, x_1(y)\, dy & (iii) & dV = \pi\left[x_0^2 - (x_0 - x_1(y))^2\right]\, dy \\ (ii) & dV = \pi x_1^2(y)\, dy & (iv) & dV = 2\pi(y_0 - y)\, x_1(y)\, dy \end{array}$$

Figure 3-6(c)

Examine rotation of element of area about lines $x = 0$, $y = 0$, $x = x_0$ and $y = y_0$. Examine the figure 3-6(c) and show the area bounded by the curves $y_1 = y_1(x)$, $y_2 = y_2(x)$ and the lines $x = a$, $x = b$, is obtained by a summation of the area element $dA = [y_1(x) - y_2(x)]\, dx$ from a to b. This summation gives the total area as

$$A = \int_a^b (y_1(x) - y_2(x))\, dx$$

If this area is rotated about a line, then a solid of revolution is created. To find the volume of the solid one can rotate the element of area about the same axis to create an element of volume which can then be summed. Consider the element of area being rotated about the axes (i) the $x-$axis, (ii) the $y-$axis, (iii) the line $x = x_0$ and (iv) the line $y = y_0$ to obtain respectively elements of volumes in the shapes of (i) a washer element, (ii) a cylindrical shell element, (iii) another cylindrical shell element and (iv) another washer element. Show these volume elements can be represented as follows.

$$(i) \quad dV = \pi \left[y_2^2(x) - y_1^2(x) \right] \, dx \qquad (iii) \quad dV = 2\pi(x_0 - x) \left[y_1(x) - y_2(x) \right] dx$$

$$(ii) \quad dV = 2\pi x \left[y_1(x) - y_2(x) \right] dx \qquad (iv) \quad dV = \pi \left[(y_0 - y_2(x))^2 - (y_0 - y_1(x))^2 \right] \, dx$$

Figure 3-6(d)

Examine rotation of element of area about lines $x = 0$, $y = 0$, $x = x_0$ and $y = y_0$. Examine the figure 3-6(d) and show the area bounded by the curves $x_1 = x_1(y)$, $x_2 = x_2(y)$ and the lines $y = \alpha$, $y = \beta$, is obtained by a summation of the area element $dA = [x_1(y) - x_2(y)] \, dy$ from α to β. This summation gives the total area

$$A = \int_\alpha^\beta [x_1(y) - x_2(y)] \, dy$$

If this area is rotated about a line, then a solid of revolution is created. The volume associated with this solid is determined by a summation of an appropriate volume elements. These volume elements can be determined by rotating the element of area about the same line from which the solid was created.

Consider the element of area rotated about the lines (i) the $x-$axis, (ii) the $y-$axis, (iii) the line $x = x_0$ and (iv) the line $y = y_0$ to obtain respectively elements of volumes in the shapes of (i) a cylindrical shell element, (ii) a washer element, (iii) another washer element and (iv) another cylindrical shell element. Show these volume elements can be represented as follows.

$$(i) \quad dV = 2\pi y \left[x_1(y) - x_2(y) \right] dy \qquad (iii) \quad dV = \pi \left[(x_0 - x_2(y))^2 - (x_0 - x_1(y))^2 \right] \, dy$$

$$(ii) \quad dV = \pi \left[x_1^2(y) - x_2^2(y) \right] \, dy \qquad (iv) \quad dV = 2\pi(y_0 - y) \left(x_1(y) - x_2(y) \right) dy$$

Example 3-29. Take the semi-circle

$$\{ \, (x, y) \mid \ x = r\cos\theta, \ y = r\sin\theta, \ 0 \le \theta \le \pi \ \}$$

as defined in the previous example and rotate it about the x-axis to form a sphere. The figure 3-7 will aid in visualizing this experiment.

Note that the vertical element of area when rotated becomes an element of volume dV in the shape of a disk with radius y and thickness dx. The volume of this disk is given by $dV = \pi y^2 \, dx$ and a summation of these volume elements from $x = -r$ to $x = r$ gives

$$V = \int_{-r}^{r} dV = \int_{-r}^{r} \pi y^2 \, dx$$

Making the same substitutions as in the previous example one finds

$$V = \int_{\pi}^{0} \pi(r\sin\theta)^2(-r\sin\theta)\,d\theta = \pi r^3 \int_0^{\pi} \sin^3\theta\,d\theta$$

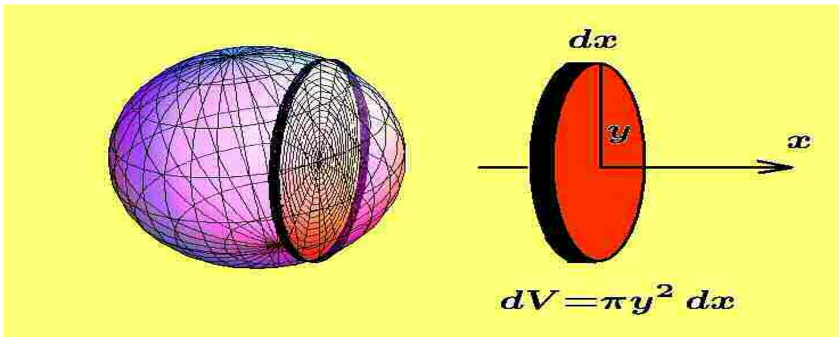


Figure 3-7.
Semi-circle rotated about x-axis creating the volume element in shape of a disk.

Now use the trigonometric identity

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

and express the volume integral in the form of an integration of trigonometric functions. After appropriate scaling, make use of the integration table to show that

$$\begin{aligned}
V =& \frac{\pi r^3}{4}\int_0^{\pi}(3\sin\theta - \sin 3\theta)\;d\theta = \frac{\pi r^3}{4}\left[3\int_0^{\pi}\sin\theta\,d\theta - \frac{1}{3}\int_0^{\pi}\sin 3\theta(3d\theta)\right]\\
V =& \frac{\pi r^3}{4}\left[3(-\cos\theta)\Big|_0^{\pi} - \frac{1}{3}(-\cos 3\theta)\Big|_0^{\pi}\right] = \frac{\pi r^3}{4}\left[-3(-1-1)] + \frac{1}{3}(-1-1)\right]\\
V =& \frac{4}{3}\pi r^3
\end{aligned}$$

This shows the volume of the sphere of radius r is $4/3$ times π times the radius cubed.

If the horizontal element of area illustrated in the example 3-24, is rotated about the x-axis a cylindrical shell element results, like the one illustrated by equation (3.92), but with x and y interchanged. The inner radius of the cylinder is y and the outer radius is $y + dy$ and the length of the cylinder is $2x$. The element of volume is given by

$$\begin{aligned}
dV =& \pi(\text{length})[(\text{outer radius})^2 - (\text{inner radius})^2]\\
dV =& \pi(2x)[(y+dy)^2 - y^2] = \pi(2x)[y^2 + 2y\,dy + (dy)^2 - y^2]\\
dV =& 2\pi(2x)y\,dy + \pi(2x)(dy)^2
\end{aligned}$$

This is an example of an equation where $(dy)^2$ is a higher ordered infinitesimal which can be neglected. Neglecting this higher ordered infinitesimal gives

$$dV = 2\pi(2x)y\,dy$$

Summation on dy from 0 to r gives the total volume

$$V = 4\pi \int_0^r xy\,dy$$

Substituting $x = r\cos\theta$, $y = r\sin\theta$ and $dy = r\cos\theta\,d\theta$ and changing the limits of integration to θ ranging from 0 to $\pi/2$, one finds

$$\begin{aligned} V =& 4\pi \int_0^{\pi/2} (r\cos\theta)(r\sin\theta)(r\cos\theta\,d\theta) \\ V =& 4\pi r^3 \int_0^{\pi/2} \cos^2\theta \sin\theta\,d\theta \end{aligned}$$

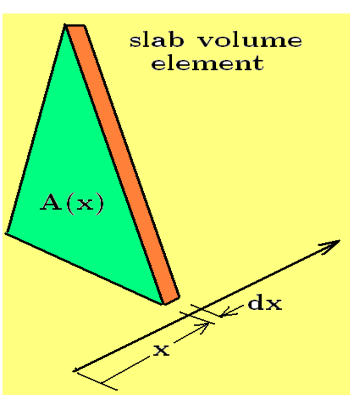
This last integral is recognized as being of the form $\displaystyle\int u^2\,du = \frac{1}{3}u^3$ where $u = \cos\theta$ and $du = -\sin\theta\,d\theta$. Perform the necessary scaling and then integrate to obtain

$$V = 4\pi r^3 \left(\frac{-1}{3}\right) (\cos\theta)^3 \Big|_0^{\pi/2} = \frac{4}{3}\pi r^3$$

for the volume of the sphere.

■

Sometimes one can place axes associated with a solid such that **plane sections** at x and $x + dx$ create a **known cross sectional area** which can be represented by a function $A = A(x)$ and consequently the plane sections produce a slab shaped volume



element given by $dV = A(x)\,dx$. The resulting volume between the planes x_0 and x_1 can then expressed as a summation of these sandwich slices

$$V = \int_{x_0}^{x_1} A(x)\,dx \tag{3.94}$$

One can also check cross sections at y and $y+dy$ to see if there results a known area $A = A(y)$. If this is the case, then the volume element associated with these plane slices are written $dV = A(y)\,dy$ and the total volume between the planes y_0 and y_1 is expressed as the summation

$$V = \int_{y_0}^{y_1} A(y)\,dy \tag{3.95}$$

Integration by Parts

Integration by parts associated with a definite integral has the form

$$\begin{aligned}
\int_a^b u(x)v'(x)\,dx =& \int_a^b d(u(x)v(x))\,dx - \int_a^b v(x)u'(x)\,dx \\
\int_a^b u(x)v'(x)\,dx =& u(x)v(x)\,\Big|_{x=a}^{x=b} - \int_a^b v(x)u'(x)\,dx \\
\int_a^b u(x)v'(x)\,dx =& u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)\,dx
\end{aligned} \tag{3.96}$$

Example 3-30. To integrate $I = \int_0^T te^{-st}\,dt$ let $u = t$ with $du = dt$ and $dv = e^{-st}\,dt$ with $v = -\frac{1}{s}e^{-st}$, then the integration by parts formula gives

$$\begin{aligned}
I = \int_0^T te^{-st}\,dt =& \frac{-t}{s}e^{-st}\Big|_0^T - \int_0^T \frac{-1}{s}e^{-st}\,dt = \frac{-T}{s}e^{-sT} - \frac{1}{s^2}e^{-st}\Big|_0^T \\
=& \frac{-T}{s}e^{-sT} - \frac{1}{s^2}[e^{-sT} - 1]
\end{aligned}$$

■

Example 3-31. Evaluate the integral $J = \int_{-\pi/2}^{\pi/2} x\sin x\,dx$

Solution Let $U = x$ and $dV = \sin x\,dx$ giving $dU = dx$ and $V = -\cos x$, so that integration by parts produces the result

$$J = -x\,\cos x\Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} -\cos x\,dx = -\frac{\pi}{2}\cos\frac{\pi}{2} + \frac{-\pi}{2}\cos\frac{-\pi}{2} + \sin x\Big|_{-\pi/2}^{\pi/2} = 2$$

■

Example 3-32. Evaluate the integral $I = \int_a^b x^2\sqrt{b-x}\,dx$, where a, b are constants satisfying $a < b$.

Solution Use integration by parts with

$$\begin{aligned}
u =& x^2 & dv =& (b-x)^{1/2}\,dx \\
du =& 2x\,dx & v =& -\frac{2}{3}(b-x)^{3/2}
\end{aligned}$$

to obtain

$$
\begin{aligned}
I =& uv \Big|_{x=a}^{x=b} - \int_a^b v\,du = -\frac{2}{3}x^2(b-x)^{3/2}\Big|_a^b + \frac{4}{3}\int_a^b x(b-x)^{3/2}\,dx \\
I =& \frac{2}{3}a^2(b-a)^{3/2} + \frac{4}{3}\int_a^b x(b-x)^{3/2}\,dx
\end{aligned}
$$

One can now apply integration by parts again on the last integral using

$$
\begin{aligned}
u =& x & dv =& (b-x)^{3/2}\,dx \\
du =& dx & v =& -\frac{2}{5}(b-x)^{5/2}
\end{aligned}
$$

to obtain

$$
\begin{aligned}
I =& \frac{2}{3}a^2(b-a)^{3/2} + \frac{4}{3}\left[-\frac{2}{5}x(b-x)^{5/2}\Big|_a^b + \frac{2}{5}\int_a^b (b-x)^{5/2}\,dx\right] \\
I =& \frac{2}{3}a^2(b-a)^{3/2} + \frac{4}{3}\left(\frac{2}{5}a(b-a)^{5/2} + \frac{2}{5}\left[-\frac{2}{7}(b-x)^{7/2}\Big|_a^b\right]\right) \\
I =& \frac{2}{3}a^2(b-a)^{3/2} + \frac{8}{15}a(b-a)^{5/2} + \frac{16}{105}(b-a)^{7/2} \\
I =& \frac{2}{105}(b-a)^{3/2}(15a^2 + 12ab + 8b^2)
\end{aligned}
$$

■

Physical Interpretation

When using definite integrals the integration by parts formula has the following physical interpretation. Consider the section of a curve C between points P and Q on the curve which can be defined by

$$
C = \{\ (x,y)\ \mid\ x = x(t),\ y = y(t),\ t_0 \le t \le t_1\ \} \tag{3.97}
$$

Here the section of the curve C is defined by a set of parametric equations $x = x(t)$ and $y = y(t)$ for $t_0 \le t \le t_1$ with the point P having the coordinates (x_0, y_0) where $x_0 = x(t_0)$ and $y_0 = y(t_0)$. Similarly, the point Q has the coordinates (x_1, y_1) where $x_1 = x(t_1)$ and $y_1 = y(t_1)$. A general curve illustrating the situation is sketched in the figure 3-8.

Examine the element of area $dA_1 = y\,dx$ and sum these elements of area from x_0 to x_1 to obtain

$$
A_1 = \int_{x_0}^{x_1} y\,dx = \int_{t_0}^{t_1} y(t)\,\frac{dx}{dt}\,dt = \text{ Area } x_0PQx_1 \tag{3.98}
$$

Similarly, if one sums the element of area $dA_2 = x\,dy$ from y_0 to y_1 there results

$$
A_2 = \int_{y_0}^{y_1} x\,dy = \int_{t_0}^{t_1} x(t)\frac{dy}{dt}\,dt = \text{ Area } y_0PQy_1 \tag{3.99}
$$

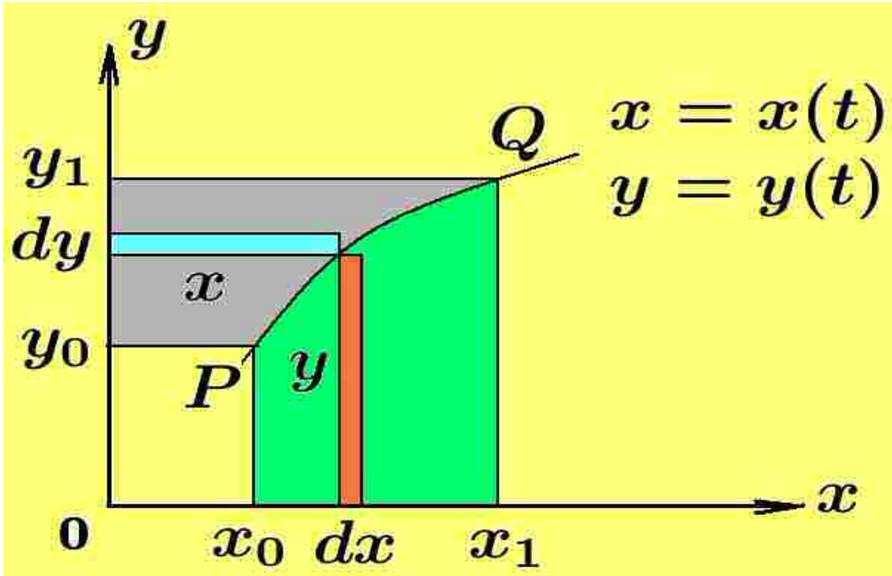


Figure 3-8. Physical interpretation for integration by parts.

Examine the figure 3-8 and verify the areas of the following rectangles

$$A_3 = \text{area rectangle } 0x_1Qy_1 = x_1y_1, \qquad A_4 = \text{area rectangle } 0x_0Py_0 = x_0y_0 \tag{3.100}$$

The integration by parts formula can then be expressed

$$\begin{aligned}\int_{t_0}^{t_1} y(t)\frac{dx}{dt}\,dt =&x(t)y(t)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} x(t)\frac{dy}{dt}\,dt\\ =&(x_1y_1 - x_0y_0) - \int_{t_0}^{t_1} x(t)\frac{dy}{dt}\,dt\end{aligned} \tag{3.101}$$

In terms of areas this result can be written

$$A_1 = A_3 - A_4 - A_2 \quad \text{or} \quad A_2 = A_3 - A_4 - A_1$$

and is interpreted as saying that the areas A_1 and A_2 are related and if one these areas is known, then the other area can also be evaluated.

Improper Integrals

Integrals of the form

$$I_1 = \int_a^\infty f(x)\,dx, \qquad I_2 = \int_{-\infty}^\infty f(x)\,dx, \qquad I_3 = \int_{-\infty}^b f(x)\,dx \tag{3.102}$$

are called **improper integrals** and are defined by the limiting processes

$$I_1 = \lim_{b\to\infty} \int_a^b f(x)\,dx, \qquad I_2 = \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f(x)\,dx, \qquad I_3 = \lim_{a\to-\infty} \int_a^b f(x)\,dx \tag{3.103}$$

if these limits exist. In general an integral of the form

$$I = \int_a^b f(x)\,dx \tag{3.104}$$

is called **an improper integral** if

(i) The lower limit a is allowed to increase or decrease without bound.

(ii) The upper limit b is allowed to increase or decrease without bound.

(iii) The lower limit a decreases without bound and the upper limit b increases without bound.

(iv) The integrand $f(x)$ is not defined at some point c between the end points a and b, then the integral is called an improper integral and one must write

$$I = \int_a^b f(x)\,dx = \lim_{\substack{\xi\to c^- \\ \xi<c}} \int_a^\xi f(x)\,dx + \lim_{\substack{\xi\to c^+ \\ \xi>c}} \int_\xi^b f(x)\,dx \tag{3.105}$$

if these limits exist.

Improper integrals occur in a variety of forms in advanced mathematics courses involving **integral transforms**. For example,

The Laplace transform of a function is written as the improper integral

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{F(t);\, t\to s\} = \int_0^\infty F(t)e^{-st}\,dt = f(s) \tag{3.106}$$

and represents a transformation of a function $F(t)$ into a function $f(s)$, if the improper integral exists. Other transforms frequently encountered are

The Fourier exponential transform is written as the improper integral

$$\mathcal{F}_e\{f(x); x\to\omega\} = \frac{1}{2\pi}\int_{-\infty}^\infty f(\xi)e^{i\omega\xi}\,d\xi = F_e(\omega) \tag{3.107}$$

The Fourier sine transform is written as the improper integral

$$\mathcal{F}_s\{f(x); x\to\omega\} = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx = F_s(\omega) \tag{3.108}$$

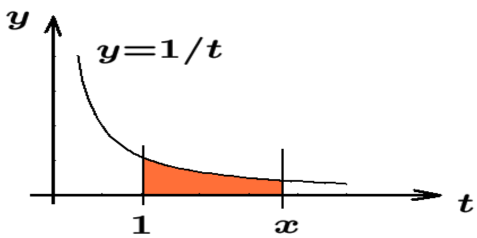
The Fourier cosine transform is written as the improper integral

$$\mathcal{F}_c\{f(x); x\to\omega\} = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx = F_c(\omega) \tag{3.109}$$

if the above integrals exist. Numerous other transforms similar to those mentioned above can be found in many advanced mathematics, physics and engineering texts.

Integrals used to define Functions

Definite integrals are frequently used to define special functions. For example, the **natural logarithm function** can be defined



$$\begin{aligned} \ln x &= \int_1^x \frac{1}{t}\,dt \\ \frac{d}{dx}\ln x &= \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x} \end{aligned} \tag{3.110}$$

The natural logarithm of x is represented as the area bounded by the curve $1/t$, the lines $t=1$, $t=x$ and the t-axis.

Properties of the natural logarithm function can be obtained from the defining integral. For example, one finds

(i) $\boldsymbol{\ln 1 = 0}$

(ii) $\ln(a\cdot b) = \int_1^{ab}\frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab}\frac{1}{t}\,dt$ In the last integral make the substitution $t = a\,u$ with $dt = a\,du$, so that when $t=a, u=1$ and when $t=ab, u=b$ and obtain

$$\ln(a\cdot b) = \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{u}\,du$$

giving $$\boldsymbol{\ln(a\cdot b) = \ln a + \ln b}$$

(iii) $\ln\left(\frac{1}{b}\right) = \int_1^{1/b}\frac{1}{t}\,dt$ Make the substitution $t=\frac{u}{b}$ with $dt = \frac{du}{b}$ with new limits on u from b to 1 and show

$$\ln\left(\frac{1}{b}\right) = \int_b^1 \frac{\frac{du}{b}}{\frac{u}{b}} = -\int_1^b \frac{du}{u}$$

giving $$\boldsymbol{\ln\left(\frac{1}{b}\right) = -\ln b}$$

(iv) Using the result from (iii) it follows that $\ln\left(\frac{a}{b}\right) = \ln\left(a\cdot\frac{1}{b}\right) = \ln a + \ln\frac{1}{b}$ giving the result $$\boldsymbol{\ln\left(\frac{a}{b}\right) = \ln a - \ln b}$$

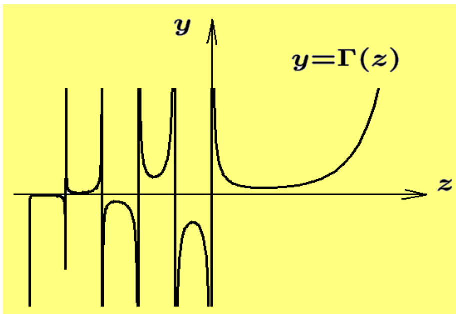
(v) $\ln(a^r) = \int_1^{a^r}\frac{1}{t}\,dt$ Make the substitution $t=u^r$ with $dt = ru^{r-1}\,du$ with new limits of integration u ranging from 1 to a to show

$$\ln(a^r) = \int_1^{a^r}\frac{1}{t}\,dt = r\int_1^a \frac{1}{u}\,du$$

showing that $$\boldsymbol{\ln(a^r) = r\ln a}$$

Other functions defined by integrals

There are many special functions which are defined as a definite integral or improper integral. For example, three functions defined by integrals which occur quite frequently are the following.



The Gamma function is defined

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \tag{3.111}$$

and integration by parts shows that

$$\Gamma(z+1) = z\Gamma(z) \tag{3.112}$$

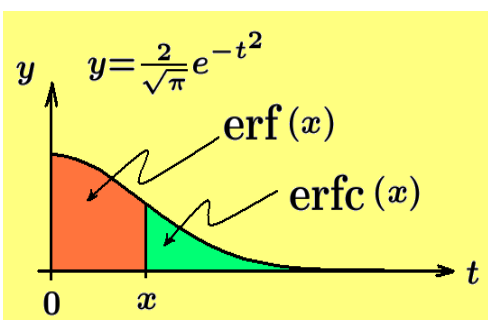
Use limits to show $\Gamma(1) = 1$ and then use equation (3.112) to show

$$\Gamma(1) = 1,\ \Gamma(2) = 1!,\ \Gamma(3) = 2\Gamma(2) = 2!, \ldots$$

and when $z = n$ is an integer the **Gamma function** reduces to the **factorial function**

$$\Gamma(n) = (n-1)! = (n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1 \quad \text{or} \quad \Gamma(n+1) = n!$$

The values $\Gamma(0),\ \Gamma(-1),\ \Gamma(-2), \ldots$ are not defined.



The error function is defined

$$\text{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{3.113}$$

The complementary error function is defined

$$\text{erfc}\,(x) = 1 - \text{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \tag{3.114}$$

The **error function**[4] erf (x) occurs in the study of the **normal probability distribution** and represents the area under the curve $\frac{2}{\sqrt{\pi}}e^{-t^2}$ from 0 to x, while the **complementary error function** is the area under the same curve from x to ∞.

The above is just a very small sampling of the many special functions which are defined by integrals.

[4] Note alternative forms for the definition of the error function are due to scaling.

Arc Length

Let $y = f(x)$ denote a continuous curve for $x \in [a,b]$ and consider the problem of assigning a length to the curve $y = f(x)$ between the points $(a, f(a)) = P_0$ and $(b, f(b)) = P_n$. Partition the interval $[a,b]$ into n-parts by defining $\Delta x = \dfrac{b-a}{n}$ and labeling the points

$$a = x_0,\ \ x_1 = x_0 + \Delta x,\ \ x_2 = x_1 + \Delta x, \ldots, x_i = x_{i-1} + \Delta x, \ldots, x_n = x_{n-1} + \Delta x = b$$

as illustrated in the figure 3-9.

Label the points $(x_i, f(x_i)) = P_i$ for $i = 0, 1, 2, \ldots, n$ and construct the line segments $\overline{P_{i-1}P_i}$ for $i = 1, 2, \ldots, n$. These line segments connect the points

$$(x_0, f(x_0)),\ \ (x_1, f(x_1)),\ \ (x_2, f(x_2)), \ldots, (x_i, f(x_i)), \ldots, (x_{n-1}, f(x_{n-1})),\ \ (x_n, f(x_n))$$

in succession and form a **polygonal line** connecting the points $(a, f(a))$ and $(b, f(b))$.

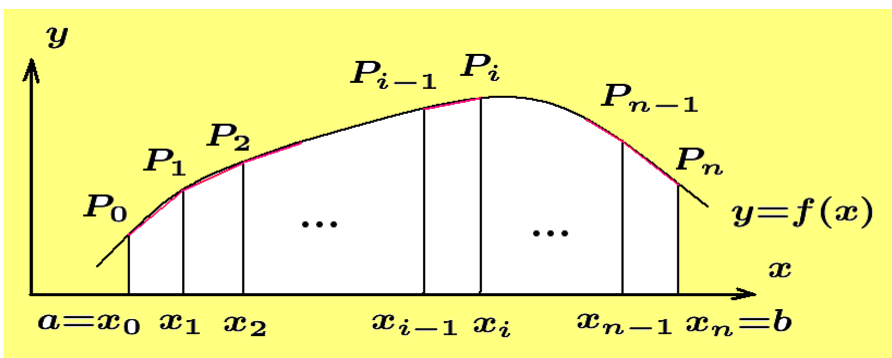


Figure 3-9.

Approximation of arc length by summation of straight line segments.

The sum of these line segments can be represented

$$s_n = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} = \sum_{i=1}^{n} \sqrt{1 + \left[\frac{f(x_i) - f(x_{i-1})}{\Delta x_i}\right]^2}\ \Delta x_i \tag{3.115}$$

where $\Delta x_i = x_i - x_{i-1}$. **This sum is an approximation to the length of the curve** $y = f(x)$ **between the points** $(a, f(a))$ **and** $(b, f(b))$. This **arc length** approximation gets better as Δx_i gets smaller or as n gets larger. If in the limit as $n \to \infty$, the above sum exists so that one can write $s = \lim_{n\to\infty} s_n$, then the curve $y = f(x)$ is said to be **rectifiable**. The limiting value s is defined to be the **arc length of the curve** $y = f(x)$ between the end points $(a, f(a))$ and $(b, f(b))$. Here $\lim_{\Delta x\to 0} \dfrac{f(x_i) - f(x_{i-1})}{\Delta x_i} = f'(x_i)$ and the infinite sum becomes a definite integral and so one can express the limiting value of the above sum as

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{3.116}$$

If $x \in [a, b]$, then define the arc length $s = s(x)$ of the curve $y = f(x)$ between the points $(a, f(a))$ and $(x, f(x))$ as

$$s = s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt \tag{3.117}$$

and define the **differential of arc length** $ds = s'(x)\, dx$. The element of arc length ds can be determined from any of the following forms

$$\begin{aligned} ds =& \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy \\ ds =& \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \end{aligned} \tag{3.118}$$

Example 3-33. Find the circumference of the circle defined by $x = r\cos\theta$, $y = r\sin\theta$ for $0 \le \theta \le 2\pi$, where r is a constant.

Solution

The element of arc length squared can be written

$$\begin{aligned} ds^2 =& dx^2 + dy^2 \qquad \text{or} \\ ds =& \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2]}\, d\theta \end{aligned}$$

Substituting in the derivatives $\dfrac{dx}{d\theta} = x'(\theta) = -r\sin\theta$ and $\dfrac{dy}{d\theta} = y'(\theta) = r\cos\theta$, the element of arc length is

$$ds = \sqrt{[-r\sin\theta]^2 + [r\cos\theta]^2}\, d\theta = r\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = r\, d\theta$$

and the total arc length is a summation of these elements

$$s = r\int_0^{2\pi} d\theta = r\,\theta\Big|_0^{2\pi} = 2\pi\, r$$

■

Example 3-34. Find the length of the line segment connecting the points (x_1, y_1) and (x_2, y_2) where $x_1 \neq x_2$.

Solution

The slope of the line through these points is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and the equation of the line through these points is $y - y_1 = m(x - x_1)$. The arc length is given by

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x_1}^{x_2} \sqrt{1+m^2}\, dx = (\sqrt{1+m^2})\, x \Big|_{x_1}^{x_2} = (\sqrt{1+m^2})(x_2 - x_1)$$

This simplifies to the well known result $s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

■

Area Polar Coordinates

The equation of a curve in polar coordinates is given by $r = f(\theta)$. To find the area bounded by the curve $r = f(\theta)$, the rays $\theta = \alpha$ and $\theta = \beta$, divide the angle $\beta - \alpha$ into n-parts by defining $\Delta\theta = \frac{\beta - \alpha}{n}$ and then defining the rays

$$\theta_0 = \alpha,\ \theta_1 = \theta_0 + \Delta\theta, \ldots, \theta_i = \theta_{i-1} + \Delta\theta, \ldots, \theta_n = \theta_{n-1} + \Delta\theta = \beta$$

The area between the rays $\theta = \theta_{i-1}$, $\theta = \theta_i$ and the curve $r = f(\theta)$, illustrated in the figure 3-10, is approximated by a circular sector with area element

$$dA_i = \frac{1}{2} r_i^2 \Delta\theta_i = \frac{1}{2} f^2(\theta_i)\, \Delta\theta_i \tag{3.119}$$

where $\Delta\theta_i = \theta_i - \theta_{i-1}$ and $r_i = f(\theta_i)$. A summation of these elements of area between the rays $\theta = \alpha$ and $\theta = \beta$ gives the approximate area

$$\sum_{i=1}^{n} dA_i = \sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta\theta_i = \sum_{i=1}^{n} \frac{1}{2} f^2(\theta_i)\, \Delta\theta_i \tag{3.120}$$

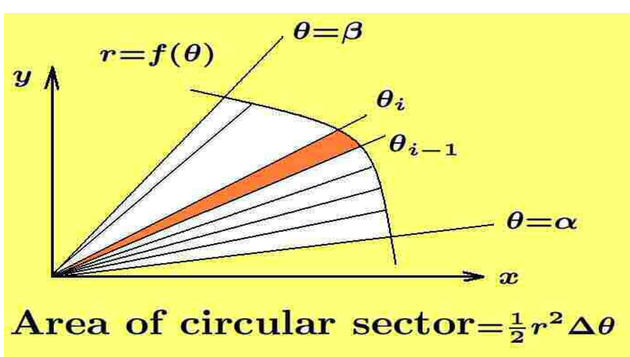


Figure 3-10.

Approximation of area by summation of circular sectors.

This approximation gets better as $\Delta\theta_i$ gets smaller. Using the fundamental theorem of integral calculus, it can be shown that in the limit as $n \to \infty$, the equation (3.119) defines the element of area $dA = \frac{1}{2}r^2\,d\theta$. A summation of these elements of area gives

$$\text{Polar Area} = \int_\alpha^\beta dA = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta = \frac{1}{2}\int_\alpha^\beta f^2(\theta)\,d\theta \tag{3.121}$$

Example 3-35. Find the area bounded by the polar curve

$$r = 2r_0\cos\theta \quad \text{for } 0 \le \theta \le \pi.$$

Solution

One finds that the polar curve $r = 2r_0\cos\theta$, for $0 \le \theta \le \pi$, is a circle of radius r_0 which has its center at the point $(r_0, 0)$ in polar coordinates. Using the area formula given by equation (3.121) one obtains

$$\text{Area} = \frac{1}{2}\int_0^\pi (2r_0\cos\theta)^2\,d\theta = 2r_0^2\int_0^\pi \cos^2\theta\,d\theta = r_0^2\int_0^\pi (\cos 2\theta + 1)\,d\theta = r_0^2\left[\frac{\sin 2\theta}{2} + \theta\right]_0^\pi = \pi r_0^2$$

Make note of the fact that polar curves sometimes sweep out a repetitive curve. For example, in the polar equation $r = 2r_0\cos\theta$, if θ varied from 0 to 2π, then the polar distance r would sweep over the circle twice. Consequently, if one performed the integration

$$\frac{1}{2}\int_0^{2\pi} (2r_0\cos\theta)^2\,d\theta$$

one would obtain twice the area or $2\pi r_0^2$. **Therefore, one should always check polar curves to see if some portions of the curve are being repeated as the independent variable θ varies.**

■

Arc Length in Polar Coordinates

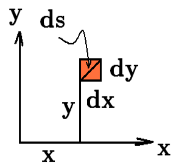
In rectangular coordinates, $ds^2 = dx^2 + dy^2$ represents the element of arc length squared. If one changes to polar coordinates using the transformation equations

$$x = x(r,\theta) = r\cos\theta \qquad \text{and} \qquad y = y(r,\theta) = r\sin\theta \tag{3.122}$$

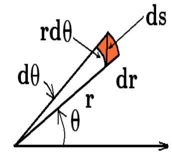
then the total differentials dx and dy are given by

$$\begin{aligned} dx &= \frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial \theta}\,d\theta \\ dx &= \cos\theta\,dr - r\sin\theta\,d\theta \end{aligned} \qquad \text{and} \qquad \begin{aligned} dy &= \frac{\partial y}{\partial r}\,dr + \frac{\partial y}{\partial \theta}\,d\theta \\ dy &= \sin\theta\,dr + r\cos\theta\,d\theta \end{aligned}$$

and represent the total differentials in terms of the variables r and θ. Squaring and adding these differentials one finds



$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \tag{3.123}$$



as the representation for the arc length squared in polar coordinates. Other forms for representing the element of arc length in polar coordinates are

$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta = \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr = \sqrt{[r'(t)]^2 + r^2[t][\theta'(t)]^2}\, dt \tag{3.124}$$

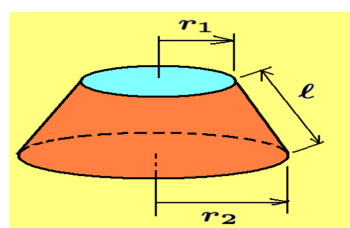
Example 3-36. Find the circumference of the circle $r = 2r_0 \cos\theta$ given in the previous example.

Solution Here an element of arc length is given by the polar coordinate representation $ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$, where $\frac{dr}{d\theta} = -2r_0 \sin\theta$. Integration of the element of arc length from 0 to π gives

$$s = \int_0^\pi \sqrt{4r_0^2 \sin^2\theta + 4r_0^2 \cos^2\theta}\, d\theta = 2r_0 \int_0^\pi d\theta = 2r_0\theta \Big|_0^\pi = 2\pi r_0$$

■

Surface of Revolution



When a curve is revolved about the x or y-axis a **surface of revolution** results. The problem of determining the surface area of the resulting surface of revolution is approached using the following arguments. First consider a right circular cone where the top has been cut off. The resulting figure is called the **frustum of a right circular cone**. The top surface is the shape of a circle with radius r_1 and the bottom surface is a circle with radius $r_2 > r_1$. The side surface has a slant height of length ℓ as illustrated in the accompanying figure. The surface area associated with the side of this figure is given by[5]

$$\text{Side surface area} = \pi(r_1 + r_2)\,\ell \tag{3.125}$$

[5] How this result is derived can be found in example 3-38

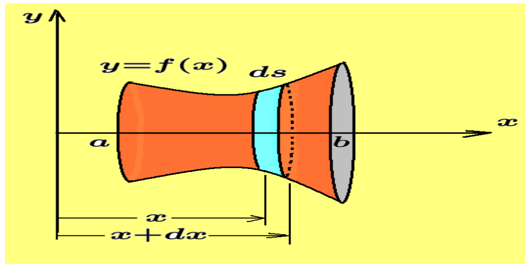


Figure 3-11.
Arc length ds rotated about x-axis to form frustum of right circular cone.

Consider next the surface of revolution obtained when a curve $y = f(x)$ is rotated about the x-axis as illustrated in the figure 3-11. Let ds denote an element of arc length in cartesian coordinates connecting the points (x, y) and $(x+dx, y+dy)$ on the curve and observe that when this element is rotated about the x-axis a frustum of a right circular cone results. The radius of one circle is y and radius of the other circle is $y+dy$. The **element of surface area** dS is the side surface area of the frustum and given by equation (3.125) and so one can write

$$dS = \pi[y + (y + dy)]\, ds \tag{3.126}$$

The product $dy\, ds$ is an infinitesimal of the second order and can be neglected so that the element of surface area can be written in the form

$$dS = 2\pi y\, ds \tag{3.127}$$

By the fundamental theorem of integral calculus a summation of these surface elements gives the total surface area of the surface of revolution. This total surface area can be expressed in different forms depending upon the representation of the arc length ds (See equations (3.118).) If $y = y(x)$, then one can write

$$S = \int_a^b 2\pi y\, ds = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{3.128}$$

In a similar fashion, a curve $x = F(y)$, for $c \le y \le d$, rotated about the y-axis would created a surface of revolution with surface area given by

$$S = 2\pi \int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \tag{3.129}$$

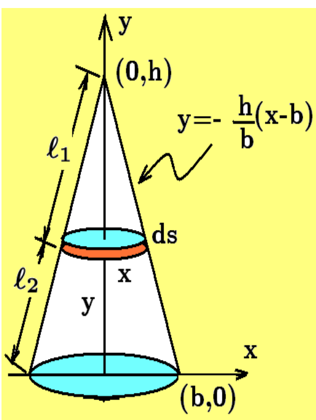
In general, if a curve is rotated about a line, one can express the element of surface area associated with a surface of revolution as $dS = 2\pi\rho\, ds$ where ds is an element of arc length on the curve expressed in an appropriate form and ρ represents the distance from the arc length element ds to the axis of revolution.

Example 3-37. Consider the upper half of the circle $x^2+y^2=r^2$ rotated about the x-axis to form a sphere. Here $2x+2y\frac{dy}{dx}=0$ or $\frac{dy}{dx}=-x/y$ so that the surface area of the sphere is given by

$$S = 2\pi \int_{-r}^{r} y\sqrt{1+\frac{x^2}{y^2}}\, dx = 2\pi \int_{-r}^{r} \sqrt{x^2+y^2}\, dx = 2\pi r \int_{-r}^{r} dx = 2\pi r x \Big|_{-r}^{r} = 4\pi r^2$$

■

Example 3-38.



The line $y = -\frac{h}{b}(x-b)$, $0 \le x \le b$ is rotated about the y-axis to form a cone. An element of arc length ds on the line is rotated about the y-axis to form an element of surface area dS given by $dS = 2\pi x\, ds$ Using $ds^2 = dx^2 + dy^2$ in the form $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$, the total surface area of a cone with height h and base radius b is given by

$$S = 2\pi \int_0^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Perform the integration and show the total surface area of the cone is given by

$$S = \pi b \ell \quad \text{where} \quad \ell^2 = b^2 + h^2, \tag{3.130}$$

with $\ell = \ell_1 + \ell_2$ representing the slant height of the cone.

As an exercise, use the above results to obtain the surface area of a frustum associated with the given right circular cone as follows. If a is the base radius of cone associated with slant height ℓ_1 and b is the base radius associated with cone of the slant height $\ell_1+\ell_2$, then show, Area of frustum $= \pi b(\ell_2+\ell_1) - \pi a \ell_1$. Then use similar triangles and simplify this result and show, Area of frustum $= \pi\ell_2(b+a)$ which agrees with equation (3.125).

■

Mean Value Theorems for Integrals

There are several **mean value theorems** associated with definite integrals which can be found under the following names.

(i) **The first mean value theorem for integrals.** *If $f(x)$ is a continuous function for $a \le x \le b$, then there is a point $x = \xi_1 \in [a,b]$ such that*

$$\int_a^b f(x)\,dx = (b-a)\,f(\xi_1) = (b-a)f(a+\theta_1(b-a)), \qquad 0 < \theta_1 < 1 \tag{3.131}$$

(ii) **The generalized first mean value theorem for integrals.**

If both $f(x)$ and $g(x)$ are continuous functions for $a \le x \le b$ and the function $g(x)$ does not change sign for $x \in [a,b]$, then there exists a point $x = \xi_2 \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(\xi_2)\int_a^b g(x)\,dx = f(a+\theta_2(b-a))\int_a^b g(x)\,dx, \qquad 0 < \theta_2 < 1 \tag{3.132}$$

(iii) **Bonnet's second mean value theorem for integrals.**

If both $f(x)$ and $g(x)$ are continuous functions for $a \le x \le b$ and the function $g(x)$ is a positive monotonic[6] *decreasing function, then there exists a point $x = \xi_3 \in [a,b]$ such that*

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^{\xi_3} f(x)\,dx \tag{3.133}$$

where $\xi_3 = a + \theta_3(b-a)$, for $0 < \theta_3 < 1$. Alternatively, if $g(x)$ is a positive monotonic increasing function, then there exists a point $x = \xi_4 \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = g(b)\int_{\xi_4}^{b} f(x)\,dx \tag{3.134}$$

where $\xi_4 = a + \theta_4(b-a)$, for $0 < \theta_4 < 1$.

(iv) **The generalized second mean value theorem for integrals.**

If both $f(x)$ and $g(x)$ are continuous functions for $a \le x \le b$ and the function $g(x)$ is a monotone increasing or monotone decreasing over the interval $[a,b]$, then there exists a point $x = \xi_5 \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^{\xi_5} f(x)\,dx + g(b)\int_{\xi_5}^{b} f(x)\,dx \tag{3.135}$$

where $\xi_5 = a + \theta_5(b-a)$, for $0 < \theta_5 < 1$.

[6] If $f(x)$ is defined on an interval $[a,b]$ and if f is such that whenever $a < x_1 < x_2 < b$, there results $f(x_1) \le f(x_2)$, then f is called a monotone increasing function over the interval $[a,b]$. If the inequality above is reversed so that $f(x_1) \ge f(x_2)$, then f is called a monotone decreasing function over the interval $[a,b]$.

Proof of Mean Value Theorems

If $f(x) > 0$ is a continuous function over the interval $a \le x \le b$, define the functions

$$\begin{aligned} G(x) =& \int_a^x g(t)\,dt, & H(x) =& \int_a^x f(t)g(t)\,dt, & P(x) =& G(x)H(b) - G(b)H(x) \\ G'(x) =& g(x), & H'(x) =& f(x)g(x), & P'(x) =& G'(x)H(b) - G(b)H'(x) \end{aligned}$$

and observe that $P(a) = P(b) = 0$ because $G(a) = H(a) = 0$ and the way $P(x)$ is defined. Consequently, it is possible to apply Rolle's[7] theorem which states that there must exist a value $x = \xi$, for $a < \xi < b$, such that $P'(\xi) = 0$. This requires

$$g(\xi)\int_a^b f(t)g(t)\,dt - f(\xi)g(\xi)\int_a^b g(t)\,dt = 0$$

which simplifies to give the generalized first mean value theorem for integrals

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$

Note the special case $g(x) = 1$ produces the first mean value theorem for integrals.

To prove Bonnet's second mean value theorem, assume $f(x) > 0$ is a continuous function for $a \le x \le b$ and consider the cases where $g(x)$ is monotone decreasing and monotone increasing over the interval $[a, b]$.

Case 1: Assume that $g(x)$ is positive and monotone decreasing over the interval $[a, b]$. Define the function $\phi(x) = g(a)\int_a^x f(x)\,dx$ which is continuous over the interval $[a, b]$ and demonstrate $\phi(a) = 0 \le \int_a^b f(x)g(x)\,dx \le \phi(b)$. An application of the intermediate value theorem shows there exists a value $x = \xi$ such that $\phi(\xi) = g(a)\int_a^\xi f(x)\,dx = \int_a^b f(x)g(x)\,dx$.

Case 2: Assume that $g(x)$ is positive and monotone increasing over the interval $[a, b]$. Define the function $\psi(x) = g(b)\int_x^b f(x)\,dx$ which is continuous over the interval $[a, b]$ and demonstrate $\psi(b) = 0 \le \int_a^b f(x)g(x)\,dx \le \psi(a)$. An application of the intermediate value theorem shows that there exists a value $x = \xi$ such that $\psi(\xi) = g(b)\int_\xi^b f(x)\,dx = \int_a^b f(x)g(x)\,dx$.

To prove the generalized second mean value theorem for integrals consider the function $F(x) = \displaystyle\int_a^x f(u)\,du$ and then evaluate the integral $\displaystyle\int_a^b f(x)g(x)\,dx$ using integration by parts to show

$$\int_a^b f(x)g(x)\,dx = g(x)F(x)\Big|_a^b - \int_a^b F(x)g'(x)\,dx = g(b)F(b) - \int_a^b F(x)g'(x)\,dx \tag{3.136}$$

---

[7] Michel Rolle (1652-1719) a French mathematician.

The last integral in equation (3.136) can be evaluated as follows. The assumption that $g(x)$ is a monotonic function implies that the derivative $g'(x)$ is of a constant sign for $x \in [a,b]$ so that by the generalized first mean value theorem for integrals the equation (3.136) can be expressed in the form

$$
\begin{aligned}
\int_a^b f(x)g(x)\,dx =& g(b)F(b) - F(\xi)\int_a^b g'(x)\,dx = g(b)F(b) - F(\xi)[g(b)-g(a)] \\
=& g(a)F(\xi) + g(b)[F(b)-F(\xi)] = g(a)\int_a^\xi f(x)\,dx + g(b)\int_\xi^b f(x)\,dx
\end{aligned}
\tag{3.137}
$$

Differentiation of Integrals

The general **Leibnitz formula** for the differentiation of a general integral, where both the lower and upper limits of integration are given by functions $\alpha(t)$ and $\beta(t)$, is given by the relation

$$
\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(t,\tau)\,d\tau = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(t,\tau)}{\partial t}\,d\tau + f(t,\beta(t))\frac{d\beta}{dt} - f(t,\alpha(t))\,\frac{d\alpha}{dt} \tag{3.138}
$$

To derive the Leibnitz differentiation formula consider the following simpler examples.

Example 3-39. Show that $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ where a is a constant.

Solution Let $F(x) = \displaystyle\int_a^x f(t)\,dt$ and use the definition of a derivative to obtain

$$
\frac{dF(x)}{dx} = \lim_{\Delta x\to 0}\frac{F(x+\Delta x)-F(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)\,dt
$$

Apply the mean value theorem for integrals and show the above reduces to

$$
\frac{dF(x)}{dx} = F'(x) = \lim_{\Delta x\to 0}\frac{1}{\Delta x} f(x+\theta\,\Delta x)\,\Delta x = f(x), \qquad \text{where} \quad 0<\theta<1
$$

Consequently, one can write

$$
\frac{d}{dx}\int_a^x f(t)\,dt = f(x), \qquad \frac{d}{d\beta}\int_0^\beta f(t)\,dt = f(\beta), \qquad \frac{d}{d\alpha}\int_0^\alpha f(t)\,dt = f(\alpha)
$$

Note that the above results imply that $\dfrac{d}{dx}\displaystyle\int_x^a f(t)\,dt = -\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = -f(x)$.

■

Example 3-40. Show that $\dfrac{d}{dx}\displaystyle\int_{\alpha(x)}^{\beta(x)} f(t)\,dt = f(\beta(x))\,\frac{d\beta}{dx} - f(\alpha(x))\,\frac{d\alpha}{dx}$

Solution Write the given integral in the form

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(t)\,dt = \frac{d}{dx}\left[\int_{\alpha(x)}^{0} f(t)\,dt + \int_{0}^{\beta(x)} f(t)\,dt\right] = \frac{d}{dx}\left[\int_{0}^{\beta(x)} f(t)\,dt - \int_{0}^{\alpha(x)} f(t)\,dt\right]$$

and use chain rule differentiation employing the results from the previous example to show

$$\begin{aligned}\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(t)\,dt =& \frac{d}{dx}\left[\int_{0}^{\beta(x)} f(t)\,dt - \int_{0}^{\alpha(x)} f(t)\,dt\right]\\ =& \frac{d}{d\beta}\left[\int_{0}^{\beta} f(t)\,dt\right]\frac{d\beta}{dx} - \frac{d}{d\alpha}\left[\int_{0}^{\alpha} f(t)\,dt\right]\frac{d\alpha}{dx}\\ \frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(t)\,dt =& f(\beta(x))\,\frac{d\beta}{dx} - f(\alpha(x))\,\frac{d\alpha}{dx}\end{aligned}$$

■

Example 3-41.

Consider the function $I = I(x, g(x), h(x))$ defined by the integral

$$I = I(x, g, h) = \int_{g(x)}^{h(x)} f(x,t)\,dt \tag{3.139}$$

where the integrand is a function of both the variables x and t and the limits of integration g and h are also functions of x. The integration is with respect to the variable t and it is assumed that the integrand f is both continuous and differentiable with respect to x. The differentiation of a function defined by an integral containing a parameter x is given by the **Leibnitz rule**

$$\frac{dI}{dx} = I'(x) = \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x}\,dt + f(x,h(x))\frac{dh}{dx} - f(x,g(x))\frac{dg}{dx}. \tag{3.140}$$

The above result follows from the definition of a derivative together with the use of chain rule differentiation.

Consider first the special case of the integral $I_1(x) = \int_g^h f(x,t)\,dt$ where g and h are constants. Calculate the difference

$$I_1(x+\Delta x) - I_1(x) = \int_g^h [f(x+\Delta x,t) - f(x,t)]\,dt$$

and then employ the mean value theorem with respect to the x-variable and write

$$f(x+\Delta x, t) - f(x,t) = \frac{\partial f(x+\theta\Delta x, t)}{\partial x}\,\Delta x, \qquad 0<\theta<1$$

to write

$$I_1(x+\Delta x) - I_1(x) = \int_g^h \frac{\partial f(x+\theta\Delta x, t)}{\partial x}\,\Delta x\,dt \qquad \text{where } 0<\theta<1.$$

Dividing both sides by Δx and letting $\Delta x \to 0$ gives the derivative

$$\frac{dI_1}{dx} = I_1'(x) = \lim_{\Delta x\to 0} \frac{I_1(x+\Delta x) - I_1(x)}{\Delta x} = \lim_{\Delta x\to 0} \int_g^h \frac{\partial f(x+\theta\Delta x, t)}{\partial x}\,dt = \int_g^h \frac{\partial f(x,t)}{\partial x}\,dt$$

In the special case that both the upper and lower limits of integration are functions of x, one can employ chain rule differentiation for functions of more than one variable and express the derivative of $I = I(x,g,h)$ as

$$\frac{dI}{dx} = \frac{\partial I}{\partial x} + \frac{\partial I}{\partial g}\frac{dg}{dx} + \frac{\partial I}{\partial h}\frac{dh}{dx}. \tag{3.141}$$

where

$$\frac{\partial I}{\partial x} = \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x}\,dt, \qquad \frac{\partial I}{\partial h} = f(x,h(x)), \qquad \frac{\partial I}{\partial g} = -f(x,g(x))$$

The equation (3.141) then simplifies to the result given by equation (3.140).

■

Double Integrals

Integrals of the form

$$I_1 = \int_a^b \int_c^d f(x,y)\,dy\,dx \qquad \text{or} \qquad I_2 = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

are called **double integrals** of the function $z = f(x,y)$ over the rectangular region R defined by

$$R = \{\ (x,y)\ \mid\ a \le x \le b,\ c \le y \le d\ \}$$

These double integrals are evaluated from the inside out and can be given the following physical interpretation. The function $z = f(x,y)$, for $(x,y) \in R$, can be thought of as a **smooth surface over the rectangle** as illustrated in the figure 3-12.

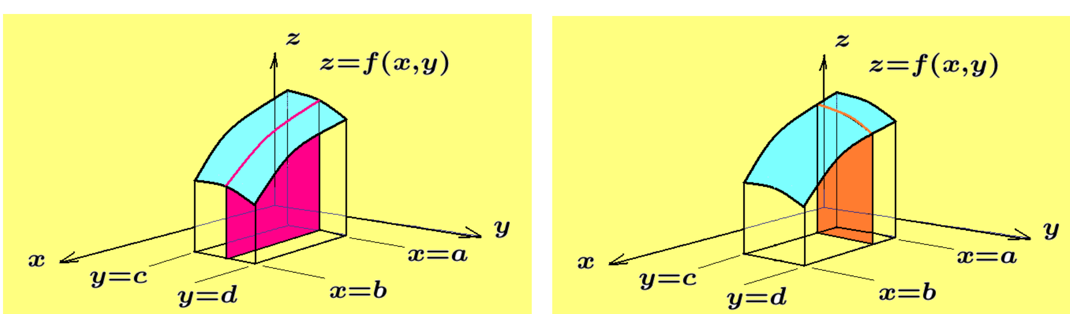


Figure 3-12.
Planes $x =$ a constant and $y =$ a constant intersecting surface $z = f(x,y)$.

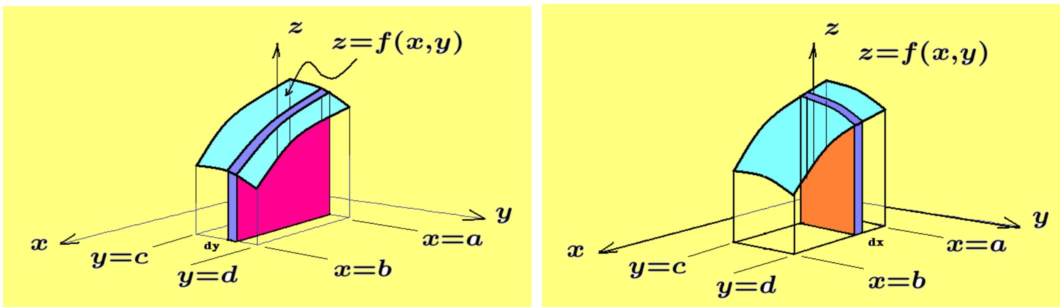


Figure 3-13.
Elements of volume in the shape of slabs.

In the left figure, **the plane, $y =$ a constant, intersects the surface $z = f(x,y)$ in the curve**

$$z = f(x,y) \qquad y = \text{ a constant}$$

and the integral

$$\int_a^b f(x,y)\,dx \qquad y = \text{a constant}$$

represents **the area under this curve.** If this area is multiplied by dy, one obtains an **element of volume dV in the shape of a slab with thickness dy** as illustrated in the figure 3-13. This element of volume dV can be expressed as an area times a thickness to obtain

$$dV = \left[\int_a^b f(x,y)\,dx\right]\,dy$$

If the element of volume is summed from $y = c$ to $y = d$, there results the total volume

$$V = \int_c^d \left[\int_a^b f(x,y)\,dx\right]\,dy$$

Here the inner integral is integrated first while holding y constant and then the result is integrated with respect to y from c to d to perform a summation representing the volume under the surface $z = f(x,y)$. Double integrals and multiple integrals in general are sometimes referred to as iterated or repeated integrals. When confronted with multiple integrals always perform the inner integral first and the outside integral last.

In a similar fashion, **the plane,** $x =$ **a constant, intersects the surface** $z = f(x, y)$ **in the curve**

$$z = f(x, y) \qquad x = \text{ a constant}$$

so that the integral

$$\int_c^d f(x, y)\, dy \qquad x = \text{ a constant}$$

represents **the plane area under the curve**. If the resulting area is multiplied by dx, one obtains a **volume element** dV **in the shape of a slab times a thickness** dx. This slab is given by

$$dV = \left[\int_c^d f(x, y)\, dy\right] dx$$

If these volume elements are summed from $x = a$ to $x = b$, then the resulting volume under the surface is given by

$$V = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx$$

Another way to interpret the previous double integrals is to first partition the interval $[a, b]$ into n parts by defining $\Delta x = \frac{b-a}{n}$ and then partition the interval $[c, d]$ into m parts by defining $\Delta y = \frac{d-c}{m}$. One can then define the points

$$\begin{aligned} a =& x_0, \ldots, x_i = a + i\Delta x, \ldots x_n = a + n\Delta x = a + n\frac{b-a}{n} = b \\ c =& y_0, \ldots, y_j = c + j\Delta y, \ldots y_m = c + m\Delta y = c + m\frac{d-c}{m} = d \end{aligned}$$

where i and j are integers satisfying $0 \le i \le n$ and $0 \le j \le m$. One can then move to a point (x_i, y_j) located within the rectangle R and construct **a parallelepiped of height** $f(x_i, y_j)$ **and base with sides** $\Delta x_i = x_{i+1} - x_i$ **and** $\Delta y_j = y_{j+1} - y_j$ as illustrated in the figure 3-14.

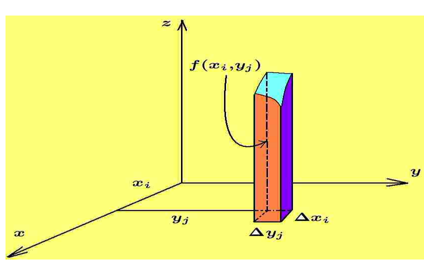


Figure 3-14. Element of volume $dV = f(x, y)\, dy\, dx$

A summation over the rectangle of these parallelepiped volume elements gives an approximation to the volume bounded by the surface $z = f(x, y)$, and the planes $x = a$, $x = b$, $y = c$, $y = d$ and $z = 0$. This approximation gets better and better as $\Delta x_i \to 0$ and $\Delta y_j \to 0$. One finds that

$$\lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\, \Delta x_i\, \Delta y_j = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy = \int_a^b \left[\int_c^d f(x, y)\, dx \right] dy$$

where the inner integrals produce a slab, either in the x or y directions and the outer integrals then represents a summation of these slabs giving the volume under the surface. Note that if the surface $z = f(x, y)$ oscillates above and below the plane $z = 0$, then the result of the double integral gives a summation of the "signed" volumes.

The orientation of the surface might be such that it is represented in the form $x = g(y, z)$, in which case the height of the surface is the distance x above the plane $x = 0$. If the surface is represented $y = h(x, z)$, then the height of the surface is the distance y above the plane $y = 0$. Hence volume integrals can be represented as double integrals having one of the forms $V_1 = \int_{a_1}^{b_1} \int_{c_1}^{d_1} f(x, y)\, dy\, dx$ if $z = f(x, y)$ describes the surface, or $V_2 = \int_{a_2}^{b_2} \int_{c_2}^{d_2} g(y, z)\, dz\, dy$ if $x = g(y, z)$ describes the surface or $V_3 = \int_{a_3}^{b_3} \int_{c_3}^{d_3} h(x, z)\, dz\, dx$ if $y = h(x, z)$ describes the surface.

Summations over nonrectangular regions

If the smooth surface $z = f(x, y)$ is defined over some nonrectangular region R where the region R can be defined

(i) by a lower curve $y = g_1(x)$ and upper curve $y = g_2(x)$ between the limits $a \le x \le b$

(ii) or by a left curve $x = h_1(y)$ and a right curve $x = h_2(y)$ between the limits $c \le y \le d$

as illustrated in the figure 3-15, then the volume is still obtained by a limiting summation of the parallelepipeds constructed with base area $dx\, dy$ and height $f(x, y)$.

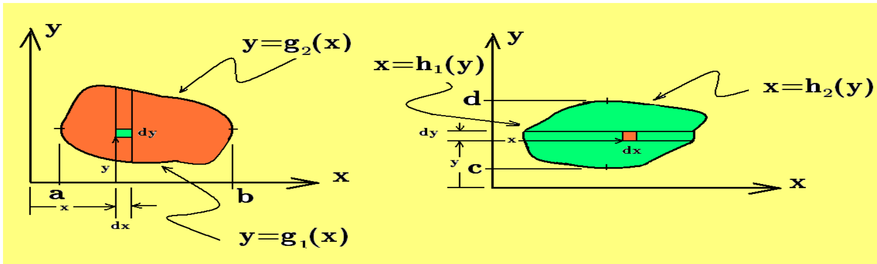


Figure 3-15. Summation of $dx\, dy$ over region R

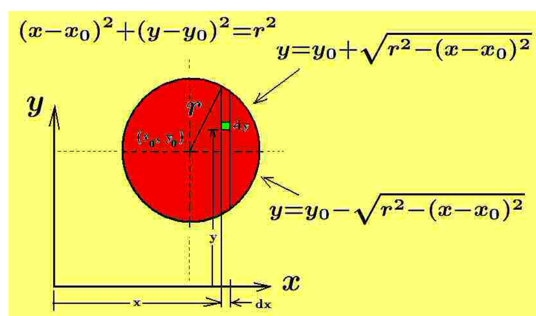
The volume under the surface is similar to the case where the region R is a rectangle, but instead the summations of the parallelepipeds are from one curve to another curve. One can write either of the volume summations

$$\iint_R f(x,y)\, dy\, dx = \int_{x=a}^{x=b} \left[\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\, dy \right] dx$$
$$\iint_R f(x,y)\, dx\, dy = \int_{y=c}^{y=d} \left[\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\, dx \right] dy$$

The first inner integral sums in the vertical direction to create a slab and the outer integral sums these slabs from a to b. The second inner integral sums in the horizontal direction to form a slab and the outer integral sums these slabs from c to d.

Example 3-42.

If the surface $(x-x_0)^2 + (y-y_0)^2 = r^2$ is a cylinder in three-dimensions. If this cylinder is cut by the planes $z=0$ and $z=h$, a finite cylinder is formed by the bounding surfaces. It is known that the total volume V of this finite cylinder is the area of the base times the height or $V = \pi r^2 h$. Derive this result using double integrals.



Solution Here the region R is a circle of radius r centered at the point (x_0, y_0) and bounded by the upper semi-circle $y = y_0 + \sqrt{r^2 - (x-x_0)^2}$ and the lower semi-circle $y = y_0 - \sqrt{r^2 - (x-x_0)^2}$. The parallelepiped element of volume is located at position (x, y) where the base area $dx\, dy$ is constructed. The height of the parallelepiped to be summed is h and so the parallelepiped element of volume is given by $dV = h\, dy\, dx$ where the element dy is written first because the inner integral is to be summed in the $y-$direction from the lower semi-circle to the upper semi-circle. Summing these volume elements first in the y-direction and then summing in the $x-$direction gives

$$V = \int_{x=x_0-r}^{x=x_0+r} \left[\int_{y=y_0-\sqrt{r^2-(x-x_0)^2}}^{y=y_0+\sqrt{r^2-(x-x_0)^2}} h\, dy \right] dx \tag{3.142}$$

Here the inner integral produces a slab and then these slab elements are summed from $x_0 - r$ to $x_0 + r$. Perform the inner integration to obtain

$$V = h \int_{x_0-r}^{x_0+r} y \Big|_{y_0-\sqrt{r^2-(x-x_0)^2}}^{y_0+\sqrt{r^2-(x-x_0)^2}} dx = 2h \int_{x_0-r}^{x_0+r} \sqrt{r^2-(x-x_0)^2}\, dx \tag{3.143}$$

Here the integrand involves a difference of squares and suggests that one make the trigonometric substitution $x - x_0 = r\cos\theta$ with $dx = -r\sin\theta\, d\theta$ to obtain

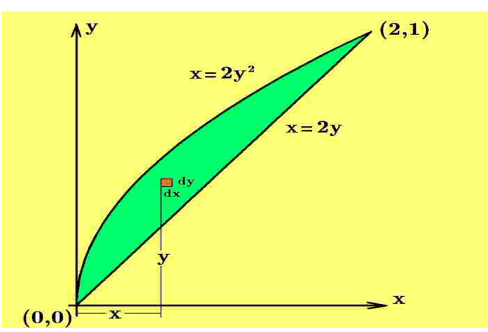
$$\begin{aligned} V =& 2h \int_{\pi}^{0} r\sqrt{1-\cos^2\theta}\,(-r\sin\theta\, d\theta) = 2hr^2 \int_0^{\pi} \sin^2\theta\, d\theta = 2hr^2 \int_0^{\pi} \frac{1}{2}(1-\cos 2\theta)\, d\theta \\ V =& hr^2 \left[\int_0^{\pi} d\theta - \frac{1}{2}\int_0^{\pi} \cos 2\theta\, 2d\theta \right] = hr^2 \left[\theta - \frac{1}{2}\sin 2\theta \right]_0^{\pi} = \pi r^2 h \end{aligned}$$

It is left as an exercise to perform the inner summation in the $x-$direction first, followed by a summation in the $y-$direction and show that the same result is obtained.

■

Example 3-43. Evaluate the iterated integral of the function $f(x, y) = xy^2$ over the region between the parabola $x = 2y^2$ and the line $x = 2y$.

Solution



Sketch the region over which the integration is to be performed and then move to a general point (x, y) within the region and construct an element of area $dx\, dy$ and determine which direction is "best" for the inner integral. For this problem the given curves intersect where $x = 2y^2 = 2y$ giving the points of intersection $(0, 0)$ and $(2, 1)$. One can then construct the figure illustrated. Let us examine the integrals

$$V = \int_{x=0}^{x=2} \left[\int_{y=x/2}^{y=\sqrt{x/2}} xy^2\, dy \right] dx \quad \text{and} \quad V = \int_{y=0}^{y=1} \left[\int_{x=2y^2}^{x=2y} xy^2\, dx \right] dy \tag{3.144}$$

where the top inner integral is a summation in the $y-$direction and the bottom inner integral represents a summation in the $x-$direction. Now select the iterated integral which you think is easiest to integrate. For this problem, both integrals are about the same degree of difficulty. For the first double integral in equation (3.144) one finds

$$V = \int_{x=0}^{x=2} \left[\int_{y=x/2}^{y=\sqrt{x/2}} xy^2\, dy \right] dx = \int_{x=0}^{x=2} \left[x\frac{y^3}{3} \right]_{y=x/2}^{y=\sqrt{x/2}} dx$$

$$V = \int_{x=0}^{x=2} \frac{x}{3} \left[\left(\frac{\sqrt{x}}{2}\right)^3 - \left(\frac{x}{2}\right)^3 \right] dx$$

$$V = \int_0^2 \left(\frac{1}{6\sqrt{2}} x^{5/2} - \frac{1}{24} x^4 \right) dx = \frac{1}{6\sqrt{2}} \left[\frac{x^{7/2}}{7/2} - \frac{1}{24}\frac{x^5}{5} \right]_0^2 = \frac{4}{35}$$

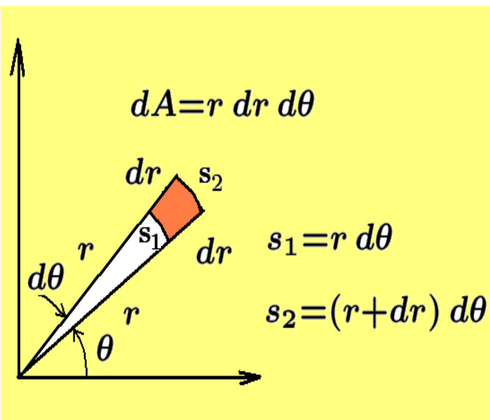
For the second double integral in equation (3.144) the inner integral is evaluated first followed by an integration with respect to the outer integral to obtain

$$V = \int_{y=0}^{y=1} \left[\int_{x=2y^2}^{x=2y} xy^2\, dx \right] dy = \int_0^1 y^2 \frac{x^2}{2} \Big|_{x=2y^2}^{x=2y} dy = \int_0^1 \frac{y^2}{2} \left[(2y)^2 - (2y^2)^2 \right] dy$$

$$V = \int_0^1 \left(2y^4 - 2y^6 \right) dy = \left(2\frac{y^5}{5} - 2\frac{y^7}{7} \right) \Big|_{y=0}^{y=1} = \frac{4}{35}$$

■

Polar Coordinates



An element of area in polar coordinates can be constructed by sketching the arbitrary rays θ and $\theta+d\theta$ together with the circular arcs of radius r and $r+dr$ as illustrated in the figure given. The element of area dA can then be expressed as the **area of the sector** with radius $r+dr$ minus the area of the sector with radius r. This can be represented

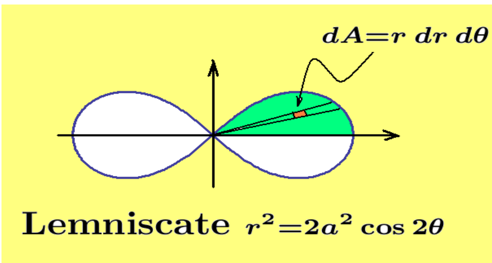
$$dA = \frac{1}{2}(r+dr)^2 d\theta - \frac{1}{2}r^2 d\theta = \frac{1}{2}\left(r^2 + 2r\,dr + dr^2 - r^2\right) d\theta = r\,dr\,d\theta + \frac{1}{2}dr^2 d\theta$$

The last term is $\frac{1}{2}dr^2 d\theta$ is an infinitesimal of higher order and so this term can be neglected. One then finds the element of area in polar coordinates is given by $dA = r\,dr\,d\theta$.

To find an area associated with a region bounded by given curves one can use cartesian coordinates and write $dA = dx\,dy$ as an element of area and perform summations in the $x-$direction and then the $y-$direction and express the total area as

$A = \iint dx\,dy$ with appropriate limits on the integrals. Alternatively, one can represent the element of area in polar coordinates as $dA = r\,dr\,d\theta$ and perform summations in the $r-$direction and then the $\theta-$direction to express the total area as $A = \iint r\,dr\,d\theta$ with appropriate limits on the integrals.

Example 3-44. Find the area bounded by the lemniscate $r^2 = 2a^2 \cos 2\theta$.



Construct an element of area $dA = r\,dr\,d\theta$ inside the lemniscate and then make use of symmetry by calculating only the area in the first quadrant. One can then represent the area in the first quadrant by the double integral

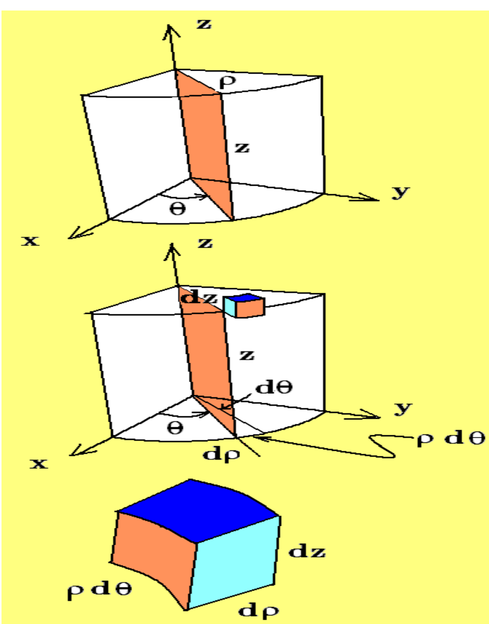
$$A = \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=\sqrt{2a^2\cos 2\theta}} r\,dr\,d\theta = \int_0^{\pi/4} \frac{1}{2} r^2 \Big|_0^{\sqrt{2a^2\cos 2\theta}} d\theta$$

$$A = a^2 \int_0^{\pi/4} \cos 2\theta\,d\theta = \frac{a^2}{2}\sin 2\theta \Big|_0^{\pi/4} = \frac{a^2}{2}$$

The total area under the lemniscate is therefore $A_{total} = 4\left(\frac{a^2}{2}\right) = 2a^2$.

■

Cylindrical Coordinates



The coordinate transformation from cartesian coordinates (x, y, z) to cylindrical coordinates (r, θ, z) is given by

$$\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \\ z &= z \end{aligned} \qquad \text{or} \qquad \begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(y/x) \\ z &= z \end{aligned}$$

where ρ is a **radial distance** from the $z-$axis, θ is an **angular displacement** measured from the $x-$axis and is called the **azimuth or azimuthal angle** and z is the **height above the plane** $z = 0$. The element of volume in cylindrical coordinates is given by $dV = \rho\,d\rho d\theta dz$

Example 3-45. Find the volume of a cylinder of radius R and height H.

Solution Use the volume element in cylindrical coordinates and express the volume as the triple integral

$$V = \int_{z=0}^{z=H} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=R} \rho \, d\rho d\theta dz$$

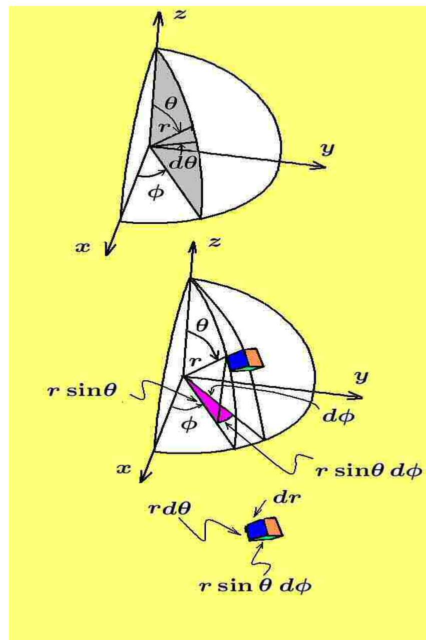
Evaluating this integral from the inside-outward gives

$$V = \int_{z=0}^{z=H} \int_{\theta=0}^{\theta=2\pi} \frac{\rho^2}{2}\Big|_{\rho=0}^{\rho=R} d\theta\, dz = \frac{R^2}{2} \int_{z=0}^{z=H} \int_{\theta=0}^{\theta=2\pi} d\theta\, dz = \frac{R^2}{2} \int_{z=0}^{z=H} \theta \Big|_{\theta=0}^{\theta=2\pi} dz$$

$$V = \pi R^2 \int_{z=0}^{z=H} dz = \pi R^2 z \Big|_{z=0}^{z=H} = \pi R^2 H$$

■

Spherical Coordinates



The coordinate transformation from cartesian coordinates (x, y, z) to spherical coordinates (r, θ, ϕ) is given by

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\cos\phi, \qquad z = r\cos\theta$$

for $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$, where r is called the **radial distance from the origin**, θ is called the **inclination from the $z-$axis** and ϕ is called the **azimuth or azimuthal angle**. The element of volume in spherical coordinates is given by

$$dV = (r\sin\theta\, d\phi)(r\, d\theta)\, dr = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Example 3-46. Find the volume of a sphere having a radius R.

Solution Using the element of volume for spherical coordinates, the volume of a sphere is given by the triple integral

$$V = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=R} r^2 dr\, \sin\theta\, d\theta\, d\phi$$

Evaluating the triple integral from the inside-outward gives

$$V = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{r^3}{3}\Big|_{r=0}^{r=R} \sin\theta\, d\theta\, d\phi$$

$$V = \frac{R^3}{3} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin\theta\, d\theta\, d\phi = \frac{R^3}{3} \int_{\phi=0}^{\phi=2\pi} (-\cos\theta)\Big|_{\theta=0}^{\theta=\pi} d\phi$$

$$V = \frac{2}{3}R^3 \int_{\phi=0}^{\phi=2\pi} d\phi = \frac{2}{3}R^3\, \phi \Big|_{\phi=0}^{\phi=2\pi} = \frac{4}{3}\pi R^3$$

■

Using Table of Integrals

The appendix C contains a table of integrals. In order to use these tables one must sometimes make appropriate substitutions in order to convert an integral into the proper form as given in the tables. For example, if it is required to evaluate the integral

$$I = \int \frac{e^{3x}\,dx}{a+be^x}$$

where a and b are constants and you look for this integral in the table, you will not find it. This is because the integral is listed under a different form. If you make the substitution $u = e^x$ with $du = e^x\,dx$ The above integral can be written in the form

$$I = \int \frac{u^2\,du}{a+bu} \quad \text{where} \quad u = e^x$$

which is a form that you can locate in the tables. One finds under the listing for integrals containing $X = a+bx$, the listing number 78 for $\int \frac{x^2\,dx}{X}$. Here the variable x in the table of integrals is just a dummy variable of integration and you can replace x by u and write

$$I = \int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln(a+bu)\right] + C$$

Use back substitution to express the integral in terms of e^x.

The Bliss Theorem

In the previous pages there were times when summations were replaced by integrals because of the fundamental theorem of integral calculus. The replacement of summations by integrals can be also be justified by using the Bliss' theorem. Gilbert Ames Bliss (1876-1961) was an American mathematician who studied mathematics his whole life. One of the results he discovered is known as Bliss' theorem[8]. This theorem relates summations and integrations and has the following geometric interpretation. Consider an interval (a,b) which is partitioned into n-subintervals by defining points

$$a = x_0 < x_1 < x_2 < \ldots < x_{i-1} < x_i < \ldots < x_n = b$$

and intervals $\Delta x_i = x_i - x_{i-1}$ for $i = 1,\ldots,n$. The spacing for the points x_i need not be uniform. Define

$$\delta = \text{maximum}\,[\Delta x_1, \Delta x_2, \ldots, \Delta x_i, \ldots, \Delta x_n\,]$$

[8] Bliss,G.A., *A substitute for Duhamel's Theorem*, Annals of Mathematics, Vol. 16 (1914-15), Pp 45-49.

as the largest subinterval associated with the selected partition. Bliss showed that if $f(x)$ and $g(x)$ are single-valued and continuous functions defined in the interval (a, b), then for each subinterval Δx_i one can select arbitrary points ξ_i and η_i inside or at the ends of the subinterval (x_{i-1}, x_i) such that for each value $i = 1, \ldots, n$ one can write

$$x_{i-1} < \xi_i < x_i \qquad \text{and} \qquad x_{i-1} < \eta_i < x_i$$

One can then form the sum

$$\sum_{i=1}^{n} f(\xi_i) g(\eta_i) \Delta x_i$$

and one can also form the integral

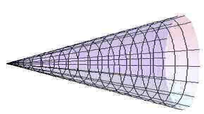
$$\int_a^b f(x) g(x)\, dx.$$

Bliss' theorem states that as $n \to \infty$ and $\delta \to 0$, then

$$\lim_{\substack{\delta \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(\xi_i) g(\eta_i) \Delta x_i = \int_a^b f(x) g(x)\, dx \tag{3.145}$$

Note that this important theorem allows one to replace summations over an interval by integrations over the interval and the fundamental theorem of integral calculus is a special case of this theorem. The above result is used quite often in developing methods for finding answers to physical problems where discrete summations become continuous integrals as the number of summations increase.

Example 3-47.



A right circular cone is obtained by revolving the line $y = \frac{r}{h} x$, $0 \le x \le h$, about the x-axis. Divide the interval $0 \le x \le h$ into n-parts each of length $\Delta x = \frac{h}{n}$ and then form circular disks at position $x_i = i \Delta x = \frac{ih}{n}$ having radius $y_i = \frac{r}{h} x_i = \frac{ir}{n}$, for $i = 1, 2, 3, \ldots, n$. Find the total volume V of the disks in the limit as $n \to \infty$.

Solution 1 Use the result from page 178, $\sum_{i=1}^{n} i^2 = \frac{1}{6}(2n^3 + 3n^2 + n)$ and write

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi y_i^2 \Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \frac{r^2}{n^2} i^2 \frac{h}{n} = \lim_{n\to\infty} \pi \frac{r^2 h}{n^3} \sum_{i=1}^{n} i^2 = \pi r^2 h \lim_{n\to\infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{\pi}{3} r^2 h$$

Solution 2 Write

$$V = \int_0^h \pi y^2\, dx = \int_0^h \pi \frac{r^2}{h^2} x^2\, dx = \frac{\pi}{3} r^2 h$$

■

Exercises

▶ **3-1.** Evaluate the given integrals.

$$(a)\ \int (2x+1)^3\,dx \qquad (c)\ \int \frac{\cos t}{\sin^2 t}\,dt \qquad (e)\ \int (\sin x)(\cos x)\,dx$$

$$(b)\ \int \sin 4x\,dx \qquad (d)\ \int (\sqrt{4-\sin 2t})\,\cos 2t\,dt \qquad (f)\ \int (ax^2+bx+c)\,dx$$

▶ **3-2.** Evaluate the given integrals.

$$(a)\ \int (2+\sin 2t)^4 \cos 2t\,dt \qquad (c)\ \int x\,4^{x^2}\,dx \qquad (e)\ \int \sin(3x+1)\,dx$$

$$(b)\ \int \frac{(3+2x)}{4+3x+x^2}\,dx \qquad (d)\ \int e^{3x}\,dx \qquad (f)\ \int x\,\cos(3x^2+1)\,dx$$

▶ **3-3.** Evaluate the given integrals.

$$(a)\ \int \sec^2(3x+4)\,dx \qquad (c)\ \int \sec(3x+4)\tan(3x+4)\,dx \qquad (e)\ \int \frac{\cos(3x^2)}{\sin(3x^2)}\,x\,dx$$

$$(b)\ \int \csc^2(3x+4)\,dx \qquad (d)\ \int (2x+1)\csc(x^2+x)\cot(x^2+x)\,dx \qquad (f)\ \int x\,e^{x^2}\,dx$$

▶ **3-4.** Evaluate the given integrals.

$$(a)\ \int \frac{x\,dx}{\sqrt{1-x^4}} \qquad (c)\ \int \frac{x\,dx}{\sqrt{1-x^2}} \qquad (e)\ \int x\cosh(x^2)\,dx$$

$$(b)\ \int \frac{x\,dx}{1+x^2} \qquad (d)\ \int x\sinh(x^2)\,dx \qquad (f)\ \int x\,\mathrm{sech}^2(x^2)\,dx$$

▶ **3-5.** Evaluate the given integrals.

$$(a)\ \int \mathrm{csch}\,(3x+1)\coth(3x+1)\,dx \qquad (c)\ \int \mathrm{csch}^2(3x+1)\,dx \qquad (e)\ \int \frac{(3x+1)\,dx}{\sqrt{(3x+1)^2-1}}$$

$$(b)\ \int \mathrm{sech}\,(3x+1)\tanh(3x+1)\,dx \qquad (d)\ \int \frac{(3x+1)\,dx}{\sqrt{1-(3x+1)^2}} \qquad (f)\ \int \frac{(3x+1)\,dx}{1-(3x+1)^2}$$

▶ **3-6.** Evaluate the given integrals.

$$(a)\ \int \left(\frac{1}{x^2}+\frac{1}{x}+5\right)dx \qquad (c)\ \int \left(a+b\sqrt{t}+\frac{c}{\sqrt{t}}\right)dt \qquad (e)\ \int u\sqrt{a+bu^2}\,du$$

$$(b)\ \int x\sqrt{x^2+25}\,dx \qquad (d)\ \int z\sqrt{2z}\,dz \qquad (f)\ \int \sqrt{a+bu}\,du$$

▶ **3-7.** Evaluate the definite integrals and give a physical interpretation of what the integral represents.

$$(a)\ \int_1^3 x^2\,dx \qquad (b)\ \int_0^{\pi} \sin x\,dx \qquad (c)\ \int_0^B \frac{H}{B}x\,dx$$

► **3-8.** If necessary use trigonometric substitution to evaluate the given integrals.

$$(a)\quad \int \frac{dx}{(3x+1)\sqrt{(3x+1)^2+1}} \qquad (c)\quad \int \frac{dx}{(3x+1)^2-1} \qquad (e)\quad \int \sqrt{a^2-u^2}\,du$$

$$(b)\quad \int \frac{dx}{(3x+1)\sqrt{1-(3x+1)^2}} \qquad (d)\quad \int \frac{dx}{\sqrt{9-(2x+1)^2}} \qquad (f)\quad \int \frac{\sqrt{a^2-t^2}}{t^2}\,dt$$

► **3-9.** For C_1, C_2 constants, explain why the following integrals are equivalent.

$$(a)\quad \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C_1, \qquad \int \frac{du}{\sqrt{1-u^2}} = -\cos^{-1} u + C_2$$

$$(b)\quad \int \frac{du}{1+u^2} = \tan^{-1} u + C_1, \qquad \int \frac{du}{1+u^2} = -\cot^{-1} u + C_2$$

$$(c)\quad \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C_1, \qquad \int \frac{du}{u\sqrt{u^2-1}} = -\csc^{-1} u + C_2$$

$$(d)\quad \int \tan u\,du = -\ln \mid \cos u \mid + C_1, \qquad \int \tan u\,du = \ln \mid \sec u \mid + C_2$$

$$(e)\quad \int \cot u\,du = \ln \mid \sin u \mid + C_1, \qquad \int \cot u\,du = -\ln \mid \csc u \mid + C_2$$

► **3-10.** Evaluate the given integrals.

$$(a)\quad \int \sin^2(3x+1)\,dx \qquad (c)\quad \int \sin^3(3x+1)\,dx \qquad (e)\quad \int \sin^4(3x+1)\,dx$$

$$(b)\quad \int \cos^2(3x+1)\,dx \qquad (d)\quad \int \cos^3(3x+1)\,dx \qquad (f)\quad \int \cos^4(3x+1)\,dx$$

► **3-11.** Use partial fractions to evaluate the given integrals.

$$(a)\quad \int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}\,dx \qquad (d)\quad \int \frac{x\,dx}{x^2+4x-5}$$

$$(b)\quad \int \frac{4x^2-8x+3}{(x-1)^2(x-2)}\,dx \qquad (e)\quad \int \frac{2x^8+5x^7+8x^6+5x^5-5x^3-2x^2-3x-1}{(x-1)(x^2+x+1)^2}\,dx$$

$$(c)\quad \int \frac{3x^2+3x+2}{(x+1)(x^2+1)}\,dx \qquad (f)\quad 2\int \frac{x^3+x^2+x}{(x^2-1)(x+2)}\,dx$$

► **3-12.** Find the function $y = y(x)$ passing through the given point (x_0, y_0) whose derivative satisfies $\frac{dy}{dx} = f(x)$, if

$$(a)\quad f(x) = x, \quad (x_0, y_0) = (1,3) \qquad (d)\quad f(x) = \tan^2(3x), \quad (x_0, y_0) = (0,1)$$

$$(b)\quad f(x) = x+1, \quad (x_0, y_0) = (1,2) \qquad (e)\quad f(x) = \sin^2(3x), \quad (x_0, y_0) = (0,1)$$

$$(c)\quad f(x) = \sin 3x, \quad (x_0, y_0) = (0,1) \qquad (f)\quad f(x) = \cos^2(3x), \quad (x_0, y_0) = (0,1)$$

Note: An equation which contains a derivative, like $\frac{dy}{dx} = f(x)$, is called a differential equation. The above problem can be restated as, "Solve the first order differential equation $\frac{dy}{dx} = f(x)$ subject to the initial condition $y(x_0) = y_0$."

▶ **3-13.** If necessary use partial fractions to evaluate the given integrals.

$$(a)\ \int \frac{x^2-x-1}{(x-2)(x-1)^2}\,dx \qquad (c)\ \int \frac{dx}{x(x^2+x+1)} \qquad (e)\ \int \frac{dx}{x(x-1)^2}$$
$$(b)\ \int \frac{x^2\,dx}{x^2+1} \qquad (d)\ \int \frac{x^2\,dx}{(x+1)^2} \qquad (f)\ \int \frac{dx}{x(a-x)}$$

▶ **3-14.** Use partial fractions to evaluate the given integrals.

$$(a)\ \int \frac{dx}{(x-a)(x-b)} \qquad (c)\ \int \frac{dx}{(x-a)^2(x-b)^2} \qquad (e)\ \int \frac{dx}{(x-a)(x^2+b^2)}$$
$$(b)\ \int \frac{dx}{(x-a)^2(x-b)} \qquad (d)\ \int \frac{dx}{(x-a)(x^2-b^2)} \qquad (f)\ \int \frac{dx}{(x^2-a^2)(x^2+b^2)}$$

▶ **3-15.** Use integration by parts to evaluate the given integrals.

$$(a)\ \int_0^1 \sin^{-1}x\,dx \qquad (c)\ \int_0^{\pi/2} x\cos x\,dx \qquad (e)\ \int_0^{\pi} x\sin 3x\,dx$$
$$(b)\ \int_0^{\pi/2} x\sin x\,dx \qquad (d)\ \int_1^{e} x\ln x\,dx \qquad (f)\ \int_1^{16} x\sqrt{x-1}\,dx$$

▶ **3-16.** Use integration by parts to evaluate the given integrals.

$$(a)\ \int x\,e^x\,dx \qquad (c)\ \int x^2\sin x\,dx \qquad (e)\ \int e^x\cos x\,dx$$
$$(b)\ \int x^2\,e^x\,dx \qquad (d)\ \int e^x\sin x\,dx \qquad (f)\ \int \sec^3 x\,dx$$

▶ **3-17.** Use integration by parts to evaluate the given integrals.

$$(a)\ \int e^{\alpha x}\sin\beta x\,dx \qquad (c)\ \int x\ln(x+1)\,dx \qquad (e)\ \int_0^1 x^2e^x\,dx$$
$$(b)\ \int \ln x\,dx \qquad (d)\ \int e^{\alpha x}\cos\beta x\,dx \qquad (f)\ \int_0^1 x^3e^{-x}\,dx$$

▶ **3-18.** Use the fundamental theorem of integral calculus and express the given sums as a definite integral.

$$(a)\quad \lim_{n\to\infty}\frac{1}{n}\left[f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n}{n})\right]$$
$$(b)\quad \lim_{n\to\infty}\frac{1}{n}\left[\sin(\frac{\pi}{n})+\sin(\frac{2\pi}{n})+\cdots+\sin(\frac{n\pi}{n})\right]$$
$$(c)\quad \lim_{n\to\infty}\frac{1}{n}\left[\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right]$$

▶ **3-19.** Sketch the curves $x=y^2-6$ and $x=4y-1$. Find the area enclosed by these curves.

▶ **3-20.** It has been found that to integrate rational functions of the sine and cosine functions, the change of variables $u = \dfrac{\sin x}{1+\cos x} = \tan\dfrac{x}{2}$ sometimes simplifies the integration problem.

(a) Show that if $u = \tan\dfrac{x}{2}$, then one obtains

$$(i)\quad dx = \frac{2du}{1+u^2}, \qquad (ii)\quad \sin x = \frac{2u}{1+u^2}, \qquad (iii)\quad \cos x = \frac{1-u^2}{1+u^2}$$

(b) Evaluate the integral $\displaystyle\int \frac{dx}{1+\cos x}$

(c) Evaluate the integral $\displaystyle\int \frac{dx}{\sin x - \cos x}$

▶ **3-21.**

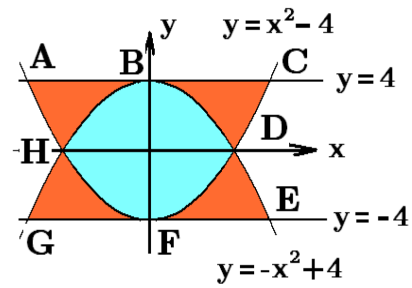
The intersection of the curves $y = x^2 - 4$, $y = -x^2 + 4$, $y = 4$ and $y = -4$ are illustrated in the figure.

(a) Find the area with vertices ABH

(b) Find the area with vertices $BDFH$

(c) Find the area with vertices $HABFGH$

(d) Find the area with vertices $ABDEFHA$



▶ **3-22.** If $\dfrac{d}{dx}\left[\tan^{-1}(3x^2) + \ln(x^4+x^2)\right] = \dfrac{6x}{1+9x^4} + \dfrac{2+4x^2}{x+x^3}$, then find

$$\int \left(\frac{6x}{1+9x^4} + \frac{2+4x^2}{x+x^3}\right) dx$$

▶ **3-23.** Evaluate the given integrals.

$$(a)\quad \int \frac{dx}{\sqrt{16+6x-x^2}} \qquad (c)\quad \int e^x \sin e^x\, dx \qquad (e)\quad \int (\ln x)^2\, dx$$

$$(b)\quad \int \left(x^2 + \frac{1}{\sqrt{1-x^2}}\right) dx \qquad (d)\quad \int x\sqrt{a^2-x^2}\, dx \qquad (f)\quad \int x e^{\alpha x}\, dx$$

▶ **3-24.** Evaluate the given integrals.

$$(a)\quad \int \frac{dt}{\sqrt{t^2+4t-3}} \qquad (c)\quad \int \frac{dx}{(x^2+9)^2} \qquad (e)\quad \int \frac{dx}{(3x)^2-9}$$

$$(b)\quad \int \frac{dx}{9-x^2} \qquad (d)\quad \int \frac{dx}{(3x)^2+9} \qquad (f)\quad \int \frac{dx}{\sqrt{9-(3x)^2}}$$

▶ **3-25.** Evaluate the given integrals.

$$(a)\ \int \frac{dx}{\sqrt{x}(1+\sqrt{x})} \qquad (c)\ \int \sin^2(3\theta)\,d\theta \qquad (e)\ \int \frac{ax+b}{x^2-c^2}\,dx$$
$$(b)\ \int \frac{e^x\,dx}{1+3e^x} \qquad (d)\ \int \frac{au+b}{u^2+c^2}\,du \qquad (f)\ \int \frac{ax+b}{\sqrt{b^2-(x+a)^2}}\,dx$$

▶ **3-26.** Evaluate the given integrals.

$$(a)\ \int \frac{dx}{e^x+e^{-x}} \qquad (c)\ \int \frac{dx}{1+\sqrt{x}} \qquad (e)\ \int \tan^{-1}\sqrt{x}\,dx$$
$$(b)\ \int \frac{\alpha\,dx}{x^3+\beta^2 x} \qquad (d)\ \int e^{\ln\sqrt{x}}\,dx \qquad (f)\ \int \sin^{-1}\sqrt{x}\,dx$$

▶ **3-27.** Evaluate the given integrals.

$$(a)\ \int \frac{dx}{a+bx} \qquad (c)\ \int (x+1)^2 e^x\,dx \qquad (e)\ \int \sec^{-1} x\,dx$$
$$(b)\ \int \frac{x\,dx}{(x-1)^2} \qquad (d)\ \int (1-x^2)^{3/2}\,dx \qquad (f)\ \int x\sin^{-1} x\,dx$$

▶ **3-28.** Find the derivative $\dfrac{dy}{dx}$ if

$$(a)\ y(x) = \int_0^{\beta(x)} f(t)\,dt \quad (b)\ y(x) = \int_{\alpha(x)}^{0} f(t)\,dt, \quad (c)\ y(x) = \int_{\alpha(x)}^{\beta(x)} f(t)\,dt$$

▶ **3-29.** Sketch the curve defined by the set of points

$$C = \{\ (x,y)\ \mid\ x = t-\sin t,\ y = 1-\cos t, \quad 0 \le t \le 2\pi\ \}$$

If the curve C is rotated about the $x-$axis a surface of revolution is generated.

(i) Show the element of surface area can be expressed $dS = 2\pi y(t)\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt$

(ii) Find the surface area produced by the rotation.

(iii) Find the volume inside the surface.

▶ **3-30.** Solve for the value of α if

$$(a)\ \int_0^{\alpha} x^2\,dx = \frac{64}{3} \qquad (b)\ \int_{\alpha}^{\alpha+1} x\,dx = \frac{3}{2} \qquad (c)\ \int_0^{\alpha} x\,dx = \frac{1}{8}\int_{\alpha}^{6} x, dx$$

▶ **3-31.** Verify the reduction formula $\displaystyle\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

► **3-32.**

(a) Let $I_n = \int x^n\, e^{\alpha x}\, dx$ and derive the reduction formula $I_n = \frac{1}{\alpha} x^n\, e^{\alpha x} - \frac{n}{\alpha} I_{n-1}$

(b) Evaluate the integral $\int x\, e^{\alpha x}\, dx$

► **3-33.**

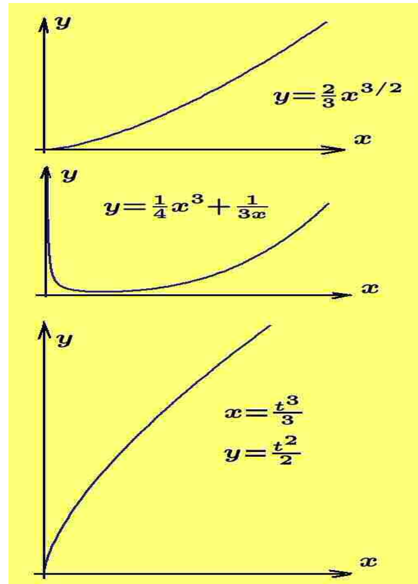
(a) Find the arc length of the curve $y = \frac{2}{3}x^{3/2}$ between the values $x=0$ and $x=3$.

(b) Find the arc length of the curve $y = \frac{1}{4}x^3 + \frac{1}{3x}$ between the values $x=1$ and $x=4$.

(c) Find the arc length of the curve defined by the parametric equations

$$x = \frac{t^3}{3} \quad \text{and} \quad y = \frac{t^2}{2}$$

between the values $t=0$ and $t=1$.



► **3-34.**

(a) Let $J_n = \int (\ln \mid \alpha x \mid)^n\, dx$ and derive the reduction formula $J_n = x\,(\ln \mid \alpha x \mid)^n - nJ_{n-1}$

(b) Evaluate the integral $\int \ln \mid \alpha x \mid\, dx$

► **3-35.**

(a) Let $I_{n,m} = \int x^m\,(\ln x)^n\, dx$ and derive the reduction formula

$$I_{m,n} = \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} I_{m,n-1}$$

(b) Evaluate the integral $\int x \ln x\, dx$

► **3-36.**

(a) Consider the area bounded by the $x-$axis, the curve $y = \sqrt{2x+1}$ and the lines $x=0$ and $x=3$. This area is revolved about the $x-$axis.

(i) Find the surface area of the solid generated.

(ii) Find the volume bounded by the surface generated.

► **3-37.** Consider the area bounded by the x-axis, the lines $x=1$ and $x=8$ and the curve $y = x^{1/3}$. This area is revolved about the $y-$axis.

(i) Find the surface area of the solid generated.

(ii) Find the volume enclosed by the surface.

▶ **3-38.** The average value of a function $y = y(x)$ over the interval $[a, b]$ is given by $\bar{y} = \dfrac{1}{b-a}\displaystyle\int_a^b y(x)\,dx$ and the weighted average of the function $y = y(x)$ is given by $\bar{y}_w = \dfrac{\displaystyle\int_a^b w(x)y(x)\,dx}{\int_a^b w(x)\,dx}$ where $w = w(x)$ is called the weight function.

(a) Find the average value of $y = y(x) = \sin x$ over the interval $[0, \pi]$.

(b) Find the weighted average of $y = y(x) = \sin x$ over the interval $[0, \pi]$ with respect to the weight function $w = w(x) = x$.

(c) Find the weighted average of $y = y(x) = \sin x$ over the interval $[0, \pi]$ with respect to the weight function $w = w(x) = \cos^2 x$

(d) Where does the weight function place the most emphasis in calculating the weighted average in parts (b) and (c)?

▶ **3-39.** Find the area under the given curves from x_0 to x_1.

$(a)\quad y = x\sqrt{x^2+1},\quad x_0 = 0,\quad x_1 = 2$

$(b)\quad y = \dfrac{x}{\sqrt{x^2+1}},\quad x_0 = 0,\quad x_1 = 2$

$(c)\quad y = \ln x,\quad x_0 = 1,\quad x_1 = e$

$(d)\quad y = \sin x,\quad x_0 = 0,\quad x_1 = \pi$

$(e)\quad y = \cos x,\quad x_0 = 0,\quad x_1 = \pi/2$

$(f)\quad y = \tan x,\quad x_0 = 0,\quad x_1 = \pi/4$

▶ **3-40.** Consider the triangular area bounded by the $x-$axis, the line $y = x,\ 0 \le x \le 1$ and the line $y = 2 - x,\ 1 \le x \le 2$. This area is revolved about the line $x = 6$ to form a solid of revolution.

(i) Find the surface area of the solid generated.

(ii) Find the volume of the solid generated.

▶ **3-41.** The line $y = r = a$ *constant*, for $0 \le x \le h$ is rotated about the $x-$axis to form a right circular cylinder of base radius r and height h.

(a) Use calculus and find the volume of the cylinder.

(b) Use calculus and find the lateral surface area of the cylinder.

▶ **3-42.** Sketch the region of integration for the double integral

$$I = \int_0^1 \int_x^{4-x} f(x, y)\,dy\,dx.$$

▶ **3-43.** The line $y = \frac{r}{h}x$ for $0 \le x \le h$ is rotated about the $x-$axis to form a right circular cone of base r and height h.

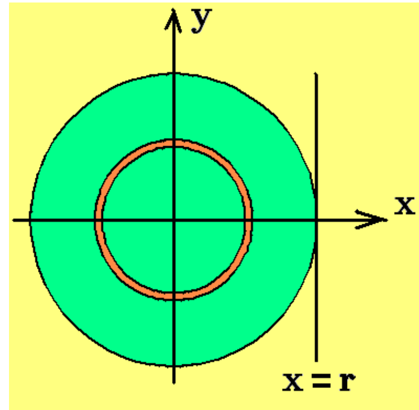
(a) Use calculus and find the volume of the cone.

(b) Use calculus and find the lateral surface area of the cone.

▶ **3-44.**

The radius r of a circle is divided into $n-$parts by defining a distance $\Delta x = r/n$ and then constructing the points

$$x_0 = 0, x_1 = \Delta r,\ x_2 = 2\Delta r, \ldots, x_i = i\,\Delta r, \ldots, x_n = n\,\Delta r = r$$



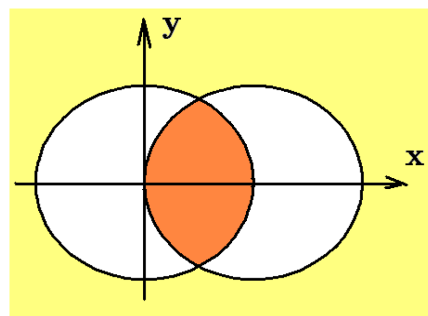
The large circle can then be thought of as being composed as a series of concentric circles. The figure on the right shows the concentric circles constructed with radii x_i and x_{i+1}.

(a) Use calculus to find the circumference of a circle with radius r.

(b) Use calculus to sum the areas between concentric circles and find the area of a circle. Hint: Show $dA = 2\pi x\,dx$

(c) Use polar coordinates and double integrals to find the area of a circle.

▶ **3-45.**



(a) Find common area of intersection associated with the circles $x^2 + y^2 = r_0^2$ and $(x - r_0)^2 + y^2 = r_0^2$

(b) Find the volume of the solid of revolution if this area is rotated about the $x-$axis.

Hint: Make use of symmetry.

▶ **3-46.** Sketch the region R over which the integration is to be performed

$$(a) \quad \int_0^1 \int_{y^2}^{y} f(x,y)\,dx\,dy \quad (b) \quad \int_0^{\pi} \int_0^{3\cos\theta} f(r,\theta)\,r\,dr\,d\theta \quad (c) \quad \int_1^4 \int_1^{x^2} f(x,y)\,dy\,dx$$

▶ **3-47.** Sketch the region of integration, change the order of integration and evaluate the integral.

$$(a) \quad \int_1^3 \int_{y-1}^{2} 12xy\,dx\,dy \quad (b) \quad \int_0^4 \int_{x/2}^{\sqrt{x}} 3xy\,dy\,dx \quad (c) \quad \int_a^b \int_c^d xy\,dx\,dy, \quad \begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix}$$

▶ **3-48.** Integrate the function $f(x,y) = \frac{3}{2}xy$ over the region R bounded by the curves

$$y = x \quad \text{and} \quad y^2 = 4x$$

(a) Sketch the region of integration.

(b) Integrate with respect to x first and y second.

(c) Integrate with respect to y first and x second.

▶ **3-49.** Evaluate the double integral and sketch the region of integration.

$$I = \int_0^1 \int_0^x 2(x+y)\,dy\,dx.$$

▶ **3-50.** For $f(x) = x$ and $b > a > 0$, find the number c such that the mean value theorem $\int_a^b f(x)\,dx = f(c)(b-a)$ is satisfied. Illustrate with a sketch the geometrical interpretation of your result.

▶ **3-51.** Make appropriate substitutions and show

$$\int \frac{dx}{\alpha^2+x^2} = \frac{1}{\alpha}\tan^{-1}\frac{x}{\alpha} + C \qquad \int \frac{dx}{b^2+(x+a)^2} = \frac{1}{b}\tan^{-1}\frac{x+a}{b} + C$$

$$\int \frac{dx}{\alpha^2-x^2} = \frac{1}{\alpha}\tanh^{-1}\frac{x}{\alpha} + C, \quad x<\alpha \qquad \int \frac{dx}{b^2-(x+a)^2} = \frac{1}{b}\tanh^{-1}\frac{x+a}{b} + C, \quad x+a<b$$

▶ **3-52.**

(a) If $f(x) = f(x+T)$ for all values of x, show that $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$

(b) If $f(x) = -f(T-x)$ for all values of x, show that $\int_a^T f(x)\,dx = -\int_0^a f(x)\,dx$

▶ **3-53.**

(a) Use integration by parts to show

$$\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx \text{ and } \int e^{ax}\cos bx\,dx = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx$$

(b) Show that

$$\int e^{ax}\sin bx\,dx = e^{ax}\frac{a\sin bx - b\cos bx}{a^2+b^2} + C$$

$$\int e^{ax}\cos bx\,dx = e^{ax}\frac{b\sin bx + a\cos bx}{a^2+b^2} + C$$

▶ **3-54.** Make an appropriate substitution to evaluate the given integrals

$$(a)\quad \int (e^{2x}+3)^m\, e^{2x}\,dx \qquad (b)\quad \int \frac{e^{4x}+e^{3x}}{e^x+e^{-x}}\,dx \qquad (c)\quad \int (e^x+1)^2\, e^x\,dx$$

▶ **3-55.** Evaluate the given integrals

$$(a)\quad \int \frac{x+a}{x^3}\,dx \qquad (b)\quad \int \frac{(x+a)(x+b)}{x^3}\,dx \qquad (c)\quad \int \frac{(x+a)(x+b)(x+c)}{x^3}\,dx$$

▶ **3-56.** Evaluate the given integrals

$$(a)\quad \int \ln(1+x)\,dx \qquad (b)\quad \int \frac{x^4+1}{x-1}\,dx \qquad (c)\quad \int (a+bx)(x+c)^m\,dx$$

▶ **3-57.** Use a limiting process to evaluate the given integrals

$$(a)\quad \int_0^\infty e^{-st}\,dt,\quad s>0 \qquad (b)\quad \int_{-\infty}^x e^t\,dt \qquad (c)\quad \int_0^\infty te^{-st}\,dt,\quad s>0$$

▶ **3-58.** Consider a function $J_n(x)$ defined by an infinite series of terms and having the representation

$$J_n(x)=\frac{x^n}{2^n}\left[\frac{1}{n!}-\frac{x^2}{2^2 1!(n+1)!}+\frac{x^4}{2^4 2!(n+2)!}+\cdots+\frac{(-1)^m x^{2m}}{2^{2m}m!(n+m)!}+\cdots\right]$$

where n is a fixed integer and m represents the mth term of the series. Here m takes on the values $m=0,1,2,\ldots$. Show that

$$\int_0^x J_1(x)\,dx = 1-J_0(x)$$

The function $J_n(x)$ is called the Bessel function of the first kind of order n.

▶ **3-59.** Determine a general integration formula for $I_n=\int x^n\,e^x\,dx$ and then evaluate the integral $I_4=\int x^4\,e^x\,dx$

▶ **3-60.** Show that if $g(x)$ is a continuous function, then $\int_0^a g(x)\,dx=\int_0^a g(a-x)\,dx$

▶ **3-61.** Let $h(x)=-h(2T-x)$ for all values of x. Show that $\int_b^{2T} h(x)\,dx=-\int_0^b h(x)\,dx$

▶ **3-62.** If for m,n positive integers one has $I_{m,n}=\int \cos^m x\sin nx\,dx$, then derive the reduction formula

$$(m+n)I_{m,n}=-\cos^m x\cos nx+m\,I_{m-1,n-1}$$

▶ **3-63.** If $f(-x)=f(x)$ for all values of x, then $f(x)$ is called an even function. Show that if $f(x)$ is an even function, then $\int_{-a}^a f(x)\,dx=2\int_0^a f(x)\,dx$

▶ **3-64.** If $g(-x)=-g(x)$ for all values of x, then $g(x)$ is called an odd function. Show that if $g(x)$ is an odd function, then $\int_{-a}^a g(x)\,dx=0$

▶ **3-65.** If $f(2T-x)=f(x)$ for all values of x, then show $\int_0^{2T} f(x)\,dx=2\int_0^T f(x)\,dx$

▶ **3-66.** Let $A = A(y)$ denote the cross-sectional area of a pond at height y measured from the bottom of the pond. If the maximum depth of the pond is h, then set up an integral to represent volume of water in the pond.

▶ **3-67.** Determine if the given improper integral exists. If the integral exists, then evaluate the integral. Assume $\beta > 0$ in parts (e) and (f).

$$(a)\quad \int_{-1}^{1} \frac{dx}{x^2} \qquad (c)\quad \int_0^1 \frac{dx}{\sqrt{x}} \qquad (e)\quad \int_0^{\beta} \frac{dx}{(\beta - x)^p} \quad \begin{cases} if\ p < 1 \\ if\ p \geq 1 \end{cases}$$

$$(b)\quad \int_0^1 \frac{dx}{\sqrt{1-x^2}} \qquad (d)\quad \int_{-\infty}^{\infty} \frac{dx}{x^2+1} \qquad (f)\quad \int_0^{\infty} \frac{dx}{(\beta + x)^p} \quad \begin{cases} if\ p > 1 \\ if\ p \leq 1 \end{cases}$$

▶ **3-68.** A particle moves around the circle $x^2 + y^2 = r_0^2$ with constant angular velocity of ω cm/s. Find the amplitude and period of the simple harmonic motion described by (a) the projection of the particles position on the x-axis. (b) the projection of the particles position on the y-axis.

▶ **3-69.** Find the angle of intersection associated with the curves $r = \sin\theta$ and $r = \cos\theta$ which occurs in the region $r > 0$ and $0 < \theta < \frac{\pi}{2}$

▶ **3-70.** Make use of symmetry when appropriate and sketch a graph of the following curves.

$$(a)\quad y = \frac{a^2}{a^2 - x^2} \qquad (b)\quad y = \frac{x}{x^2 + a^2} \qquad (c)\quad y = \frac{x^2}{x^2 - a^2}$$

▶ **3-71.** Let $A_1 = \int_0^b \sin\left(\frac{\pi x}{2b}\right)\, dx$ and $A_2 = \int_0^b \sin\left(\frac{\pi x}{b}\right)\, dx$

(a) Sketch the representation of the area A_1 and evaluate the integral to find the area A_1.

(b) Sketch the representation of the area A_2 and evaluate the integral to find the area A_2.

(c) Which area is larger?

▶ **3-72.** A plane cuts a sphere of radius r forming a spherical cap of height h. Show the volume of the spherical cap is $V = \dfrac{\pi}{3} h^2 (3r - h)$

▶ **3-73.** A solid sphere $x^2 + y^2 + z^2 = r^2$ is placed in a drill press and a cylindrical hole is drilled through the center of the sphere. Find the volume of the resulting solid if the diameter of the drill is αr, where $0 < \alpha < 1/4$

▶ **3-74.** Show that $\displaystyle\int \frac{dx}{\sqrt{x(a-x)}} = 2\sin^{-1}\sqrt{\frac{x}{a}} = 2\cos^{-1}\sqrt{\frac{a-x}{a}} = 2\tan^{-1}\sqrt{\frac{x}{a-x}}$

Chapter 4

Sequences, Summations and Products

There are many different types of functions that arise in the application of mathematics to real world problems. In chapter two many of the basic functions used in mathematics were investigated and derivatives of these functions were calculated. In chapter three integration was investigated and it was demonstrated that definite integrals can be used to define and represent functions. Many of the functions previously introduced can be represented in a variety of ways. An infinite series is just one of the ways that can be used to represent functions. Some of the functions previously introduced are easy to represent as a series while others functions are very difficult to represent. In this chapter we investigate selected methods for representing functions. We begin by examining summation methods and multiplication methods to represent functions because these methods are easy to understand. In order to investigate summation and product methods to represent functions, one must know about sequences.

Sequences

A **sequence** is defined as a **one-to-one correspondence** between the set of positive integers $n = 1, 2, 3, \ldots$ and a set of real or complex quantities $u_1, u_2, u_3, \ldots$, which are given or defined in some specific way. Such a sequence of terms is often expressed as $\{u_n\}$, $n = 1, 2, 3, \ldots$ or alternatively by $\{u_n\}_{n=1}^{\infty}$ or for short just by $\{u_n\}$. The set of real or complex quantities $\{u_n\}$, for $n = 1, 2, 3, \ldots$ is called an infinite sequence. The indexing or numbering for the terms in the sequence can be selected to begin with any convenient number. For example, there may be times when it is convenient to examine sequences such as

$$\{u_n\},\ n = 0, 1, 2, 3, \ldots \qquad \text{or} \qquad \{u_n\},\ n = \nu, \nu + 1, \nu + 2, \ldots$$

where ν is some convenient starting index.

A sequence of real numbers can be represented as a **function or mapping from the set of integers to the set of real numbers** $f : N \to R$ and is sometimes represented as $f(1), f(2), \ldots$ or $f_1, f_2, \ldots$. Similarly, a sequence of complex numbers can be represented as a function or mapping from the set of integers to the set of complex numbers $f : N \to C$.

Example 4-1. The following are some examples of sequences.

$$(a)\quad \left\{\frac{1}{n}\right\}_{n=1}^{\infty}, \qquad u_n=\frac{1}{n}, \qquad \left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots,\frac{1}{n},\ldots\right\}$$

$$(b)\quad \left\{\frac{1}{n!}\right\}_{n=0}^{\infty}, \qquad u_n=\frac{1}{n!}, \qquad \left\{1,1,\frac{1}{2},\frac{1}{6},\ldots,\frac{1}{n!},\ldots\right\}$$

$$(c)\quad \left\{\frac{4n}{n-2}\right\}_{n=3}^{\infty}, \qquad u_n=\frac{4n}{n-2}, \qquad \left\{12,8,\frac{20}{3},6,\ldots,\frac{4n}{n-2},\ldots\right\}$$

$$(d)\quad \left\{(-1)^n\sin\frac{n\pi}{6}\right\}_{n=1}^{\infty}, \qquad u_n=(-1)^n\sin\frac{n\pi}{6}, \qquad \left\{-\frac{1}{2},\frac{\sqrt{3}}{3},-1,\frac{\sqrt{3}}{2},\ldots,(-1)^n\sin\frac{n\pi}{6},\ldots\right\}$$

■

Limit of a Sequence

The limit of a sequence, if it exists, is the problem of determining the value of the sequence $\{u_n\}$ as the index n increases without bound. A sequence such as $u_n=3+\frac{4}{n}$ for $n=1,2,3,\ldots$ has the values

$$u_1=7,\ u_2=5,\ u_3=13/3,\ u_4=4,\ u_5=19/5,\ u_6=11/3,\ldots$$

and as n increases without bound one can write

$$\lim_{n\to\infty} u_n=\lim_{n\to\infty}\left(3+\frac{4}{n}\right)=3$$

This limit statement means that **for n sufficiently large the values u_n are as close as desired to the value 3**.

Some sequences $\{u_n\}$ do not have a limit as the index n increases without bound. For example, the sequence $u_n=(-1)^n$ for $n=1,2,3,\ldots$ **oscillates between the values $+1$ and -1** and does not have a limit. Another example of a sequence which does not have a limit is the sequence $\{v_n\}$ where $v_n=3^n$ for $n=1,2,3,\ldots$. Here the values v_n **increase without bound as n increases**.

Convergence of a sequence

A sequence $\{u_n\}$, $n=1,2,3,\ldots$, where u_n can be a real or complex number, is said to converge to a number ℓ or have a limit ℓ, if to each small positive number $\epsilon>0$ there exists an integer N such that

$$|u_n-\ell|<\epsilon \quad \text{for every integer } n > N \tag{4.1}$$

If the sequence converges, then the limit is written

$$\lim_{n\to\infty} u_n=\ell \tag{4.2}$$

If the sequence does not converge it is said to diverge.

Divergence of a sequence

A sequence $\{u_n\}$ is said to diverge in the positive direction if for every number $M > 0$, there exists a number $N > 0$, such that $u_n > M$ for all integers $n > N$. This is sometimes expressed $\lim_{n\to\infty} u_n = \infty$. Similarly, a sequence $\{v_n\}$ is said to diverge in the negative direction if there exists numbers $M > 0$ and $N > 0$ such that $v_n < -M$ for all integers $n > N$. This is sometimes expressed $\lim_{n\to\infty} v_n = -\infty$.

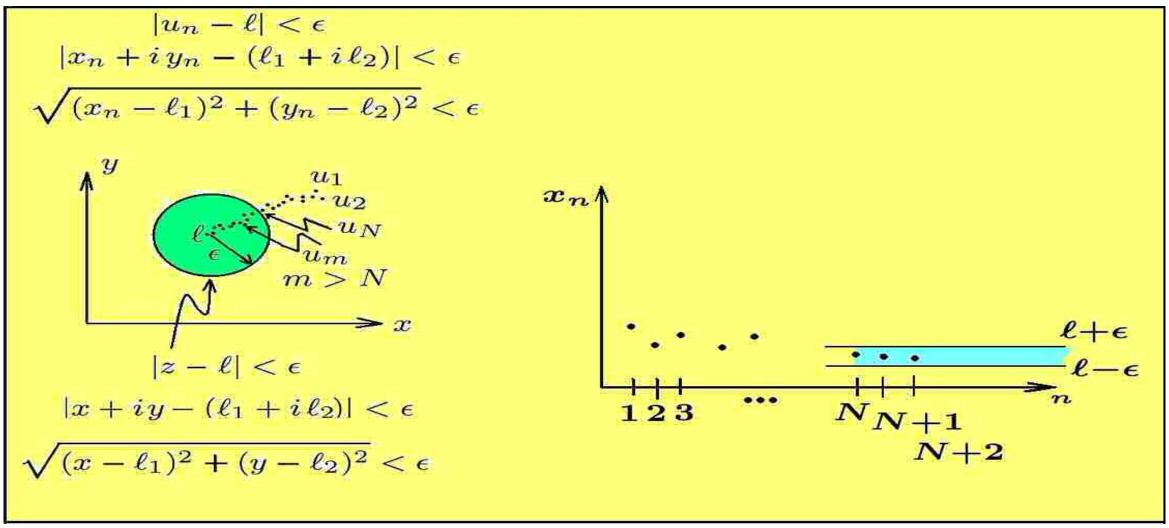


Figure 4-1. Interpretation of convergence of a sequence $\{u_n\}$.

When the sequence converges, the elements u_n of the sequence tend to **concentrate themselves around the point ℓ for large values of the index** n. The terms u_n do not have to approach ℓ at any specified rate nor do they have to approach ℓ from a particular direction. However, there may be times where u_n approaches ℓ only from the left and there may be other times when u_n approaches ℓ only from the right. These are just special cases associated with the more general definition of a limit given above.

There are two geometric interpretations associated with the above limit statement. The first whenever $u_n = x_n + i\,y_n$ is a set of complex numbers and the second geometric interpretation arises whenever $u_n = x_n$ represents a sequence of real numbers. These geometric interpretations are illustrated in the figure 4-1. In the case the sequence $\{u_n\}$ is a sequence of complex numbers, the quantity $|z-\ell|<\epsilon$ represents **an open disk centered at the point** $\ell=\ell_1+i\,\ell_2$ and the statement $\lim_{n\to\infty} u_n=\ell$ can be interpreted to mean that there exists an integer N, such that for all integers $m>N$, the terms u_m are **trapped inside the circular disk of radius** ϵ **centered at** ℓ**.** In the case where the terms $u_n=x_n$, $n=1,2,3,\dots$ are real quantities, the interpretation of convergence is that for all integers $m>N$, the terms u_m are **trapped inside the interval** $(\ell-\epsilon,\ell+\epsilon)$. These regions of entrapment can be made arbitrarily small by making the quantity $\epsilon>0$ small. In either case, there results an infinite number of terms inside the disk or interval illustrated in the figure 4-1. A sequence which is not convergent is called divergent or non-convergent.

Relation between Sequences and Functions

There is a definite relation associated with limits of sequences and limits of functions. For example, if $\{u_n\}$ is a sequence and $f=f(x)$ is a continuous function defined for all $x\ge 1$ with the property that $u_n=f(n)$ for all integers $n\ge 1$, then if ℓ exists, the following limit statements are equivalent

$$\lim_{n\to\infty} u_n=\ell, \qquad \text{and} \qquad \lim_{x\to\infty} f(x)=\ell$$

One can make use of this property to find the limits of certain sequences.

Example 4-2. Evaluate the limit $\lim_{n\to\infty} u_n$ where $u_n=\dfrac{\ln n}{\sqrt{n}}$

Solution

Let $f(x)=\dfrac{\ln x}{\sqrt{x}}$ and use L´Hôpital's rule to show

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{1/2\sqrt{x}}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

■

The limit properties for functions also apply to sequences. For example, if the sequences $\{u_n\}$ and $\{v_n\}$ are convergent sequences and $\lim_{n\to\infty} v_n\ne 0$, then one can write

$$\begin{aligned}\lim_{n\to\infty}(u_n\pm v_n)&=\lim_{n\to\infty} u_n\pm\lim_{n\to\infty} v_n\\ \lim_{n\to\infty}(u_n\,v_n)&=\left(\lim_{n\to\infty} u_n\right)\,\left(\lim_{n\to\infty} v_n\right)\\ \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)&=\frac{\lim_{n\to\infty} u_n}{\lim_{n\to\infty} v_n}\end{aligned}$$

If k is a constant and $\lim_{n\to\infty} u_n=U$ exists, then the limit $\lim_{n\to\infty} k\,u_n=k\,U$ also exists.

Establish Bounds for Sequences

A real sequence $\{u_n\}$, $n = 1, 2, 3, \ldots$ is said to be **bounded** if there exists numbers m and M such that $m \le u_n \le M$ for all integers n. The number M is called **an upper bound** and the number m is called a **lower bound** for the sequence.

The previous squeeze theorem[1] from chapter 1 can be employed if there exists three sequences $\{f_j\}, \{g_j\}, \{h_j\}$, $j = 1, 2, 3, \ldots$ where the sequences $\{f_j\}$ and $\{h_j\}$ have the same limit so that

$$\lim_{j\to\infty} f_j = \ell \quad \text{and} \quad \lim_{j\to\infty} h_j = \ell$$

If one can verify the inequalities $f_j \le g_j \le h_j$, for all values of the index j, then the terms g_j are sandwiched in between the values for f_j and h_j for all values j and consequently one must have $\lim_{j\to\infty} g_j = \ell$.

Example 4-3. Evaluate the limit $\lim_{n\to\infty} u_n$ where $u_n = \dfrac{n!}{n^n}$, where $n!$ is n-factorial.

Solution

An examination of u_n for $n = 1, 2, 3, \ldots, m, \ldots$ shows that

$$u_1 = 1, \quad u_2 = \frac{2\cdot 1}{2^2}, \quad u_3 = \frac{3\cdot 2\cdot 1}{3^3}, \quad \cdots \quad u_m = \frac{m\cdot(m-1)\cdots 3\cdot 2\cdot 1}{m^m}, \quad \cdots$$

Observe that the u_m term is positive and can be written as

$$u_m = \frac{m\cdot(m-1)\cdots 3\cdot 2\cdot 1}{\underbrace{m\cdot m\cdot m\cdots m}_{\text{m times}}} = \left(\frac{m}{m}\frac{(m-1)}{m}\cdots\frac{3}{m}\frac{2}{m}\right)\frac{1}{m}$$

where the term within the parentheses is less than 1. Consequently, one can say u_m is bounded with

$$0 < u_m \le \frac{1}{m}$$

Here $\frac{1}{m} \to 0$ as m increases without bound and so by the sandwich principle one can say that $u_m \to 0$ as m increases without bound.

■

Every **convergent sequence** $\{u_n\}$ is bounded. If $\lim_{n\to\infty} u_n = \ell$, then there exists neighborhoods N_ϵ given by

(i) A **circular neighborhood about** ℓ given by $|z - \ell| < \epsilon$. This occurs when the values u_n are complex numbers and there are an infinite number of values u_n from the sequence inside the circle and a finite number of points u_n outside the circle.

[1] Also known as the pinching lemma or the sandwich lemma

(ii) An **interval neighborhood** given by $\ell - \epsilon < u_n < \ell + \epsilon$. Here the values u_n are real numbers where there is an infinite number of points u_n inside the interval and only a finite number of points outside the interval.

These neighborhoods N_ϵ represent **bounded sets**. By increasing the radius of the bounded sets N_ϵ to encompass the finite number of points outside N_ϵ, a new set can be constructed which will still be bounded and contain all the terms of the sequence.

If the real sequence $\{x_j\}$, $j = 1, 2, 3, \ldots$ is a convergent sequence, then

(i) The sequence must be bounded and

(ii) The limit of the sequence is unique.

Make note that it is sometimes convenient to replace the small quantity ϵ used in the definition of convergence by some other small quantity such as ϵ^2, $\frac{\epsilon}{2}$, or $\frac{\epsilon}{M}$, ($M > 0$ constant). The small quantity used is usually selected so that when many applications of the inequality (4.1) are made, the final results add up to some convenient number.

To show the sequence must be bounded, use the definition of convergence of a sequence with a fixed value for ϵ. If the limit of the sequence is ℓ, then there exists an integer N such that for integers n satisfying $n > N$

$$|x_n| = |x_n - \ell + \ell| \le |x_n - \ell| + |\ell| < \epsilon + |\ell| = M_1 \quad \text{for } n > N$$

That is, if the sequence has a limit, then there exists a value N such that each term of the sequence $x_{N+1}, x_{N+2}, \ldots$ are less then M_1 in absolute value. Let us now examine the terms $x_1, x_2, \ldots, x_N$. Let $M_2 = \max\{|x_1|, |x_2|, \ldots, |x_N|\}$. By selecting $M = \max\{M_1, M_2\}$, it follows that $|x_j| < M$ for all values of j and so the sequence is bounded. If the sequence is not bounded, then it diverges.

To show the limit of the sequence is **unique**, assume that there are two limits, say ℓ and ℓ'. Now use the method of **reductio ad absurdum** to show this assumption is false. The assumption of convergence of the sequence insures that for every $\epsilon > 0$, there exists a value N_1 such that for all integers $n > N_1$ there results $|x_n - \ell| < \frac{\epsilon}{2}$. If ℓ' is also a limit, then there must exist an integer N_2 such that for all $n > N_2$ there results $|x_n - \ell'| < \frac{\epsilon}{2}$. Here the selection $\epsilon/2$ has been used as the small quantity in the definition of convergence rather than ϵ. The reason for this change is to make the final answer come out in terms of the quantity ϵ. Define, $N = \max\{N_1, N_2\}$, then for all integers $n > N$ it follows that

$$|\ell - \ell'| = |\ell - x_n + x_n - \ell'| \le |\ell - x_n| + |x_n - \ell'| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and since ϵ can be made arbitrarily small, then ℓ must equal ℓ'. Hence our original assumption was false.

Additional Terminology Associated with Sequences

1. A real sequence $u_1, u_2, u_3, \ldots$ is said to be **monotone increasing** if $u_1 \le u_2 \le u_3 \le \ldots$. Alternatively, define a monotone increasing sequence $\{u_n\}$ as one where $u_{n+1} \ge u_n$ for all values of n greater than some fixed integer N.

2. A real sequence $u_1, u_2, u_3, \ldots$ is said to be **monotone decreasing** if $u_1 \ge u_2 \ge u_3 \ge \ldots$. An alternative definition of a monotone decreasing sequence $\{u_n\}$ is that $u_{n+1} \le u_n$ for all values of n greater than some fixed integer N.

3. Consider an infinite sequence $u_1, \ldots, u_{j_1}, \ldots, u_{j_2}, \ldots, u_{j_3}, \ldots, u_{n-1}, u_n, u_{n+1}, \ldots$ where $j_1 < j_2 < j_3 < \cdots < j_n < \cdots$ represent a selected subset from the real numbers N. The sequence of numbers $u_{j_1}, u_{j_2}, u_{j_3}, \ldots$ or $\{u_{j_k}\}_{k=1}^{\infty}$ is called a **subsequence** of the given sequence.

4. A sequence is called **oscillating**, either **finite oscillatory** or **infinitely oscillatory**, depending upon whether the terms are **bounded or unbounded**. For example, the sequence $\{\cos \frac{n\pi}{3}\}$, for $n = 1, 2, 3, \ldots$, is said to oscillate finitely, because the terms remain bounded. In contrast, consider the sequence of terms $\{(-1)^n n^2\}$ for the values $n = 1, 2, 3, \ldots$ This sequence is said to oscillate infinitely, because the terms become unbounded. In either case the sequence is called a nonconvergent sequence. A finite oscillatory sequence $\{x_n\}$, $n = 1, 2, 3, \ldots$ is a sequence of real numbers which bounce around between finite limits and does not converge. An example of a finite oscillatory sequence are the numbers $\{-1, 0, 1, -1, 0, 1, -1, 0, 1, \ldots\}$ with the pattern repeating forever. Oscillatory sequences occur in certain applied mathematics problems quite frequently.

5. A number L is called a **limit point of the sequence** $\{u_n\}$, $n = 1, 2, 3, \ldots$, if for every $\epsilon > 0$, $\mid u_n - L \mid < \epsilon$ for infinitely may values of n. A sequence may have more than one limit point. For example, the sequence $1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots$ with the pattern $1, 2, 3$ repeating forever, has the limit points $1, 2, 3$. Note the following special cases. (i) A finite set cannot have a limit point. (ii) An infinite set may or may not have a limit point.

6. A real sequence $\{u_n\}$, $n = 1, 2, 3, \ldots$, is called a **null sequence** if for every small quantity $\epsilon > 0$ there exists an integer N such that $|u_n| < \epsilon$ for all values of $n > N$.

7. **Every bounded, monotonic sequence converges.** That is, if the sequence is increasing and bounded above, it must converge. Similarly, if the sequence is decreasing and bounded below, it must converge. Another way of examining these situations is as follows.

If $\{u_n\}$ is an increasing sequence and the u_n are bounded above, then the set $S_L = \{\ u_n \ \mid\ n \geq 1\ \}$ has a least upper bound L. Similarly, if $\{v_n\}$ is a decreasing sequence and the v_n are bounded below, the set of values $S_G = \{\ v_n \ \mid\ n \geq 1\ \}$ has greatest lower bound G. One can then show

$$\lim_{n\to\infty} u_n = L \qquad \text{and} \qquad \lim_{n\to\infty} v_n = G$$

since the sequences are monotonic.

8. Cauchy's convergence criteria

The existence of a limit for the sequence $\lim\limits_{n\to\infty} u_n$, can be established by using the **Cauchy condition for convergence**. A sequence $\{u_j\}$, $j = 1, 2, 3, \ldots$, is said to be **Cauchy convergent** or to satisfy the **Cauchy convergence criteria** if for every small quantity $\epsilon > 0$, there exists an integer $N > 0$, such that if one selects any two terms from the sequence, say u_m and u_n, where $n > N$ and $m > n > N$, then $|u_m - u_n| < \epsilon$.

The Cauchy criteria is another way of saying that for large values of m and n the terms of the sequence will always stay close together. Also observe that in using the Cauchy criteria it is not necessary to know the exact limit in order to demonstrate convergence.

There are alternative ways for expressing the Cauchy convergence condition. Some of these representations are the following.

(i) A sequence of points $u_1 = (x_1, y_1)$, $u_2 = (x_2, y_2), \ldots, u_m = (x_m, y_m), \ldots, u_n = (x_n, y_n)$ tends to a limit point if and only if, for every $\epsilon > 0$ there exists an integer $N = N(\epsilon)$ such that $\sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} < \epsilon$ whenever $m > N$ and $n > N$. That is, the distance between the points u_m and u_n can be made as small as desired if m and n are selected large enough.

(ii) A necessary condition for the existence of the limit $\lim\limits_{n\to\infty} u_n$, is for every $\epsilon > 0$, there exists an integer N such that for all integers $n > N$ and for every integer $k > 0$, the condition $|u_{n+k} - u_n| < \epsilon$ is satisfied.

The Cauchy condition follows from the following argument. If the limit of the sequence $\{u_n\}$ is ℓ, then for every $\epsilon > 0$ one can find an integer N such that for integers n and m both greater than N one will have

$$|u_n - \ell| < \frac{\epsilon}{2} \quad \text{and} \quad |u_m - \ell| < \frac{\epsilon}{2}$$

and consequently one can add and subtract ℓ to obtain

$$|u_n - u_m| = |(u_n - \ell) + (\ell - u_m)| \le |u_n - \ell| + |u_m - \ell| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

In more advanced mathematics courses one can show that the Cauchy convergence condition is both **a necessary and sufficient condition** for the existence of a limit for the sequence $\{u_n\}$.

Example 4-4. If $\{u_n\}$ is a monotone increasing sequence with

$$u_1 \le u_2 \le u_3 \le \cdots \le u_n \le \cdots$$

and for each value of the index n one can show $u_n \le K$ where K is some constant, then one can state that the sequence $\{u_n\}$ is a convergent sequence and is such that $\lim_{n\to\infty} u_n \le K$.

To prove the above statement let $S_L = \{\ x \ \mid \ x = u_n\ \}$ and select for the set S_L a least upper bound and call it L so that one can state $u_n \le L$ for all values $n = 1, 2, 3, \ldots$. Note that if L is the least upper bound, then every number $K > L$ is also an upper bound to the set S_L. If $\epsilon > 0$ is a small positive number, then one can state that $L - \epsilon$ is not an upper bound of S_L, but $L + \epsilon$ is an upper bound of the set S_L. Let N denote an integer such that $L - \epsilon < u_N$. Such an integer N exists since the infinite set $\{u_n\}$ is monotone increasing. Once N is found, one can write that for all integers $n > N$ one must have $L - \epsilon < u_N < u_n < L + \epsilon$, which can also be written as the statement

$$\text{for all integers } n > N \text{ the inequality} \quad |u_n - L| < \epsilon$$

is satisfied. But this is the meaning of the limit statement $\lim_{n\to\infty} u_n = L$. Consequently, one can state that the sequence is convergent and if each $u_n \le K$, then $\lim_{n\to\infty} u_n \le K$.

■

Stolz -Cesàro Theorem

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ denote sequences of real numbers such that the terms b_n are strictly increasing and unbounded. The Stolz[2]-Cesàro[3] theorem states that if the limit

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell$$

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[2] Otto Stolz (1842-1905) an Austrian mathematician.
[3] Ernesto Cesàro (1859-1906) an Italian mathematician.

exits, then the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$$

will also exists with the limit ℓ. This result is sometimes referred to as the **L'Hôpital's rule for sequences**.

A proof of the Stolz -Cesàro theorem is along the following lines. If the limit $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \ell$ exists, then for every $\epsilon > 0$ there must exists an integer N such that for all $n > N$ there results the inequality

$$\left| \frac{a_{n+1}-a_n}{b_{n+1}-b_n} - \ell \right| < \epsilon \quad \text{or} \quad \ell - \epsilon < \frac{a_{n+1}-a_n}{b_{n+1}-b_n} < \ell + \epsilon$$

By hypothesis, $\{b_n\}$ is strictly increasing so that $b_{n+1} - b_n \neq 0$ and consequently one can write

$$(\ell - \epsilon)(b_{n+1} - b_n) < a_{n+1} - a_n < (\ell + \epsilon)(b_{n+1} - b_n) \tag{4.3}$$

Let K denote a large number satisfying $K > N$ and then sum each term in equation (4.3) from N to K and show

$$(\ell - \epsilon) \sum_{n=N}^{K} (b_{n+1} - b_n) < \sum_{n=N}^{K} (a_{n+1} - a_n) < (\ell + \epsilon) \sum_{n=N}^{K} (b_{n+1} - b_n)$$

which simplifies to

$$(\ell - \epsilon)(b_{K+1} - b_N) < a_{K+1} - a_N < (\ell + \epsilon)(b_{K+1} - b_N)$$

By dividing each term by b_{K+1} one obtains

$$(\ell - \epsilon)\left(1 - \frac{b_N}{b_{K+1}}\right) < \frac{a_{K+1}}{b_{K+1}} - \frac{a_N}{b_{K+1}} < (\ell + \epsilon)\left(1 - \frac{b_N}{b_{K+1}}\right)$$

For N fixed and large enough values of K, the above inequality reduces to

$$(\ell - \epsilon) < \frac{a_{K+1}}{b_{K+1}} < (\ell + \epsilon) \tag{4.4}$$

because the other sequences are null sequences. The final equation (4.4) implies that $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$.

Examples of Sequences

The following table gives some examples of sequences.

Table 4.1 Example of Sequences		
	Example Sequence	Comments
1	$1, 0, 1, 0, 1, 0, 1, 0, \ldots$	Divergent sequence with limit points 0 and 1 A bounded oscillating sequence
2	$\frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{4}{5}, \ldots, \frac{1}{n}, \frac{n-1}{n}, \ldots$	Divergent sequence with limit points 0 and 1.
3	$1, 2, 3, 4, \ldots$	Divergent sequence.
4	$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$	Convergent sequence with limit 1.
5	$0, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9} \ldots$	Convergent subsequence of previous sequence.
6	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$	A null sequence.
7	$\{r^n\}_{n=0}^{\infty}$	If $\mid r \mid < 1$, sequence converges. If $r = 1$, constant sequence with limit 1. If $r > 1$ or $r \le -1$, sequence diverges.

Infinite Series

Consider the infinite series

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_m + \cdots \tag{4.5}$$

where the terms $u_1, u_2, u_3, \ldots$ are called the first, second, third,... terms of the series. The set of terms $\{u_n\}$ usually represents a set of real numbers, complex numbers or functions. In the discussions that follow it is assumed that the terms of the series are represented by one of the following cases.

(i) u_m are real numbers $\quad u_m = \alpha_m$

(ii) u_m are complex numbers $\quad u_m = \alpha_m + i\,\beta_m$

(iii) u_m are functions of a real variable $\quad u_m = u_m(x)$

(iv) u_m are functions of a complex variable $\quad u_m = u_m(z)$ for $z = x + i\,y$

for $m = 1, 2, 3, \ldots$. Examine the cases (i) and (ii) above, where the terms of the infinite series are constants.

The infinite series given by equation (4.5) is sometimes represented in the forms $\sum u_n$ or $\sum\limits_{n\in N} u_n$, where N denotes the set of integers $\{1,2,3,\ldots\}$. This is done as a shorthand representation of the series and is a way of referring to the formal series after it has been properly defined and no confusion arises as to its meaning. In equation (4.5) the index n is called a dummy summation index. This summation index can be changed to any other symbol and it is sometimes shifted by making a change of variable. For example, by making the substitution $n = k - m$, where m is some constant, the summation index is shifted so that when $n = 1$, k takes on the value $m + 1$ and the series given by equation (4.5) can be represented in the alternative form $\sum\limits_{k=m+1}^{\infty} u_{k-m}$. Because k is a dummy index it is possible to replace k by the original index n to obtain the equivalent representation

$$\sum_{n=m+1}^{\infty} u_{n-m} = u_1 + u_2 + u_3 + \cdots, \quad m \text{ is a constant integer.} \tag{4.6}$$

The indexing for an infinite series can begin with any convenient indexing. For example, it is sometimes more advantages to consider series of the form

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots, \qquad \text{or} \qquad \sum_{n=\nu}^{\infty} u_n = u_\nu + u_{\nu+1} + u_{\nu+2} + \cdots \tag{4.7}$$

where ν is some convenient starting index.

Sequence of Partial Sums

Given an infinite series $\sum\limits_{j=1}^{\infty} u_j = u_1 + u_2 + u_3 + \cdots$, form the sequence of terms $U_1, U_2, U_3, \ldots$ defined by

$$U_1 = u_1, \qquad U_2 = u_1 + u_2, \quad \cdots \quad U_n = u_1 + u_2 + u_3 + \cdots + u_n$$

where the finite sum $U_m = \sum\limits_{j=1}^{m} u_j = u_1 + u_2 + \cdots + u_m$ represents the summation of the first m terms from the infinite series. The sequence of terms $\{U_m\}$, $m = 1,2,3,\ldots$ is called the **sequence of partial sums** associated with the infinite series given by equation (4.5). The notation of capital letters with subscripts or Greek letters with subscripts is used to denote partial sums. For example, if the given infinite series is $\sum_{j=1}^{\infty} a_j$, then the sequence of partial sums is denoted by the sequence $\{A_n\}$ where $A_n = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j$ for $n = 1,2,\ldots$ or alternatively use the notation $\alpha_n = \sum_{j=1}^{n} a_j$.

Convergence and Divergence of a Series

The infinite series $\sum_{j=1}^{\infty} u_j$ is said to converge to a limit U, or is said to have a sum U, whenever the sequence of partial sums $\{U_n\}$, has a finite limit, in which case one can write $\lim_{n\to\infty} U_n = U$. If the sequence of partial sums $\{U_n\}$ becomes unbounded, is oscillatory or the limit $\lim_{n\to\infty} U_n$ does not exist, then the infinite series is said to diverge.

Example 4-5. Divergent finite oscillatory and infinite oscillatory

Consider the series $\sum_{n=0}^{\infty}(-1)^n = 1-1+1-1+1-1+\cdots$ which has the partial sums $U_1 = 1, U_2 = 0, U_3 = 1, U_4 = 0, \cdots$. These partial sums oscillate between the finite values of 0 and 1 and so the sequence of partial sums is called a finite oscillatory sequence. In this case the series is said to diverge.

In contrast consider the series $\sum_{n=0}^{\infty}(-1)^n 2^n = 1-2+4-8+16-32+\cdots$ which has the partial sums $U_1 = 1, U_2 = -1, U_3 = 3, U_4 = -5, U_5 = 11, \cdots$. Here the sequence of partial sums become infinite oscillatory and so the series is said to diverge.

■

Example 4-6. Harmonic series

Consider the harmonic series $H = \sum_{m=1}^{\infty} \frac{1}{m}$ which has the nth partial sum[4]

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

Nicole Oresme (1323-1382), a French mathematician and scholar who studied infinite series, examined this series. His analysis considers the above finite sum, using the value $n = 2^m$, where he demonstrated that the terms within the partial sum H_n can be grouped together and expressed in the form

$$H_n = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \cdots + \left(\frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \cdots + \frac{1}{2^m}\right)$$

[4] The nth partial sum of the harmonic series occurs in numerous areas of mathematics, statistics and probability theory and no simple formula has been found to represent the sum H_n. The sums H_n are also known as harmonic numbers, with $H_0 = 0$. A complicated formula for H_n is given by $H_n = \gamma + \frac{d}{dz}\text{Log}\,[\Gamma(z)]$ where γ is the Euler-Mascheroni constant and $\Gamma(z)$ is the Gamma function.

Observe that using a term by term comparison of the above finite sums one can state $H_n > h_n$, where $n = 2^m$. The finite sum h_n, becomes unbounded and so the harmonic series diverges[5]. Express the nth partial sum of harmonic series as $H_n = \sum_{m=1}^{n} \frac{1}{m}$ and express the harmonic series using the partial sums $H_2, H_4, H_8, H_{16}, \ldots$ by writing

$$
\begin{aligned}
H =& 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \left(\frac{1}{17} + \cdots + \frac{1}{32}\right) + \cdots \\
H =& \, H_2 + (H_4 - H_2) + (H_8 - H_4) + (H_{16} - H_8) + \cdots + (H_{2n} - H_n) + \cdots
\end{aligned}
$$

where

$$
H_{2n} - H_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}_{\text{n terms}} = \frac{1}{2}
$$

Hence, the 2^nth partial sum can be written

$$
\begin{aligned}
H_{2^n} =& H_2 + (H_4 - H_2) + (H_8 - H_4) + \cdots + (H_{2^n} - H_{2^{n-1}}) \\
H_{2^n} >& \frac{3}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \quad n \text{ terms} \\
H_{2^n} >& \frac{3}{2} + \frac{1}{2}(n-1) = 1 + \frac{1}{2}n
\end{aligned}
$$

and $\frac{1}{2}n$ increases without bound with increasing n and by comparison H_{2^n} also increases without bound as n increases.

Note that the **harmonic mean of two numbers** n_1 **and** n_2 is defined as $\bar{n} = \dfrac{2}{\frac{1}{n_1} + \frac{1}{n_2}}$. The harmonic series gets its name from the fact that every term of the series, after the first term, is **the harmonic mean of its neighboring terms**. For example, examining the three consecutive sums $\dfrac{1}{m-1} + \dfrac{1}{m} + \dfrac{1}{m+1}$ from the harmonic series, one can show that the harmonic mean of $n_1 = \frac{1}{m-1}$ and $n_2 = \frac{1}{m+1}$ is given by $\bar{n} = \frac{1}{m}$.

■

Example 4-7. Convergent Series

Triangular numbers, illustrated in the figure 4-2, were known to the Greeks. The first few triangular numbers are $1, 3, 6, 10, \ldots$ and one can verify that the nth triangular number is given by the formula $\dfrac{n(n+1)}{2}$.

---

[5] The harmonic series is a very slowly diverging series. For example, it would take a summation of over $1.509(10)^{43}$ terms before the sum reached 100.

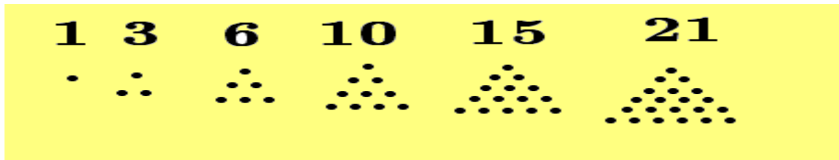


Figure 4-2. Triangular numbers.

Consider the infinite sum $\sum_{n=1}^{\infty} \frac{2}{n(n+1)} = \frac{2}{1\cdot 2} + \frac{2}{2\cdot 3} + \frac{2}{3\cdot 4} + \cdots + \frac{2}{m(m+1)} + \cdots$ which represents **the sum of the reciprocal of the triangular numbers**. The mth partial sum of this series is given by

$$U_m = \frac{2}{1\cdot 2} + \frac{2}{2\cdot 3} + \frac{2}{3\cdot 4} + \cdots + \frac{2}{m(m+1)}$$

Observe that by partial fractions one can write $\frac{2}{m(m+1)} = \frac{2}{m} - \frac{2}{m+1}$ so that

$$U_m = (2-1) + (1 - \frac{2}{3}) + (\frac{2}{3} - \frac{2}{4}) + \cdots + (\frac{2}{m} - \frac{2}{m+1})$$

This is called a **telescoping series** because of the way the terms add up. The resulting sum is

$$U_m = 2 - \frac{2}{m+1} \quad \text{with limit} \quad \lim_{m\to\infty} U_m = \lim_{m\to\infty} (2 - \frac{2}{m+1}) = 2$$

Consequently, the infinite series converges with sum equal to 2.

■

In general, one should examine the nth partial sum $U_n = \sum_{j=1}^{n} u_j$ of a given series such as (4.5) to determine if the limit of the sequence of partial sums $\lim_{n\to\infty} U_n$ is infinite, becomes finite oscillatory or infinite oscillatory or the limit does not exist, then the series $\sum u_n$ is said to be a divergent series. Whenever the limit $\lim_{n\to\infty} U_n$ exists with a value U, then U is called the sum of the series and the series is called convergent. The convergence of the series can be represented in one of the forms

$$\sum_{n=0}^{\infty} u_n = \lim_{n\to\infty} U_n = U = \lim_{N\to\infty} \sum_{j=0}^{N} u_j$$

Comparison of Two Series

Consider two infinite series which differ only in their starting values

$$\sum_{m=1}^{\infty} u_m = u_1 + u_2 + \cdots + u_\nu + u_{\nu+1} + \cdots \quad \text{and} \quad \sum_{n=\nu}^{\infty} u_n = u_\nu + u_{\nu+1} + \cdots \tag{4.8}$$

where $\nu > 1$ is an integer. Observe that it follows from the above definition for convergence that if one of the series in equation (4.8) converges, then the other series must also converge. Similarly, if one of the series from equation (4.8) diverges, then the other series must also diverge. In dealing with an infinite series there are many times where it is convenient to chop off or truncate the series after a finite number of terms counted from the beginning of the series. One can then deal with the remaining part. This is because the portion chopped off is a finite number of terms representing some constant being added to the series. Consequently, it is possible to add or remove a finite number of terms to or from the beginning of an infinite series without affecting the convergence or divergence of the series.

Test For Divergence

If the infinite series $\sum_{n=1}^{\infty} u_n$ converges to a sum U, then **a necessary condition for convergence** is that the nth term of the series approach zero as n increases without bound. This necessary condition is expressed $\lim_{n\to\infty} u_n = 0$. This requirement follows from the following arguments.

If $U_n = \sum_{i=1}^{n} u_i$ is the nth partial sum, and $U_{n-1} = \sum_{i=1}^{n-1} u_i$ is the $(n-1)$st partial sum, then for convergence, both of these partial sums must approach a limit U as n increases without bound and so

$$\lim_{n\to\infty} U_n = U \qquad \text{and} \qquad \lim_{n\to\infty} U_{n-1} = U \tag{4.9}$$

By subtracting the nth and $(n-1)$st partial sums one obtains $U_n - U_{n-1} = u_n$ and consequently

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (U_n - U_{n-1}) = U - U = 0 \tag{4.10}$$

The condition $\lim_{n\to\infty} u_n = 0$ is **a necessary condition for convergence of the infinite series** $\sum_{i=1}^{\infty} u_i$. If this condition is not satisfied, then one can say the infinite series $\sum_{i=1}^{\infty} u_i$ diverges. The nth term of a series approaching zero as n increases without bound is a necessary condition for any series to converge, **but it doesn't guarantee convergence of the series**. If the nth term of the series approaches zero, then additional testing must be done to determine if the series converges or diverges. Take for example the harmonic series

$$H = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots$$

considered earlier, one can observe that the nth term $\frac{1}{n}$ approaches zero, but further investigation shows that the series diverges

Cauchy Convergence

The **Cauchy convergence condition** associated with an infinite series examines the sequence of partial sums $\{U_j\}$ for $j = 1, 2, 3, \ldots$ and requires for convergence that for every given small number $\epsilon > 0$, there exists an integer N such that for any two integers m and n satisfying $n > m > N$, one can show that $|U_n - U_m| < \epsilon$. Observe that the sequence of partial sums U_n and U_m are written

$$U_n = \sum_{j=1}^{n} u_j \qquad \text{and} \qquad U_m = \sum_{j=1}^{m} u_j, \qquad n > m$$

so that for Cauchy convergence of the sequence of partial sums it is required that $|U_n - U_m| = |u_{m+1} + u_{m+2} + \cdots + u_n|$ be less than the given small quantity $\epsilon > 0$. This test holds because if $\lim\limits_{n\to\infty} U_n = \ell$ and $\lim_{m\to\infty} U_m = \ell$, then by definition of a limit one can select integer values for n and m so large that one can write

$$| U_n - \ell | < \frac{\epsilon}{2} \qquad \text{and} \qquad | U_m - \ell | < \frac{\epsilon}{2}$$

where $\epsilon > 0$ is any small positive quantity and m and n are sufficiently large, say both m and n are greater than N. It follows then that

$$| U_n - U_m | = | (U_n - \ell) - (U_m - \ell) | \leq | |U_n - \ell | + | U_m - \ell | < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The Cauchy test is an important test for convergence because it allows one to test for convergence without actually finding the limit of the sequence.

Example 4-8. **Geometric series**

Consider the geometric series $a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots$ where a and r are nonzero constants. The sequence of partial sums is given by

$$\begin{aligned}
A_1 =& a \\
A_2 =& a + a\,r \\
\vdots \quad & \quad \vdots \\
A_n =& a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^{n-1}
\end{aligned}$$

Recall that by multiplying A_n by r and subtracting the result from A_n one obtains

$$(1-r)A_n = a - a\,r^n \qquad \text{or} \qquad A_n = \frac{a}{1-r} - \frac{a\,r^n}{1-r} \tag{4.11}$$

The convergence or divergence of the sequence $\{A_n\}$ depends upon the sequence $\{r^n\}$. Skipping the trivial case where $r=0$, the sequence $\{A_n\}$ converges if the sequence $\{r^n\}$ converges. Consider the sequence $\{r^n\}$, for $n = 0, 1, 2, 3, \dots$ in the following cases $|r| < 1$, $r > 1$, $r = 1$, $r < -1$ and $r = -1$.

(i) If $|r| < 1$, write $|r| = \frac{1}{1+\alpha}$ where $\alpha > 0$. Using the binomial expansion show $\dfrac{1}{r^n} = (1+\alpha)^n > 1 + n\alpha$ for $n > 2$, or $r^n = \dfrac{1}{(1+\alpha)^n} < \dfrac{1}{1+n\alpha}$ and consequently for a given $\epsilon > 0$, with $0 < \epsilon < 1$, write

$$|r^n| = |r|^n = \frac{1}{(1+\alpha)^n} < \frac{1}{1+n\alpha} < \epsilon \quad \text{for all } n > N > \frac{1-\epsilon}{\epsilon\alpha}$$

Here $\frac{1}{1+n\alpha} \to 0$ as $n \to \infty$. This is an example of the sandwich theorem and demonstrates $\lim_{n\to\infty} r^n = 0$.

(ii) If $r > 1$, then r^n for $n = 1, 2, 3, \dots$ increases without bound. Consequently, for any given positive number M, $r^n > M$ for all integers $n > \dfrac{\ln M}{\ln r}$ and so the sequence $\{r^n\}$ diverges.

(iii) If $r = 1$, then $r^n = 1$ for all integers n and the sequence for $\{A_n\}$ diverges.

(iv) If $r < -1$ the sequence $\{r^n\}$ diverges since it becomes infinitely oscillatory with $r^{2n} \to +\infty$ and $r^{2n+1} \to -\infty$.

(v) The special case $r = -1$ also gives a finite oscillating sequence since $r^{2n} = 1$ and $r^{2n+1} = -1$ for $n = 1, 2, 3, \dots$ and so in this case the sequence $\{r^n\}$ diverges.

In summary, the geometric series has the finite sum given by equation (4.11) and the infinite sum

$$A = \lim_{n\to\infty} A_n = a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \frac{a}{1-r}, \quad \text{for } |r| < 1$$

otherwise, the geometric series diverges.

■

The Integral Test for Convergence

Assume $f(x)$ is a given function satisfying the following properties.

(i) ***$f(x)$ is defined and continuous for $x \geq M$ for some positive integer M.***

(ii) ***$f(x)$ has the property that $f(n) = u_n$ for all integers $n \geq M$.***

(iii) ***$f(x)$ decreases as x increases which implies $u_n = f(n)$ is a monotonic decreasing function.***

(iv) ***$f(x)$ satisfies $\lim_{x\to\infty} f(x) = 0$.***

One can then say that the infinite series $\sum_{n=1}^{\infty} u_n$

(a) converges if the integral $\int_M^{\infty} f(x)\,dx$ converges.

(b) diverges if the integral $\int_M^{\infty} f(x)\,dx$ diverges.

$$\textit{where } \int_M^{\infty} f(x)\,dx = \lim_{T\to\infty} \int_M^{T} f(x)\,dx \textit{ is an improper integral.}$$

The integral test for convergence of an infinite series compares the area under the curve $y = f(x)$ with overestimates and underestimates for this area. The following is a proof of the integral test in the case $M = 1$. Given the infinite series $\sum_{n=1}^{\infty} u_n$, with $u_n > 0$ for all values of n, one tries to find a continuous function $y = f(x)$ for $1 \le x < \infty$ which decreases as x increases and is such that $f(n) = u_n$ for all values of n. It is then possible to compare the summation of the infinite series with the area under the curve $y = f(x)$ using rectangles. This comparison is suggested by examining the rectangles sketched in the figure 4-3.

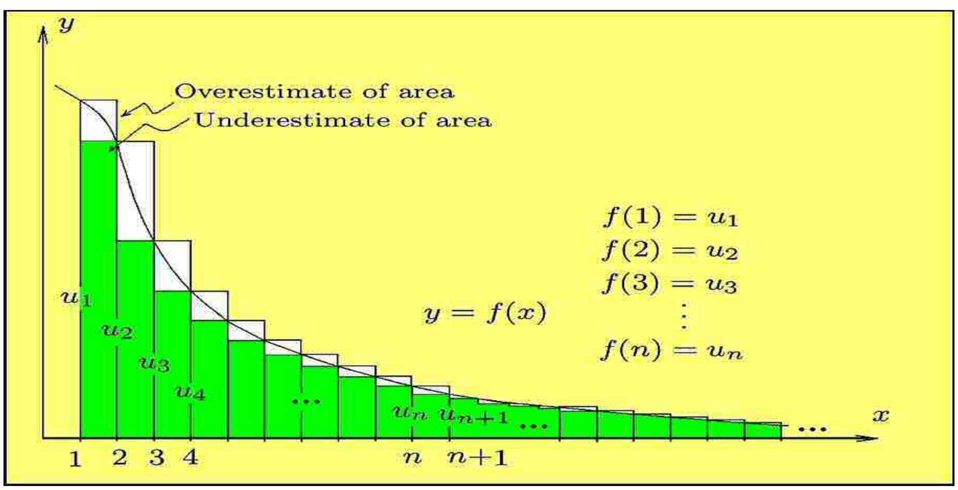


Figure 4-3.

Overestimates and underestimates for area under curve $y = f(x)$.

Assume there exists a function $f(x) > 0$ which decreases as x increases with the property $f(n) = u_n$ and that $\lim_{x\to\infty} f(x) = 0$. The integral $\displaystyle\int_n^{n+1} f(x)\,dx$ represents the area under the curve $y = f(x)$ bounded by the x-axis and the lines $x = n$ and $x = n+1$. Using the mean value theorem for integrals the value of this integral is $f(\xi)$ for $n < \xi < n+1$.

The assumption $f(x)$ decreases as x increases implies the inequality

$$u_n = f(n) \geq \int_n^{n+1} f(x)\,dx = f(\xi) \geq f(n+1) = u_{n+1} \quad \text{for all values of } n. \tag{4.12}$$

The inequality (4.12) can now be applied to each interval $(n, n+1)$ to calculate overestimates and underestimates for the area under the curve $y = f(x)$, for x satisfying $n \leq x \leq n+1$ and for $n = 1, 2, 3, \ldots$. A summation of the inequalities given by equation (4.12), for $n = 1, 2, \ldots, N-1$, gives a summation of areas under the curve and produces the inequality

$$u_2 + u_3 + \cdots + u_N \leq \int_1^N f(x)\,dx \leq u_1 + u_2 + u_3 + \cdots + u_{N-1} \tag{4.13}$$

A graphic representation of this inequality is given in the following figure.

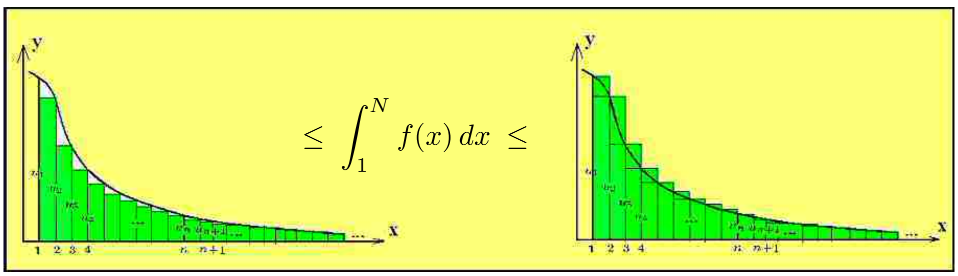


Figure 4-4. Pictorial representation of the inequality (4.13)

The inequality (4.13) gives an underestimate and overestimate for the area under the curve $y = f(x)$ of figure 4-3. Consider the following cases.

Case 1: If in the limit as N increases without bound the integral $\lim_{N\to\infty} \int_1^N f(x)\,dx$ exists, say with value S, then the left-hand side of equation (4.13) indicates the sequence of partial sums is monotonic increasing and bounded above by S and consequently the infinite series must converge.

Case 2: If in the limit as N increases without bound the integral $\lim_{N\to\infty} \int_1^N f(x)\,dx$ is unbounded or the integral does not exist, then the right-hand side of the inequality (4.13) indicates that the infinite series diverges.

Example 4-9. **The p-series**

Consider the p-series which is defined $\quad H = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$

(Case I) In the case $p = 1$, the above series is called the harmonic series or the p-series of order 1. Sketch the curve $y = f(x) = 1/x$ and construct rectangular overestimates for the nth partial sum. One can then verify that the nth partial sum of the harmonic series satisfies

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i} \geq \int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1)$$

In the limit as n increases without bound the logarithm function diverges and so the harmonic series diverges using the integral test.

(Case II) In the case $p \leq 0$, the p-series diverges because it fails the test of the nth term approaching zero as n increases without bound.

(Case III) In the case p is positive and $p \neq 1$ use the function $f(x) = \dfrac{1}{x^p}$ and show

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{T\to\infty} \int_1^{T} \frac{1}{x^p}\, dx = \lim_{T\to\infty} \left[\frac{1}{p-1} \left(1 - \frac{1}{T^{p-1}} \right) \right] \tag{4.14}$$

If $p > 1$ the right-hand limit from equation (4.14) has the value $\frac{1}{p-1}$ and so the integral exists and consequently the p-series converges. If $0 \leq p < 1$ the limit on the right-hand side of equation (4.14) diverges and so by the integral test the p-series also diverges.

In conclusion, the p-series

$$\boldsymbol{H = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}$$

converges for $p > 1$ and diverges for $p \leq 1$ ■

Example 4-10. **Estimation of error**

Let $\sum_{n=1}^{\infty} u_n$ denote a given infinite series which converges by the integral test. If U denotes the true sum of this series and U_n denotes its nth partial sum, then $|U - U_n| = R_n$ is the remainder term that was omitted and represents the true error in using U_n as an approximate value for the sum of the series.

Let E_n denote an estimate for the error associated with the truncation of an infinite series after the nth term where the nth partial sum U_n is being used as an estimate for the true value U of the series. The function $f(x)$ used in the integral test for convergence is a decreasing function of x and so

$$|R_n| \le \int_n^\infty f(x)\,dx$$

If one sets $E_n = \int_n^\infty f(x)\,dx$, then the magnitude of the remainder $|R_n| \le E_n$ or the true error is less than the estimated error E_n.

Here $u_n = f(n)$ so that the partial sum $U_N = \sum_{n=1}^N u_n = \sum_{k=1}^N f(k)$ can be used as an approximation to the infinite sum. A better approximation for the sum is given by $U_N^* = \sum_{k=1}^N f(k) + \int_N^\infty f(x)\,dx$ since the integral representing the tail end of the area under the curve in figure 4-3 is a good approximation to the sums neglected.

■

Example 4-11.

Sum 100 terms of the infinite series $S = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{m^2} + \cdots$ and estimate the error associated with this sum.

Solution: Let S_n denote the nth partial sum

$$S_n = \sum_{m=1}^n \frac{1}{m^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

Use a computer and verify that $S_{100} = 1.63498$ so that an estimate for the error between the finite sum and the infinite sum is given by

$$E_{100} = \int_{100}^\infty \frac{1}{x^2}\,dx = 0.01$$

Therefore, one can state that the difference between the true sum and approximate sum is $|S - S_{100}| < 0.01$ or $1.62498 < S < 1.64498$. The exact value for S is known to be $\pi^2/6 = 1.64493...$ and this exact value can be compared with our estimate. Observe that the value $S_{100} + \int_{100}^\infty \frac{1}{x^2}\,dx$ gives a better estimate for the sum.

It is important that you make note of the fact that the integral test does not give the sum of the series. For example,

$$S = \sum_{m=1}^\infty \frac{1}{m^2} = \frac{\pi^2}{6} \qquad \text{and} \qquad \int_1^\infty \frac{1}{x^2}\,dx = 1$$

■

Alternating Series Test

An alternating series has the form

$$\sum_{j=1}^{\infty}(-1)^{j+1}u_j = u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots, \qquad u_j > 0 \tag{4.15}$$

where each term of the series is positive, but the sign in front of each term alternates between plus and minus. An alternating series converges if the following two conditions are satisfied.

(i) For a large enough integer N, the terms u_n of the series are decreasing in absolute value so that $|u_{n+1}| \le |u_n|$, for all values of $n > N$.

(ii) The nth term approaches zero as n increases without bound so that one can write $\lim_{n\to\infty} u_n = 0$ or $\lim_{n\to\infty} |u_n| = 0$.

To prove the above statement one can examine the sequence of partial sums U_N, starting with $N = 1$, and make use of the fact that $u_{n+1} \le u_n$ to obtain the situation illustrated in the figure 4-5.

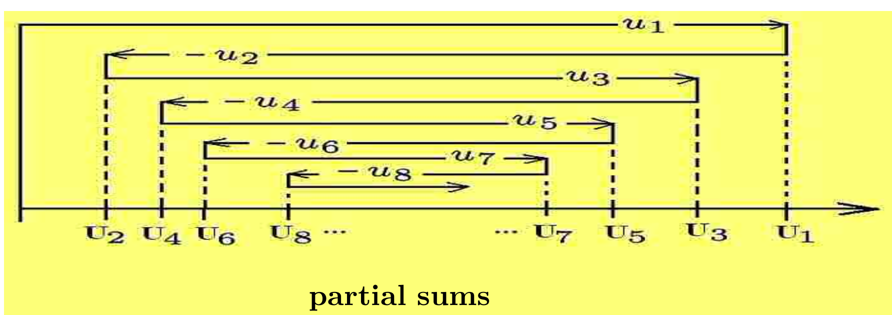


Figure 4-5.
Sequence of partial sums for alternating series.

The even sequence of partial sums $U_2, U_4, U_6, \ldots$ forms a bounded monotone increasing sequence. The odd sequence of partial sums $U_1, U_3, U_5, \ldots$ forms a bounded monotone decreasing sequence. Both the even $\{U_{2n}\}$ and odd $\{U_{2n+1}\}$ sequence of partial sums converge and consequently

$$\lim_{n\to\infty} U_{2n} = U \qquad \text{and} \qquad \lim_{n\to\infty} U_{2n+1} = V$$

Note also that for n large, the term u_{2n+1} of the series must approach zero and consequently

$$\lim_{n\to\infty} u_{2n+1} = \lim_{n\to\infty}(U_{2n+1} - U_{2n}) = \lim_{n\to\infty} U_{2n+1} - \lim_{n\to\infty} U_{2n} = V - U = 0$$

which implies that $U = V$ and the alternating series converges.

Example 4-12. **Alternating series**

Consider the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$. For $n+1 > n$, then $\ln(n+1) > \ln n$ and consequently $\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$ indicating the terms $u_n = \frac{1}{\ln n}$ decrease as n increases or equivalently, the sequence $\{u_n\}$ is monotonic decreasing. In addition, $\lim_{n\to\infty} \frac{1}{\ln n} = 0$. These two conditions guarantee the convergence of the given alternating series.

■

Example 4-13. **Estimation of error**

Denote by U_m the mth partial sum of a convergent alternating series and let U denote the true sum of the series. For $n > m$ examine the difference between the nth partial sum and mth partial sum by writing

$$|U_n - U_m| = \left|\sum_{i=1}^{n}(-1)^{i+1}u_i - \sum_{i=1}^{m}(-1)^{i+1}u_i\right| = \left|\sum_{i=m+1}^{n}(-1)^{i+1}u_i\right|$$

Using an appropriate selection of the value n (i.e. being either even or odd and greater than m) one can write

$$\left|\sum_{i=m+1}^{n}(-1)^{i+1}u_i\right| = u_{m+1} - (u_{m+2} - u_{m+3}) - (u_{m+4} - u_{m+5}) - \cdots - (u_{n-1} - u_n) < u_{m+1}$$

This inequality is independent of the value of n so that

$$|U - U_n| = \left|\sum_{i=1}^{\infty}(-1)^{i+1}u_i - \sum_{i=1}^{n}(-1)^{i+1}u_i\right| < u_{n+1}$$

This last inequality is sometimes referred to as the Leibniz condition and implies that by selecting an error E_n as the absolute value of the $(n+1)$ st term of an alternating series, then one can write $|R_n| = |U - U_n| \le E_n$. Hence, to obtain the sum of an alternating series accurate to within some small error $\epsilon > 0$, one must find an integer value n such that $|u_{n+1}| = E_{n+1} < \epsilon$, then it can be stated with confidence that the nth partial sum U_n and the true sum U of the alternating series satisfies the inequality $U_n - \epsilon < U < U_n + \epsilon$.

■

Bracketing Terms of a Convergent Series

Let $A = a_0 + a_1 + a_2 + \cdots + a_m + \cdots = \sum_{n=0}^{\infty} a_n$ denote a convergent series and define the sequence of terms $\{j_\nu\}_{\nu=1}^{\infty}$ which is a strictly increasing sequence of nonnegative integers. That is the terms $j_1, j_2, \ldots$ are positive integers satisfying

$$j_1 < j_2 < \cdots < j_n < j_{n+1} < \cdots$$

The series for A can then be bracketed into nonoverlapping groups or partitions as follows

$$A = (a_0 + \cdots + a_{j_1}) + (a_{j_1+1} + \cdots + a_{j_2}) + (a_{j_2+1} + \cdots + a_{j_3}) + \cdots + (a_{j_n+1} + \cdots + a_{j_{n+1}}) + \cdots$$

where there is a finite number of terms in each group. This is equivalent to defining the infinite series

$$B = \sum_{n=0}^{\infty} b_n \qquad \text{where} \qquad \begin{aligned} b_0 =& a_0 + a_1 + \cdots + a_{j_1} \\ b_1 =& a_{j_1+1} + a_{j_1+2} + \cdots + a_{j_2} \\ \vdots \quad & \quad \vdots \\ b_n =& a_{j_n+1} + a_{j_n+2} + \cdots + a_{j_{n+1}} \\ \vdots \quad & \quad \vdots \end{aligned}$$

The partial sums $B_n = b_0 + b_1 + \cdots + b_n$ of the bracketed series, by definition, must equal the partial sum $A_{j_{n+1}}$ of the original series and consequently the sequence of partial sum $\{B_n\}$ is a subsequence of the partial sums $\{A_n\}$. In advanced calculus it is shown that every subsequence of a convergent sequence converges and from this result it can be concluded that

If a convergent series A has its terms bracketed into nonoverlapping groups to form a new series B, then the series B converges to the sum of the original series.

The converse of the above statement is not true. For example, the bracketed series $(1-1) + (1-1) + (1-1) + \cdots$ converges, but the unbracketed series has partial sums which oscillate and hence is divergent.

Example 4-14.

Consider the infinite series

$$A = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{m-1}\frac{1}{m} + \cdots$$

and say this series is bracketed into groups of two terms as follows

$$B = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n)}$$

Here both series converge to the same value as the bracketing does not effect the convergence of a converging series.

■

Comparison Tests

Consider two infinite series, say $\sum_{n=1}^{\infty} u_n$ ***and*** $\sum_{n=1}^{\infty} v_n$ ***where the terms of the series*** u_n ***and*** v_n ***are nonnegative. Let*** M ***denote a positive integer, then***

(i) if $u_n \le v_n$ ***for all integers*** $n > M$ ***and the infinite series*** $\sum_{n=1}^{\infty} v_n$ ***converges, then the infinite series*** $\sum_{n=1}^{\infty} u_n$ ***is also convergent.***

(ii) if $u_n \ge v_n \ge 0$ ***for all integers*** $n > M$ ***and if the infinite series*** $\sum_{n=1}^{\infty} v_n$ ***diverges, then the infinite series*** $\sum_{n=1}^{\infty} u_n$ ***must also diverge.***

To prove statement (i) above, let $\sum_{n=1}^{\infty} v_n$ denote a convergent series with sum V and let

$$U_n = u_1 + u_2 + \cdots + u_n, \quad \text{and} \quad V_n = v_1 + v_2 + \cdots + v_n$$

denote the nth partial sums associated with the infinite series $\sum u_n$ and $\sum v_n$ respectively. If $u_n \le v_n$ and V is the value of the converging series, then the sequence of partial sums $\{U_n\}$ and $\{V_n\}$ satisfy $U_n \le V_n \le V$ and consequently the sequence $\{U_n\}$ is an increasing bounded sequence which must converge.

If the series $\sum_{n=1}^{\infty} v_n$ diverges, then the sequence of partial sums $\{V_n\}$ increases without bound. If $u_n \ge v_n$ for all n, then $U_n \ge V_n$ for all n and so U_n must also increase without bound, indicating that the series $\sum_{n=1}^{\infty} u_n$ must also diverge.

Ratio Comparison Test

If $\sum_{n=1}^{\infty} u_n$ ***is a series to be compared with a known convergent series*** $\sum_{n=1}^{\infty} c_n$, ***then if the ratios of the*** $(n+1)$***st term to the*** n***th term satisfies***

$$\frac{u_{n+1}}{u_n} \le \frac{c_{n+1}}{c_n}, \tag{4.16}$$

then the series $\sum_{n=1}^{\infty} u_n$ ***is a convergent series.***

If $\sum_{n=1}^{\infty} u_n$ ***is a series to be compared with a known divergent series*** $\sum_{n=1}^{\infty} d_n$, ***then if the ratios of the*** $(n+1)$***st term to the*** n***th term satisfies***

$$\frac{u_{n+1}}{u_n} \ge \frac{d_{n+1}}{d_n}, \tag{4.17}$$

then the series $\sum_{n=1}^{\infty} u_n$ ***is a divergent series.***

To prove the above statements, make the assumption that the inequalities hold for all integers $n \ge 0$. The proofs can then be modified to consider the cases where the inequalities hold for all integers $n \ge N$. The proof of the above statements follows by listing the inequalities (4.16) and (4.17) for the values $n = 0, 1, 2, \ldots, (m-1)$. This produces the listings

$$\begin{array}{ccc}
\frac{u_1}{u_0} \le \frac{c_1}{c_0} & \qquad & \frac{u_1}{u_0} \ge \frac{d_1}{d_0} \\
\frac{u_2}{u_1} \le \frac{c_2}{c_1} & & \frac{u_2}{u_1} \ge \frac{d_2}{d_1} \\
\frac{u_3}{u_2} \le \frac{c_3}{c_2} & & \frac{u_3}{u_2} \ge \frac{d_3}{d_2} \\
\vdots \qquad \vdots & & \vdots \qquad \vdots \\
\frac{u_m}{u_{m-1}} \le \frac{c_m}{c_{m-1}} & & \frac{u_m}{u_{m-1}} \ge \frac{d_m}{d_{m-1}}
\end{array} \tag{4.18}$$

Multiply the terms on the left-hand sides and right-hand sides of the above listings and then simplify the result to show

$$u_m \le \frac{u_0}{c_0}\, c_m, \qquad \text{and} \qquad u_m \ge \frac{u_0}{d_0}\, d_m \tag{4.19}$$

A summation of the terms on both sides of the above inequalities produces a comparison of the given series with known convergent or divergent series multiplied by some constant.

Example 4-15. Comparison test (set up an inequality)

Two known series used quite frequently for comparison with other series are the geometric series $\sum_{n=0}^{\infty} ar^n$ and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. These series are used in comparison tests because inequalities involving powers of known quantities are easy to construct. For example, to test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence, use the comparison test with the known p-series. The inequality $n^2+1 > n^2$ implies that $\frac{1}{n^2+1} < \frac{1}{n^2}$ and consequently by summing both sides of this inequality one finds $\sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$. It is known that the p-series, with $p=2$, converges and so by comparison the given series converges.

As another example, consider the series $\sum_{n=1}^{\infty} \frac{a}{3^n+b}$ where $a>0$ and $b>0$ are constants and compare this series with the geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$. Since $3^n+b > 3^n$, then $\frac{a}{3^n+b} < \frac{a}{3^n}$ and consequently, $\sum_{n=1}^{\infty} \frac{a}{3^n}$ converges, so that the given series must also converge.

■

Example 4-16. Estimation of error

Let $\sum_{n=1}^{\infty} u_n$ denote a convergent infinite series, then use the nth partial sum U_n to estimate the true value U of the series. The true error in using the nth partial sum as an estimated value for the sum is given by

$$R_n = |U - U_n| = \sum_{i=n+1}^{\infty} u_n$$

where R_n is the remainder after the summation of n-terms of the series.

If there exists a known comparison series $\sum_{n=1}^{\infty} v_n$ such that for all integers n greater than some fixed integer N there results the inequality $|u_n| \le v_n$, then it follows that

$$\begin{aligned} |u_{n+1} + u_{n+2} + \cdots + u_{n+m}| \le& |u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+m}| \\ \le& v_{n+1} + v_{n+2} + \cdots + v_{n+m} \le \sum_{i=n+1}^{\infty} v_i \end{aligned}$$

which implies that for all $n > N$, $|R_n| = |U - U_n| = \lim_{m\to\infty} \left| \sum_{i=n=1}^{n+m} u_i \right| \le \sum_{i=n+1}^{\infty} v_i$. Consequently if one selects $E_n = \sum_{i=n+1}^{\infty} v_i$ as an error estimate for the series sum, then write $|R_n| \le E_n$ which says the true error must be less than the estimated error E_n obtained by summation of terms greater than v_n from the known comparison series.

■

Absolute Convergence

Consider the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \mid a_n \mid$ where the second series has terms which are the absolute value of the corresponding terms in the first series. By definition **the series $\sum_{n=1}^{\infty} a_n$ is called an absolutely convergent series if the series of absolute values $\sum_{n=1}^{\infty} \mid a_n \mid$ is a convergent series.**

Example 4-17.

(a) The series $S_1 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is called the alternating harmonic series. By the alternating series test it is a convergent series. It is not an absolutely convergent series because the series of absolute values is the harmonic series which is a known divergent series.

(b) The series $S_2 = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is an absolutely convergent series because the corresponding series of absolute values is the p-series or order 2 which is a known convergent series.

■

Example 4-18.

If $\sum_{j=1}^{\infty} |u_j|$ is an absolutely convergent series, then the series $\sum_{j=1}^{\infty} u_j$ must also be a convergent series.

To prove the above statement examine the sequence $\{\overline{U}_n\}$ where $\overline{U}_n$ denotes the nth partial sum associated with the series of absolute values

$$\overline{U}_n = |u_1| + |u_2| + \cdots + |u_n| = \sum_{j=1}^{n} |u_j| \quad \text{for } n = 1, 2, 3, \ldots$$

For convergence of the series of absolute values the Cauchy convergence criteria requires that there exist an integer N such that for all integers n and m satisfying, $n > m > N$, one has

$$|\overline{U}_n - \overline{U}_m| = ||u_{m+1}| + |u_{m+2}| + \cdots + |u_n|| < \epsilon \tag{4.20}$$

where $\epsilon > 0$ is any small number. Select the value N large enough that one can apply the Cauchy convergence criteria to the infinite series $\sum_{j=1}^{\infty} u_j$ for the same given value of $\epsilon > 0$ using the same values of m and n. For Cauchy convergence of the series $\sum_{j=1}^{\infty} u_j$, it is required that the difference of the mth partial sum $U_m = \sum_{j=1}^{m} u_j$ and nth partial sum, $U_n = \sum_{j=1}^{n} u_j$, for $n > m$, must satisfy $|U_n - U_m| = |u_{m+1} + u_{m+2} + \cdots + u_n| < \epsilon$. Using the generalized triangle inequality, the absolute value of a sum is less than or equal to the sum of the absolute values. That is,

$$|u_{m+1} + u_{m+2} + \cdots + u_n| \le ||u_{m+1}| + |u_{m+2}| + \cdots + |u_n|| < \epsilon. \tag{4.21}$$

But this is the Cauchy condition which is required for convergence of the infinite series $\sum_{j=1}^{\infty} |u_j|$.

Another proof is to consider the two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} |u_n|$ with partial sums

$$\begin{aligned} U_n =& u_1 + u_2 + u_3 + \cdots + u_n \\ \text{and} \qquad \overline{U}_n =& |u_1| + |u_2| + |u_3| + \cdots |u_n| \end{aligned}$$

Our hypothesis is that the sequence of partial sums $\{\overline{U}_n\}$ converges. Our problem is to show that the sequence of partial sums $\{U_n\}$ also converges. Assume the series $\sum_{n=1}^{\infty} u_n$ has both positive and negative terms so that the partial sum $\{U_n\}$ can be written as $U_n = P_n - N_n$ where P_n is the sum of the positive terms and N_n is the sum of the absolute values of the negative terms within the nth partial sum. This implies that the partial sum $\overline{U}_n$ can be expressed $\overline{U}_n = P_n + N_n$. Here the sequence $\{\overline{U}_n\}$ is a bounded increasing sequence with some limit $\lim_{n\to\infty} \overline{U}_n = \overline{U}$ which implies the inequalities

$$P_n \le \overline{U}_n \le \overline{U} \qquad \text{and} \qquad N_n \le \overline{U}_n \le \overline{U}$$

so that the limit $\overline{U}$ is an upper bound for the sequences $\{P_n\}$ and $\{N_n\}$. Both the sequences $\{P_n\}$ and $\{N_n\}$ are monotone increasing and bounded sequences and must converge to some limiting values. If these limiting values are called P and N, then one can employ the limit theorem from calculus to write

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty}(P_n - N_n) = \lim_{n\to\infty} P_n - \lim_{n\to\infty} N_n = P - N$$

which shows the infinite series $\sum_{n=1}^{\infty} u_n$ is a convergent series.

If $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series, then write

$$\lim_{N\to\infty}\left|\sum_{n=1}^{N} a_n\right| \le \lim_{N\to\infty}\sum_{n=1}^{N} |a_n|$$

as this is just a limit associated with the generalized triangle inequality.

Also note the terms within an absolutely convergent series can be rearranged with the resulting series also being absolutely convergent with the same sum as the original series.

■

Slowly Converging or Slowly Diverging Series

In using the comparison test to determine whether a series converges or diverges, one should be aware that some series converge very slowly while other series diverge very slowly. To estimate the value of a very slowly convergent series to within some error bound, it is sometimes necessary to sum an excessive number of terms.

If $\sum a_n$ and $\sum b_n$ are two convergent series, one says that the series $\sum a_n$ converges at a slower rate than the series $\sum b_n$ if the condition $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ is satisfied.

In a similar fashion

If $\sum a_n$ and $\sum b_n$ are two divergent series, one says that the series $\sum a_n$ diverges at a slower rate than the series $\sum b_n$ if the condition $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ is satisfied.

Example 4-19.

The series $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ will converge at a slower rate than the series $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^2}$ because $\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n(\ln n)^2}} = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = 0$

This result follows by using L'Hôpital's rule to evaluate

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2(\ln x)\cdot x^{-1}}{1} = \lim_{x\to\infty} \frac{2x^{-1}}{1} = 0$$

■

Example 4-20.

The series $\displaystyle\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges at a slower rate than the comparison series $\displaystyle\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n}$, because $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$
As an exercise estimate how many terms of each series needs to be summed for the result to be greater than 3. Verify that the answer is $\sum_{n=2}^{8718} a_n > 3$ and $\sum_{n=2}^{31} b_n > 3$

■

Ratio Test

The following tests investigate the ratio of certain terms in an infinite series as the index of the terms increases without bound. The ratio test is sometimes referred to as the d'Alembert's test, after Jean Le Rond d'Alembert (1717-1783) a French mathematician. The ratio test examines the absolute value of the ratio of the $(n+1)$st term divided by the nth term of the series $\displaystyle\sum_{i=1}^{\infty} u_i$ as the index n increases without bound.

d'Alembert ratio test

If the terms u_n are different from zero for $n = 1, 2, 3, \ldots$ and the limit $\displaystyle\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = q$ exists[6], then

(i) The series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent if $q < 1$.

(ii) The series $\sum_{n=1}^{\infty} u_n$ diverges if $q > 1$.

(iii) The test fails if $q = 1$.

The above result can be proven using the geometric series. Assume that for all values of n greater than some value N it is possible to show that

$$\left|\frac{u_{n+1}}{u_n}\right| \le r < 1, \tag{4.22}$$

Define the quantities

$$v_0 = u_{N+1}, \quad v_1 = u_{N+2}, \quad v_2 = u_{N+3}, \quad \cdots$$

and use the inequality given by equation (4.22) to show $|v_m| \le r|v_{m-1}|$. This is accomplished by setting $n = N+1, N+2, \ldots$ in equation (4.22) to obtain the inequalities

[6] If all the terms of the series are nonnegative, then the absolute value sign can be removed

$$\begin{aligned}|v_1| &\le r|v_0| \\ |v_2| &\le r|v_1| \le \; r^2 v_0 \\ &\vdots \qquad\qquad \vdots \\ |v_m| &\le r|v_{m-1}| \le \; r^m v_0\end{aligned} \tag{4.23}$$

The original series can then be split into two parts. The first part $\sum_{j=1}^{N} |u_j|$ is a finite series leaving the series $\sum_{j=N+1}^{\infty} |u_j|$ representing the second part which can be compared with a geometric series. The second part satisfies the inequality

$$\sum_{j=N+1}^{\infty} |u_j| \le \sum_{i=0}^{\infty} |v_i| \le |v_0|(1+r+r^2+r^3+\cdots) \le \frac{|v_0|}{1-r}, \qquad |r|<1 \tag{4.24}$$

and so the series is absolutely convergent by the comparison test.

If for all values of n greater than some value N it is possible to show

$$\left|\frac{u_{n+1}}{u_n}\right| = q > 1 \tag{4.25}$$

then the terms of the series $\sum u_n$ become a monotonic increasing sequence of positive numbers and consequently the nth term cannot approach zero as n increases without bound. Under these conditions the given series is divergent.

Example 4-21. **Ratio test**

Consider the series $\displaystyle\sum_{m=1}^{\infty} \frac{m+1}{2^m} = \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots + \frac{n+1}{2^n} + \cdots$

Using the ratio test one finds $\displaystyle\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{n+2}{2^{n+1}}}{\frac{n+1}{2^n}} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+2}{n+1}\right) = \frac{1}{2} < 1$ and so the given series is absolutely convergent. ■

Example 4-22. **Ratio test**

Test the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ for convergence using the ratio test. The ratio test produces the limit $\displaystyle\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$ and so the ratio test fails and so some other test must be used to investigate convergence or divergence of the series. ■

Example 4-23. **Ratio test**

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$ to determine if the series converges. Using the ratio test one finds $\displaystyle\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$ and so the given series converges. ■

Root Test

If the limit $\lim_{n\to\infty} \sqrt[n]{|u_n|} = L$ exists, then the series $\sum_{n=1}^{\infty} u_n$ is

(i) absolutely convergent if $L < 1$

(ii) is divergent if $L > 1$

(iii) The test fails if $L = 1$

Note that if $\sqrt[n]{|u_n|} < q < 1$ for all $n > N$, the $|u_n| < q^n < 1$ so that the series $\sum_{i=N}^{\infty} |u_i|$ converges by comparison with the geometric series. If $\sqrt[n]{|u_n|} > q > 1$, the nth term of the series does not approach zero and so the series diverges.

Certain Limits

Three limits that prove to be very useful are the following.

1. If $\alpha > 0$ and β is any real number, then $\lim_{n\to\infty} \frac{n^\beta}{(1+\alpha)^n} = 0$. This limit follows by examining the function $f(x) = \frac{x^\beta}{(1+\alpha)^x} = \frac{x^\beta}{e^{x\ln(1+\alpha)}} = \left(\frac{x}{e^{\frac{x}{\beta}\ln(1+\alpha)}}\right)^\beta$.

2. If β is any real number and $\alpha > 0$ is real, then $\lim_{n\to\infty} \frac{(\ln n)^\beta}{n^\alpha} = 0$. This limit follows by examining the function $g(x) = \frac{(\ln x)^\beta}{x^\alpha} = \left(\frac{\ln x}{x^{\alpha/\beta}}\right)^\beta$.

3. A consequence of the ratio test is that if all the terms of the sequence $\{u_n\}$ are such that $u_n \neq 0$ and the limit $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$, then the series $\sum u_n$ is absolutely convergent, which implies that $\lim_{n\to\infty} u_n = 0$ as this is a necessary condition for convergence. Hence, the ratio test can be used to investigate certain limits which approach zero.

Example 4-24. Limits

If $u_n = \frac{n^{1000}}{(1+\epsilon)^n}$, investigate the infinite series $\sum_{n=1}^{\infty} u_n$ and find the limit $\lim_{n\to\infty} \frac{n^{1000}}{(1+\epsilon)^n}$ where $\epsilon > 0$ is a constant.

Solution: If one tests the ratio

$$\frac{u_{n+1}}{u_n} = \frac{\frac{(n+1)^{1000}}{(1+\epsilon)^{n+1}}}{\frac{n^{1000}}{(1+\epsilon)^n}} = \left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{(1+\epsilon)} = \left(1 + \frac{1}{n}\right)^{1000} \cdot \frac{1}{(1+\epsilon)}$$

in the limit as n increases without bound, one finds $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{(1+\epsilon)} < 1$ and so the sum $\sum u_n$ converges. The convergence of the series implies that the nth term approaches zero, so that $\lim_{n\to\infty} \frac{n^{1000}}{(1+\epsilon)^n} = 0$, provided $\epsilon > 0$.

■

Power Series

A series of the form

$$\sum_{n=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_m x^m + \cdots$$

is called a power series centered at the origin. Here x is a variable and the terms $c_0, c_1, \ldots, c_m, \ldots$ are constants called the coefficients of the power series. If x is assigned a constant value, the power series then becomes a series of constant terms and it can be tested for convergence or divergence. One finds that in general power series converge for some values of x and diverge for other values of x. If the power series converges for $|x| < R$, then R is called the radius of convergence of the power series.

A series having the form

$$\sum_{n=1}^{\infty} c_n (x-x_0)^n = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \cdots + c_m(x-x_0)^m + \cdots$$

is called a power series in $(x - x_0)$ centered at the point x_0 with coefficients c_m for $m = 0, 1, 2, \ldots$.

Example 4-25. Power series

Consider the power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

where the coefficients a_n, for $n = 0, 1, 2, \ldots$ are constants and independent of the value selected for x. This series converges for certain values of x and diverges for other values of x. For convergence, the ratio test is required to satisfy the limiting condition

$$\lim_{n\to\infty} \left| \frac{u_n}{u_{n-1}} \right| = \lim_{n\to\infty} \left| \frac{a_n(x-x_0)^n}{a_{n-1}(x-x_0)^{n-1}} \right| = \lim_{n\to\infty} \left| \frac{a_n(x-x_0)}{a_{n-1}} \right| = \frac{|x-x_0|}{R} < 1$$

where $R = \lim_{n\to\infty} \left| \frac{a_{n-1}}{a_n} \right|$. Consequently, the power series is absolutely convergent whenever $|x-x_0| < R$ and is divergent whenever x satisfies $|x-x_0| > R$. For real variables, the power series converges for x satisfying $-R < x - x_0 < R$ and the interval $(x_0 - R, x_0 + R)$ is called the interval of convergence for the power series and R is called the radius of convergence of the power series. For complex variables the region of convergence is the interior of the circle $|z - z_0| < R$ in the complex plane.

■

A power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_m(x-x_0)^m + \cdots \tag{4.26}$$

will satisfy one of the following conditions.

(i) If $\lim_{n\to\infty} \left| \frac{a_{n-1}}{a_n} \right| = 0$, then the infinite series (4.26) converges only for $x = x_0$ and diverges for all other values of x.

(ii) If $\lim_{n\to\infty} \left| \frac{a_{n-1}}{a_n} \right| = \infty$, then the infinite series (4.26) converges absolutely for all values of x.

(iii) If $\lim_{n\to\infty} \left| \frac{a_{n-1}}{a_n} \right| = R$ exists and is nonzero, then the infinite series (4.26) converges for $\mid x - x_0 \mid < R$ and diverges for $\mid x - x_0 \mid > R$.

In general, if a power series

$$f(x) = \sum_{i=0}^{\infty} a_i (x-x_0)^i = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

exists and converges for $\mid x - x_0 \mid < R$, then one can say

(i) The function $f(x)$ is a continuous function on the interval $|x - x_0| < R$.

(ii) The function $f(x)$ has the derivative

$$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots + na_n(x-x_0)^{n-1} + \cdots$$

and this series for the derivative converges over the same interval $|x - x_0| < R$.

(iii) The function $f(x)$ has the integral given by

$$\int f(x)\,dx = a_0(x-x_0) + \frac{a_1}{2}(x-x_0)^2 + \frac{a_2}{3}(x-x_0)^3 + \cdots + \frac{a_n}{n+1}(x-x_0)^{n+1} + \cdots$$

plus some arbitrary constant of integration can be added to this result. The series for the integral also converges over the interval $|x - x_0| < R$.

Operations with Power Series

Two power series given by $f(x) = \sum_{n=0}^{\infty} f_n (x-x_0)^n$ with radius of convergence R_f and $g(x) = \sum_{n=0}^{n} g_n (x-x_0)^n$ with radius of convergence R_g can be added

$$a(x) = f(x) + g(x) = \sum_{n=0}^{\infty} f_n (x-x_0)^n + \sum_{n=0}^{\infty} g_n (x-x_0)^n$$

or they can be subtracted

$$b(x) = f(x) - g(x) = \sum_{n=0}^{\infty} f_n(x-x_0)^n - \sum_{n=0}^{\infty} g_n(x-x_0)^n$$

by adding or subtracting like powers of $(x-x_0)$. The resulting series has a radius of converge $R =$ smaller of $\{R_f, R_g\}$.

The product of the power series for $f(x)$ and $g(x)$ can be expressed

$$\begin{aligned} f(x)g(x) &= \left[\sum_{n=0}^{\infty} f_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} g_n(x-x_0)^n\right] \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f_n g_k (x-x_0)^{n+k} \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} f_j g_{n-j}\right)(x-x_0)^n \\ &= \sum_{n=0}^{\infty} c_n (x-x_0)^n \end{aligned}$$

where $c_n = \sum_{j=0}^{n} f_j g_{n-j}$ is the Cauchy product, sometimes referred to as the convolution of the sequences f_n and g_n.

The two power series for $f(x)$ and $g(x)$ can be divided and written

$$\frac{f(x)}{g(x)} = \frac{\sum_{n=0}^{\infty} f_n(x-x_0)^n}{\sum_{n=0}^{\infty} g_n(x-x_0)^n} = \sum_{n=0}^{\infty} h_n(x-x_0)^n$$

where the coefficients h_n are related to the coefficients f_n and g_n by comparison of the coefficients of like powers of $(x-x_0)$ on both sides of the expression

$$\sum_{n=0}^{\infty} f_n(x-x_0)^n = \left[\sum_{n=0}^{\infty} g_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} h_n(x-x_0)^n\right]$$

Maclaurin Series

Let $f(x)$ denote a function which has derivatives of all orders and assume the function and each of its derivatives has a value at $x = 0$. Also assume that the function $f(x)$ can be represented within some interval of convergence by an infinite series of the form

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

where $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants to be determined. If the above equation is to be an identity, then it must be true for all values of x. Substituting $x = 0$ into the equation gives $f(0) = c_0$. The series can be differentiated on a term by term basis as many times as desired. For example, one can write

$$\begin{aligned}
f'(x) =& c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots \\
f''(x) =& 2!c_2 + 3!c_3 x + 4 \cdot 3c_4 x^2 + 5 \cdot 4x^3 + \cdots \\
f'''(x) =& 3!c_3 + 4!c_4 x + 5 \cdot 4 \cdot 3x^2 + \cdots \\
\vdots \qquad & \qquad \vdots \\
f^{(n)}(x) =& n!c_n + (n+1)!c_{n+1} x + \cdots \\
\vdots \qquad &
\end{aligned}$$

Substituting $x = 0$ into each of the above derivative equations gives the results

$$c_1 = f'(0), \quad c_2 = \frac{f''(0)}{2!}, \qquad c_3 = \frac{f'''(0)}{3!}, \qquad \cdots \qquad , \; c_n = \frac{f^{(n)}(0)}{n!}, \; \cdots$$

This shows that $f(x)$ can be represented in the form

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(x)}{n!}x^n + \cdots \tag{4.27}$$

or

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!}\, x^m \tag{4.28}$$

where $f^{(0)}(0) = f(0)$ and $0! = 1$ by definition. The series (4.27) is known as a Maclaurin[7] series expansion of the function $f(x)$ in powers of x. This type of series is useful in determining values of $f(x)$ in the neighborhood of the point $x = 0$ since if $|x|$ is less than 1, then the successive powers x^n get very small for large values of n.

In the special case $f(x) = g(x + h)$ one finds, for h constant, the derivatives $f'(x) = g'(x + h)$, $f''(x) = g''(x + h)$, etc. Evaluating these derivatives at $x = 0$ gives $f(0) = g(h)$, $f'(0) = g'(h)$, $f''(0) = g''(h)$, etc., so that the equation (4.27) takes on the form

$$g(x + h) = g(h) + g'(h)x + g''(h)\frac{x^2}{2!} + g'''(h)\frac{x^3}{3!} + \cdots + g^{(n)}(h)\frac{x^n}{n!} + \cdots \tag{4.29}$$

which is called Taylor's form for Maclaurin's results.

[7] Colin Maclaurin (1698-1746) a Scottish mathematician.

Example 4-26. Some well known Maclaurin series expansions are the following.

$$
\begin{aligned}
\sin x =& x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots & |x| < \infty \\
\cos x =& 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots & |x| < \infty \\
\sinh x =& x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \cdots & |x| < \infty \\
\cosh x =& 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \cdots & |x| < \infty \\
e^x = \text{Exp}(x) =& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots & |x| < \infty \\
a^x = e^{x \ln a} =& 1 + x\, \ln a + \frac{(x\, \ln a)^2}{2!} + \frac{(x\, \ln a)^3}{3!} + \cdots & -\infty < x < \infty \\
\ln(1+x) =& x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots & -1 \le x < 1 \\
\frac{1}{1-x} =& 1 + x + x^2 + x^3 + x^4 + \cdots & |x| < 1 \\
(1+x)^\beta =& 1 + \beta x + \beta(\beta-1)\frac{x^2}{2!} + \beta(\beta-1)(\beta-2)\frac{x^3}{3!} + \cdots & |x| < 1 \\
\sin^{-1} x =& x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots & -1 < x < 1 \\
\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x =& \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \right) & -1 < x < 1
\end{aligned}
$$

■

Note that many functions do not have a Maclaurin series expansion. This occurs whenever the function $f(x)$ or one of its derivatives cannot be evaluated at $x = 0$. For example, the functions $\ln x$, $x^{3/2}$, $\cot x$ are examples of functions which do not have a Maclaurin series expansion.

Example 4-27. The following are series expansions of selected special functions occurring in advanced mathematics, engineering, mathematical physics and the sciences.

$J_\nu(x)$ The Bessel function of the first kind of order ν

$$
J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}\, k!\, \Gamma(\nu + k + 1)}
$$

where $\Gamma(x)$ is the gamma function.

$Y_\nu(x)$ The Bessel function of the second kind of order ν

$$
Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}
$$

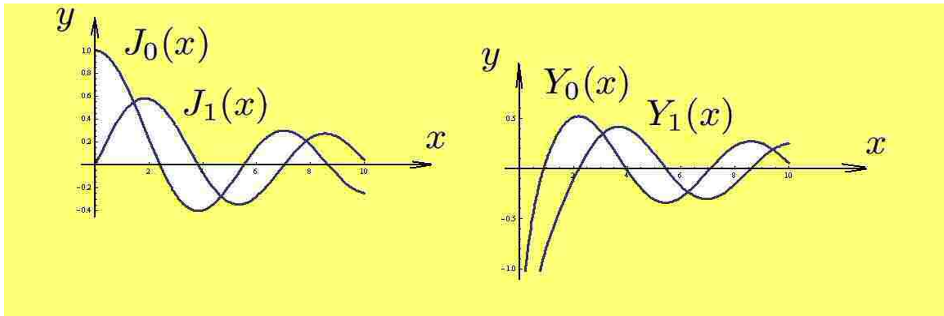
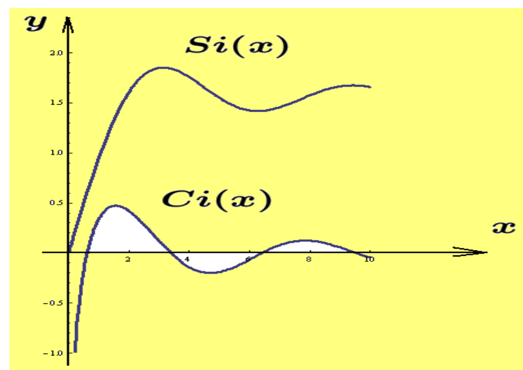


Figure 4-6. The Bessel functions $J_0(x),\ Y_0(x),\ J_1(x),\ Y_1(x)$

The sine integral $Si(x) = \int_0^x \frac{\sin t}{t}\,dt$

$$Si(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$



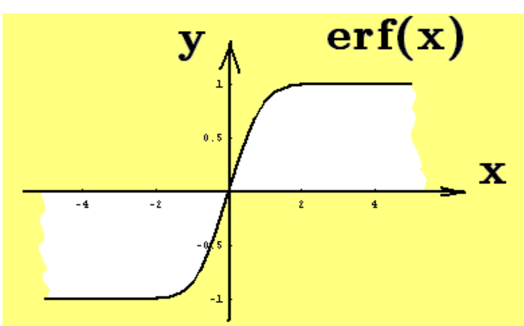
The cosine integral

$Ci(x) = \gamma + \ln x + \int_0^x \frac{\cos t - 1}{t}\,dt, \qquad |\arg x| < \pi$

$$Ci(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n(2n)!}$$

where $\gamma = \lim_{n\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right] = 0.57721\ldots$ is called the Euler-Mascheroni constant.

The error function $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$



$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)}$$

The hypergeometric function $F(\alpha, \beta\,;\gamma\,;x)$

$$F(\alpha, \beta\,;\gamma\,;x) = \sum_{k=0}^{\infty} \frac{\alpha^{\overline{k}}\,\beta^{\overline{k}}}{\gamma^{\overline{k}}}\frac{x^k}{k!}$$

where $a^{\overline{0}} = 1$ and $a^{\overline{k}} = a(a+1)(a+2)\cdots(a+k-1)$ for k a nonnegative integer, is called the **factorial rising function**[8] or **upper factorial**. In expanded form the hypergeometric function is written

$$\begin{aligned} F(\alpha, \beta\,;\gamma\,;x) =& 1 + \frac{\alpha\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots \\ &+ \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)}\frac{x^n}{n!} + \cdots \end{aligned}$$

The hypergeometric function is related to many other functions. Some example relationships are the following.

$$\begin{aligned} F(1,1\,;2\,;x) =& -\frac{\ln(1-x)}{x} & F(\alpha,\beta\,;\beta\,;x) =& \frac{1}{(1-x)^\alpha} \\ F(\frac{1}{2},1\,;\frac{3}{2}\,;x) =& \frac{\tan^{-1}x}{x} & \lim_{\alpha\to\infty} F(\alpha,\beta\,;\beta\,;\frac{x}{\alpha}) =& e^x \\ F(-\alpha,\alpha\,;\frac{1}{2}\,;\sin^2 x) =& \cos(2\alpha x) & F(\frac{1}{2},-1\,;\frac{1}{2}\,;x^2) =& 1-x^2 \end{aligned}$$

■

Taylor and Maclaurin Series

Brook Taylor (1685-1731) an English mathematician and Colin Maclaurin (1698-1746) a Scottish mathematician both studied the representation of functions $f(x)$ in terms of a series expansion in powers of the independent variable x. If $f(x)$ is a real or complex function which is infinitely differentiable in the neighborhood of a fixed point x_0, then $f(x)$ can be represented in the form

$$f(x) = P_n(x, x_0) + R_n(x, x_0) \tag{4.30}$$

where $P_n(x, x_0)$ is called a nth degree Taylor polynomial centered at x_0 and $R_n(x, x_0)$ is called a remainder term. The Taylor polynomial of degree n has the form

$$P_n(x, x_0) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \tag{4.31}$$

[8] There is a factorial falling function or lower factorial defined by $a^{\underline{k}} = a(a-1)(a-2)\cdots(a-(k-1))$ for k a nonnegative integer. There are alternative notations to represent the factorial rising and falling functions. Some texts use the notation $x^{(n)}$ for the rising factorial function and the notation $(x)_n$ for the falling factorial function. In terms of gamma functions one can write $x^{(n)} = x^{\overline{n}} = \dfrac{\Gamma(x+n)}{\Gamma(x)}$ and $(x)_n = x^{\underline{n}} = \dfrac{\Gamma(x+1)}{\Gamma(x-n+1)}$.

and the remainder term is represented

$$R_n(x, x_0) = \frac{1}{n!} \int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)\, dt \tag{4.32}$$

If the point $x_0 = 0$, then the series is called a Maclaurin series.

One method of deriving the above series expansion is to assume that a given function $f(x)$ has derivatives of all orders and these derivatives all have a finite value at some point x_0. Also assume the function $f(x)$ can be represented by a convergent infinite series of the form

$$f(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_n(x - x_0)^n + \cdots \tag{4.33}$$

where $c_0, c_1, \ldots$ are constants to be determined. Substituting $x = x_0$ into this equation gives $f(x_0) = c_0$. The assumed series representation can be differentiated on a term by term basis as many times as desired. The first n-derivatives are

$$\begin{aligned}
f'(x) =& c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + 4c_4(x - x_0)^3 + \cdots \\
f''(x) =& 2!c_2 + 3!c_3(x - x_0) + 4 \cdot 3c_4(x - x_0)^2 + \cdots \\
f'''(x) =& 3!c_3 + 4!c_4(x - x_0) + \cdots \\
\vdots \quad & \quad \vdots \\
f^{(n)}(x) =& n!c_n + (n+1)!c_{n+1}(x - x_0) + \cdots
\end{aligned}$$

Substituting the value $x = x_0$ into each of the above derivatives produces the results

$$c_1 = f'(x_0), \qquad c_2 = \frac{f''(x_0)}{2!}, \qquad c_3 = \frac{f'''(x_0)}{3!}, \quad \cdots \quad , \; c_n = \frac{f^{(n)}(x_0)}{n!}, \quad \cdots$$

This shows that $f(x)$ can be represented in the form

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots \tag{4.34}$$

or

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m \tag{4.35}$$

which is known as a Taylor series expansion of $f(x)$ about the point x_0. Note that when $x_0 = 0$ the Taylor series expansion reduces to the Maclaurin series expansion.

The validity of the infinite series expansions given by the Maclaurin and Taylor series is related to the convergence properties of the resulting infinite series. In general, the Taylor series given by equation (4.33) will satisfy one of the following conditions (i) The infinite series converges for all values of x (ii) the series converges only when $x = x_0$ or (iii) The infinite series converges for x satisfying $| x - x_0 | < R$ and diverges for $| x - x_0 | > R$, where $R > 0$ is a real number called the radius of convergence of the power series. Note that in the case where there is a radius of convergence R and x is an endpoint of the interval $(x_0 - R, x_0 + R)$, then the infinite series may or may not converge. Usually the ratio test, and the root test are used to determine the radius of convergence of the infinite series. The endpoints of the interval of convergence must be tested separately to determine convergence or divergence of the series.

Using the mean value theorem for integrals the remainder term can be reduced to one of the forms

$$R_n(x, x_0) = f^{(n+1)}(\xi_1)\frac{(x - x_0)^{n+1}}{(n+1)!}, \tag{4.36}$$

or

$$R_n(x, x_0) = \frac{f^{(n+1)}(\xi_2)(x - \xi_2)^n (x - x_0)}{n!} \tag{4.37}$$

where ξ_1, ξ_2 are constants satisfying $x_0 < \xi_1 < x$ and $x_0 < \xi_2 < x$. The equation (4.36) is known as the Lagrange form of the remainder term and equation (4.37) is known as the Cauchy form for the remainder term.

Another method to derived the above results involves integration by parts. Consider the integral

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\, dt \tag{4.38}$$

where x_0 and x are held constant. An integration of the right-hand side is performed using integration by parts with $U = f'(t),\ dU = f''(t)\, dt$ and $dV = dt$ and $V = t - x$. Here $-x$ is treated as a constant of integration so that

$$\begin{aligned}\int_{x_0}^{x} f'(t)\, dt =& f'(t)(t - x)\Big|_{x_0}^{x} - \int_{x_0}^{x} (t - x)\, f''(t)\, dt \\ \int_{x_0}^{x} f'(t)\, dt =& f'(x_0)(x - x_0) + \int_{x_0}^{x} (x - t)\, f''(t)\, dt\end{aligned} \tag{4.39}$$

Now evaluate the integral on the right-hand side of equation (4.39) using integration by parts to show

$$\int_{x_0}^{x} f'(t)\, dt = f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + \int_{x_0}^{x} \frac{(x - t)^2}{2!}\, f'''(t)\, dt \tag{4.40}$$

Continue to use integration by parts n-times to obtain

$$\int_{x_0}^{x} f'(t)\,dt = f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots + f^{(n)}(t)\frac{(x-x_0)^n}{n!} + R_n(x,x_0) \qquad (4.41)$$

where the remainder term is given by

$$R_n(x,x_0) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n\, f^{(n+1)}(t)\,dt \qquad (4.42)$$

Example 4-28. Some additional Series Expansions

$$\begin{aligned}
\tan x =& x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots + \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!} + \cdots \quad |x| < \frac{\pi}{2}\\
\cot x - \frac{1}{x} =& -\frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots - \frac{(-1)^{n-1}2^{2n}B_{2n}x^{2n-1}}{(2n)!} - \cdots \quad 0 < |x| < \pi\\
\sec x =& 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{(-1)^n E_{2n}x^{2n}}{(2n)!} + \cdots \quad |x| < \frac{\pi}{2}\\
\csc x - \frac{1}{x} =& \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15,120} + \cdots + \frac{(-1)^{n-1}2(2^{2n-1}-1)B_{2n}x^{2n-1}}{(2n)!} + \cdots \quad 0 < |x| < \pi\\
\tanh x =& x - \frac{x^3}{3} + \frac{2x^5}{15} + \cdots + \frac{2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!} + \cdots \quad |x| < \frac{\pi}{2}
\end{aligned}$$

Where B_n are the Bernoulli numbers and E_n are the Euler numbers. These numbers are defined[9] from the expansions

$$\begin{aligned}
\frac{x}{e^x - 1} =& 1 + \frac{B_1 x}{1!} + \frac{B_2 x^2}{2!} + \frac{B_4 x^4}{4!} + \frac{B_6 x^6}{6!} + \cdots + \frac{B_{2n}x^{2n}}{(2n)!} + \cdots\\
\frac{x}{e^x - 1} =& 1 - \frac{1}{2}x + \frac{1}{6}\frac{x^2}{2!} - \frac{1}{30}\frac{x^4}{4!} + \frac{1}{42}\frac{x^6}{6!} - \cdots\\
\frac{2e^x}{e^{2x}+1} =& E_0 + \frac{E_1 x}{1!} + \frac{E_2 x^2}{2!} + \frac{E_3 x^3}{3!} + \cdots\\
\frac{2e^x}{e^{2x}+1} =& 1 - \frac{x^2}{2!} + 5\frac{x^4}{4!} - 61\frac{x^6}{6!} + 1385\frac{x^8}{8!} - \cdots
\end{aligned}$$

and produce the numbers

$$\begin{aligned}
B_0 =& 1,\ B_1 = -1/2,\ B_2 = \frac{1}{6},\ B_4 = \frac{-1}{30},\ B_6 = \frac{1}{42}, \cdots, B_{2n+1} = 0 \text{ for } n > 1\\
E_0 =& 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61, \cdots, E_{2n+1} = 0 \text{ for } n = 0,1,2,\ldots
\end{aligned}$$

■

[9] There are alternative definitions of the Bernoulli and Euler numbers which differ by subscripting notation, signs and scale factors.

Taylor Series for Functions of Two Variables

Using the above results it is possible to derive a Taylor series expansion associated with a function of two variables $f = f(x,y)$. Assume the function $f(x,y)$ is defined in a region about a fixed point (x_0, y_0), where the points (x_0, y_0) and (x,y) can be connected by a straight line. Such regions are called **connected regions.** Further, let $f(x,y)$ possess nth-order partial derivatives which also exist in the region which surrounds the fixed point (x_0, y_0). The Taylor's series expansion of $f(x,y)$ about the point (x_0, y_0) is given by

$$\begin{aligned} f(x_0+h, y_0+k) =& f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}h + \frac{\partial f(x_0,y_0)}{\partial y}k \\ &+ \frac{1}{2!}\left[\frac{\partial^2 f(x_0,y_0)}{\partial x^2}h^2 + 2\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y}hk + \frac{\partial^2 f(x_0,y_0)}{\partial y^2}k^2\right] + \cdots \end{aligned} \tag{4.43}$$

where $h = x - x_0$ and $k = y - y_0$. This expansion can be represented in a simpler form by defining the **differential operator**

$$D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}, \qquad h \text{ and } k \text{ are constants.}$$

The Taylor series can then be represented in the form

$$f(x_0+h, y_0+k) = \sum_{j=0}^{n} \frac{1}{j!} D^j f(x,y) + R_{n+1}, \tag{4.44}$$

where all the derivatives are evaluated at the point (x_0, y_0). The remainder term can be expressed as

$$R_{n+1} = \frac{1}{(n+1)!} D^{(n+1)} f(x,y), \qquad \text{to be evaluated at } (x,y) = (\xi, \eta) \tag{4.45}$$

where the point (ξ, η), lies somewhere on the straight line connecting the points (x_0+h, y_0+k) and (x_0, y_0).

The equation (4.43) or (4.44) is derived by introducing a new independent variable t which is the parameter for the straight line defined by the equations

$$x = x_0 + ht, \qquad y = y_0 + kt, \quad \text{with} \quad \frac{dx}{dt} = h, \quad \text{and} \quad \frac{dy}{dt} = k$$

where h and k are constants and $0 \le t \le 1$. Consider the function of the single variable t defined by

$$F(t) = f(x,y) = f(x_0 + ht, y_0 + kt)$$

which is a composite function of the single variable t. The composite function can be expanded in a Maclaurin series about $t = 0$ to obtain

$$F(t) = F(0) + F'(0)t + F''(0)\frac{t^2}{2!} + \cdots + F^{(n)}(0)\frac{t^n}{n!} + F^{(n+1)}(\xi)\frac{t^{(n+1)}}{(n+1)!}, \quad 0 < \xi < t. \tag{4.46}$$

Evaluation of equation (4.46) at $t = 1$ gives $f(x_0+h, y_0+k)$.

The first n derivatives of the function $F(t)$ are calculated using chain rule differentiation. The first derivative is

$$\begin{aligned} F'(t) &= \frac{\partial f(x,y)}{\partial x}\frac{dx}{dt} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial f(x,y)}{\partial x}h + \frac{\partial f(x,y)}{\partial y}k. \end{aligned} \tag{4.47}$$

By differentiating this expression, the second derivative can be determined as

$$\begin{aligned} F''(t) = & \left[\frac{\partial^2 f(x,y)}{\partial x^2}h + \frac{\partial^2 f(x,y)}{\partial y\,\partial x}k\right]\frac{dx}{dt} \\ & + \left[\frac{\partial^2 f(x,y)}{\partial x\,\partial y}h + \frac{\partial^2 f(x,y)}{\partial y^2}k\right]\frac{dy}{dt} \end{aligned}$$

or

$$F''(t) = \frac{\partial^2 f(x,y)}{\partial x^2}h^2 + 2\frac{\partial^2 f(x,y)}{\partial x\,\partial y}hk + \frac{\partial^2 f(x,y)}{\partial y^2}k^2. \tag{4.48}$$

Continuing in this manner, higher derivatives of $F(t)$ can be calculated. For example, the third derivative is

$$\begin{aligned} F'''(t) = & \left[\frac{\partial^3 f}{\partial x^3}h^2 + 2\frac{\partial^3 f}{\partial x^2 \partial y}hk + \frac{\partial^3 f}{\partial x \partial y^2}k^2\right]\frac{dx}{dt} \\ & + \left[\frac{\partial^3 f}{\partial x^2 \partial y}h^2 + 2\frac{\partial^3 f}{\partial x \partial y^2}hk + \frac{\partial^3 f}{\partial y^3}k^2\right]\frac{dy}{dt} \end{aligned}$$

or

$$F'''(t) = \frac{\partial^3 f}{\partial x^3}h^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}h^2k + 3\frac{\partial^3 f}{\partial x \partial y^2}hk^2 + \frac{\partial^3 f}{\partial y^3}k^3, \tag{4.49}$$

Using the operator $D = h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}$ a pattern to these derivatives can be constructed

$$\begin{aligned} F'(t) &= Df(x,y) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y) \\ F''(t) &= D^2 f(x,y) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) \\ F'''(t) &= D^3 f(x,y) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x,y) \\ &\ \vdots \qquad\qquad\qquad \vdots \\ F^{(n)}(t) &= D^n f(x,y) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y). \end{aligned}$$

Here the operator $D^n = \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^n$ can be expanded just like the binomial expansion and

$$\begin{aligned} F^{(n)}(t) = D^n f(x,y) = \ & h^n \frac{\partial^n f}{\partial x^n} + \binom{n}{1} h^{n-1} k \frac{\partial^n f}{\partial x^{n-1}\partial y} + \binom{n}{2} h^{n-2} k^2 \frac{\partial^n f}{\partial x^{n-2}\partial y^2} \\ & + \cdots + \binom{n}{n-1} h k^{n-1} \frac{\partial^n f}{\partial x \partial y^{n-1}} + k^n \frac{\partial^n f}{\partial y^n}, \end{aligned} \tag{4.50}$$

where $\dbinom{n}{m} = \dfrac{n!}{m!(n-m)!}$ are **the binomial coefficients.**

In order to calculate the Maclaurin series about $t = 0$, each of the derivatives must be evaluated at the value $t = 0$ which corresponds to the point (x_0, y_0) on the line. Substituting these derivatives into the Maclaurin series produces the result given by equation (4.43), where all derivatives are understood to be evaluated at the point (x_0, y_0).

In order for the Taylor series to exist, all the partial derivatives of f through the nth order must exist at the point (x_0, y_0). In this case, write $f \in C^n$ over the connected region containing the points (x_0, y_0) and (x, y). The notation $f \in C^n$ is read, "f belongs to the class of functions which have all partial derivatives through the nth order, and further, these partial derivatives are continuous functions in the connected region surrounding the point (x_0, y_0). "

In a similar fashion it is possible to derive the Taylor series expansion of a function $f = f(x, y, z)$ of three variables. Assume the Taylor series expansion is to be about the point (x_0, y_0, z_0), then show the Taylor series expansion has the form

$$f(x_0 + h, y_0 + k, z_0 + \ell) = \sum_{j=0}^{n} \frac{1}{j!} D^j f(x, y, z) + R_{n+1} \tag{4.51}$$

where

$$Df = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + \ell\frac{\partial}{\partial x}\right) f = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \ell\frac{\partial f}{\partial x}\right) \tag{4.52}$$

is a differential operator and $h = x - x_0$, $k = y - y_0$ and $\ell = z - z_0$. After expanding the derivative operator $D^j f$ for $j = 0, 1, 2, \ldots$, each of the derivatives are to be evaluated at the point (x_0, y_0, z_0). The term R_{n+1} is the remainder term given by

$$R_{n+1} = \frac{1}{(n+1)!} D^{(n+1)} f(x, y, z) \ \Big|_{(x,y,z)=(\xi,\eta,\zeta)} \tag{4.53}$$

where the point (ξ, η, ζ) is some unknown point on the line connecting the points (x_0, y_0, z_0) and $(x_0 + h, y_0 + k, z_0 + \ell)$.

Functions of n-variables $f = f(x_1, x_2, \ldots, x_n)$ have their Taylor series expansions derived in a manner similar to the above by employing a differential operator of the form

$$D = \left(h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} + \cdots + h_h \frac{\partial}{\partial x_n} \right) \tag{4.54}$$

where $h_1 = x_1 - x_{10}, h_2 = x_2 - x_{20}, \ldots, h_n = x_n - x_{n0}$.

Example 4-29. **Hypergeometric series**

The hypergeometric series defines a power series in x in terms of three parameters a, b, c as

$$F(a, b\,; c\,; x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots + \frac{a^{\overline{n}}\, b^{\overline{n}}}{c^{\overline{n}}}\frac{x^n}{n!} + \cdots$$

where

$$a^{\overline{n}} = a(a+1)(a+2)(a+3)\cdots(a+n-1) = \prod_{i=0}^{n-1}(a+i)$$

is called the rising factorial function and the symbol $\prod$ is used to denote a product of terms as the index i ranges from 0 to $n-1$. Apply the ratio test and examine the ratio of successive terms and show

$$\frac{u_{n+1}}{u_n} = \frac{\dfrac{a^{\overline{n+1}}\, b^{\overline{n+1}}}{c^{\overline{n+1}}}\dfrac{x^{n+1}}{(n+1)!}}{\dfrac{a^{\overline{n}}\, b^{\overline{n}}}{c^{\overline{n}}}\dfrac{x^n}{n!}} = \frac{(a+n)(b+n)}{(c+n)}\frac{x}{(n+1)}$$

Divide the numerator and denominator by n^2 and show

$$\frac{u_{n+1}}{u_n} = \frac{\left(1 + \frac{a}{n}\right)\left(1 + \frac{b}{n}\right)}{\left(1 + \frac{c}{n}\right)\left(1 + \frac{1}{n}\right)}\, x$$

and in the limit as n increases without bound one obtains the limit x for the ratio of successive terms. Hence, in order for the series to converge it is required that $|x| < 1$.

■

Alternative Derivation of the Taylor Series

The above results for the representation of $f(x)$ as a power series can also be derived by considering the definite integral

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\, dt \tag{4.55}$$

where x_0 and x are held constant. An integration of the right-hand side is performed using integration by parts with $U = f'(t),\ dU = f''(t)\,dt$ and $dV = dt$ and $V = t - x$. Here $-x$ is treated as a constant of integration so that

$$\begin{aligned}\int_{x_0}^{x} f'(t)\,dt =& f'(t)(t-x)\Big|_{x_0}^{x} - \int_{x_0}^{x} (t-x)\,f''(t)\,dt\\ \int_{x_0}^{x} f'(t)\,dt =& f'(x_0)(x-x_0) + \int_{x_0}^{x} (x-t)\,f''(t)\,dt\end{aligned} \tag{4.56}$$

Now evaluate the integral on the right-hand side of equation (4.56) using integration by parts to show

$$\int_{x_0}^{x} f'(t)\,dt = f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \int_{x_0}^{x} \frac{(x-t)^2}{2!}\,f'''(t)\,dt \tag{4.57}$$

Continue to use integration by parts n-times to obtain

$$\int_{x_0}^{x} f'(t)\,dt = f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots + f^{(n)}(t)\frac{(x-x_0)^n}{n!} + R_n(x,x_0) \tag{4.58}$$

or

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots + f^{(n)}(t)\frac{(x-x_0)^n}{n!} + R_n(x,x_0) \tag{4.59}$$

where $R_n(x,x_0)$ is called the remainder term and is given by

$$R_n(x,x_0) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n\,f^{(n+1)}(t)\,dt \tag{4.60}$$

Remainder Term for Taylor Series

Use the generalized mean value theorem for integrals

$$\int_{x_0}^{x} F(t)G(t)\,dt = F(\xi)\int_{x_0}^{x} G(t)\,dt, \qquad x_0 < \xi < x \tag{4.61}$$

to evaluate the integral used in the representation of the remainder term as given by equation (4.60). Let $F(t) = f^{(n+1)}(t)$ and $G(t) = \frac{(x-t)^n}{n!}$ in equation (4.61) and show

$$R_n(x,x_0) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)\,dt = f^{(n+1)}(\xi_1)\int_{x_0}^{x} \frac{(x-t)^n}{n!}\,dt = f^{(n+1)}(\xi_1)\,\frac{(x-x_0)^{n+1}}{(n+1)!}$$

where $x_0 < \xi_1 < x$. This is the Lagrange form of the remainder term associated with a Taylor series expansion. Alternatively, substitute $F(t) = \frac{f^{(n+1)}(t)(x-t)^n}{n!}$ and $G(t) = 1$ into the equation (4.61) to obtain

$$\begin{aligned}R_n(x,x_0) =& \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)\,dt = \frac{f^{(n+1)}(\xi_2)(x-\xi_2)^n}{n!}\int_{x_0}^{x} 1\,dt\\ R_n(x,x_0) =& \frac{f^{(n+1)}(\xi_2)(x-\xi_2)^n}{n!}(x-x_0)\end{aligned} \tag{4.62}$$

where $x_0 < \xi_2 < x$. This is the Cauchy form for the remainder term associated with a Taylor series expansion.

Schömilch and Roche Remainder Term

Still another form for the remainder term associated with the Taylor series expansion is obtained from the following arguments. Let $f(x), f'(x), \ldots, f^{(n+1)}(x)$ all be defined and continuous on the interval $[x_0, x_0+h]$ and construct the function

$$F(x) = f(x) + \sum_{m=1}^{n} \frac{(x_0+h-x)^m}{m!} f^{(m)}(x) + (x_0+h-x)^{p+1} A \tag{4.63}$$

where A and p are nonzero constants. Select the constant A such that

$$\begin{aligned} F(x_0+h) =& f(x_0+h) \\ \text{and} \qquad F(x_0) =& f(x_0) + \sum_{m=1}^{n} \frac{h^m}{m!} f^{(m)}(x) + h^{p+1} A = f(x_0+h), \end{aligned} \tag{4.64}$$

then $F(x)$ satisfies all the conditions of Rolle's theorem so there must exist a point $x = \xi = x_0 + \theta h$, $0 < \theta < 1$, such that $F'(\xi) = 0$. Differentiate the equation (4.63) and show

$$F'(x) = f'(x) + \sum_{m=1}^{n} \left[\frac{(x_0+h-x)^m}{m!} f^{(m+1)}(x) - \frac{m(x_0+h-x)^{m-1}}{m!} f^{(m)}(x) \right] - (p+1)(x_0+h-x)^p A$$

which can be simplified as follows.

$$\begin{aligned} F'(x) =& \sum_{m=1}^{n} \frac{(x_0+h-x)^m}{m!} f^{(m+1)}(x) - \sum_{m=2}^{n} \frac{m(x_0+h-x)^{m-1}}{m!} f^{(m)}(x) - (p+1)(x_0+h-x)^p A \\ F'(x) =& \sum_{m=1}^{n} \frac{(x_0+h-x)^m}{m!} f^{(m+1)}(x) - \sum_{i=1}^{n-1} \frac{(x_0+h-x)^i}{i!} f^{(i+1)}(x) - (p+1)(x_0+h-x)^p A \\ F'(x) =& \frac{(x_0+h-x)^n}{n!} f^{(n+1)}(x) - (p+1)(x_0+h-x)^p A \end{aligned} \tag{4.65}$$

At $x = \xi = x_0 + \theta h$ it follows that

$$F'(\xi) = \frac{(h-\theta h)^n}{n!} f^{(n+1)}(\xi) - (p+1)(h-\theta h)^p A = 0$$

a condition which requires that

$$A = \frac{[h(1-\theta)]^{n-p}}{(p+1)\, n!} f^{(n+1)}(\xi) \tag{4.66}$$

Substituting A from equation (4.66) into the equation (4.64) produces the result

$$f(x_0+h) = f(x_0) + \sum_{m=1}^{n} \frac{h^m}{m!} f^{(m)}(x_0) + \frac{h^{p+1}[h(1-\theta)]^{n-p}}{(p+1)\, n!} f^{(n+1)}(\xi)$$

Let $x = x_0 + h$ and write the above equation in the form

$$f(x) = f(x_0) + \sum_{m=1}^{n} \frac{(x-x_0)^m}{m!} f^{(m)}(x_0) + R_n(x, x_0) \tag{4.67}$$

where $R_n(x, x_0)$ is the Schlömilch[10] and Roche[11] form of the remainder term given by

$$R_n(x, x_0) = \frac{(x-x_0)^{p+1}(x-\xi)^{n-p}}{(p+1)\, n!} f^{(n+1)}(\xi) \tag{4.68}$$

where $\xi = x_0 + \theta h$, for $0 < \theta < 1$ and p is a constant satisfying $0 \le p \le n$. Note that in the special case $p = 0$ there results the Cauchy form for the remainder and when $p = n$ there results the Lagrange form for the remainder.

No one can sum an infinite number of terms on the computer. In order to use a Taylor series expansion to represent a function for computational purposes, one must chop of the infinite series or truncated it after n-terms. Knowing and controlling the error associated with the part of the infinite series that is thrown away is very important in applications and use of Taylor series when summation with a computer is used. If the remainder term is known, then it is possible to work backwards by first specifying an error tolerance and then determining the number of terms n required to achieve this error tolerance for the values of x being used in the application of the Taylor series.

Example 4-30. **(Alternative derivation of L´Hôpital's rule)**

Assume the functions $f(x)$ and $g(x)$ have the Taylor series representations

$$\begin{aligned} f(x) =& f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots \\ g(x) =& g(x_0) + g'(x_0)(x-x_0) + \frac{g''(x_0)}{2!}(x-x_0)^2 + \cdots \end{aligned}$$

which are convergent series in some neighborhood of the point x_0. One can then express the limit $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ in the form

$$\lim_{x\to x_0} \frac{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots}{g(x_0) + g'(x_0)(x-x_0) + \frac{g''(x_0)}{2!}(x-x_0)^2 + \cdots} \tag{4.69}$$

The L´Hôpital's rule can be derived by considering the following cases.

---

[10] Oscar Xaver Schlömilch (1823-1901) a German mathematician.

[11] Edouard Albert Roche (1820-1883) a French mathematician.

Case 1 If $f(x_0)=0$ and $g(x_0)=0$ but $g'(x_0)\neq 0$, then equation (4.69) reduces to

$$\lim_{x\to x_0}\frac{f'(x_0)+\dfrac{f''(x_0)}{2!}(x-x_0)+\dfrac{f'''(x_0)}{3!}(x-x_0)^2+\cdots}{g'(x_0)+\dfrac{g''(x_0)}{2!}(x-x_0)+\dfrac{g'''(x_0)}{3!}(x-x_0)^2+\cdots}=\frac{f'(x_0)}{g'(x_0)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)} \tag{4.70}$$

and so L´Hôpital's rule takes on the form

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

Case 2 If $f(x_0)=0$, $f'(x_0)=0$, $g(x_0)=0$ and $g'(x_0)=0$, but $g''(x_0)\neq 0$, then equation (4.69) reduces to the form

$$\lim_{x\to x_0}\frac{\dfrac{f''(x_0)}{2!}+\dfrac{f'''(x_0)}{3!}(x-x_0)+\cdots}{\dfrac{g''(x_0)}{2!}+\dfrac{g'''(x_0)}{3!}(x-x_0)+\cdots}=\frac{f''(x_0)}{g''(x_0)}=\lim_{x\to x_0}\frac{f''(x)}{g''(x)} \tag{4.71}$$

and so in this case the L´Hôpital's rule takes on the form

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=\lim_{x\to x_0}\frac{f''(x)}{g''(x)}$$

The above examples illustrate that one can reapply L´Hôpital's rule whenever the ratio of derivatives gives an indeterminate form.

■

Indeterminate forms $0\cdot\infty,\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^\infty$

If the limit $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{0}{0}$ or $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{\infty}{\infty}$, then the limits are said to have indeterminate forms and are calculated using the L´Hôpital's rule

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

if the limit exists.

Other indeterminate forms are

$$\lim_{x\to x_0} f(x)g(x)=0\cdot\infty \qquad \text{or} \qquad \lim_{x\to x_0} f(x)g(x)=\infty\cdot 0$$

$$\lim_{x\to x_0}[f(x)-g(x)]=\infty-\infty$$

$$\lim_{x\to x_0} f(x)^{g(x)}=0^0$$

$$\lim_{x\to x_0} f(x)^{g(x)}=\infty^0$$

$$\lim_{x\to x_0} f(x)^{g(x)}=1^\infty$$

The general procedure used to investigate these other indeterminate forms is to use some algebraic or trigonometric transformation that reduces these other indeterminate forms to the basic forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that L´Hôpital's rule can then be applied. The following examples illustrate some of these techniques.

Example 4-31. Evaluate the limit $\lim_{x\to 0}[\csc x \cdot \ln(1+x)]$

Solution If $x=0$ is substituted into the functions given one obtains $\infty \cdot 0$ which is an indeterminate form. Products of functions can be written in alternative forms using algebra. For example,

$$f(x)\cdot g(x) = \frac{f(x)}{\frac{1}{g(x)}} \qquad \text{or} \qquad f(x)\cdot g(x) = \frac{g(x)}{\frac{1}{f(x)}}$$

If the given limit is expressed in the form $\lim_{x\to 0}\frac{\ln(1+x)}{\sin x}$, then one can use L´Hôpital's rule to investigate the limit. One finds

$$\lim_{x\to 0}\frac{\ln(1+x)}{\sin x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{\cos x} = 1$$

■

Example 4-32. Evaluate the limit $\lim_{x\to\pi/2}(\sec x - \tan x)$

Solution Investigating this limit one finds that it depends upon how x approaches $\pi/2$. One finds $\lim_{x\to\pi/2}(\sec x - \tan x) = \lim_{x\to\pi/2}\sec x - \lim_{x\to\pi/2}\tan x = \pm(\infty-\infty)$ which is an indeterminate form. Using appropriate trigonometric identities the given limit can be expressed in an alternative form where L´Hôpital's rule applies. One can write

$$\lim_{x\to\pi/2}(\sec x - \tan x) = \lim_{x\to\pi/2}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to\pi/2}\frac{1-\sin x}{\cos x} = \lim_{x\to\pi/2}\frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0$$

■

Example 4-33. Evaluate the limit $\lim_{x\to 0}|x|^x$

Solution This limit gives the indeterminate form 0^0. One can use the identity $|x| = e^{\ln|x|}$ and write

$$\lim_{x\to 0}|x|^x = \lim_{x\to 0}e^{x\cdot\ln|x|} = e^{\lim_{x\to 0}x\cdot\ln|x|}$$

Here $\lim_{x\to 0}x\cdot\ln|x|$ gives the indeterminate form $0\cdot(-\infty)$. Writing

$$\lim_{x\to 0}x\cdot\ln|x| = \lim_{x\to 0}\frac{\ln|x|}{\frac{1}{x}}$$

one can apply L´Hôpital's rule and show

$$\lim_{x\to 0}x\cdot\ln|x| = \lim_{x\to 0}\frac{\ln|x|}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0}(-x) = 0$$

Consequently, one can write $\lim_{x\to 0}|x|^x = \lim_{x\to 0}e^{x\cdot\ln|x|} = e^0 = 1$

■

Example 4-34. Evaluate the limit $\lim_{x\to\infty}(x+9)^{1/x^2}$

Solution This limit gives the indeterminate form ∞^0. One can use the identity $(x+9)=e^{\ln(x+9)}$ and write

$$\lim_{x\to\infty}(x+9)^{1/x^2}=\lim_{x\to\infty}e^{\frac{1}{x^2}\ln(x+9)}=e^{\lim_{x\to\infty}\frac{\ln(x+9)}{x^2}}$$

Apply L´Hôpital's rule to this last limit and show

$$\lim_{x\to\infty}\frac{\ln(x+9)}{x^2}=\lim_{x\to\infty}\frac{\frac{1}{x+9}}{2x}=\lim_{x\to\infty}\frac{1}{2x(x+9)}=0$$

Therefore

$$\lim_{x\to\infty}(x+9)^{1/x^2}=e^{\lim_{x\to\infty}\frac{\ln(x+9)}{x^2}}=e^0=1$$

■

Example 4-35. Evaluate the limit $\lim_{x\to 0}(2-3^x)^{1/x}$

Solution This limit gives the indeterminate form 1^∞. Write

$$\lim_{x\to 0}(2-3^x)^{1/x}=\lim_{x\to 0}e^{\frac{1}{x}\ln(2-3^x)}=e^{\lim_{x\to 0}\frac{\ln(2-3^x)}{x}}$$

Recall that

$$\frac{d}{dx}3^x=\frac{d}{dx}e^{x\ln 3}=e^{x\ln 3}\cdot\ln 3=3^x\cdot\ln 3$$

and consequently when L´Hôpital's rule is applied to the above limit, one finds

$$\lim_{x\to 0}\frac{\ln(2-3^x)}{x}=\lim_{x\to 0}\frac{\frac{1}{2-3^x}\cdot(0-3^x\cdot\ln 3)}{1}=-\ln 3$$

Therefore,

$$\lim_{x\to 0}(2-3^x)^{1/x}=e^{\lim_{x\to 0}\frac{\ln(2-3^x)}{x}}=e^{-\ln 3}=3^{-1}=\frac{1}{3}$$

■

Modification of a Series

Let $u_n\ge 0$ for all n and let $\{v_n\}$ denote a bounded sequence satisfying $|v_n|<K$, where K is a constant. If the infinite series $\sum_{n=1}^{\infty}u_n$ is a convergent series, then the series $\sum_{n=1}^{\infty}u_nv_n$ will also be a convergent series.

This follows from an analysis of the Cauchy condition for convergence. Select an integer value N so large that for all integer values $n > m > N$ the Cauchy convergence condition satisfies

$$|U_n - U_m| = |u_{m+1} + u_{m+2} + \cdots + u_n| \le \sum_{i=m+1}^{n} |u_i| < \frac{\epsilon}{K}$$

then write $\left|\sum_{i=m+1}^{n} u_i v_i\right| \le \sum_{i=m+1}^{n} |u_i v_i|$ and since the terms v_i are bounded, it follows that $|u_i v_i| = u_i |v_i| \le u_i K$ so that the Cauchy condition for convergence becomes

$$\left|\sum_{i=1}^{n} u_i v_i - \sum_{i=1}^{m} u_i v_i\right| \le \sum_{i=m+1}^{n} |u_i v_i| \le K \sum_{i=m+1}^{n} |u_i| < K\left(\frac{\epsilon}{K}\right) = \epsilon$$

so that the infinite series $\sum_{i=1}^{\infty} u_i v_i$ is convergent.

Conditional Convergence

An infinite series $\sum_{n=1}^{\infty} u_n$ is called a conditionally convergent series or semi-convergent series, if the given series is convergent but the series of absolute values is not convergent.

As an example, consider the alternating series given by

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

This alternating series converges, however it is not absolutely convergent because the series of absolute values turns into the harmonic series which diverges. The series is therefore said to be conditionally convergent.

In dealing with an absolutely convergent series, the rearrangement of terms does not affect the sum of the series. However, in dealing with a conditionally convergent series, the value of the sum can be changed by using some special rearrangement of terms and some rearrangements of terms can even make the series diverge. Conditionally convergent series are very sensitive to any changes made to the summation process.

Algebraic Operations with Series

Examine the series operations of addition, subtraction and multiplying a series by a constant term together with the operation of multiplying two infinite series as these are operations that occur quite frequently when dealing with infinite series.

Addition and Subtraction

Two convergent series can be added or subtracted if one is careful to maintain parenthesis. That is, given two series $A = \sum_{n=0}^{\infty} a_n$ and $B = \sum_{n=0}^{\infty} b_n$ then these series can be added or subtracted on a term by term basis to obtain

$$
\begin{aligned}
S = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (a_n + b_n) =& (a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots \\
D = \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (a_n - b_n) =& (a_0 - b_0) + (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + \cdots
\end{aligned}
$$

The use of parentheses is important because the b_n terms may be negative and in such cases the removal of parenthesis is not allowed. That is, the addition or subtraction of two infinite series is on a term by term basis with parenthesis being used to group terms. The partial sums are given by $A_n = \sum_{m=0}^{n} a_m$ and $B_n = \sum_{m=0}^{n} b_m$ so that the sum S and difference D can be expressed

$$
\begin{aligned}
\lim_{n\to\infty} S_n =& \lim_{n\to\infty} \sum_{m=0}^{n} (a_m + b_m) \qquad & \lim_{n\to\infty} D_n =& \lim_{n\to\infty} \sum_{m=0}^{n} (a_n - b_n) \\
S =& \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n & D =& \lim_{n\to\infty} A_n - \lim_{n\to\infty} B_n \\
S =& \quad A \ + \ B & D =& \quad A \ - \ B
\end{aligned}
$$

Multiplication by a Constant

A series $\sum_{n=0}^{\infty} a_n$ can be multiplied by a nonzero constant c to obtain the series

$$
c \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (c\, a_n) = c\, a_0 + c\, a_1 + c\, a_2 + c\, a_3 + \cdots
$$

The multiplication of each term by a nonzero constant does not affect the convergence or divergence of the series.

Cauchy Product

If the infinite series $\displaystyle\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$ and the infinite series $\displaystyle\sum_{n=0}^{\infty} b_n = b_0 + b_1 + b_2 + b_3 + \cdots$ are multiplied, then the product series can be written

$$
\begin{aligned}
&a_0b_0 + a_0b_1 + a_0b_2 + a_0b_3 + \cdots + a_0b_n + \cdots \\
&+ a_1b_0 + a_1b_1 + a_1b_2 + a_1b_3 + \cdots + a_1b_n + \cdots \\
&+ a_2b_0 + a_2b_1 + a_2b_2 + a_2b_3 + \cdots + a_2b_n + \cdots \\
&+ a_3b_0 + a_3b_1 + a_3b_2 + a_3b_3 + \cdots + a_3b_n + \cdots \\
&+ \cdots \\
&+ a_nb_0 + a_nb_1 + a_nb_2 + a_nb_3 + \cdots + a_nb_n + \cdots \\
&+ \cdots
\end{aligned}
\tag{4.72}
$$

and this result can be grouped into a summation in any convenient way. The Cauchy method of grouping is to use a summation of terms on a diagonal starting in the upper left corner of the sum given by (4.72) and drawing diagonal lines from column n to row n and then summing the results. This gives the elements $\{c_n\}$ from the double array defined as the diagonal elements

$$
\begin{aligned}
c_0 =&a_0b_0 \\
c_1 =&a_1b_0 + a_0b_1 \\
c_2 =&a_2b_0 + a_1b_1 + a_0b_2 \\
c_3 =&a_3b_0 + a_2b_1 + a_1b_2 + a_0b_3 \\
\vdots \quad & \vdots \\
c_n =&a_nb_0 + a_{n-1}b_1 + a_{n-2}b_2 + \cdots + a_1b_{n-1} + a_0b_n = \sum_{i=0}^{n} a_ib_{n-i}
\end{aligned}
$$

and consequently the product series, called **the Cauchy product**, can be represented

$$
\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_ib_{n-i}\right) \tag{4.73}
$$

The Cauchy product is often used in multiplying power series because the result is also a power series. The Cauchy product is just one of several different definitions which can be used for the representation of a multiplication of two infinite series.

If the summation of the series begins with the index 1, instead of 0, then

$$
\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} a_ib_{n+1-i}\right) \tag{4.74}
$$

Bernoulli Numbers

The sequence of numbers $\{B_n\}$ defined by the coefficients of the Maclaurin series expansion

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \qquad |x|<2\pi$$

are called **Bernoulli**[12] **numbers**. Multiply by e^x-1 and use the Maclaurin series for the exponential function along with the Cauchy product to show

$$x=\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}\right]-\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}B_j\frac{x^n}{j!(n-j)!}-\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}$$

Now equate coefficients of like powers of x above and show $B_0=1$ and

$$B_n=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}B_j=\sum_{j=0}^{n}\binom{n}{j}B_j \quad \text{for } n\ge 2 \tag{4.75}$$

Verify that $B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_3=0, B_4=-\frac{1}{30},\dots$ Alternatively, expand the function $\dfrac{x}{e^x-1}$ in a Maclaurin series about $x=0$ and show

$$\frac{x}{e^x-1}=1-\frac{1}{2}x+\frac{1}{6}\frac{x^2}{2!}-\frac{1}{30}\frac{x^4}{4!}+\frac{1}{42}\frac{x^6}{6!}-\frac{1}{30}\frac{x^8}{8!}+\frac{5}{66}\frac{x^{10}}{10!}-\frac{691}{2730}\frac{x^{12}}{12!}+\frac{7}{6}\frac{x^{14}}{14!}-\frac{3617}{510}\frac{x^{16}}{16!}+\frac{43867}{798}\frac{x^{18}}{18!}-\cdots$$

These expansions produce the Bernoulli numbers

n	0	1	2	4	6	8	10	12	14	16	$\cdots$
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\cdots$

Here the odd Bernoulli numbers are given by $B_{2n+1}=0$ for $n\ge 1$.

Euler Numbers

The sequence of numbers $\{E_n\}$ defined by the coefficients of the Maclaurin series expansion

$$f(x)=\frac{2e^x}{e^{2x}+1}=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}, \qquad |x|<\frac{\pi}{2}$$

are called **Euler**[13] **numbers**. The function $f(x)$ is an even function of x which implies that the odd Euler numbers satisfy $E_{2n+1}=0$ for all $n\ge 0$.

---

[12] Named after Jakob Bernoulli (1654-1705) a Swiss mathematician. Due to scaling, indexing and sign conventions, there are alternative definitions for the Bernoulli numbers, sometimes denoted $\mathfrak{B}_n$ (see table of integrals).

[13] Named after Leonhard Euler (1701-1783) a Swiss mathematician. Due to scaling, indexing and sign conventions, there are alternative definitions for the Euler numbers, sometimes denoted $\mathfrak{E}_n$ (see table of integrals).

Consequently,

$$f(x) = \frac{2e^x}{e^{2x}+1} = \frac{2}{e^x+e^{-x}} = \operatorname{sech} x = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sum_{m=0}^{\infty} E_{2m} \frac{x^{2m}}{(2m)!}$$

A multiplication by $e^{2x}+1$ and a Maclaurin series expansion of e^{2x} together with an application of the Cauchy product formula demonstrates that

$$2\sum_{n=0}^{\infty} \frac{x^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}\right] \left[\sum_{n=0}^{\infty} E_n \frac{x^n}{n!}\right] + \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{2^{n-k}\, n!}{k!(n-k)!} E_k \frac{x^n}{n!} + \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}$$

By equating like powers of x show $E_0 = 1$ and

$$E_n = 2 - \sum_{k=0}^{n} \frac{n! 2^{n-k}}{k!(n-k)!} E_k, \quad \text{for } n \ge 1$$

Alternatively, expand the function $f(x) = \frac{2e^x}{e^{2x}+1}$ in a Maclaurin series and show

$$f(x) = 1 - \frac{x^2}{2!} + 5\frac{x^4}{4!} - 61\frac{x^6}{6!} + 1385\frac{x^8}{8!} - 50521\frac{x^{10}}{10!} + 2702765\frac{x^{12}}{12!} - 199360981\frac{x^{14}}{14!} + \cdots$$

These expansions produce the Euler numbers.

n	0	2	4	6	8	10	12	14	$\cdots$
E_n	1	-1	5	-61	1385	-50521	2702765	-199360981	$\cdots$

Here the odd Euler numbers are zero and $E_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$.

Example 4-36. Additional Series Expansions

One can verify the following infinite series expansions.

$$\begin{aligned}
\tan x =& x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n x^{2n-1}}{(2n)!} + \cdots && |x| < \frac{\pi}{2} \\
\cot x - \frac{1}{x} =& -\frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} + \cdots + \frac{2^{2n} B_n x^{2n-1}}{(2n)!} + \cdots && 0 < |x| < \pi \\
\sec x =& 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots + \frac{E_n x^{2n}}{(2n)!} + \cdots && |x| < \frac{\pi}{2} \\
\csc x - \frac{1}{x} =& \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15,120} + \cdots + \frac{2(2^{2n-1} B_n x^{2n-1}}{(2n)!} + \cdots && 0 < |x| < \pi \\
\tanh x =& x - \frac{x^3}{3} + \frac{2x^5}{15} - \cdots (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)B_n x^{2n-1}}{(2n)!} + \cdots && |x| < \frac{\pi}{2} \\
\coth x - \frac{1}{x} =& \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \cdots + \frac{(-1)^{n-1} 2^{2n}(2^{2n}-1)B_n x^{2n-1}}{(2n)!} + \cdots && 0 < |x| < \pi \\
\operatorname{sech} x =& 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots + \frac{(-1)^n E_n x^{2n}}{(2n)!} + \cdots && |x| < \frac{\pi}{2} \\
\operatorname{csch} x - \frac{1}{x} =& -\frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15,120} + \cdots + \frac{(-1)^n 2(2^{2n-1}-1)B_n x^{2n-1}}{(2n)!} + \cdots && 0 < |x| < \pi
\end{aligned}$$

where B_n are the Bernoulli numbers and E_n are the Euler numbers.

■

Functions Defined by Series

If $\{f_n(x)\}$, $n = 0, 1, 2, \ldots$, denotes an infinite sequence of functions defined over an interval $[a, b]$, then other functions can be constructed from these functions.

Many functions $F(x)$ are **defined by an infinite series** having the form

$$F(x) = \sum_{j=0}^{\infty} c_j f_j(x) \tag{4.76}$$

where $c_0, c_1, c_2, \ldots$ are constants. To study the convergence or divergence of such series one should consider the sequence of finite sums $\{F_n(x)\}$ where

$$F_n(x) = \sum_{j=0}^{n} c_j f_j(x) = c_0 f_0(x) + c_1 f_1(x) + \cdots + c_n f_n(x) \tag{4.77}$$

for $n = 0, 1, 2, \ldots$. The sequence $\{F_n(x)\}$ is called the **sequence of partial sums** associated with the infinite series (4.76). The infinite series is said to converge if the sequence of partial sums converges. If the sequence of partial sums diverges, then the infinite series (4.76) is said to diverge.

The sequence is said to converge uniformly on an interval $a \le x \le b$ to a function $F(x)$, if for every $\epsilon > 0$ there exists an integer N such that

$$|F_n(x) - F(x)| < \epsilon, \qquad \text{for all } n > N \text{ and for all } x \in [a, b] \tag{4.78}$$

Example 4-37.

(a) From the sequence of functions $\{\sin nx\}$ one can define the Fourier sine series expansions

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{4.79}$$

where the b_n coefficients are constants.

(b) From the sequence of functions $\{\cos nx\}$ one can define the Fourier cosine series expansions

$$G(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \tag{4.80}$$

where the a_n coefficients are constants. The study of Fourier series expansions has many applications in advanced mathematics courses.

■

Generating Functions

Any function $g(x,t)$ which has a **power series expansion in the variable** t having the form

$$g(x,t)=\sum_{n=0}^{\infty}\phi_n(x)\,t^n=\phi_0(x)+\phi_1(x)\,t+\phi_2(x)\,t^2+\cdots+\phi_m(x)\,t^m+\cdots \tag{4.81}$$

is called **a generating function** which **defines the set of functions** $\{\phi_n(x)\}$ for the values $n=0,1,2,\ldots$. In the above definition **scaling of the terms** sometimes occurs. For example, the starting index $n=0$ can be changed to some other value and sometimes t^n is replaced by $\dfrac{t^n}{n!}$. Some examples of generating functions are the following.

(i) $g(x,t)=\dfrac{1}{1-xt}=\displaystyle\sum_{n=0}^{\infty}x^n\,t^n$

(ii) $g(x,t)=\dfrac{1-t\cos\theta}{1-2t\cos\theta+t^2}=\displaystyle\sum_{n=0}^{\infty}(\cos n\theta)\,t^n$

(iii) $g(x,t)=\dfrac{t\sin\theta}{1-2t\cos\theta+t^2}=\displaystyle\sum_{n=1}^{\infty}(\sin n\theta)\,t^n$

(iv) $g(x,t)=\dfrac{1}{1-te^x}=\displaystyle\sum_{n=0}^{\infty}(e^{nx})\,t^n$

(v) $g(x,t)=(1-2xt+t^2)^{-1/2}=\displaystyle\sum_{n=0}^{\infty}P_n(x)\,t^n$ Legendre polynomials $\{P_n(x)\}$

(vi) $g(x,t)=(1-t)^{-1}\exp\left(\dfrac{-xt}{1-t}\right)=\displaystyle\sum_{n=0}^{\infty}L_n(x)\,t^n$ Laguerre polynomials $\{L_n(x)\}$

There are many other special functions which can be defined by special generating functions.

Functions Defined by Products

Given a sequence $\{f_n\}$ of numbers or functions, one can define the finite product $\displaystyle\prod_{i=1}^{n}f_i=f_1f_2f_3\cdots f_n$ and then the infinite product is written

$$\prod_{i=1}^{\infty}f_i=f_1f_2f_3\cdots \quad \text{where} \quad \prod_{i=1}^{\infty}f_i=\lim_{n\to\infty}\prod_{i=1}^{n}f_i=\lim_{n\to\infty}f_1f_2\cdots f_n$$

if this limit exists. Let S_n denote the finite product $S_n=\displaystyle\prod_{i=1}^{n}f_i$ and take the logarithm of both sides to obtain

$$\ln S_n=\ln\prod_{i=1}^{n}f_i=\sum_{i=1}^{n}\ln f_i \tag{4.82}$$

One can then say that the infinite product $\prod_{i=1}^{\infty} f_i$ is convergent or divergent depending upon whether the infinite sum $\sum_{i=1}^{\infty} \ln f_i$ is convergent or divergent.

Example 4-38. **Some examples of infinite products**

(a) The French mathematician and astronomer François Viéte (1540-1603) discovered the representation $\dfrac{2}{\pi} = \dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{2+\sqrt{2}}}{2} \cdot \dfrac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$

(b) The English mathematician John Wallis (1616-1703) discovered the representation $\dfrac{\pi}{2} = \dfrac{2}{1} \cdot \dfrac{2}{3} \cdot \dfrac{4}{3} \cdot \dfrac{4}{5} \cdot \dfrac{6}{5} \cdot \dfrac{6}{7} \cdot \dfrac{8}{7} \cdot \dfrac{8}{9} \cdots$

(c) The German mathematician Karl Theodor Wilhelm Weierstrass (1815-1897) represented the Gamma function as the infinite product

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right]$$

where $\gamma = 0.577215665\ldots$ is known as the **Euler-Mascheroni constant**.

(d) Euler represented the function $\sin\theta$ as the infinite product

$$\sin\theta = \theta \left(1 - \frac{\theta^2}{\pi^2}\right) \left(1 - \frac{\theta^2}{2^2\pi^2}\right) \left(1 - \frac{\theta^2}{3^2\pi^2}\right) \cdots$$

(e) One definition of the Riemann zeta function is $\zeta(z) = \sum_{n=1}^{\infty} \dfrac{1}{n^z}$. Another form is $\zeta(z) = \prod_{n=1}^{\infty} \dfrac{1}{1 - p_n^{-z}}$ where $\{p_n\}$ denotes the sequence of prime numbers. The Riemann zeta function has many uses in number theory.

■

Continued Fractions

Continued fractions occasionally arise in the representation of various kinds of mathematically quantities. A continued fraction has the form

$$f = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cfrac{b_5}{a_5 + \cdots}}}}} \tag{4.83}$$

where the coefficients $a_0, a_1, \ldots, b_1, b_2, \ldots$ can be real or complex quantities. They can be constants or functions of x.

In general, when using the continued fraction representation[14] given by equation (4.83) the coefficients a_0, $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, 3, \ldots$ can be constants or functions of x and these coefficients can be finite in number or infinite in number. The pattern of numerator over denominator can go on forever or the ratios can terminate after a finite number of terms. A finite continued fraction has the form

$$f_n = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cdots + \cfrac{b_n}{a_n}}}}} \tag{4.84}$$

which terminates with the ratio $\dfrac{b_n}{a_n}$.

Terminology

(i) The numbers $b_1, b_2, b_3, \ldots$ are called the **partial numerators.**

(ii) The numbers $a_1, a_2, a_3, \ldots$ are called the **partial denominators.**

(iii) If the partial numerators b_i, for $i = 1, 2, 3, \ldots$ are all equal to 1 and all the a_i coefficients have integer values, then the continued fraction is called a **simple or regular continued fraction.** A simple continued fraction is sometimes represented using the shorthand **list notation** $f = [a_0 \,; a_1, a_2, a_3, \ldots]$ where the a_i, $i = 0, 1, 2, \ldots$ are called the **quotients of the regular continued fraction**.

(iv) The continued fraction is called **generalized** if the terms a_i and b_i for $i = 1, 2, 3, \ldots$ do not have any restrictions as to their form.

(v) The ratio of terms notation as illustrated by the equations (4.83) and (4.84) is awkward and takes up too much space in typesetting and is often abbreviated to the shorthand Pringsheim[15] notation

$$f_n = a_0 + \frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \cdots + \frac{b_n|}{|a_n} \tag{4.85}$$

for a finite continued fraction terminating with the $\frac{b_n}{a_n}$ term and in the form

$$f = a_0 + \frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \cdots + \frac{b_n|}{|a_n} + \cdots \tag{4.86}$$

[14] Take note that the starting index is zero. Some notations use a different starting index which can lead to confusion at times.

[15] Alfred Israel Pringsheim (1850-1941) a German mathematician.

for an infinite continued fraction. Historically, the shorthand notation originally used for representing an infinite continued fraction was of the form

$$f = a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{b_3}{a_3+}\cdots \tag{4.87}$$

where the three dots indicates that the ratios continue on forever.

(vi) If the continued fraction is truncated after the nth term, the quantity f_n is called the **nth convergent.**

(vii) The continued fraction is called convergent if the sequence of partial convergents $\{f_n\}$ converges, otherwise it is called a divergent continued fraction.

Evaluation of Continued Fractions

Consider a regular continued fraction which has been truncated after the nth ratio

$$f_n = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots + \frac{1|}{|a_{n-1}} + \frac{1|}{|a_n} \tag{4.88}$$

To evaluate this continued fraction start at the bottom and **calculate backwards** through the continued fraction. For example calculate the sequence of rational numbers

$$r_1 = a_{n-1} + \frac{1}{a_n},\ r_2 = a_{n-2} + \frac{1}{r_1},\ \cdots, r_{n-2} = a_2 + \frac{1}{r_{n-3}},\ r_{n-1} = a_1 + \frac{1}{r_{n-2}},\ r_n = f_n = a_0 + \frac{1}{r_{n-1}}$$

For example, consider the continued fraction

$$f_5 = 1 + \frac{1|}{|2} + \frac{1|}{|3} + \frac{1|}{|4} + \frac{1|}{|5} + \frac{1|}{|6}$$

and start at the bottom and calculate the ratios

$$\begin{aligned} r_1 =& 5 + \frac{1}{6} = \frac{31}{6}, \quad r_2 = 4 + \frac{6}{31} = \frac{130}{31}, \quad r_3 = 3 + \frac{31}{130} = \frac{421}{130}, \\ r_4 =& 2 + \frac{130}{421} = \frac{972}{421}, \quad r_5 = 1 + \frac{421}{972} = \frac{1393}{972} \end{aligned}$$

Continued fractions have a long history of being used to approximate numbers and functions. In 1655 John Wallis discovered an iterative scheme for calculating the partial convergents of a continued fraction **in the forward direction.** His iterative scheme can be written as follows. Define

$$A_{-1} = 1, \quad A_0 = a_0, \quad B_{-1} = 0, \quad B_0 = 1 \tag{4.89}$$

and for $j = 1, 2, 3, 4, \ldots$ define the **recursion relations**

$$A_j = a_j A_{j-1} + b_j A_{j-2}, \qquad B_j = a_j B_{j-1} + b_j B_{j-2} \tag{4.90}$$

or the matrix equivalent

$$\begin{bmatrix} A_j \\ B_j \end{bmatrix} = \begin{bmatrix} A_{j-1} & A_{j-2} \\ B_{j-1} & B_{j-2} \end{bmatrix} \begin{bmatrix} a_j \\ b_j \end{bmatrix} \tag{4.91}$$

then the ratio $f_n = \dfrac{A_n}{B_n}$ is the nth partial convergent and represents the continued fraction after having been truncated after the $\dfrac{b_n}{a_n}$ term. A proof of the above assertion is a proof by mathematical induction. For $j = 1$ one obtains

$$\begin{aligned} A_1 =& a_1 A_0 + b_1 A_{-1} = a_1 a_0 + b_1 \\ B_1 =& a_1 B_0 + b_1 B_{-1} = a_1 \end{aligned}$$

so that the ratio f_1 is given by

$$f_1 = \frac{A_1}{B_1} = \frac{a_1 a_0 + b_1}{a_1} = a_0 + \frac{b_1}{a_1}$$

Similarly, for $j = 2$ one finds

$$\begin{aligned} A_2 =& a_2 A_1 + B_2 A_0 = a_2(a_1 a_0 + b_1) + b_2 a_0 \\ B_2 =& a_2 B_1 + b_2 B_0 = a_2 a_1 + b_2 \end{aligned}$$

so that the second partial convergent is written

$$f_2 = \frac{A_2}{B_2} = \frac{a_0 a_1 a_2 + a_0 b_2 + a_2 b_1}{a_1 a_2 + b_2} = a_0 + \frac{a_2 b_1}{a_1 a_2 + b_2} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2}}$$

Hence, the recursion relations hold for $j = 1$ and $j = 2$. Assume the recursion relations holds for $j = n$ such that

$$f_n = \frac{A_n}{B_n} = \frac{a_n A_{n-1} + b_n A_{n-2}}{a_n B_{n-1} + b_n B_{n-2}} \tag{4.92}$$

Observe that the partial convergent for f_{n+1} is obtained from the partial convergent for f_n by replacing a_n by $a_n + \dfrac{b_{n+1}}{a_{n+1}}$. Making this substitution in equation (4.92) one obtains

$$\begin{aligned} \frac{\left(a_n + \frac{b_{n+1}}{a_{n+1}}\right) A_{n-1} + b_n A_{n-2}}{\left(a_n + \frac{b_{n+1}}{a_{n+1}}\right) B_{n-1} + b_n B_{n-2}} =& \frac{a_n A_{n-1} + b_n A_{n-2} + \frac{b_{n+1}}{a_{n+1}} A_{n-1}}{a_n B_{n-1} + b_n B_{n-2} + \frac{b_{n+1}}{a_{n+1}} B_{n-1}} \\ =& \frac{A_n + \frac{b_{n+1}}{a_{n+1}} A_{n-1}}{B_n + \frac{b_{n+1}}{a_{n+1}} B_{n-1}} = \frac{a_{n+1} A_n + b_{n+1} A_{n-1}}{a_{n+1} B_n + b_{n+1} B_{n-1}} = f_{n+1} \end{aligned}$$

and so the truth of the nth proposition implies the truth of the $(n+1)$st proposition.

Convergent Continued Fraction

Examine the sequence of partial convergents $f_n = \dfrac{A_n}{B_n}$ associated with a given continued fraction. If the limit $\lim_{n\to\infty} f_n = \lim_{n\to\infty} \dfrac{A_n}{B_n} = f$ exists, then the continued fraction is called convergent. Otherwise, it is called a divergent continued fraction.

Regular Continued Fractions

Regular continued fractions of the form

$$f = a_0 + \frac{1 |}{|a_1} + \frac{1 |}{|a_2} + \cdots + \frac{1 |}{|a_n} + \cdots \tag{4.93}$$

are the easiest to work with and are sometimes represented using the list notation

$$f = [a_0\,; a_1, a_2, a_3, \ldots, a_n, \ldots] \tag{4.94}$$

Example 4-39. (Continued fraction)

The representation of a number x as a regular continued fraction of the form of equation (4.93) is accomplished using the following algorithm to calculate the partial denominators a_i for $i = 1, 2, 3, \ldots, n$.

$$\begin{aligned}
a_0 =&[x], \qquad x_1 = \frac{1}{x - a_0}\\
a_1 =&[x_1], \qquad x_2 = \frac{1}{x_1 - a_1}\\
\vdots\ &\ \vdots\\
a_n =&[x_n], \qquad x_{n+1} = \frac{1}{x_n - a_n}
\end{aligned} \tag{4.95}$$

where $[x]$ is the greatest integer in x function. For example, to represent the number $x = \pi = 3.14159265358979323 85\ldots$ as a continued fraction one finds

$$\begin{aligned}
a_0 =&[x] = 3, \qquad x_1 = \frac{1}{\pi - 3} = 7.0625133059310457698\ldots\\
a_1 =&[x_1] = 7, \qquad x_2 = \frac{1}{x_1 - 7} = 15.9965944066857199\ldots\\
a_2 =&[x_2] = 15, \quad x_3 = \frac{1}{x_2 - a_2} = 1.0034172310133726\ldots\\
a_3 =&[x_3] = 1, \qquad x_4 = \frac{1}{x_3 - a_3} = 292.63459101440\ldots
\end{aligned}$$

and so one representation of π as a continued fraction has the list form given by $f = \pi = [3\,; 7, 15, 1, 292, \ldots]$ which gives the following rational number approximations for π.

$$f_1 = 3, \quad f_2 = \frac{22}{7}, \quad f_3 = \frac{333}{106}, \quad f_4 = \frac{355}{113}, \quad f_5 = \frac{103993}{33102}, \quad \cdots$$

Continue the above algorithm and show

$$f = \pi = [3\,;7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \ldots]$$

A generalized continued fraction expansion for π can be obtained from the $\arctan x$ function evaluated at $x = 1$ to obtain the representation

$$\frac{\pi}{4} = \frac{1|}{|1} + \frac{1|}{|3} + \frac{4|}{|5} + \frac{9|}{|7} + \frac{16|}{|9} + \frac{25|}{|11} + \frac{36|}{|13} + \cdots$$

where all the partial numerators after the first term are squares and the partial denominators are all odd numbers.

Other examples of mathematical constants represented by regular continued fractions are

$$\begin{aligned} e =&[2\,;1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, \ldots] \\ \gamma =&[0\,;1, 1, 2, 1, 2, 1, , 4, 3, 13, 5, 1, 1, 8, 1, 2, 4, 40, 1, \ldots] \end{aligned}$$

■

Euler's Theorem for Continued Fractions

Euler showed that the infinite series $U = u_1 + u_2 + u_3 + \cdots$ can be represented by the continued fraction

$$U = \frac{u_1|}{|1} - \frac{u_2|}{|u_1+u_2} - \frac{u_1u_3|}{|u_2+u_3} - \frac{u_2u_4|}{|u_3+u_4} - \frac{u_3u_5|}{|u_4+u_5} - \cdots - \frac{u_nu_{n+2}|}{|u_{n+1}+u_{n+2}} - \cdots \tag{4.96}$$

The convergence or divergence of this continued fraction is then closely related to the convergence or divergence of the infinite series which it represents.

Gauss Representation for the Hypergeometric Function

Carl Fredrich Gauss (1777-1855) a famous German mathematician showed that the hypergeometric function could be represented by the continued fraction

$$\frac{{}_2F_1(a+1, b\,; c+1\,; z)}{{}_2F_1(a, b\,; c\,; z)} = \frac{1|}{|1} + \frac{\frac{(a-c)b}{c(c+1)}z|}{|1} + \frac{\frac{(b-c-1)(a+1)}{(c+1)(c+2)}z|}{|1} + \frac{\frac{(a-c-1)(b+1)}{(c+2)(c+3)}z|}{|1} + \frac{\frac{(b-c-2)(a+2)}{(c+3)(c+4)}z|}{|1} + \cdots$$

Representation of Functions

There are many areas of mathematics where functions $f(x)$ are represented in the form of **an infinite generalized continued fraction** having the form

$$f(x) = a_0(x) + \cfrac{b_1(x)}{a_1(x) + \cfrac{b_2(x)}{a_2(x) + \cfrac{b_3(x)}{a_3(x) + \cdots + \cfrac{b_n(x)}{a_n(x) + r_{n+1}(x)}}}} \tag{4.97}$$

where $r_{n+1}(x) = \dfrac{b_{n+1}(x)}{a_{n+1}(x) + r_{n+2}(x)}$. This continued fraction is often expressed in the more compact form

$$f(x) = a_0(x) + \frac{b_1(x)|}{|a_1(x)} + \frac{b_2(x)|}{|a_2(x)} + \frac{b_3(x)|}{|a_3(x)} + \cdots + \frac{b_n(x)|}{|a_n(x)} + \cdots \tag{4.98}$$

in order to conserve space in typesetting. It is customary to select the functions $a_0(x), a_i(x)$ and $b_i(x)$, $i = 1, 2, 3, \ldots$ as simple functions such as some linear function of x or a constant, but this is not a requirement for representing a function. If one selects the functions $b_i(x)$ and $a_i(x)$, for $i = 1, 2, 3, \ldots$ as polynomials, then whenever the continued fraction is truncated, the resulting function $f_n(x)$ becomes a rational function of x. The converse of this statement is that if $f(x)$ is a rational function of x, then it is always possible to construct an equivalent continued fraction.

Observe also that the reciprocal function is given by

$$\frac{1}{f(x)} = \frac{1|}{|a_0(x)} + \frac{b_1(x)|}{|a_1(x)} + \frac{b_2(x)|}{|a_2(x)} + \frac{b_3(x)|}{|a_3(x)} + \cdots + \frac{b_n(x)|}{|a_n(x)} + \cdots \tag{4.99}$$

Example 4-40. (Arctangent function)

Assume the function $\arctan x$ has the continued fraction expansion

$$\arctan x = \cfrac{x}{a_1 + \cfrac{x^2}{a_2 + \cfrac{4x^2}{a_3 + \cfrac{9x^2}{a_4 + \cfrac{16x^2}{a_5 + \cfrac{25x^2}{a_6 + \cdots}}}}}} \tag{4.100}$$

where $a_1, a_2, \ldots$ are constants to be determined. Note

$$\frac{1}{a_1} = \lim_{x\to 0} \frac{\arctan x}{x} = 1$$

and the continued fraction (4.100) has the form

$$\begin{aligned} \arctan x =& \frac{x}{a_1 + r_1} \qquad \Longrightarrow \qquad r_1 = \frac{x}{\arctan x} - 1 = \frac{1}{3}x^2 - \frac{4}{45}x^4 + \cdots \\ r_1 =& \frac{x^2}{a_2 + r_2} \qquad \Longrightarrow \qquad r_2 = \frac{x^2}{r_1} - 3 = \frac{4}{5}x^2 - \frac{36}{175}x^4 + \cdots \end{aligned}$$

Continuing this iterative processes one obtains

$$\begin{aligned}
r_2 =& \frac{4x^2}{a_3 + r_3} \qquad \Longrightarrow \qquad r_3 = \frac{4x^2}{r_2} - 5 = \frac{9}{7}x^2 - \frac{16}{49}x^4 + \cdots \\
r_3 =& \frac{9x^2}{a_4 + r_4} \qquad \Longrightarrow \qquad r_4 = \frac{9x^2}{r_3} - 7 = \frac{16}{9}x^2 - \frac{400}{891}x^4 + \cdots \\
\vdots \quad & \quad \vdots \\
r_n =& \frac{(nx)^2}{a_{n+1} + r_{n+1}} \qquad \Longrightarrow \qquad r_{n+1} = \frac{(nx)^2}{r_n} - [2(n+1) - 1] = \frac{(n+1)^2}{2(n+1)+1}x^2 + \cdots
\end{aligned}$$

Observe that $\lim_{x\to 0} r_i(x) = 0$ so it is possible to calculate the coefficients $a_i,\ i = 1, 2, 3, \ldots$ and show

$$(a_1, a_2, a_3, a_4, a_5, \ldots, a_n, \ldots) = (1, 3, 5, 7, 9, \ldots, 2n+1, \ldots)$$

Evaluating the arctangent function at $x = 1$ gives the continued fraction

$$\frac{\pi}{4} = \frac{1|}{|1} + \frac{1|}{|3} + \frac{4|}{|5} + \frac{9|}{|7} + \frac{16|}{|\,9} + \frac{25|}{|11} + \frac{36|}{|13} + \cdots$$

given in the previous example 4-29.

■

Fourier Series

Consider two functions $f = f(x)$ and $g = g(x)$ which are continuous over the interval $a \le x \le b$. The **inner product of f and g with respect to a weight function** $r = r(x) > 0$ is written (f, g) or (g, f) and is defined

$$(f, g) = (g, f) = \int_a^b r(x) f(x) g(x)\, dx \tag{4.101}$$

The inner product of a function f with itself is called a **norm squared** and written $\| f \|^2$. The norm squared is defined

$$\| f \|^2 = (f, f) = \int_a^b r(x) f^2(x)\, dx \tag{4.102}$$

with norm given by $\| f \| = \sqrt{(f, f)}$. If the inner product of two functions f and g with respect to a weight function r is zero, then the functions f and g are said to be **orthogonal functions**.

Example 4-41. The set of functions $\{1, \sin x, \cos x\}$ are orthogonal functions over the interval $(0, \pi)$ with respect to the weight functions $r = r(x) = 1$. This is because the various combinations of inner products satisfy

$$(1, \sin x) = \int_0^\pi (1) \sin x \, dx = 0$$
$$(1, \cos x) = \int_0^\pi (1) \cos x \, dx = 0$$
$$(\sin x, \cos x) = \int_0^\pi \sin x \cos x \, dx = 0$$

The given functions have the norm squared values

$$(1, 1) = \| \, 1 \, \|^2 = \int_0^\pi (1)^2 \, dx = \pi$$
$$(\sin x, \sin x) = \| \sin x \|^2 = \int_0^\pi \sin^2 x \, dx = \frac{\pi}{2}$$
$$(\cos x, \cos x) = \| \cos x \|^2 = \int_0^\pi \cos^2 x \, dx = \frac{\pi}{2}$$

■

A set or sequence of functions $\{f_1(x), f_2(x), \ldots, f_n(x), \ldots, f_m(x), \ldots\}$ is said to be **orthogonal over an interval** (a, b) **with respect to a weight function** $r(x) > 0$ if for all integer values of n and m, with $n \neq m$, the inner product of f_m with f_n satisfies

$$(f_m, f_n) = (f_n, f_m) = \int_a^b r(x) f_m(x) f_n(x) \, dx = 0 \qquad m \neq n. \tag{4.103}$$

Here the inner product is zero for all combinations of m and n values with $m \neq n$. If the sequence of functions $\{f_n(x)\}, \; n = 0, 1, 2, \ldots$ is an **orthogonal sequence** one can write for integers m and n that the inner product satisfies the relations

$$(f_m, f_n) = (f_n, f_m) = \int_a^b r(x) f_n(x) f_m(x) \, dx = \begin{cases} 0, & m \neq n \\ \| \, f_n \, \|^2, & m = n \end{cases}$$

This result can be expressed in the more compact form

$$(f_m, f_n) = ||f_n||^2 \delta_{mn} = \begin{cases} 0 & m \neq n \\ ||f_n||^2 & m = n \end{cases} \tag{4.104}$$

where $||f_n||^2$ is the norm squared and δ_{mn} is the Kronecker delta defined to have a value of unity when m and n are equal and to have a value of zero when m and n are unequal.

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{4.105}$$

In the special case where $||f_n||^2 = 1$, for all values of n, the sequence of functions $\{f_n(x)\}$ is said to be **orthonormal** over the interval (a, b).

Example 4-42. If the set of functions $\{g_n(x)\}$ is an orthogonal set of functions over the interval (a,b) with respect to some given weight function $r(x) > 0$, then the set of functions $f_n(x) = \frac{g_n(x)}{\|g_n\|}$ is an orthonormal set. This result follows since

$$(f_n, f_m) = (f_m, f_n) = \int_a^b r(x) \frac{g_n(x)}{\| g_n \|} \cdot \frac{g_m(x)}{\| g_m \|}\, dx = \frac{1}{\| g_n \| \cdot \| g_m \|}(g_n, g_m)$$

since the norm squared values are constants. The above inner product representing (f_n, f_m) is zero if $m \neq n$ and has the value 1 if $m = n$.

■

Example 4-43. Show the set of functions $\{1,\ \sin\frac{n\pi x}{L},\ \cos\frac{n\pi x}{L}\}$ is an orthogonal set over the interval $-L \le x \le L$ with respect to the weight function $r = r(x) = 1$.

Solution Using the definition of an inner product one can show

$$\begin{aligned}
(1, \sin\frac{n\pi x}{L}) =& \int_{-L}^{L} \sin\frac{n\pi x}{L}\, dx = 0 \quad \text{for } n = 1, 2, 3, \ldots \\
(1, \cos\frac{n\pi x}{L}) =& \int_{-L}^{L} \cos\frac{n\pi x}{L}\, dx = 0 \quad \text{for } n = 1, 2, 3, \ldots \\
(\sin\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}) =& \int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = 0 \quad n \neq m \\
(\cos\frac{n\pi x}{L}, \cos\frac{m\pi x}{L}) =& \int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = 0 \quad n \neq m \\
(\cos\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}) =& \int_{-L}^{L} \cos\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = 0 \quad \text{for all } n, m \text{ values.}
\end{aligned} \tag{4.106}$$

The given set of functions have the norm-squared values

$$\begin{aligned}
(1,1) = ||1||^2 =& \int_{-L}^{L} dx = 2L \\
(\sin\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}) =|| \sin\frac{n\pi x}{L}||^2 =& \int_{-L}^{L} \sin^2\frac{n\pi x}{L}\, dx = L \quad \text{for all values of } n \\
(\cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}) =|| \cos\frac{n\pi x}{L}||^2 =& \int_{-L}^{L} \cos^2\frac{n\pi x}{L}\, dx = L \quad \text{for all values of } n
\end{aligned} \tag{4.107}$$

■

A Fourier[16] trigonometric series representation of a function $f(x)$ is expressing $f(x)$ in a series having the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{where} \quad -L \le x \le L \tag{4.108}$$

[16] Jean Baptgiste Joseph Fourier (1768-1830) A French mathematician.

with a_0 and a_n, b_n for $n = 1, 2, 3, \ldots$ are constants called the **Fourier coefficients.** If the Fourier coefficients are properly defined, then $f(x)$ is said be represented in the form of a **trigonometric Fourier series expansion over the interval** $(-L, L)$**.** The interval $(-L, L)$ is called the **full Fourier interval** associated with the series expansion.

One can make use of the orthogonality properties of the set $\{1, \sin\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}\}$ to obtain formulas for determining the Fourier coefficients of the Fourier trigonometric expansion. For example, if one integrates both sides of equation (4.108) from $-L$ to L one finds

$$\int_{-L}^{L} f(x)\,dx = a_0 \int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\frac{n\pi x}{L}\,dx + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L}\,dx$$

and this result can be expressed in terms of inner products as

$$(1, f(x)) = a_0 \parallel 1 \parallel^2 + \sum_{n=1}^{\infty} a_n (1, \cos\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n (1, \cos\frac{n\pi x}{L})$$

By the above orthogonality properties one finds

$$a_0 = \frac{(1, f(x))}{\parallel 1 \parallel^2} = \frac{\int_{-L}^{L} f(x)\,dx}{\int_{-L}^{L} dx} = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \tag{4.109}$$

If one multiplies both sides of equation (4.108) by $\sin\frac{m\pi x}{L}$ and then integrates both sides of the resulting equation from $-L$ to L, the result can be expressed in terms of inner products as

$$(f(x), \sin\frac{m\pi x}{L}) = \frac{a_0}{2}(1, \sin\frac{m\pi x}{L}) + \sum_{n=1}^{\infty} a_n (\cos\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}) + \sum_{n=1}^{\infty} b_n (\sin\frac{n\pi x}{L}, \sin\frac{m\pi x}{L})$$

and by the orthogonality of these functions one finds the above equation reduces to

$$(f(x), \sin\frac{m\pi x}{L}) = b_m \parallel \sin\frac{m\pi x}{L} \parallel^2$$

because the only nonzero inner product occurs when the summation index n takes on the value m. This shows that the coefficients b_m, for $m = 1, 2, 3, \ldots$ can be determined from the relations

$$b_m = \frac{(f(x), \sin\frac{m\pi x}{L})}{\parallel \sin\frac{m\pi x}{L} \parallel^2} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \quad \text{for } m = 1, 2, 3, \ldots \tag{4.110}$$

Similarly, if one multiplies both sides of equation (4.108) by $\cos\frac{m\pi x}{L}$ and then integrates both sides of the resulting equation from $-L$ to L, one can make use of inner products and orthogonality properties to show

$$a_m = \frac{(f(x), \cos\frac{m\pi x}{L})}{\|\cos\frac{m\pi x}{L}\|^2} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \quad \text{for } m = 1, 2, 3, \dots \tag{4.111}$$

In summary, the equations (4.109), (4.110), (4.111) demonstrate that the Fourier coefficients can be determined from an appropriate inner product divided by a norm squared

$$a_0 = \frac{(1, f)}{\|1\|^2}, \qquad a_n = \frac{(\cos(\frac{n\pi x}{L}), f)}{\|\cos(\frac{n\pi x}{L})\|^2}, \qquad b_n = \frac{(\sin(\frac{n\pi x}{L}), f)}{\|\sin(\frac{n\pi x}{L})\|^2} \tag{4.112}$$

Note that the set of functions $\{1, \sin\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}\}$ are **periodic functions** with period $2L$ and consequently the Fourier trigonometric series will produce a periodic function for all values of x. The notation $\tilde{f}(x)$ is introduced to define **the periodic extension of $f(x)$ outside the full Fourier interval** $(-L, L)$. One can write

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{where} \quad -L \le x \le L$$

or

$$\tilde{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{where} \quad -\infty < x < \infty$$

The above definitions are introduced due to the fact that $f(x) \ne \tilde{f}(x)$ because the original function $f(x)$ need only be defined over the full Fourier interval and $f(x)$ is not necessarily a periodic function, whereas the function $\tilde{f}(x)$ is periodic and satisfies $\tilde{f}(x + 2L) = \tilde{f}(x)$ for all values of x.

Example 4-44. (Fourier Series.)

Represent the exponential function as a Fourier series

$$e^x = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \text{ over the interval } (-L, L)$$

Solution

One must find the Fourier coefficients a_0, a_n, b_n, $n = 1, 2, 3, \dots$ associated with the exponential function e^x. The Fourier coefficients are calculated from the relations (4.109), (4.110) and (4.111). One finds

$$\begin{aligned}
a_0 =& \frac{(e^x, 1)}{||1||^2} = \frac{1}{2L}\int_{-L}^{L} e^x\,dx = \frac{1}{L}\sinh L\\
a_n =& \frac{(e^x, \cos\frac{n\pi x}{L})}{||\cos\frac{n\pi x}{L}||^2} = \frac{1}{L}\int_{-L}^{L} e^x \cos\frac{n\pi x}{L}\,dx = \frac{2L(-1)^n \sinh L}{L^2 + n^2\pi^2}\\
b_n =& \frac{(e^x, \sin\frac{n\pi x}{L})}{\|\sin\frac{n\pi x}{L}\|^2} = \frac{1}{L}\int_{-L}^{L} e^x \sin\frac{n\pi x}{L}\,dx = \frac{-2n\pi(-1)^n \sinh L}{L^2 + n^2\pi^2}
\end{aligned}$$

which gives the Fourier trigonometric series representation of e^x as

$$\widetilde{e^x} = \frac{\sinh L}{L} + \sum_{n=1}^{\infty} \left(\frac{2L(-1)^n \sinh L}{L^2+n^2\pi^2} \cos \frac{n\pi x}{L} - \frac{2n\pi(-1)^n \sinh L}{L^2+n^2\pi^2} \sin \frac{n\pi x}{L} \right) \tag{4.113}$$

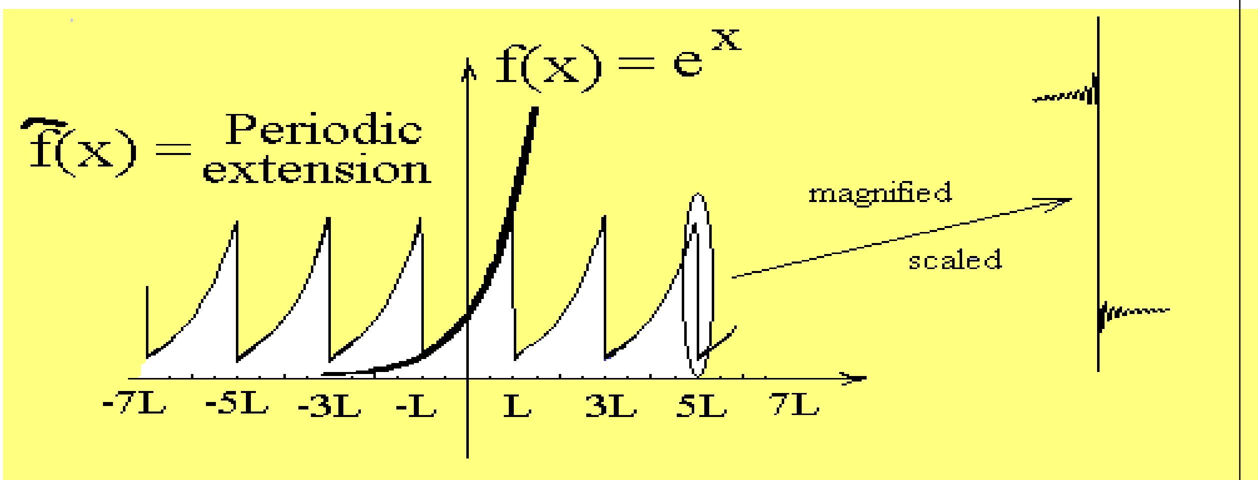


Figure 4-7.
Fourier trigonometric representation of the function e^x compared with e^x

The figure 4-7 illustrates a graphical representation of two curves. The first curve plotted illustrates the given function $f(x) = e^x$ for all values of x while the second curve plotted illustrates $\tilde{f}(x) = \widetilde{e^x}$, the Fourier trigonometric series representation. Note that because the set of functions $\{1, \sin\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}\}$ are periodic of period $2L$ the Fourier series given by equation (4.113) only represents e^x on the interval $(-L, L)$. The Fourier series does not represent e^x for all values of x. The interval $(-L, L)$ is called the full Fourier interval. Outside the full Fourier interval the Fourier series gives the periodic extension of the values of $f(x)$ inside the full Fourier interval.

■

Properties of the Fourier trigonometric series

Conditions for the existence of a Fourier series are: (i) $f(x)$ must be single-valued and piecewise continuous over the interval $(-L, L)$. (ii) The function $f(x)$ is bounded with a finite number of maxima and minima and a finite number of discontinuities over the interval $(-L, L)$. (iii) The integrals defining the Fourier coefficients must exist.

The Fourier series, when it exists, represents $f(x)$ on the interval $(-L, L)$ which is called the full Fourier interval. The Fourier series evaluated at points x outside the full Fourier interval gives the periodic extension of $f(x)$ defined over the full Fourier interval.

In order for a function $f(x)$ to have a Fourier series representation one must be able to calculate the Fourier coefficients a_0, a_n, b_n given by the equations (4.109), (4.110)and (4.111). Consequently, some functions will not have a Fourier series. For example, the functions $\frac{1}{x}$, $\frac{1}{x^2}$ are examples of functions which do not have a Fourier trigonometric series representation over the interval $(-L, L)$. Note that these functions are unbounded over the interval.

If the functions $f(x)$ and $f'(x)$ are piecewise continuous over the interval $(-L, L)$ then the Fourier series representation for $f(x)$ (a) Converges to $f(x)$ at points where $f(x)$ is continuous. (b) Converges to the periodic extension of $f(x)$ if x is outside the full Fourier interval $(-L, L)$. (c) At points x_0 where there is a finite jump discontinuity, the Fourier trigonometric series converges to $\frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right]$ which represents the average of the left and right-hand limits associated with the jump discontinuity.

The function $S_N(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right)$ is called the Nth partial sum associated with the Fourier series and represents a truncation of the series after N terms of both the sine and cosine terms are summed. One usually plots the approximating function $S_N(x)$ when representing the Fourier series $\tilde{f}(x)$ graphically. Whenever the function $f(x)$ being approximated has a point where **a jump discontinuity occurs**, then the approximating function $S_N(x)$ has **oscillations in the neighborhood of the jump discontinuity** as well as an **"overshoot" of the jump in the function**. These effects are known as **the Gibb's[17] phenomenon**. The Gibb's phenomenon always occurs whenever one attempts to use a series of continuous functions to represent a discontinuous function. The Gibb's phenomenon is illustrated in the figure 4-7. **These effects are not eliminated by increasing the value of N in the partial sum.**

Fourier Series of Odd Functions

If $f(-x) = -f(x)$ for all values of x, then $f(x)$ is called an odd function of x and $f(x)$ is symmetric about the origin. In this special case the Fourier series of $f(x)$ reduces to the **Fourier sine series**

[17] Josiah Willard Gibbs (1839-1903) An American mathematician.

$$\tilde{f}(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{4.114}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \tag{4.115}$$

Fourier Series of Even Functions

If $f(-x) = f(x)$ for all values of x, then $f(x)$ is called an even function of x and $f(x)$ is symmetric about the $y-$axis. In this special case the Fourier series of $f(x)$ reduces to a **Fourier cosine series**

$$\tilde{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \tag{4.116}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{for } n = 1, 2, 3, \ldots \tag{4.117}$$

Options

If you are only interested in the function $f(x)$ defined on the interval $0 \le x \le L$, then you can represent this function in three different ways. (1) You can extend $f(x)$ to the full Fourier interval by making it into an odd function. This extension produces a Fourier sine series. (2) You can extend $f(x)$ to the full Fourier interval by making into an even function. This extension produces a Fourier cosine series. (3) You can extend $f(x)$ is some arbitrary fashion so $f(x)$ is neither even nor odd, then one obtains the full Fourier trigonometric series for the Fourier expansion of $f(x)$.

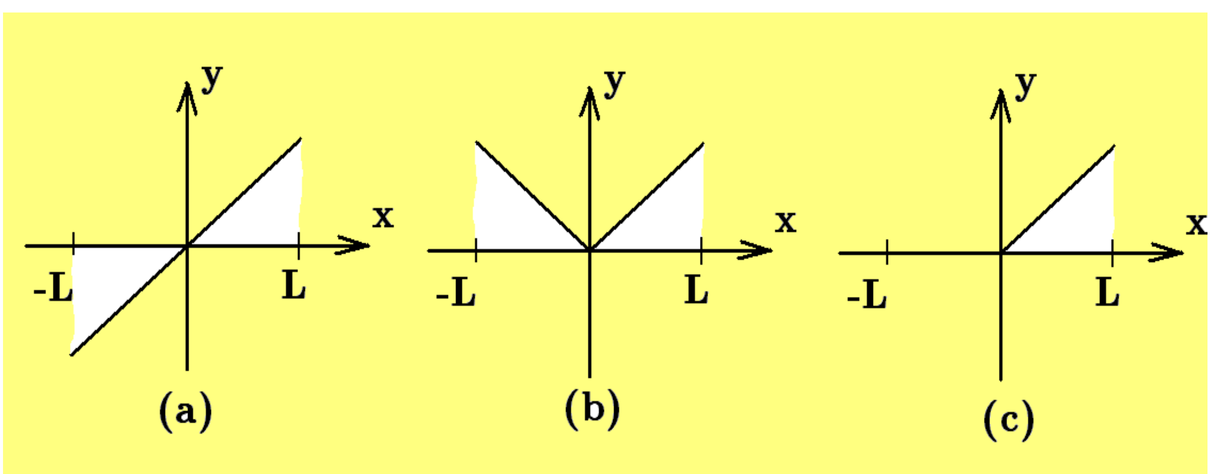


Figure 4-8.

Function $f(x)$ extended as (a) an odd function (b) an even function (c) neither

Example 4-45. Given the function $f(x) = x$ for $0 < x < L$. Extend this function to the full Fourier interval $(-L, L)$ and express $f(x)$ as (i) a Fourier sine series (ii) a Fourier cosine series (c) a Fourier trigonometric series.

Solution

(a) If $f(x)$ is extended as an odd function, then $f(x) = x$ for $-L < x < L$ so that the Fourier trigonometric series

$$\tilde{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) \tag{4.118}$$

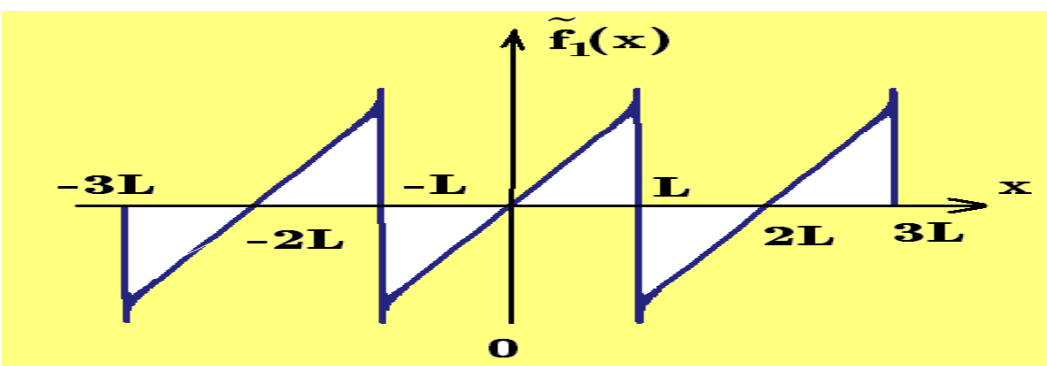
reduces to a Fourier sine series since

$$\begin{aligned}
a_0 &= \frac{(1, f)}{\| 1 \|^2} = \frac{1}{L} \int_{-L}^{L} x\, dx = 0 \\
a_n &= \frac{(\cos(\frac{n\pi x}{L}), f)}{\| \cos(\frac{n\pi x}{L}) \|^2} = \frac{1}{L} \int_{-L}^{L} x \cos(\frac{n\pi x}{L})\, dx = 0 \\
b_n &= \frac{(\sin(\frac{n\pi x}{L}), f)}{\| \sin(\frac{n\pi x}{L}) \|^2} = \frac{1}{L} \int_{-L}^{L} x \sin(\frac{n\pi x}{L})\, dx = -2(-1)^n \frac{L}{n\pi}
\end{aligned}$$

This gives the Fourier sine series

$$\tilde{f}_1(x) = \sum_{n=1}^{\infty} -2(-1)^n \frac{L}{n\pi} \sin(\frac{n\pi x}{L})$$

A graph of $\tilde{f}_1(x)$ over the interval $(-3L, 3L)$ is illustrated in the following figure.



Note that $\tilde{f}_1(x)$ is periodic and has jump discontinuities at the points $-3L, -L, L$ and $3L$ where the Gibb's phenomena is readily observed.

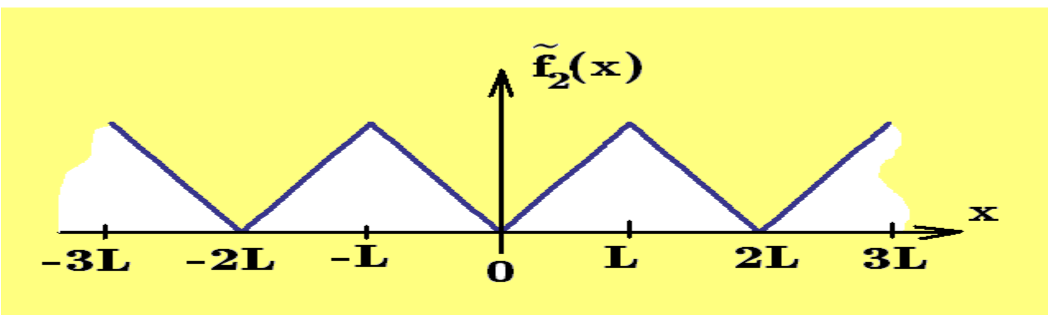
(b) If $f(x)$ is extended to the full Fourier interval as an even function, then it can be represented as $f(x) = \begin{cases} x, & 0 < x < L \\ -x, & -L < x < 0 \end{cases}$ and the Fourier trigonometric series (4.118) reduces to a Fourier cosine series since

$$\begin{aligned} a_0 =& \frac{(1, f)}{\| 1 \|^2} = \frac{1}{2L}\left(2\int_0^L x\,dx\right) = \frac{L}{2} \\ a_n =& \frac{(\cos(\frac{n\pi x}{L}), f)}{\| \cos(\frac{n\pi x}{L}) \|^2} = \frac{1}{L}\left(2\int_0^L x\cos(\frac{n\pi x}{L})\,dx\right) = \frac{2L}{n^2\pi^2}(-1+(-1)^n) \\ b_n =& \frac{(\sin(\frac{n\pi x}{L}), f)}{\| \sin(\frac{n\pi x}{L}) \|^2} = 0 \end{aligned}$$

This gives the Fourier cosine series

$$\tilde{f}_2(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}(-1+(-1)^n)\cos(\frac{n\pi x}{L})$$

A graph of $\tilde{f}_2(x)$ over the interval $(-3L, 3L)$ is illustrated in the following figure.



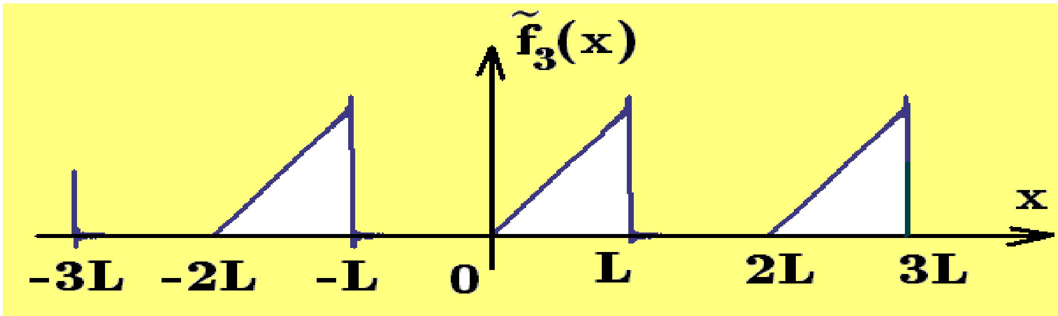
(c) If $f(x)$ is defined $f(x) = \begin{cases} x, & 0 < x < L \\ 0, & -L < x < 0 \end{cases}$, then $f(x)$ is neither an odd nor even function and so there results a Fourier trigonometric series with coefficients

$$\begin{aligned} a_0 =& \frac{(1, f)}{\| 1 \|^2} = \frac{1}{2L}\int_0^L x\,dx = \frac{L}{4} \\ a_n =& \frac{(\cos(\frac{n\pi x}{L}), f)}{\| \cos(\frac{n\pi x}{L}) \|^2} = \frac{1}{L}\int_0^L x\cos(\frac{n\pi x}{L})\,dx = \frac{L}{n^2\pi^2}(-1+(-1)^n) \\ b_n =& \frac{(\sin(\frac{n\pi x}{L}), f)}{\| \sin(\frac{n\pi x}{L}) \|^2} = \frac{1}{L}\int_0^L x\sin(\frac{n\pi x}{L})\,dx = \frac{-(-1)^n}{n\pi} \end{aligned}$$

This gives the Fourier series

$$\tilde{f}_3(x) = \frac{L}{4} + \sum_{n=1}^{\infty} \frac{L}{n^2\pi^2}(-1+(-1)^n)\cos(\frac{n\pi x}{L}) - \sum_{n=1}^{\infty}(-1)^n\frac{L}{n\pi}\sin(\frac{n\pi x}{L})$$

A graph of $\tilde{f}_3(x)$ over the interval $(-3L, 3L)$ is given in the following figure.



Note the Gibb's phenomena results because of the jump discontinuity in the periodic extension of the function. Also note that $\tilde{f}_3(x) = \frac{1}{2}\left[\tilde{f}_1(x) + \tilde{f}_2(x)\right]$.

■

Some Spectacular Results

1. Summation of positive powers

For m a positive integer

$$1^m + 2^m + 3^m + \cdots + n^m = \frac{(B+n+1)^{m+1} - B^{m+1}}{m+1}$$

where the right-hand side of the above equation is evaluated as follows.

(a) Expand $(B+n+1)^{m+1}$ in a binomial series.

(b) In the binomial expansion replace B^k by the Bernoulli number B_k

2. Summation of negative powers

For m a positive integer with B_{2m} a Bernoulli number and $\zeta(m)$ the Riemann zeta function, then

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = 1 + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \frac{1}{4^{2m}} + \cdots \;=\; \frac{(-1)^m (2\pi)^{2m} B_{2m}}{2(2m)!} = \zeta(2m)$$

Euler showed that

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(6) = \frac{\pi^6}{945}$$

One just doesn't sit down and come up with wonderful formulas like the ones above. It takes a lot of work to make a discovery. If you don't have a lot of information about a subject, then you don't know what questions to ask about the subject. Therefore, one can say that the more information you have about different subjects, a better understanding of interrelationships between subjects can be developed. Results like the above come about by a person getting deeply involved in the subject matter and investigating simple ideas which in turn lead to complicated results.

Exercises

► **4-1.** Examine the given sequence $\{u_n\}$ and determine if it converges or diverges. If the sequence converges, then find its limit.

$$(a)\quad u_n = \frac{3n}{4n-3} \qquad (c)\quad u_n = 1 + (\frac{1}{2})^n \qquad (e)\quad u_n = \frac{n^2}{e^n}$$
$$(b)\quad u_n = \frac{n^2}{2n^2-1} \qquad (d)\quad u_n = \frac{\ln n}{\sqrt{n}} \qquad (f)\quad u_n = \left(1+\frac{1}{n}\right)^n$$

► **4-2.** Examine the given sequence $\{v_n\}$ and determine if it converges or diverges. If the sequence converges, then find its limit.

$$(a)\quad v_n = \frac{n}{1-2n} \qquad (c)\quad v_n = 1 + (-1)^n \qquad (e)\quad v_n = \sin(n\pi/2)$$
$$(b)\quad v_n = (-1)^n \frac{2n^2+3n+4}{n^2+n+1} \qquad (d)\quad v_n = \frac{1+(-1)^n}{n} \qquad (f)\quad v_n = \frac{2^n}{3^n}$$

► **4-3.** Find the sum of the given series

$$(a)\quad \sum_{n=1}^{50}(3+5n) \qquad (c)\quad \sum_{n=1}^{10}(2+7n) \qquad (e)\quad \sum_{n=1}^{100}(0.02)^{n-1}$$
$$(b)\quad \sum_{j=1}^{20}(\frac{3}{2}+\frac{5}{2}j) \qquad (d)\quad \sum_{n=1}^{10}4(3)^{n-1} \qquad (f)\quad \sum_{m=1}^{10}(2+\sqrt{2})\left(\frac{3-\sqrt{2}}{2+\sqrt{2}}\right)^{m-1}$$

► **4-4.** Find the sum of the given geometric series.

$$S_1 = \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$$
$$S_2 = 0.6+0.06+0.006+0.0006+\cdots$$
$$S_3 = (2-\sqrt{3})+(7-4\sqrt{3})+(26-15\sqrt{3})+(97-56\sqrt{3})+\cdots$$
$$S_4 = \sqrt{6}-\sqrt{2}+\frac{1}{3}\sqrt{6}-\frac{1}{3}\sqrt{2}+\cdots$$

► **4-5.** Find the sum of the given series.

$$S_1 = \sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^i \qquad S_3 = \sum_{i=1}^{\infty}\left[\left(\frac{1}{3}\right)^i + \left(\frac{1}{4}\right)^i\right] \qquad S_5 = \sum_{n=1}^{\infty}\left[\frac{1}{n+x} - \frac{1}{n+x+1}\right]$$
$$S_2 = \sum_{i=1}^{\infty}\left(\frac{1}{4}\right)^i \qquad S_4 = \sum_{i=1}^{\infty}\left[\left(\frac{1}{3}\right)^i - \left(\frac{1}{4}\right)^i\right] \qquad S_6 = \sum_{n=1}^{N}\ln\left(1+\frac{1}{n}\right)$$

► **4-6.** Determine values of x for which the given series converges.

$$(a)\quad \sum_{n=1}^{\infty}\frac{x^n}{2n^2-n} \qquad (b)\quad \sum_{n=1}^{\infty}\frac{nx^n}{2^n} \qquad (c)\quad \sum_{n=1}^{\infty}\frac{1}{nx^{2n}}$$

► **4-7. Probability Theory**

Assume that a random variable X can take on any of the values $\{1, 2, 3, \ldots, k, \ldots\}$, where k is an integer. If p_k is the probability that X takes on the value k, then the probabilities $p_1, p_2, \ldots, p_k, \ldots$ must be selected such that

$$(i) \quad \text{Each } p_k \geq 0 \quad \text{and} \quad (ii) \quad \sum_{k=1}^{\infty} p_k = 1 \qquad 7(a)$$

In statistics the quantity $E(X)$ is called the expected value of X and is defined

$$E(X) = \sum_{k=1}^{\infty} k\, p_k \qquad 7(b)$$

provided the series converges.

Show that the given probabilities satisfy each of the conditions 7(a) and then calculate the expected value given by equation 7(b).

$$(i) \quad p_k = \frac{1}{2^k} \qquad (ii) \quad p_k = \frac{1}{2}\left(\frac{2}{3}\right)^k \qquad (iii) \quad p_k = 3\left(\frac{3}{4}\right)^k$$

► **4-8.** For constants a, b and r with $a > 0$ and $b > 0$ the given series are known to converge. Find their sums and required condition for convergence.

$$S_1 = \sum_{i=1}^{\infty} \frac{1}{a^{i/2}} \qquad S_3 = \sum_{k=0}^{N} ar^k$$

$$S_2 = \sum_{i=1}^{\infty} \frac{a^{2i}+1}{b^i} \qquad S_4 = \sum_{k=0}^{N} \left(ar^k + bkr^{k-1}\right)$$

► **4-9.** Use partial fractions and convert the given series to telescoping series and find their sums.

$$(a) \quad \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} + \cdots$$

$$(b) \quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-2)(n-1)} + \cdots$$

$$(c) \quad \frac{1}{3^2} + \frac{2}{15^2} + \frac{3}{35^2} + \cdots + \frac{n}{(4n^2-1)^2} + \cdots$$

► **4-10.** Examine the Nth partial sum associated with the given infinite series and determine if the series converge. If the given series converges, find its sum.

$$(a) \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \qquad (c) \quad \sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \qquad (d) \quad \sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)}$$

Hint: Use partial fractions.

▶ **4-11.** Use the integral test to determine convergence or divergence of the given series.

$$(a)\ \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \qquad (c)\ \sum_{n=1}^{\infty}\frac{\ln n}{n^2} \qquad (e)\ \sum_{n=1}^{\infty}\frac{3n+4}{n^2}$$

$$(b)\ \sum_{n=1}^{\infty}\frac{\ln n}{n} \qquad (d)\ \sum_{n=1}^{\infty}\frac{1}{n\ln n} \qquad (f)\ \sum_{n=1}^{\infty}\frac{n}{e^n}$$

▶ **4-12.** Assume that $f(x)$ is a given function satisfying the following properties.

(i) The function $f(x)$ is a continuous function such that $f(x)>0$ for all values of x.

(ii) For $p>0$ the limit $\lim_{n\to\infty} n^P f(n)$ exists and the limit is different from zero.

Show that $\sum_{n=1}^{\infty} f(n)$ converges if $p>1$ and diverges for $0<p\le 1$.

Hint: See modification of a series.

▶ **4-13.** Use the comparison test to determine convergence or divergence of the given series.

$$(a)\ \sum_{n=1}^{\infty}\frac{1}{n(n+3)(n+6)} \qquad (c)\ \sum_{n=1}^{\infty}\frac{1}{3+2\sqrt{n}} \qquad (e)\ \sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2+1}$$

$$(b)\ \sum_{n=1}^{\infty}\frac{1}{3+2^n} \qquad (d)\ \sum_{n=1}^{\infty}\frac{1}{3n^2+2n+1} \qquad (f)\ \sum_{n=1}^{\infty}\frac{1}{n^2\ln n}$$

▶ **4-14.**

(a) Verify that the given series converge.

(b) Find the sum of the first four terms of each series and give an estimate for the error between the exact solution and your calculated value.

(c) Find the sum of the first eight terms of each series and give an estimate for the error between the exact solution and your calculated value.

$$(i)\ \sum_{n=1}^{\infty}\frac{1}{n^3} \qquad (ii)\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3} \qquad (iii)\ \sum_{n=1}^{\infty}\frac{1}{n^4} \qquad (iv)\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$$

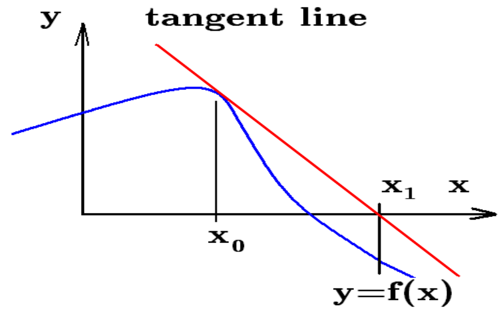
▶ **4-15.** Show that the given series converge and determine which series converges at the slower rate.

$$(i)\quad A=\sum_{n=1}^{\infty}\frac{1}{n\,3^n} \qquad (ii)\quad B=\sum_{n=1}^{\infty}\frac{n}{5^n}$$

▶ **4-16.** Show that the given series diverge and determine which series diverges at the slower rate.

$$(i)\quad A=\sum_{n=1}^{\infty}\frac{1}{n} \qquad (ii)\quad B=\sum_{n=1}^{\infty}\frac{1}{\ln n}$$

▶ **4-17. Newton's root finding method** To determine where a given curve $y = f(x)$ crosses the x-axis one can select an initial guess x_0 and if $f(x_0) \neq 0$ one can then calculate $f'(x_0)$. From the values $f(x_0)$ and $f'(x_0)$ one can construct the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$. This tangent line given by $y - f(x_0) = f'(x_0)(x - x_0)$.



(a) Show the tangent line intersects the x-axis at the point $x_1 = x_0 - f(x_0)/f'(x_0)$

(b) Form the sequence $\{x_n\}$ where $x_n = x_{n-1} - f(x_{n-1})/f'(x_{n-1})$ for $n = 1, 2, 3, \dots$

(c) Give a geometric interpretation to what this sequence is doing. Hint: What has been done once can be done again.

(d) If $y = f(x) = x^2 - 3x + 1$ and $x_0 = 1$, find using a calculator x_1, x_2, x_3 and x_4

(e) If $y = f(x) = x^2 - 3x + 1$ and $x_0 = 2$, find using a calculator x_1, x_2, x_3 and x_4

(f) Sketch the curve $y = f(x) = x^2 - 3x + 1$ and find the roots of the equation $f(x) = 0$.

(g) What happens if the initial guess x_0 is bad? Say $x_0 = 3/2$ for the above example.

▶ **4-18.** Let $f_n(x) = \dfrac{x^n}{n(n+1)}$

(a) Show that $\displaystyle\sum_{n=1}^{\infty} f_n(9/10)$ converges. (b) Show that $\displaystyle\sum_{n=1}^{\infty} f_n(10/9)$ diverges.

▶ **4-19.** Given the infinite series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,[\ln n]^p}$, with $p > 0$.

(a) Show the series converges for $p > 1$.

(b) Show the series diverges for $p \le 1$.

Hint: Let $f_0(t) = t,\ f_1(t) = \ln f_0(t),\ f_2(t) = \ln f_1(t), \dots, f_{n+1}(t) = \ln f_n(t)$ and show

$$\int \frac{dt}{f_0(t) f_1(t) f_2(t) \cdots f_{m-1}(t)\,[f_m(t)]^p} = \begin{cases} f_{m+1}(t), & p = 1 \\ \frac{-1}{(p-1)}\,[f_m(t)]^{p-1}, & p \neq 1 \end{cases}$$

and then examine $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,[\ln n]^p} = \sum_{n=2}^{\infty} \frac{1}{f_0(n)[f_1(n)]^p}$

▶ **4-20.** Show that if the series $\displaystyle\sum_{n=1}^{\infty} u_n$ converges, then the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{u_n}$ diverges.

▶ **4-21.** If $U_n = \frac{1}{3} n^{\overline{3}} = \frac{1}{3} n(n+1)(n+2)$, show $\Delta U_k = U_{k+1} - U_k = (k+1)(k+2)$ and find the sum of the series $S_n = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n\cdot(n+1) = \displaystyle\sum_{k=0}^{n-1} (k+1)(k+2) = \sum_{k-0}^{n-1} \Delta U_k$

▶ **4-22. Reversion of series**

(a) Given the series

$$y = y(x) = 1 - (x-1) + \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!} + \cdots \qquad 22(a)$$

and it is required that you solve for $x-1$ in terms of y to obtain a series of the form

$$(x-1) = A_1(y-1) + A_2(y-1)^2 + A_3(y-1)^3 + A_4(y-1)^4 + \cdots \qquad 22(b)$$

where $A_1, A_2, A_3, A_4 \ldots$ are constants to be determined. Substitute equation 22(b) into equation 22(a) and expand all terms. Equate like powers of $(y-1)$ and obtain a system of equations from which the constants $A_1, A_2, A_3, A_4, \ldots$ can be determined.

(b) If the original series in part(a) is $y = e^{-(x-1)}$ and the series obtained by reversion is the function $(x-1) = -\ln y$, expanded in a series about $y = 1$, then examine the approximation for $x-1$ by truncation of your series after the A_4 term. Let $E = E(y)$ denote the error in using this truncated series to solve for $x-1$. Plot a graph of the error $E = E(y)$ for $1 \le y < 2$.

▶ **4-23.** Examine the given alternating series to determine if they converge.

$$(a)\ \sum_{n=1}^{\infty} \frac{1}{(-3)^{n-1}} \qquad (c)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \qquad (e)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$(b)\ \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n} \qquad (d)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad (f)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$$

▶ **4-24.** Use the root test to determine if the given series converge.

$$(a)\ \sum_{n=2}^{\infty} \frac{1}{[\ln n]^n} \qquad (c)\ \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad (e)\ \sum_{n=1}^{\infty} \frac{1}{n^n}$$

$$(b)\ \sum_{n=1}^{\infty} \frac{n^n}{2^{4n}} \qquad (d)\ \sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^n \qquad (f)\ \sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)^n$$

▶ **4-25.** Test the following series to determine convergence or divergence.

$$(a)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \qquad (c)\ \sum_{n=1}^{\infty} \sin^n(\pi/3) \qquad (e)\ \sum_{n=1}^{\infty} \frac{n^3}{3^n}$$

$$(b)\ \sum_{n=1}^{\infty} \frac{n^{10}}{(1.001)^n} \qquad (d)\ \sum_{n=1}^{\infty} \frac{n}{2^n} \qquad (f)\ \sum_{n=1}^{\infty} \frac{3^n}{n!}$$

▶ **4-26.** Determine whether the given series converge or diverge.

$(a)\quad \dfrac{1}{2}+\dfrac{2}{3}+\dfrac{3}{4}+\dfrac{4}{5}+\cdots+\dfrac{n}{n+1}+\cdots$

$(b)\quad \dfrac{3}{19}+\dfrac{5}{35}+\dfrac{7}{51}+\cdots+\dfrac{2n+1}{16n+1}+\cdots$

$(c)\quad \sin(1)+2\sin(1/2)+3\sin(1/3)+\cdots+n\sin(1/n)+\cdots$

$(d)\quad \displaystyle\sum_{n=1}^{\infty} n\cos(\frac{1}{n})$

$(e)\quad \displaystyle\sum_{n=1}^{\infty} \frac{2}{n}$

▶ **4-27.** Determine the convergence or divergence of the given series.

$(a)\quad \displaystyle\sum_{n=1}^{\infty}\frac{n}{\sqrt{n+1}} \qquad (c)\quad \sum_{n=1}^{\infty}\frac{1}{n^{3/2}} \qquad (e)\quad \sum_{n=1}^{\infty}\frac{1}{2^n+4^n}$

$(b)\quad \displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!} \qquad (d)\quad \sum_{n=1}^{\infty}\frac{1}{n^3+n} \qquad (f)\quad \sum_{n=1}^{\infty}\frac{2^n}{n^2}$

▶ **4-28.** Determine if the given series is (a) conditionally convergent, (b) absolutely convergent or (c) divergent.

$(a)\quad \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}} \qquad (c)\quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \qquad (e)\quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,\ln n}{n^2}$

$(b)\quad \displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^2} \qquad (d)\quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{6n+3} \qquad (f)\quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,n^n}{n!}$

▶ **4-29.** Determine if the given series is (a) conditionally convergent, (b) absolutely convergent or (c) divergent.

$(a)\quad \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{3n+2}} \qquad (c)\quad \sum_{n=2}^{\infty}\frac{(-1)^n}{2\ln n} \qquad (e)\quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{3^n}$

$(b)\quad \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{(n+1)(n+2)}} \qquad (d)\quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1} \qquad (f)\quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{n^2+1}$

▶ **4-30.** Find the interval where the power series converges absolutely.

$(a)\quad \displaystyle\sum_{n=1}^{\infty}\frac{x^{2n}}{n\,2^n} \qquad (c)\quad \sum_{n=1}^{\infty}\frac{n\,(x-1)^n}{3^n} \qquad (e)\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{3^n}$

$(b)\quad \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^n}{n} \qquad (d)\quad \sum_{n=1}^{\infty}\frac{(3x)^n}{\ln(n+1)} \qquad (f)\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(3x+2)^n}{4^n}$

▶ **4-31.** Let $y=f(x)=\dfrac{x|}{|1}+\dfrac{x|}{|1}+\dfrac{x|}{|1}+\cdots$ and show that

$$\frac{dy}{dx}=f'(x)=\frac{1|}{|1}+\frac{2x|}{|\,1}+\frac{x|}{|1}+\frac{x|}{|1}+\cdots$$

Hint: Show that $y=\dfrac{x}{1+y}$

▶ **4-32.** Explain the difference between (a) the limit of a sequence and (b) the limit point of a sequence.

▶ **4-33.** Examine the binomial series for the expansion of $(a+b)^n$ when n is an integer.

$$
\begin{aligned}
(a+b)^n =& a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n \\
(a+b)^n =& \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-1}b^2 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n \\
(a+b)^n =& \sum_{j=0}^{\infty}\binom{n}{j}a^{n-j}b^j
\end{aligned}
\tag{33(a)}
$$

where $\binom{n}{m} = \begin{cases} \frac{n!}{m!\,(n-m)!} & m \le n \\ 0, & m > n \end{cases}$ are the binomial coefficients.

(a) Show that $(a+b)^n = \displaystyle\sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^j = \sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}$

(b) Newton generalized the binomial expansion to

$$
\begin{aligned}
(a+b)^r =& \sum_{k=0}^{\infty}\binom{r}{k}a^{r-k}b^k \\
(a+b)^r =& a^r + ra^{r-1}b + \frac{r(r-1)}{2!}a^{r-2}b^2 + \frac{r(r-1)(r-2)}{3!}a^{r-3}b^3 + \cdots
\end{aligned}
\tag{33(b)}
$$

where r represents an arbitrary real number.

(i) Show that when r is a nonnegative integer, the equation 33(b) reduces to equation 33(a).

(ii) (**Difficult problem**) Write equation 33(b) in the form $a^r(1+x)^r$ where $x = b/a$. Examine the series expansion for $f(x) = (1+x)^r$. Then use the Lagrange and Cauchy forms of the remainder R_n to show the equation 33(b) converges if $|a| > |b|$ and diverges if $|a| \le |b|$, where $x = b/a$.

▶ **4-34.** Let $y = g(x) = x + \frac{1|}{|x} + \frac{1|}{|x} + \frac{1|}{|x} + \cdots$ and show that

$$\frac{dy}{dx} = g'(x) = \frac{1|}{|2} - \frac{x|}{|x} + \frac{1|}{|x} + \cdots + \frac{1|}{|x}\cdots \qquad \text{Hint: Show that } y = x + \frac{1}{y}$$

▶ **4-35.** Let $y = h(x) = \frac{\sin x|}{|\ 1} + \frac{\cos x|}{|\ 1} + \frac{\sin x|}{|\ 1} + \frac{\cos x|}{|\ 1} + \cdots$ and show that

$$\frac{dy}{dx} = h'(x) = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}$$

Hint: Show that $y = \dfrac{\sin x}{1 + \frac{\cos x}{1+y}}$

▶ **4-36.** The continued fraction function

$$y_n = y_n(x) = \frac{P_n(x)}{Q_n(x)} = \alpha_0 + \frac{1|}{|\alpha_1} + \frac{1|}{|\alpha_2} + \cdots + \frac{1|}{|\alpha_n} + \frac{1|}{|x}$$

where $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ are constants, represents a rational function of x.

(a) Show that $\dfrac{dy_1}{dx} = \dfrac{1}{[Q_1(x)]^2}$

(b) Show that $\dfrac{dy_2}{dx} = \dfrac{-1}{[Q_2(x)]^2}$

(c) Show that $\dfrac{dy_3}{dx} = \dfrac{1}{[Q_3(x)]^2}$

(d) Can you show that in general $\dfrac{dy_n}{dx} = (-1)^n \dfrac{1}{[Q_n(x)]^2}$

▶ **4-37.** Euler used the product formula $\sin\theta = \theta \left[1 - \left(\frac{\theta}{\pi}\right)^2\right]\left[1 - \left(\frac{\theta}{2\pi}\right)^2\right]\cdots\left[1 - \left(\frac{\theta}{n\pi}\right)^2\right]\cdots$ to represent sine of θ. Differentiate this relation and show $\theta\cot\theta = 1 + 2\theta^2 \displaystyle\sum_{n=1}^{\infty} \frac{1}{\theta^2 - n^2\pi^2}$

▶ **4-38.** Assume that $\sum u_n$ and $\sum v_n$ are two infinite series of positive terms and that there exists an integer N such that for all $n > N$ the inequality $u_n \le K v_n$ for some positive constant K.

(i) If the series $\sum v_n$ converges, prove the series $\sum u_n$ converges.

(ii) If the series $\sum u_n$ diverges, prove the series $\sum v_n$ diverges.

▶ **4-39.** Show that the alternating p-series $\displaystyle\sum_{n=1}^{\infty} (1)^{n+1} \frac{1}{n^p}$ converges if $p > 0$.

▶ **4-40.** Consider the geometric series $\dfrac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots$ where $z = re^{i\theta}$, $|z| < 1$ and $i^2 = -1$. Show that by equating real and imaginary parts

$$\begin{aligned}\frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} =& 1 + r\cos\theta + r^2\cos 2\theta + \cdots + r^n\cos n\theta + \cdots \\ \frac{r\sin\theta}{1 - 2r\cos\theta + r^2} =& r\sin\theta + r^2\sin 2\theta + \cdots + r^n\sin n\theta + \cdots\end{aligned}$$

Hint: Use Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$

▶ **4-41.**

(a) Show $\{\sin nx\}$, $n = 1, 2, 3, \ldots$ is an orthogonal sequence over the interval $(0, \pi)$ with respect to the weight function $r = 1$.

(b) Scale the above sequence to construct an orthonormal sequence over the given interval.

▶ **4-42.** Calculate the inner products and norm squared values associated with the given sequence of functions $\{f_n(x)\}$ using the given interval (a,b) and weight function $r(x)$, for $n = 1, 2, 3, \ldots$.

$$\begin{array}{ll}(a) & \{f_n\} = \{\sin \frac{n\pi x}{L}\},\ (0, L),\ r = 1 \\ (b) & \{f_0, f_n\} = \{1, \cos \frac{n\pi x}{L}\},\ (0, L),\ r = 1 \\ (c) & \{f_0, f_{2n}, f_{2n-1}\} = \{1, \cos \frac{n\pi x}{L}, \sin \frac{n\pi x}{L}\},\ (-L, L),\ r = 1 \\ (d) & \{f_0, f_1, f_2\} = \{1, 1-x, x^2 - 4x + 2\},\ (0, \infty),\ r = e^{-x}\end{array}$$

▶ **4-43. Even and Odd Functions**

(a) If $G_o(-x) = -G_o(x)$ for all values of x, show that $\int_{-L}^{L} G_o(x)\,dx = 0$

(b) If $G_e(-x) = G_e(x)$ for all values of x, show that $\int_{-L}^{L} G_e(x)\,dx = 2\int_0^L G_e(x)\,dx$

(c) Let $F_o(x)$ denote an odd function of x and $F_e(x)$ denote an even function of x. Similarly, let $G_o(x)$ denote an odd function of x and $G_e(x)$ denote an even function of x, show that

(i) $H(x) = F_o(x)G_o(x)$ is an even function of x

(ii) $H(x) = F_o(x)G_e(x)$ is an odd function of x

(iii) $H(x) = F_e(x)G_e(x)$ is an even function of x

(d) Determine which functions are even or odd.

$$\begin{array}{llll}(i) & F_e(x)\cos(\frac{n\pi x}{L}) & (iii) & F_o(x)\cos(\frac{n\pi x}{L}) \\ (ii) & F_e(x)\sin(\frac{n\pi x}{L}) & (iv) & F_o(x)\sin(\frac{n\pi x}{L})\end{array}$$

(e) Use the above properties to simplify the Fourier series representation of $f(x)$ over the interval $(-L, L)$, as given by equation (4.108), if

(i) The function $f(x)$ is an even function.

(ii) The function $f(x)$ is an odd function.

▶ **4-44. (Newton's method for nonlinear system)**

To solve the system of simultaneous nonlinear equations

$$f(x, y) = 0, \qquad g(x, y) = 0$$

in the two unknowns x and y, one can use Newton's method which is described as follows.

Start with an initial guess of the solution and call it x_0 and y_0. Now expand f and g in Taylor series expansions about the point (x_0, y_0). These expansions can be written

$$\begin{aligned} f(x_0+h, y_0+k) =& \ f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}h + \frac{\partial f(x_0,y_0)}{\partial y}k \\ &+ \frac{1}{2!}\left[f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2\right] + \cdots \\ g(x_0+h, y_0+k) =& \ g(x_0,y_0) + \frac{\partial g(x_0,y_0)}{\partial x}h + \frac{\partial g(x_0,y_0)}{\partial y}k \\ &+ \frac{1}{2!}\left[g_{xx}h^2 + 2g_{xy}hk + g_{yy}k^2\right] + \cdots. \end{aligned}$$

Usually the initial guess (x_0, y_0) is such that $f(x_0, y_0)$ and $g(x_0, y_0)$ are not zero. It is desired to find values h and k such that the equations

$$f(x_0+h, y_0+k) = 0 \quad \text{and} \quad g(x_0+h, y_0+k) = 0$$

are satisfied simultaneously. Now assume that the values h and k to be selected are small corrections to x_0 and y_0 so that second-order terms h^2, hk, k^2, and higher order product terms are small and can consequently be neglected in the above Taylor series expansion. These assumptions produce the linear system of equations

$$\begin{aligned} f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}h + \frac{\partial f(x_0,y_0)}{\partial y}k = 0 \\ g(x_0,y_0) + \frac{\partial g(x_0,y_0)}{\partial x}h + \frac{\partial g(x_0,y_0)}{\partial y}k = 0 \end{aligned}$$

which can then be solved to determine the correction terms h and k.

(a) Show by letting $h = x_1 - x_0$ and $k = y_1 - y_0$ that an improved estimate for the solution to the simultaneous equations $f(x, y) = 0$ and $g(x, y) = 0$, is given by

$$\begin{aligned} x_1 = x_0 + h = x_0 + \frac{\alpha}{\Delta} \\ y_1 = y_0 + k = y_0 + \frac{\beta}{\Delta}, \end{aligned}$$

where

$$\alpha = \begin{vmatrix} -f(x_0,y_0) & \frac{\partial f(x_0,y_0)}{\partial y} \\ -g(x_0,y_0) & \frac{\partial g(x_0,y_0)}{\partial y} \end{vmatrix} \quad \text{and} \quad \beta = \begin{vmatrix} \frac{\partial f(x_0,y_0)}{\partial x} & -f(x_0,y_0) \\ \frac{\partial g(x_0,y_0)}{\partial x} & -g(x_0,y_0) \end{vmatrix}$$

and Δ is the determinant of the coefficients given by $\Delta = \begin{vmatrix} \frac{\partial f(x_0,y_0)}{\partial x} & \frac{\partial f(x_0,y_0)}{\partial y} \\ \frac{\partial g(x_0,y_0)}{\partial x} & \frac{\partial g(x_0,y_0)}{\partial y} \end{vmatrix}$.

(b) Illustrate Newton's method by solving the nonlinear system of equations

$$f(x,y) = 2x^2 - 3y + 1 = 0 \qquad g(x,y) = 8x + 11 - 3y^2 = 0$$

Hint: Nonlinear equations may have multiple solutions, a unique solution, or no solutions at all. Sometimes a graph is helpful in estimating a solution if one exists.

▶ **4-45.**

Verify the Fourier series representation for the functions illustrated. In each graph assume the maximum amplitude of each function is +1 and the minimum amplitude of each function is either zero or -1 depending upon the graph.

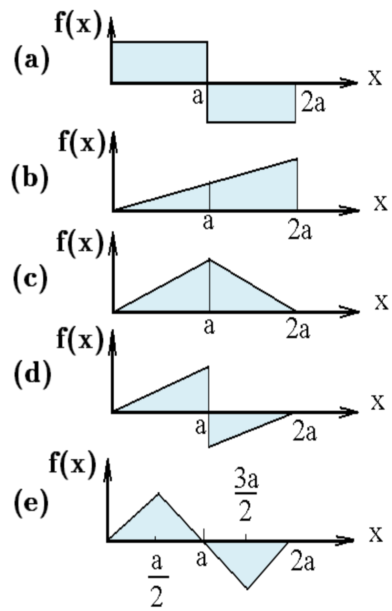
$$(a)\quad f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1}\sin\left[\frac{(2n+1)\pi x}{a}\right]$$

$$(b)\quad f(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left[\frac{n\pi x}{a}\right]$$

$$(c)\quad f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\cos\left[\frac{(2n+1)\pi x}{a}\right]$$

$$(d)\quad f(x) = \frac{-2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin\left[\frac{n\pi x}{a}\right]$$

$$(e)\quad f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin(\frac{n\pi}{2})\sin(\frac{n\pi x}{a})$$



▶ **4-46.** Show the Fourier trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{where} \quad -L \le x \le L$$

can also be expressed in the form $f(x) = a_0 + \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L} + \beta_n)$ by finding the values c_n and β_n.

▶ **4-47.** Let $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$ denote the Fourier series representation of $f(x)$ over the full Fourier interval $(-L, L)$.

(a) Use the Euler formulas

$$e^{in\pi x/L} = \cos\frac{n\pi x}{L} + i\,\sin\frac{n\pi x}{L} \quad \text{and} \quad e^{-in\pi x/L} = \cos\frac{n\pi x}{L} - i\,\sin\frac{n\pi x}{L}$$

and show

$$\cos\frac{n\pi x}{L} = \frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2} \quad \text{and} \quad \sin\frac{n\pi x}{L} = \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i}$$

(b) Define $C_0 = \frac{a_0}{2}$, $C_n = \frac{1}{2}(a_n - ib_n)$, $C_n = \frac{1}{2}(a_n + ib_n)$ and show the Fourier series can be represent in the complex form

$$f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/L} \quad \text{where} \quad C_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx$$

▶ **4-48.** You are sick and your doctor prescribes medication XYZ to be taken τ-times a day based upon the concentration of XYZ. Find τ. First get over the shock of being asked such a question. To solve the problem you must make some assumptions such as the following.

(i) At time $\tau = 0$ you take medication XYZ and this produces a concentration C_0 of XYZ in your blood stream.

(ii) The concentration C_0 decays exponentially with time so that after a time τ the concentration of XYZ in your blood is $C_0e^{-k\tau}$, where k is called the decay constant.

(iii) At times $\tau, 2\tau, 3\tau, \ldots, n\tau$ you take the medication XYZ and consequently you build up a certain residual concentration of XYZ in your blood stream given by

$$C_0e^{-k\tau} + C_0e^{-2k\tau} + C_0e^{-3k\tau} + \cdots + C_0e^{-kn\tau}$$

(a) If you continue the prescribed dosage forever, then the residual concentration would be

$$C = \sum_{m=1}^{\infty} C_0e^{-mk\tau} = C_0e^{-k\tau}\sum_{m=0}^{\infty}\left(e^{-k\tau}\right)^m$$

Sum this series and determine the residual concentration C.

(b) If C_s denotes the maximum safe concentration of XYZ that the human body can stand, then show that τ must be selected to satisfy

$$C_0 + C = C_0\left(1 + \frac{e^{-k\tau}}{1 - e^{-k\tau}}\right) \;\le\; C_s$$

(c) Assume k, C_0, C_s are known values and solve for τ.

▶ **4-49.**

(a) Verify the well known result $\dfrac{1}{1+\xi} = 1 - \xi + \xi^2 - \xi^3 + \xi^4 - \xi^5 + \xi^6 - \xi^7 + \cdots$ and memorize this result.

(b) Assume $y = f(x) = \tan^{-1} x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$ where $a_0, a_1, a_2, \ldots$ are constants to be determined. Show

$$\frac{dy}{dx} = f'(x) = \frac{1}{1+x^2} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

(c) Use the well known result from part (a) to expand $f'(x) = \dfrac{1}{1+x^2}$ and compare the expansion in part (b) with the expansion in part (c) to determine the coefficients $a_1, a_2, a_3, \ldots$

(d) Pick a particular value of x whereby you can determine a_0 and then give the series expansion for $\tan^{-1} x$

▶ **4-50.** Apply the method outlined in the previous problem to determine the series expansion for $\sin^{-1} x$.

▶ **4-51.** Show that

(a) $\quad a^x = 1 + x \ln a + \dfrac{x^2}{2!}(\ln a)^2 + \dfrac{x^3}{3!}(\ln a)^3 + \cdots$

(b) $\quad \sqrt{x+h} = \sqrt{x} + \dfrac{h}{2\sqrt{x}} - \dfrac{h^2}{2^3\, x^{3/2}} + \dfrac{h^3}{2^4\, x^{5/2}} - \dfrac{5h^4}{2^6\, x^{7/2}} + \cdots$

▶ **4-52.**

(a) Assume the series expansion $\quad y = a^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \qquad$ 52-(a)

and show $\quad \dfrac{dy}{dx} = a^x \ln a = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \qquad$ 52-(b)

(b) Substitute equation 52-(a) into equation 52-(b) and compare coefficients to show

$$a^x = 1 + x \ln a + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \frac{x^4}{4!}(\ln a)^4 + \frac{x^5}{5!}(\ln a)^5 + \cdots$$

▶ **4-53.** If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots}}}}$ show that $\dfrac{dy}{dx} = \dfrac{\cos x}{2y - 1}$

▶ **4-54.** Show that $e^{x \sin x} = 1 + x^2 + \dfrac{1}{3}x^4 + \dfrac{1}{120}x^6 + \cdots$

▶ **4-55.** Show that $e^{x \cos x} = 1 + x + \dfrac{1}{2}x^2 - \dfrac{1}{3}x^3 - \dfrac{11}{24}x^4 - \dfrac{1}{5}x^5 + \dfrac{61}{720}x^6 + \cdots$

▶ **4-56.** Show that $e^{x \tan x} = 1 + x^2 + \dfrac{5}{6}x^4 + \dfrac{19}{30}x^6 + \cdots$

Chapter 5

Applications of Calculus

Selected problems from various areas of physics, chemistry, engineering and the sciences are presented to illustrate applications of the differential and integral calculus. Many of these selected topics require knowledge of basic background material, such as terminology and fundamentals, associated with the area of application. Consequently, much of this chapter gives a presentation of selected basic material from areas of engineering, physics, chemistry and the sciences which is required knowledge for the understanding of many scientific applications of the differential and integral calculus.

Related Rates

The **rate of change of a quantity** $Q = Q(t)$ with respect to time t is denoted by the derivative $\frac{dQ}{dt}$. Problems which involve **rates of change** of two or more time dependent variables are referred to as **"related rate problems"**. The general procedure for solving related rate problems is something like the following.

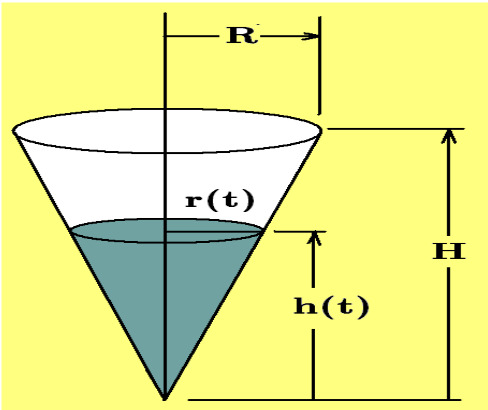
1. If necessary, **define the variables of the problem** and **make note of the units of measurement** being used. For example, one could write $[Q]$ = cubic centimeters which is read[1] "The dimension of Q is cubic centimeters".
2. Find **how the variables of the problem are related for all values of time** t being considered.
3. Determine if the variables of the problem, or their derivatives, have **known values at some particular instant of time.**
4. Find the **rate of change relation** between the variables by differentiating the relation or relations found in step 2 above.
5. Evaluate the results in step 4 **at the particular instant of time specified**.

Example 5-1. Consider a large inverted right circular cone with altitude H and base radius R where water runs into the cone at the rate of 3 cubic feet per second. How fast is the water level rising when the water level, as measured from the vertex of the cone, is 4 feet? Here the base radius R and height H of the cone are considered as fixed constants.

[1] Notation introduced by J.B.J. Fourier, theorie analytique de la chaleur, Paris 1822.

Solution Let $r = r(t)$, $[r] =$ feet, denote the radius of the water level at time t and let $h = h(t)$, $[h] =$ feet, denote the height of the water level at time t, $[t] =$ minutes. One can then express the volume V of water in the cone at time t as

$$V = V(t) = \frac{\pi}{3} r^2 h, \qquad [V] = \text{cubic feet} \tag{5.1}$$



Using **similar triangles** one finds that there is a relation between the variables r and h given by

$$\frac{r}{h} = \frac{R}{H} \qquad \text{or} \qquad r = \frac{R}{H} h \tag{5.2}$$

The given problem states that $\frac{dV}{dt} = 3$, where $[\frac{dV}{dt}] = \text{ft}^3/\text{min}$ and it is required to find $\frac{dh}{dt}$, when $h = 4$ feet. Differentiating equation (5.1) with respect to time t gives

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2 \frac{dh}{dt} + 2r \frac{dr}{dt} h\right] \tag{5.3}$$

and differentiating equation (5.2) with respect to t gives

$$\frac{dr}{dt} = \frac{R}{H}\frac{dh}{dt} \tag{5.4}$$

since R and H are constants. Substituting the results from the equations (5.2) and (5.4) into the equation (5.3) gives

$$\frac{dV}{dt} = \frac{\pi}{3}\left[\left(\frac{R^2}{H^2} h^2\right)\frac{dh}{dt} + 2\left(\frac{R}{H} h\right)\left(\frac{R}{H}\frac{dh}{dt}\right) h\right]$$

which simplifies to

$$\frac{dV}{dt} = \pi \frac{R^2}{H^2} h^2 \frac{dh}{dt} \tag{5.5}$$

Now one can evaluate the equation (5.5) when $h = 4$ to obtain

$$3 = \pi \frac{R^2}{H^2}(4)^2 \frac{dh}{dt} \qquad \text{or} \qquad \frac{dh}{dt} = \frac{3}{16\pi}\frac{H^2}{R^2}, \qquad \left[\frac{dh}{dt}\right] = \text{ft/min} \tag{5.6}$$

Alternatively, one could have substituted the equation (5.2) into the equation (5.1) to obtain

$$V = \frac{\pi}{3}\frac{R^2}{H^2} h^3 \tag{5.7}$$

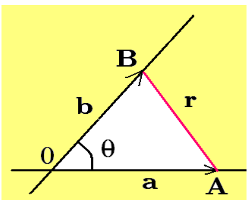
and then differentiate the equation (5.7) with respect to time t to obtain

$$\frac{dV}{dt} = \frac{\pi}{3}\frac{R^2}{H^2}\left(3h^2 \frac{dh}{dt}\right)$$

and evaluating this last equation when $h = 4$ gives the same result as equation (5.6). ∎

Example 5-2. Two roads intersect at point 0 at an angle of 60 degrees. Assume car A moves away from 0 on one road at a speed of 50 miles per hour and a second car B moves away from point 0 at 60 miles per hour on the other road. Let a denote the distance from point 0 for car A and let b denote the distance from point 0 for car B. How fast is the distance between the cars changing when $a = 1$ mile and $b = 2$ miles.

Solution



Let r denote the distance between the cars A and B and use the **law of cosines** to show

$$r^2 = a^2 + b^2 - 2ab\cos\theta$$

Differentiate this relation with respect to time t and show

$$2r\frac{dr}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} - 2\left[a\frac{db}{dt} + \frac{da}{dt}b\right]\cos\theta, \qquad \theta \text{ is a constant}$$

and then solve for the rate of change $\frac{dr}{dt}$ to obtain

$$\frac{dr}{dt} = \frac{a\frac{da}{dt} + b\frac{db}{dt} - \left[a\frac{db}{dt} + \frac{da}{dt}b\right]\cos\theta}{\sqrt{a^2 + b^2 - 2ab\cos\theta}}$$

Substitute into this equation the values

$$a = 1 \text{ mi} \qquad b = 2 \text{ mi} \qquad \theta = \frac{\pi}{3}$$
$$\frac{da}{dt} = 50 \text{ mi/hr} \qquad \frac{db}{dt} = 60 \text{ mi/hr}$$

and show $\frac{dr}{dt} = 30\sqrt{3}$ miles per hour.

■

Example 5-3. Boyle's[2] law resulted from a study of an ideal compressed gas at a constant temperature. Boyle discovered the relation $PV = C = constant$, where P represents pressure, $[P] =$ Pascal, abbreviated Pa, and V represents volume, $[V] = \text{cm}^3$ and C is a constant. If at some instant the pressure is P_0 and the volume of the gas has the value V_0 and the pressure is increasing at the rate r_0, $[r_0] =$ Pa/min, then at what rate is the volume decreasing at this instant?

[2] Robert Boyle (1627-1691) an Irish born chemist/mathematician.

Solution Here **Boyle's law is $\boldsymbol{PV = P_0 V_0} =$ constant**, where the pressure and volume are changing with respect to time. Differentiating this relation with respect to time t gives the relation

$$P\frac{dV}{dt} + \frac{dP}{dt}V = \frac{d}{dt}(P_0V_0) = 0 \tag{5.8}$$

Evaluating the equation (5.8) at the instant where $\frac{dP}{dt} = r_0$, $P = P_0$ and $V = V_0$, one finds

$$P_0\frac{dV}{dt} + r_0V_0 = 0 \qquad \text{or} \qquad \frac{dV}{dt} = -r_0\frac{V_0}{P_0}$$

The minus sign indicates that the volume is decreasing and the volume rate of change has dimension, $[\frac{dV}{dt}] = \text{cm}^3/\text{min}$.

Note that Boyle's law is a special case of the more general gas law given by $\frac{PV}{T} = C = \mathit{Constant}$ relating pressure P, volume V and temperature T all having appropriate units of measurements.

■

Newton's Laws

Isaac Newton used his new mathematical knowledge of calculus to formulate basic principles of physics in studying the motion of objects and particles. The following are known as **Newton's laws of motion**.

(i) **Newton's First Law**

A body at rest tends to stay at rest or a body in a uniform straight line motion tends to stay in motion unless acted upon by an external force.

(ii) **Newton's Second Law**

The time rate of change of momentum[3] of a body is proportional to the resultant force that acts upon it.

(iii) **Newton's Third Law**

For every action there is an equal and opposite reaction.

In the following discussions the symbols F, x, v, a, m, p, t are used to denote force, distance, velocity, acceleration, mass, momentum and time. Time t is measured in units of seconds, abbreviated (s). The symbol F is to denote force, measured in units of Newton's abbreviated[4] (N). The quantity x denotes distance, measured in meters, abbreviated (m). The velocity is denoted $v = \frac{dx}{dt}$ and represents the

[3] Momentum is defined as mass times velocity.

[4] If a unit of measurement is named after a person, then the unit is capitalized, otherwise it is lower case.

change in distance with respect to time. The velocity is measured in units of meters per second, abbreviated (m/s). The second derivative of distance $\frac{d^2x}{dt^2} = \frac{dv}{dt} = a$ or derivative of the velocity with respect to time t, is called the acceleration, which is measured in units (m/s^2). The symbol m denotes the mass[5] of a body, measured in units called kilograms, abbreviated (kg) and the momentum $p = mv$ is defined as the mass times the velocity and is measured in units (kg $\cdot$ m/s).

The first law concerns the **inertia** of a body. A body at rest, unless acted upon by an external force, will remain at rest. In terms of the above symbols one can write the first law as $F = 0$ or $\Delta v = 0$. That is, if the body is at rest, then it has no forces acting on it and if the body is in a state of uniform motion, then there is no change in the velocity. An external force is required to change the state of rest or uniform motion.

The **momentum** p of a body is defined as **the mass times the velocity** and written $p = mv$. Consequently, the second law can be expressed $F =\propto \frac{d}{dt}(mv)$, where $\propto$ is a proportionality sign. The units of measurement for force, mass, distance, velocity and time **are selected to make the proportionality constant unity** so that one can write Newton's second law as

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} \tag{5.9}$$

If the mass is constant and does not change with time, then the second law can be expressed

$$F = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = ma \tag{5.10}$$

The **units of measurement** used for the representation of Newton's laws are either the meter-kilogram-second system (MKS), the centimeter-gram-second system (CGS) or the foot-pound-second system (FPS) where

$$\begin{Bmatrix} MKS \\ F \text{ in N} \\ m \text{ in kg} \\ a \text{ in m/s}^2 \end{Bmatrix} \qquad \begin{Bmatrix} FPS \\ F \text{ in lb} \\ m \text{ in slugs} \\ a \text{ in ft/s}^2 \end{Bmatrix} \qquad \begin{Bmatrix} CGS \\ F \text{ in dynes} \\ m \text{ in gm} \\ a \text{ in cm/s}^2 \end{Bmatrix}$$

[5] Note the subtle distinction between the notation used to denote mass (m) and meters (m).

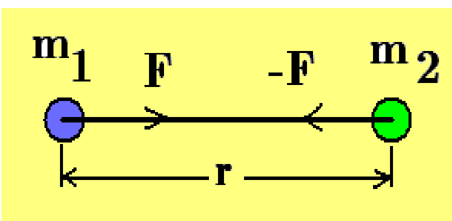
$$1\,\text{N} = 10^5\,\text{dynes} = \ 0.2248\,\text{lbs-force}$$
$$1\,\text{Kg} = 6.852\,(10)^{-2}\,\text{slugs} = \ 1000\,\text{gm}$$
$$9.807\,\text{m/s}^2 = 32.17\,\text{ft/s}^2 = \ 980.7\,\text{cm/s}^2$$

Here F denotes a summation of the forces acting in the direction of motion. Note that if the sum of the forces or **resultant force** is zero, then the object is said to be in **translational equilibrium**. If the velocity of a body is constant, but its mass is changing, then the equation of motion (5.9) becomes

$$F = v\frac{dm}{dt} \tag{5.11}$$

In terms of symbols, the third law can be expressed by examining two bodies, call them body A and body B. If body A exerts a force F_{AB} on body B, then body B exerts a force F_{BA} on body A and the third law requires that $F_{AB} = -F_{BA}$, that is **the forces are equal and opposite.**

Newton's Law of Gravitation



Newton's **law of gravitation** states that the centers of mass associated with two solids m_1 and m_2 experience an inverse square law force F of attraction given by

$$F = \frac{Gm_1m_2}{r^2} \tag{5.12}$$

where r is the distance between the centers of mass and $G = 6.673\,10^{-11}\,\text{m}^3/\text{kg}\cdot s^2$ is a proportionality constant called the **gravitational constant.**

If $m_1 = m_e$ is the mass of the Earth and $m_2 = m$ is the mass of an object at a height h above the surface of the Earth, then the force of gravity between these masses is given by

$$F_g = \frac{Gm_e\,m}{(r_e+h)^2} = m\left[\frac{Gm_e}{(r_e+h)^2}\right] \tag{5.13}$$

where r_e denotes the radius of the Earth[6]. Write the quantity in brackets as

$$\left[\frac{Gm_e}{(r_e+h)^2}\right] = \frac{Gm_e}{r_e^2}\left(1+\frac{h}{r_e}\right)^{-2} \approx \frac{Gm_e}{r_e^2} \tag{5.14}$$

[6] The radius of the Earth is approximately $6400\,km \approx 4000\,mi$ and the mass of the Earth is approximately $6.035\,(10)^{24}\,kg$.

since h is much less than the radius of the Earth r_e. The equation (5.14) can be used to define the following terms.

The **acceleration of gravity** g is defined

$$g = \frac{Gm_e}{r_e^2} \tag{5.15}$$

and the **weight** W of an object of mass m due to gravity is defined

$$W = F_g = m\,g \tag{5.16}$$

That is, the weight of an object is the force (force of gravity), by which an object of mass m is pulled vertically downward toward the center of the Earth. The dimensions of g and W are given by $[g] = \text{m/s}^2$, and $[W] = \text{kg} \cdot \text{m/s}^2 = N$. The acceleration of gravity varies slightly over the surface of the Earth because the radius of the Earth is not constant everywhere. If r_e is assumed to be constant, then the acceleration of gravity is found to have the following values in the MKS, FPS and CGS system of units

$$g = 9.807\,\text{m/s}^2, \qquad g = 32.17\,\text{ft/s}^2, \qquad g = 980.7\,\text{cm/s}^2 \tag{5.17}$$

Example 5-4. **Approximating Value of Escape Velocity**

A rocket launched straight upward from the surface of the Earth will fall back down if it doesn't achieve the correct velocity. Let $r = r(t)$ denote the distance of the rocket measured from the center of the Earth and let $m = m(t)$ denote the mass of the rocket which changes with time. The forces acting on the rocket as it moves upward are the thrust from the engines, the pull of gravity and resistance due to air friction called a drag force. By Newton's second law one can write

$$\frac{d}{dt}(mv) = \frac{d}{dt}\left(m\frac{dr}{dt}\right) = F_{total} = F_{thrust} - F_{gravity} - F_{drag} \tag{5.18}$$

where $v = v(t) = \dfrac{dr}{dt}$ is the velocity of the rocket. This is an equation, called a differential equation, which describes the motion of the rocket. When you learn more about aerodynamics you will learn how to represent the thrust force and drag forces on the rocket and then you can solve the resulting differential equation.

Instead, let us solve a much simpler problem created by **making assumptions** which will reduce the equation (5.18) to a form which is tractable[7]

[7] When confronted with a very difficult problem to solve, one can always make assumptions to simplify the problem to a form which can be solved. Many times an analysis of the simplified solution produces an incite into how to go about solving the more difficult original problem.

(i) Neglect the thrust force and drag force and consider only the gravitational force.

(ii) Assume the mass of the rocket remains constant.

(iii) Assume that at time $t=0$, the initial velocity of the rocket is v_0 and the position of the rocket is given by $r(0)=r_e$, where r_e is the radius of the Earth.

These assumptions greatly simplify the differential equation (5.18) to the form

$$m\frac{dv}{dt} = -\frac{Gm_e\,m}{r^2} \qquad \text{where} \quad \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v \tag{5.19}$$

and one obtains after simplification the differential equation

$$v\frac{dv}{dr} = -\frac{Gm_e}{r^2} \tag{5.20}$$

One can separate the variables and express equation (5.20) in the form

$$v\,dv = -Gm_e\frac{dr}{r^2} \tag{5.21}$$

An integration of both sides of this separated equation gives the result

$$\int v\,dv = -Gm_e\int\frac{dr}{r^2} \qquad \Longrightarrow \qquad \frac{v^2}{2} = \frac{Gm_e}{r} + C \tag{5.22}$$

where C is a constant of integration. The constant C is selected such that the initial conditions are satisfied. This requires

$$\frac{v_0^2}{2} = \frac{Gm_e}{r_e} + C \qquad \Longrightarrow \qquad C = \frac{v_0^2}{2} - \frac{Gm_e}{r_e} \tag{5.23}$$

Substitute this value for C into the equation (5.22) and simplify the result to obtain

$$v^2 = v_0^2 + 2Gm_e\left(\frac{1}{r} - \frac{1}{r_e}\right) \tag{5.24}$$

In equation (5.24) the term $\dfrac{2Gm_e}{r}$ is always positive so that if it is required that

$$v_0^2 - \frac{2Gm_e}{r_e} \geq 0$$

then one can say that the velocity of the rocket will always be positive. This condition can be written

$$v_0 \geq \sqrt{\frac{2Gm_e}{r_e}} = \left(2\frac{Gm_e}{r_e^2}r_e\right)^{1/2} = \sqrt{2g\,r_e} \approx 11200\,\text{m/s} \approx 7\,\text{miles per second}$$

where $g = \dfrac{Gm_e}{r_e^2}$ is the acceleration of gravity. This value is a good approximation of the velocity necessary to overcome the gravitational forces pulling the rocket back to Earth. This velocity is called **the escape velocity.**

■

Work

Let W denote the **work done** by a constant force F, which has moved an object in a straight line a distance x in the direction of the force. Here W is defined[8] as the scalar quantity

$$\text{Work} = \text{Force times distance} \qquad W = F\,x \tag{5.25}$$

If the force $F = F(x)$ varies continuously as the distance x changes, then if the object is moved in a straight line an increment dx, the **increment of work done** dW is expressed

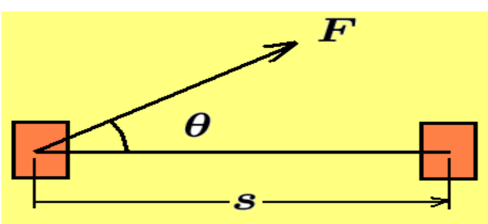
$$dW = F(x)\,dx$$

and the total work done in moving an object from x_1 to x_2 in a straight line is given by the integral

$$\boldsymbol{W = \int_{x_1}^{x_2} F(x)\,dx} \tag{5.26}$$

The equation (5.26) tells us that **the work done is nothing more than the area under the curve $F = F(x)$ between the values x_1 and x_2.**

Example 5-5.



If the constant force F acts at an angle to the direction of motion, then the component of force in the direction of motion is $F\cos\theta$ and the work done in moving an object is the component of force in direction of the displacement times the displacement or $W = (F\cos\theta)\,s$

If the force $F = F(s)$ varies as a function of displacement s, then the increment of work done in moving an object the incremented distance ds is

$$dW = (F(s)\cos\theta)\,ds$$

and the total work done moving an object from s_1 to s_2 is

$$W = \int_{s_1}^{s_2} F(s)\cos\theta\,ds$$

[8] If there are many discrete forces acting on a body at different times, then one can define the work as the average force times the displacement.

Recall that force is measured in **units called Newtons**, where $1\,\text{N} = 1\,\text{kg}\cdot\text{m/s}^2$. Displacement is measured in meters (m) so that work is force times distance and is measured in units of **Newton-meters** or $(\text{N}\cdot\text{m})$ and one can write $[W] = \text{N}\cdot\text{m}$, which is read, "The dimension of work is Newton-meter". By definition $1\,\text{N}\cdot\ \text{m} = 1\,\text{Joule}$, where Joule is abbreviated (J).

Energy

In the language of science the term **energy** is a scalar measure of a physical systems **ability to do work.** There are many different kinds of energy. A few selected types of energy you might have heard of are **chemical energy, kinetic energy, various kinds of potential energy, internal energy, elastic energy due to stretching or twisting, heat energy, light energy and nuclear energy.**

Kinetic Energy E_k

The energy associated with **a body in motion** is called **kinetic energy** and is denoted by E_k. The kinetic energy is defined $E_k = \frac{1}{2}mv^2$, where m is the mass of the body, $[m] = \text{kg}$ and v is the velocity of the body, $[v] = \text{m/s}$. Kinetic energy is a positive scalar quantity measured in the same units as work. One can verify that $[E_k] = \text{kg}\cdot\text{m}^2/\text{s}^2 = (\text{kg}\cdot\text{m/s}^2)\cdot\text{m} = \text{N}\cdot\ \text{m} = \text{J}$

Example 5-6. The work done by a constant force F moving an object in a straight line through a distance s during a time t is given by the integral

$$W = \int_0^s F\,ds \tag{5.27}$$

Let s denote distance traveled during a time t with $\frac{ds}{dt} = v$ denoting the velocity and $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ denoting the acceleration. Using Newton's second law of motion one can write

$$F = ma = m\frac{dv}{dt} = m\frac{d^2s}{dt^2}, \qquad \text{where } \frac{ds}{dt} = v \text{ and } \frac{d^2s}{dt^2} = \frac{dv}{dt} = a \tag{5.28}$$

Substituting the equation (5.28) into the equation (5.27) gives

$$W = \int_0^s \left(m\frac{dv}{dt}\right) ds = \int_0^s m\,\frac{d^2s}{dt^2}\,ds = \int_0^t m\,\frac{d^2s}{dt^2}\,\frac{ds}{dt}\,dt \tag{5.29}$$

Observe the equation (5.29) is written as an integration with respect to time by using the relations $v = \frac{ds}{dt}$ and $\frac{dv}{dt} = \frac{d^2s}{dt^2}$. If the object has an initial velocity v_0 at time $t = 0$, then the integration (5.29) can be expressed in the form

$$W = \int_0^t m\frac{dv}{dt}\,v\,dt = m\int_0^t d\left(\frac{1}{2}v^2\right) = \frac{1}{2}mv^2\Big|_0^t = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \tag{5.30}$$

The equation (5.30) is a representation of the **work-energy relation**

"The work done by forces acting on a body

equals the change in kinetic energy of the body"

■

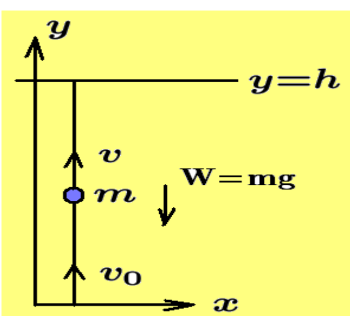
Potential Energy E_p

The energy associated with a body **as a result of its position with respect to some reference line** is called the **potential energy** and is defined $E_p = mgh$, where m is the mass of the body, $[m] = \text{kg}$, g is the acceleration of gravity, $[g] = \text{m/s}^2$ and h is the height of the body above the reference line, $[h] = \text{m}$. The potential energy is sometimes called the **gravitational potential energy.** The potential energy is measured in units of $\text{kg}\cdot\text{m/s}^2\cdot\ \text{m} = \text{N}\cdot\text{m} = J$ and has the same units of measurement as work. The work done against gravity in lifting a weight from a height h_1 to a height h_2 is given by

$$W = \int_{h_1}^{h_2} -F_g\,dx = \int_{h_1}^{h_2} -mg\,dx = -mg\,x\,\Big|_{h_1}^{h_2} = -(mg\,h_2 - mg\,h_1) = -\Delta E_p$$

where $-F_g = -W$ is the weight acting downward. One can say the work done equals the change in potential energy.

Example 5-7.



Consider a ball of mass m which is thrown vertically upward with an initial velocity v_0. Neglect air resistance so that the only force acting on the ball is the force due to gravity and construct a coordinate system with the origin placed at the point where the ball is released. Here the upward direction is taken as positive and by Newton's second law one can write

$$m\frac{dv}{dt} = -mg \tag{5.31}$$

since the weight of the ball is mg and this force is acting downward. Separate the variables in equation (5.31) and then integrate to obtain

$$\int_{v_0}^{v} m\,dv = -mg\int_0^t dt \qquad \text{or} \qquad mv - mv_0 = -mgt \tag{5.32}$$

If y denotes the distance of the ball above the reference axis, then the velocity of the ball is given by $v = \frac{dy}{dt}$. The equation (5.32) can now be expressed in the form

$$m\frac{dy}{dt} = mv_0 - mgt \tag{5.33}$$

since the velocity $v = \frac{dy}{dt}$ represents the change in the height of the ball as a function of time. Multiply equation (5.33) by dt and integrate to obtain

$$m\int_0^y dy = \int_0^t [mv_0 - mgt]\, dt \qquad \text{or} \qquad my = mv_0 t - \frac{1}{2}mgt^2 \tag{5.34}$$

Solve equation (5.32) for the variable t and substitute for t in equation (5.34) and then simplify to show

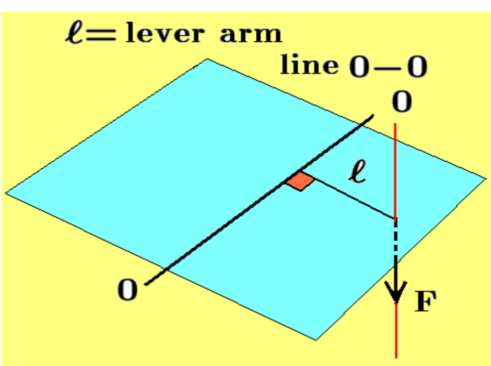
$$\frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2 \tag{5.35}$$

which can be interpreted as stating that **the sum of the kinetic energy plus the potential energy of the ball always has a constant value.** Note that when the ball reaches its maximum height, where $y = h$, the velocity of the ball is zero, and at this time the equation (5.35) shows that the initial kinetic energy of the ball equals the potential energy of the ball at its maximum height.

■

There are many more types of energy and all these energy types obey the **law of conservation of energy** which states that there is no change in the total energy in the Universe. Another way of saying this is to state that energy can be transformed, but it cannot be created or destroyed.

First Moments and Center of Gravity



Consider a force F acting perpendicular to a plane containing a line $0-0$. The **first moment of a force** F, also called **a torque**, is defined

$$\begin{aligned} \textbf{Moment} &= \textbf{(Force)(Lever arm distance)} \\ M &= F\ell \end{aligned} \tag{5.36}$$

where the lever arm ℓ is understood to represent the **shortest perpendicular distance from the line** $0-0$ **to the line of action of the force** F. The moment is **a measure of the ability of the force to produce a rotation** about the line $0-0$. In general a quantity times a distance to a point, or times a distance to a line, or times a distance to a plane, is called a moment of that quantity with respect to a point, line or plane.

Centroid and Center of Mass

In the figure 5-1 the x-axis is considered as a see-saw with weights W_1 and W_2 placed at the positions $(x_1, 0)$ and $(x_2, 0)$ respectively. Consider the problem of determining where one would place a fulcrum so that the see-saw would balance.

Let $(\bar{x}, 0)$ denote the point where the fulcrum is placed and let $\ell_2 = x_2 - \bar{x}$ denote a lever arm associated with the weight W_2 and let $\ell_1 = \bar{x} - x_1$ denote the lever arm associated with the weight W_1. The see-saw will balance if $\bar{x}$ is selected such that the sum of the moments[9] about the fulcrum equals zero. This requires that

$$M_1 = \ell_1 W_1 = \ell_2 W_2 = M_2 \qquad \text{or} \qquad (\bar{x} - x_1) W_1 = (x_2 - \bar{x}) W_2 \tag{5.37}$$

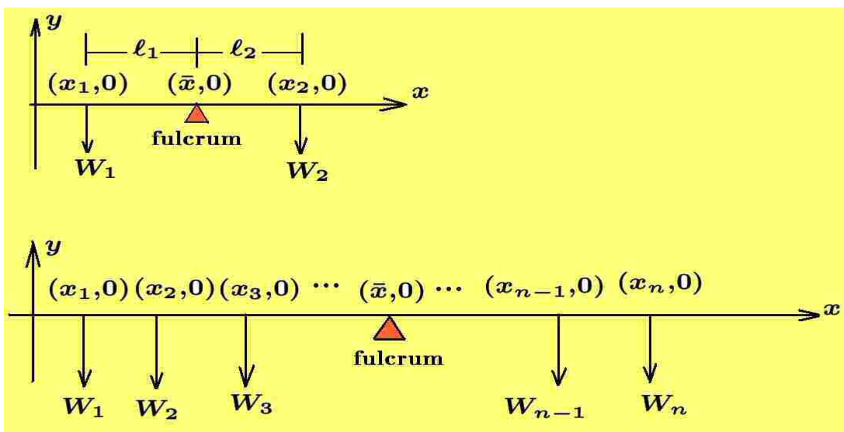


Figure 5-1. Balancing of weights using moments.

Another way to express the balancing of the see-saw is to examine the distances $\bar{x} - x_1$ and $\bar{x} - x_2$. One distance is positive and the other is negative and the product $(\bar{x} - x_1)W_1$ gives a positive moment and the product $(\bar{x} - x_2)W_2$ gives a negative moment. One can then say that the moments produced by the weights balance if $\bar{x}$ is selected such that the sum of the moments is zero or

$$\sum_{i=1}^{2} (\bar{x} - x_i) W_i = 0 \qquad \text{or} \qquad \bar{x} = \frac{W_1 x_1 + W_2 x_2}{W_1 + W_2} \tag{5.38}$$

The point $(\bar{x}, 0)$ is then called the **center of gravity or centroid of the system.**

---

[9] By placing the fingers of the right-hand in the direction of the force and letting the fingers move in the direction of rotation produced by the force, then the thumb points in a positive or negative direction. If the z-axis comes out of the page toward you, then this is the positive direction assigned to the moment. The moment $M_1 = \ell_1 W_1$ is then said to be a positive moment and the moment $M_2 = \ell_2 W_2$ is called a negative moment. The sum of the moments equal to zero is then written $-\ell_2 W_2 + \ell_1 W_1 = 0$.

If there are n-weights $W_1, W_2, \ldots, W_n$ placed at the positions $(x_1,0),(x_2,0),\ldots,(x_n,0)$ respectively, then **the centroid of the system** is defined as that point $(\bar{x},0)$ where the sum of the moments produces zero or

$$\sum_{i=1}^{n}(\bar{x}-x_i)W_i = 0 \qquad \text{or} \qquad \bar{x} = \frac{W_1x_1+W_2x_2+\cdots+W_nx_n}{W_1+W_2+\cdots+W_n} = \frac{\sum_{i=1}^{n} W_i x_i}{\sum_{i=1}^{n} W_i} \tag{5.39}$$

If $W=\sum_{i=1}^{n} W_i$ is the total sum of the weights, then equation (5.39) can be written as

$$W\,\bar{x} = W_1x_1+W_2x_2+\cdots+W_nx_n \tag{5.40}$$

and this equation has the following interpretation. Imagine a three-dimensional right-handed xyz Cartesian system of axes with the z-axis in figure 5-1 coming out of the page. Each weight W_i then produces a moment $M_i = W_i x_i$ about the z-axis. That is, for each value $i=1,2,\ldots,n$, the distance x_i denotes the lever arm and W_i denotes the force. The total sum of these moments gives the right-hand side of equation (5.40). The left-hand side of equation (5.40) is then interpreted as stating that if the sum of the weights W was placed at the position $(\bar{x},0)$, it would create a moment $M = W\,\bar{x}$ equivalent to summing each individual moment produced by all the weights. The position $\bar{x}$ is then called the **center of gravity or centroid of the system**. Another interpretation given to equation (5.39) is that the numerator is a weighted sum of the x-values and the denominator is the sum of the weights so that $\bar{x}$ is then **a weighted average of the x-values**.

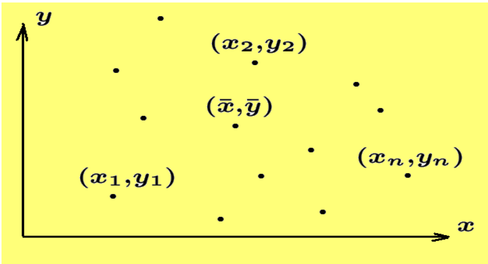
One can **generalize the definition of a first moment** by defining a first moment associated with just about any quantity. For example, one can define first moments such as

$$\begin{aligned} M_f =&(force)(lever\ arm) & M_v =&(volume)(lever\ arm)\\ M_m =&(mass)(lever\ arm) & \vdots\quad & \quad\vdots\\ M_a =&(area)(lever\ arm) & M_q =&(quantity)(lever\ arm)\end{aligned} \tag{5.41}$$

where the lever arm is understood to represent the shortest perpendicular distance from some reference point, line or plane to the quantity.

Example 5-8. Let $m_1, m_2, \ldots, m_n$ denote n point masses located respectively at the points $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$. Find the center of mass $(\bar{x},\bar{y})$ of this system of masses.

Solution



Use the x and y-axes as the lines about which one can take first moments associated with the given point masses. If $m = \sum_{i=1}^{n} m_i$ is the total sum of all the point masses, then if this mass were placed at the point $(\bar{x}, \bar{y})$ it would produce first moments about these axes given by

$$M_x = m\bar{y} \qquad \text{and} \qquad M_y = m\bar{x}$$

These moments must be equivalent to the sum of the first moments produced by each individual mass so that one can write

$$M_y = m\bar{x} = \sum_{i=1}^{n} m_i x_i \qquad \text{and} \qquad M_x = m\bar{y} = \sum_{i=1}^{n} m_i y_i$$

The **center of mass of the system** then has the coordinates $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{M_y}{m} \qquad \text{and} \qquad \bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} = \frac{M_x}{m}$$

Here the center of mass of the system of masses has coordinates $(\bar{x}, \bar{y})$ where $\bar{x}$ is a weighted sum of the x_i values and $\bar{y}$ is a weighted sum of the y_i values for positions ranging from $i = 1, 2, \ldots, n$

■

Centroid of an Area

Moments can be used to find the **centroid of an area** bounded by the curve $y = f(x) > 0$, the x-axis and the lines $x = a$ and $x = b$. Partition the interval $[a, b]$ into n equal parts with

$$a = x_0, \;\; x_1 = x_0 + \Delta x, \;\; x_2 = x_0 + 2\Delta x, \ldots, x_n = x_0 + n\Delta x = b \quad \text{where} \quad \Delta x = \frac{b-a}{n}$$

Consider the center of the rectangular element of area illustrated in the figure 5-2 which has the coordinates (ξ_i, y_i), where $\xi_i = x_{i-1} + \frac{\Delta x}{2}$ and $y_i = \frac{1}{2} f(\xi_i)$. The center of this element of area has a first moment about the y-axis given by

$$\Delta M_y = (lever\ arm)(area) = (\xi_i)[f(\xi_i)\,\Delta x_i]$$

and it also has a first moment about the x-axis given by

$$\Delta M_x = (lever\ arm)(area) = \left(\frac{1}{2} f(\xi_i)\right) [f(\xi_i)\,\Delta x_i]$$

A summation of the first moments associated with each rectangle produces a sum from 1 to n giving the total moments

$$M_y = \sum_{i=1}^{n} \xi_i f(\xi_i)\Delta x_i \qquad\qquad M_x = \sum_{i=1}^{n} \frac{1}{2}[f(\xi_i)]^2\,\Delta x_i$$

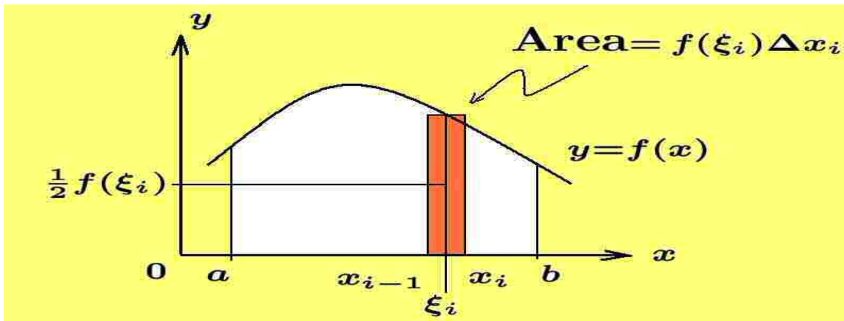


Figure 5-2. Moments for Centroid of an area.

Neglecting infinitesimals of higher order and using the **fundamental theorem of integral calculus** one finds that in the limit as $\Delta x_i \to 0$, the above sums become the definite integrals

$$M_y = \int_a^b x f(x)\,dx \qquad\qquad M_x = \int_a^b \frac{1}{2}[f(x)]^2\,dx \tag{5.42}$$

The total area under the curve $y = f(x)$ is given by the definite integral

$$A = \int_a^b f(x)\,dx$$

and if this total area were concentrated and placed at the point $(\bar{x}, \bar{y})$ it would produce moments about the x and y-axes given by $M_x = A\bar{y}$ and $M_y = A\bar{x}$. The centroid is that point $(\bar{x}, \bar{y})$ where

$$M_x = A\bar{y} = \int_a^b \frac{1}{2}[f(x)]^2\,dx \qquad \text{and} \qquad M_y = A\bar{x} = \int_a^b x f(x)\,dx \tag{5.43}$$

from which one can solve for $\bar{x}$ and $\bar{y}$ to obtain $\bar{y} = \dfrac{M_x}{A}$ and $\bar{x} = \dfrac{M_y}{A}$.

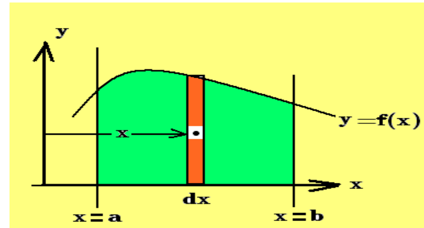
In a similar fashion one can use **the fundamental theorem of integral calculus** to show the lever arms associated with the first moments of the center point of an element of area can be expressed in terms of the x and y coordinates associated with the element of area. One can then verify the following lever arm equations associated with the elements of area illustrated.

1.) For the center point of the element of area

$dA = y\,dx$

lever arm to y-axis is x

lever arm to x-axis is $y/2$

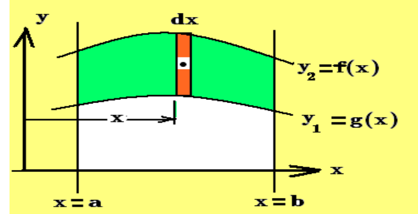


2.) For the center point of the element of area

$dA = (y_2 - y_1)\,dx$

lever arm to y-axis is x

lever arm to x-axis is $\frac{1}{2}(y_1 + y_2)$

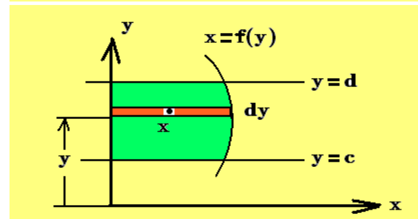


3.) For the center point of the element of area

$dA = x\,dy$

lever arm to y-axis is $x/2$

lever arm to x-axis is y

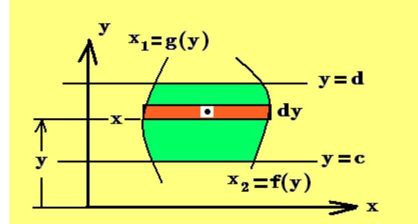


4.) For the center point of the element of area

$dA = (x_2 - x_1)\,dy$

lever arm to y-axis is $\frac{1}{2}(x_1 + x_2)$

lever arm to x-axis is y



Note that in determining the above lever arm distances the **infinitesimals of higher order have been neglected.**

For example, associated with the last figure there is an element of area given by $dA = (x_2 - x_1)\,dy = [g(y) - f(y)]\,dy$ and the total area is given by

$$A = \int_c^d [g(y) - f(y)]\,dy$$

This element of area has a moment about the x-axis given by

$$dM_x = (lever\ arm)(area) = y\,dA = y[g(y) - f(y)]\,dy$$

and a moment about the y-axis given by

$$dM_y = (lever\ arm)(area) = \frac{1}{2}[g(y) + f(y)][g(y) - f(y)]\,dy = \frac{1}{2}\left[g^2(y) - f^2(y)\right]\,dy$$

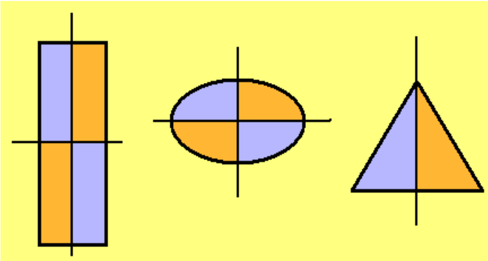
Summing these moments one finds

$$M_x = \int_c^d y[g(y) - f(y)]\,dy \qquad \text{and} \qquad M_y = \int_c^d [g^2(y) - f^2(y)]\,dy$$

with the centroid $(\bar{x}, \bar{y})$ found from the relations

$$\bar{x} = \frac{M_y}{A} \qquad \text{and} \qquad \bar{y} = \frac{M_x}{A}$$

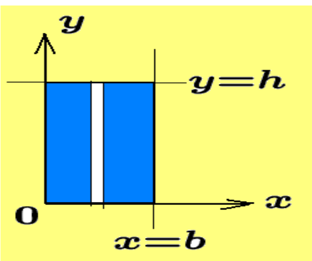
Symmetry



If an object has an **axis of symmetry**, then the **centroid of the object must lie on this line**. For example, if an area has a line of symmetry, then when the area is rotated 180° about this line the area has its same shape. Examine the rectangle when rotated about a line through its center and find out the rectangle is unchanged. One can say the centroid for the rectangle is at its geometric center.

Example 5-9. Use the equations (5.43) and find the centroid of a rectangle of height h and base b.

Solution



Here $y = f(x) = h$ is a constant and so one can write

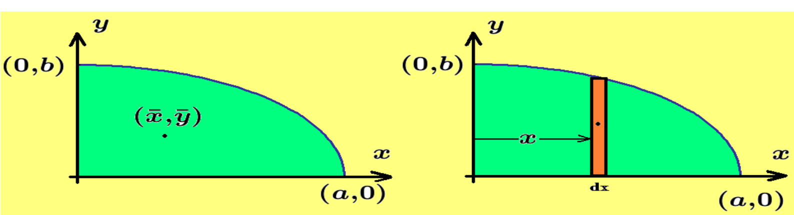
$$M_y = \int_0^b x f(x)\, dx = \int_0^b xh\, dx = \frac{1}{2}hb^2$$
$$M_x = \int_0^b \frac{1}{2}[f(x)]^2\, dx = \int_0^b \frac{1}{2}h^2\, dx = \frac{1}{2}bh^2$$

The total area of the rectangle is $A = bh$ and so the centroid $(\bar{x}, \bar{y})$ is determined by the equations

$$\bar{x} = \frac{M_y}{A} = \frac{b}{2} \qquad \text{and} \qquad \bar{y} = \frac{M_x}{A} = \frac{h}{2}$$ ■

Example 5-10. Find the centroid of the area bounded by the x-axis, the y-axis and the ellipse defined by the parametric equations $x = a\cos\theta$, $y = b\sin\theta$, for $0 \le \theta \le \pi/2$ and $a > b > 0$ constants.

Solution



The area to be investigated is the upper quadrant of an ellipse. Move out a distance x from the origin and construct an element of area $dA = y\, dx$. and substitute $y = b\sin\theta$, $x = a\cos\theta$ with $dx = -a\sin\theta\, d\theta$ and show the total area is

$$A = \int_0^a y\, dx = \int_{\pi/2}^0 b\sin\theta\, (-a\sin\theta)\, d\theta = ab \int_0^{\pi/2} \sin^2\theta\, d\theta$$
$$A = ab \int_0^{\pi/2} \frac{1}{2}[1 - \cos 2\theta]\, d\theta = \frac{\pi}{4}ab$$

The element of area dA has a moment about the y-axis given by

$$M_y = \int_0^a xy\,dx = \int_{\pi/2}^0 (a\cos\theta)(b\sin\theta)(-a\sin\theta)\,d\theta = a^2b\int_0^{\pi/2}\sin^2\theta\,\cos\theta\,d\theta = \frac{1}{3}a^2b$$

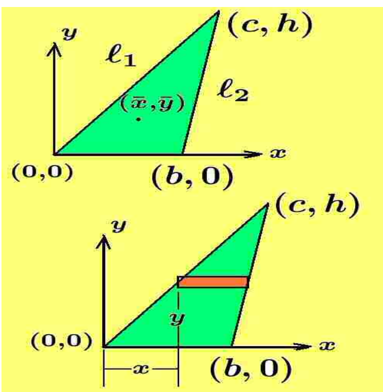
The element of area has a moment about the x-axis given by

$$M_x = \int_0^a \frac{1}{2}y(y\,dx) = \int_{\pi/2}^0 \frac{1}{2}b^2\sin^2\theta\,(-a\sin\theta)\,d\theta$$

$$M_x = \frac{ab^2}{2}\int_0^{\pi/2}\sin^3\theta\,d\theta = \frac{ab^2}{2}\int_0^{\pi/2}\left[\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right]\,d\theta = \frac{1}{3}ab^2$$

The centroid $(\bar{x},\bar{y})$ is given by $\quad \bar{x} = \dfrac{M_y}{A} = \dfrac{4}{3}\dfrac{a}{\pi} \quad$ and $\quad \bar{y} = \dfrac{M_y}{A} = \dfrac{4}{3}\dfrac{b}{\pi}$ ∎

Example 5-11. Find the centroid of the triangle with vertices $(0,0)$, $(b,0)$, (c,h)



The equation of the line ℓ_1 with slope h/c is given by $y = (h/c)x$. The equation of the line ℓ_2 with slope $h/(c-b)$ is given by $y = [h/(c-b)]\,(x-b)$ Construct a horizontal element of area

$$dA = (x_2 - x_1)\,dy$$

and show

$$dA = \left[b + \frac{(c-b)}{h}y - \frac{c}{h}y\right]\,dy = \left[b - \frac{b}{h}y\right]\,dy$$

and after summing these elements of area one finds

$$A = \int_0^h \left[b - \frac{b}{h}y\right]\,dy = \frac{1}{2}bh$$

This element of area has a moment about the x-axis given by

$$M_x = \int_0^h y\left[b - \frac{b}{h}y\right]\,dy = \frac{1}{6}bh^2$$

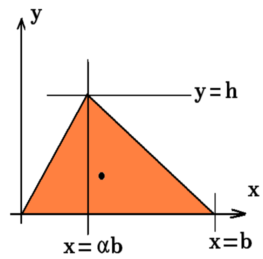
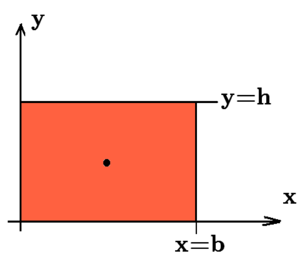
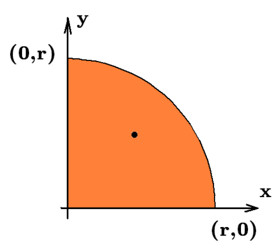
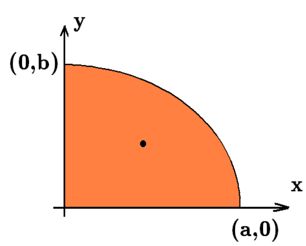
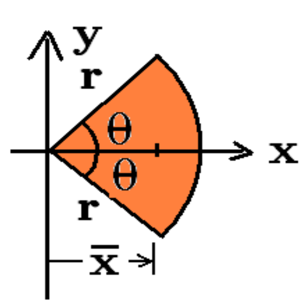
and moment about the y-axis given by

$$M_y = \int_0^h \frac{1}{2}\left[\left(b + \frac{(c-b)}{h}y\right)^2 - \frac{c^2}{h^2}y^2\right]\,dy = \frac{1}{6}hb(b+c)$$

The centroid for the given triangle is $(\bar{x},\bar{y})$ where

$$\bar{x} = \frac{M_y}{A} = \frac{1}{3}(b+c) \qquad \text{and} \qquad \bar{y} = \frac{M_x}{A} = \frac{h}{3}$$

∎

Table 5-1 Centroids of Some Simple Shapes				
Shape		Area	$\bar{x}$	$\bar{y}$
Triangle		$\frac{1}{2}bh$	$\frac{1}{3}(1+\alpha)b$	$\frac{1}{3}h$
Rectangle		bh	$\frac{b}{2}$	$\frac{h}{2}$
Quadrant of circle		$\frac{\pi}{4}r^2$	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$
Quadrant of ellipse		$\frac{\pi}{4}ab$	$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$
Wedge		θr^2	$\frac{2}{3}r\frac{\sin\theta}{\theta}$	0

Centroids of composite shapes

If an area is composed of some combination of simple shapes such as triangles, rectangles, circles or some other shapes where the centroids of each shape have known centroids, then the resultant moment about an axis is the **algebraic sum of the moments** of the component shapes and the centroid of the composite shape is given by $\bar{x} = \frac{M_y}{A}$ and $\bar{y} = \frac{M_x}{A}$, where A is the total area of the composite shape. Whenever the centroids of all the individual shapes which make up the total shape are known, then integration is not required.

Example 5-12. If the composite shape is composed of n known shapes having

$$\begin{array}{l} \text{area } A_1 \text{ with centroid } (\bar{x}_1, \bar{y}_1) \\ \text{area } A_2 \text{ with centroid } (\bar{x}_2, \bar{y}_2) \\ \quad\vdots \\ \text{area } A_n \text{ with centroid } (\bar{x}_n, \bar{y}_n) \end{array}$$

then the total area of the composite shape is

$$A = A_1 + A_2 + A_3 + \cdots + A_n$$

The total moment produced about the $y-$axis from each area is

$$M_y = A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 + \cdots + A_n\bar{x}_n$$

The total moment produced about the $x-$axis from each area is

$$M_x = A_1\bar{y}_1 + A_2\bar{y}_2 + A_3\bar{y}_3 + \cdots + A_n\bar{y}_n$$

The centroid of the composite shape is therefore

$$\bar{x} = \frac{M_y}{A} \qquad \text{and} \qquad \bar{y} = \frac{M_x}{A}$$

■

Centroid for Solid of Revolution

Consider the area bounded by the curve $y = f(x) > 0$ and the lines $x = a$, $x = b > a$ and the x-axis which is revolved about the x-axis to form a solid of revolution. One can construct an element of volume dV for this solid using the disk generated when dA is rotated about the x-axis. The volume element associated with this disk is

$$dV = \pi y^2\, dx \qquad \text{with} \qquad V = \int_a^b \pi y^2\, dx$$

where the integral represents a summation of the volume elements. The moment of the disk about the plane perpendicular to the axis of rotation which passes through the origin is

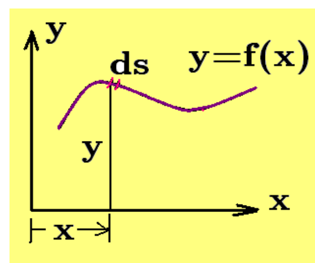
$$dM = x\,dV = \pi x\,y^2\,dx \qquad \text{and} \qquad M = \int_a^b \pi x y^2\,dx$$

where x is the lever arm distance from the plane to the volume element and M is a summation of these moments. The above integral is called **the first moment of the solid of revolution** with respect to the plane through the origin and perpendicular to the axis of rotation. The centroid $\bar{x}$ is then defined as

$$\bar{x} = \frac{M}{V} = \frac{\displaystyle\int_a^b xy^2\,dx}{\displaystyle\int_a^b y^2\,dx} \tag{5.44}$$

and by symmetry the position of $\bar{x}$ is on the axis of rotation.

Centroid for Curve



Let $y = f(x)$ define a smooth continuous curve for $a \le x \le b$. At the position (x, y) on the curve construct the element of arc length $ds = \sqrt{dx^2 + dy^2}$.

The first moments about the x and y-axes **associated with the curve**

$$C = \{\ (x, y)\ \mid\ y = f(x),\ a \le x \le b\ \}$$

are defined as **a summation of the first moments associated with the element of arc length** ds. One can define

$$dM_x = (\text{lever arm})(\text{element of arc length}) = y\,ds$$
$$dM_y = (\text{lever arm})(\text{element of arc length}) = x\,ds$$

A summation of these first moments gives

$$M_x = \int_a^b y\,ds \qquad \text{and} \qquad M_y = \int_a^b x\,ds$$

If the given curve C has an arc length s given by

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx,$$

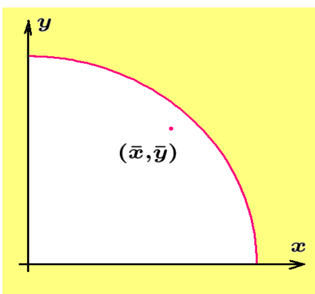
then the centroid of the curve C is defined as the point $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{s} \qquad \text{and} \qquad \bar{y} = \frac{M_x}{s} \tag{5.45}$$

That is, if the arc length s could be concentrated and placed at a point $(\bar{x}, \bar{y})$ called the centroid, then the centroid is selected so that the first moment $\bar{x}\,s$ is the same as that produced by the summation of individual moments about the y-axis and the first moment $\bar{y}\,s$ is the same as that produced by the summation of individual moments about the x-axis.

Example 5-13. Consider the arc of a circle which lies in the first quadrant. This curve is defined $C = \{\ (x,y)\ \mid\ x = r\cos\theta,\ y = r\sin\theta,\ 0 \le \theta \le \pi/2\ \}$ where r is the radius of the circle. Find the centroid associated with this curve.

Solution



The element of arc length squared is given by $ds^2 = dx^2 + dy^2$ so that one can write

$$s = \int_0^{\pi/2} ds = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$s = \int_0^{\pi/2} ds = \int_0^{\pi/2} r\, d\theta = r\,\theta\,\Big|_0^{\pi/2} = \frac{1}{2}\pi\, r$$

Use the equations (5.45) and calculate the first moments of the curve about the x and y-axes to obtain

$$M_x = \int_0^{\pi/2} y\, ds = \int_0^{\pi/2} r\sin\theta\, r\, d\theta = r^2(-\cos\theta)\,\Big|_0^{\pi/2} = r^2$$

$$M_y = \int_0^{\pi/2} x\, ds = \int_0^{\pi/2} r\cos\theta\, r\, d\theta = r^2(\sin\theta)\,\Big|_0^{\pi/2} = r^2$$

The centroid $(\bar{x}, \bar{y})$ is then $\qquad \bar{x} = \dfrac{M_y}{s} = \dfrac{2}{\pi} r \qquad \text{and} \qquad \bar{y} = \dfrac{M_x}{s} = \dfrac{2}{\pi} r$

■

Higher Order Moments

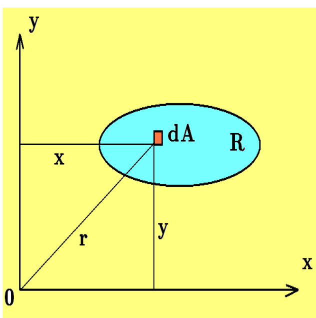
The various **first moments** defined by the relation (5.41) can be further generalized to **second moments** by replacing the lever arm by **the lever arm squared**. Second moments about a line ℓ are denoted $I_{\ell\ell}$ and one can write

$$\begin{aligned} I_{\ell\ell} &= (force)(lever\ arm)^2 & I_{\ell\ell} &= (volume)(lever\ arm)^2 \\ I_{\ell\ell} &= (mass)(lever\ arm)^2 & &\vdots \qquad \vdots \\ I_{\ell\ell} &= (area)(lever\ arm)^2 & I_{\ell\ell} &= (quantity)(lever\ arm)^2 \end{aligned} \tag{5.46}$$

where the lever arm is understood to represent a perpendicular distance from some reference line ℓ. Second moments are referred to as **moments of inertia.** In the study of rotational motion of rigid bodies it is found that the moment of inertia is **a measure of how mass distribution affects changes to the angular motion of a body as it rotates about an axis.**

Third order moments would involve the lever arm cubed and nth order moments would involve the lever arm raised to the nth power. Third order and higher order moments arise in the study of statistics, mechanics and physics.

Example 5-14.



Given a region R one can construct at a general point $(x,y) \in R$ an element of area dA. This element of area has a second moment of inertia about the $y-$axis given by

$$dI_{yy} = x^2\, dA$$

and summing these second moments of inertia over the region R gives the total second moment about the y-axis as

$$I_{yy} = \iint_R x^2\, dA$$

In a similar fashion, the second moment of inertia of the element dA about the $x-$axis is given by

$$dI_{xx} = y^2\, dA$$

and a summation of these second moments over the region R gives the total second moment about the x-axis as

$$I_{xx} = \iint_R y^2\, dA$$

If the moment axis is perpendicular to the plane in which the region R lies, say a line through the origin and perpendicular to the x and y axes, then the second moment with respect to this line is called a **polar moment of inertia** and is written

$$J_{00} = \iint_R r^2\, dA = \iint_R (x^2+y^2)\, dA = \iint_R x^2\, dA + \iint_R y^2\, dA = I_{yy} + I_{xx}$$

which shows the polar moment of inertia about the line through the origin is the sum of the moments of inertia about the x and y axes.

An examination of the Figure 5-3 shows that if ℓ_y is the line $x = x_0$ parallel to the $y-$axis, then an element of area dA has a second moment with respect to the line ℓ_y given by

$$I_{\ell_y \ell_y} = \iint_R |x - x_0|^2 \, dA = \iint_R (x - x_0)^2 \, dA \tag{5.47}$$

If ℓ_x is the line $y = y_0$ which is parallel to the $x-$axis, then the element of area dA has the second moment with respect to line ℓ_x given by

$$I_{\ell_x \ell_x} = \iint_R |y - y_0|^2 \, dA = \iint_R (y - y_0)^2 \, dA \tag{5.48}$$

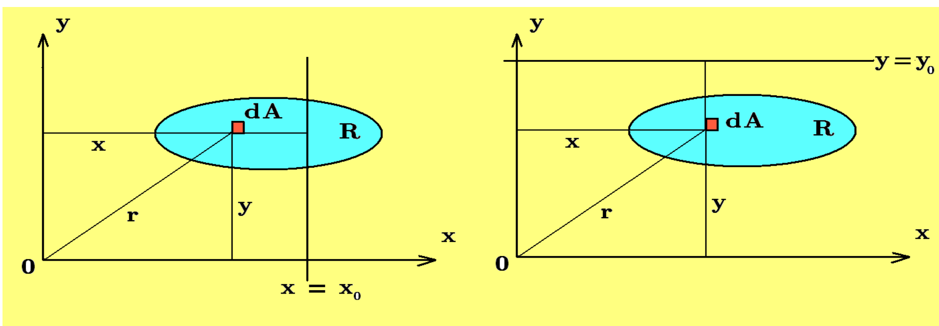


Figure 5-3.

Second moments with respect to lines parallel to the x and y axes.

Expanding equation (5.47) one finds

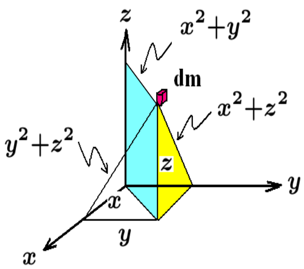
$$\begin{aligned} I_{\ell_y \ell_y} =& \iint_R (x^2 - 2x x_0 + x_0^2) \, dA \\ =& \iint_R x^2 \, dA - 2x_0 \iint_R x \, dA + x_0^2 \iint_R dA \\ I_{\ell_y \ell_y} =& I_{yy} - 2x_0 M_y + x_0^2 A \end{aligned} \tag{5.49}$$

Similarly, if one expands the equation (5.48) one finds that

$$I_{\ell_x \ell_x} = I_{xx} - 2y_0 M_x + y_0^2 A \tag{5.50}$$

The results given by the equations (5.49) and (5.50) are known as the basic equations for representing the **parallel axes theorem from mechanics.** This theorem states that if you know the area of a region and the first and second moments of the region about one of the coordinate axes, then you can find the second moment about any axis parallel to the coordinate axes by using one of the above results.

Example 5-15. Let $d\tau = dx\,dy\,dz$ denote an element of volume and $\rho\,d\tau = dm$ denote an element of mass, where ρ is the density of the solid. The second moments of mass with respect to the x, y and z axes are given by

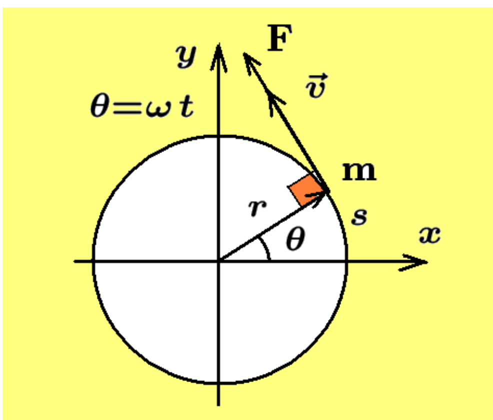


$$I_{xx} = \iiint (y^2+z^2)\rho\,d\tau$$
$$I_{yy} = \iiint (x^2+z^2)\rho\,d\tau$$
$$I_{zz} = \iiint (x^2+y^2)\rho\,d\tau$$

where integrations are over the volume defining the solid.

■

Example 5-16.



Consider a particle P with constant mass m rotating in a circle of radius r about the origin having a tangential force F producing the motion. Let θ denote the angular displacement of the particle, $\omega = \dfrac{d\theta}{dt}$ the angular velocity of the particle, $\dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2} = \alpha$, the angular acceleration of the particle. Newton's second law of motion can then be expressed $F = m\,\dfrac{dv}{dt}$.

If $s = r\theta$ is the distance traveled by the particle, then $\dfrac{ds}{dt} = v = r\dfrac{d\theta}{dt} = r\omega$ is the change in distance with respect to time or speed[10] of the particle, so that Newton's second law can be expressed

$$F = m\frac{dv}{dt} = m\frac{d}{dt}\left(r\frac{d\theta}{dt}\right) = mr\frac{d^2\theta}{dt^2} \tag{5.51}$$

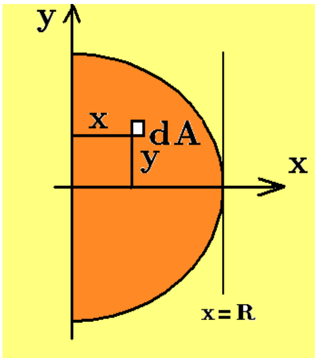
Note that if F is the tangential force acting on the particle, then $M = Fr$ is the torque or first moment of the force about the origin. Consequently, multiplying equation (5.51) on both sides by r one finds $Fr = mr^2\,\dfrac{d^2\theta}{dt^2}$. Here $Fr = M$ is the first moment or torque about the origin, $mr^2 = I$ is the mass times the lever arm squared or moment of inertia of the mass about the origin. The Newton's law for rotational motion can therefore be expressed in the form

$$M = I\alpha \qquad \text{or} \qquad M = I\frac{d^2\theta}{dt^2} \tag{5.52}$$

■

[10] Speed is a scalar quantity representing the magnitude of velocity which is a vector quantity.

Example 5-17. Find the centroid and moments of inertia about the x and $y-$ axes associated with the semi-circle $x^2+y^2=R^2$ for $x>0$.



Solution The area inside the semi-circle is $A=\frac{\pi}{2}R^2$. At a general point (x,y) within the semi-circle construct an element of area $dA=dx\,dy$, then the first moment about the $y-$axis is given by

$$dM_y = x\,dA = x\,dx\,dy$$

and a summation over all elements dA within the semi-circle gives the first moment

$$M_y=\int_{x=0}^{x=R}\int_{y=-\sqrt{R^2-x^2}}^{y=\sqrt{R^2-x^2}} x\,dy\,dx \qquad\qquad M_y=\int_0^R 2x\sqrt{R^2-x^2}\,dx$$

Make the substitution $u=R^2-x^2$ with $du=-2x\,dx$ and show the integration produces the result $M_y=\dfrac{2}{3}R^3$, so that $\bar{x}=\dfrac{M_y}{A}=\dfrac{2R^3/3}{\pi R^2/2}=\dfrac{4R}{3\pi}$. By symmetry, the centroid must lie on the line $y=0$ so that the centroid of the semi-circle lies at the point $(\bar{x},\bar{y})=(\dfrac{4R}{3\pi},0)$.

The second moments of the area element about the x and $y-$axes gives

$$dI_{xx}=y^2\,dA=y^2\,dy\,dx \qquad \text{and} \qquad dI_{yy}=x^2\,dA=x^2\,dx\,dy$$

A summation over the area of the semi-circle gives the integrals

$$I_{xx}=\int_{x=0}^{x=R}\int_{y=-\sqrt{R^2-x^2}}^{y=\sqrt{R^2-x^2}} y^2\,dy\,dx=\frac{2}{3}\int_0^R (R^2-x^2)^{3/2}\,dx=\frac{\pi}{8}R^4$$

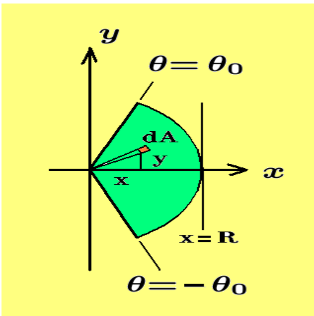
$$I_{yy}=\int_{x=0}^{x=R}\int_{y=-\sqrt{R^2-x^2}}^{y=\sqrt{R^2-x^2}} x^2\,dy\,dx=\int_0^R 2x^2(R^2-x^2)^{1/2}\,dx=\frac{\pi}{8}R^4$$

where the substitution $x=R\sin\theta$ can be used to aid in evaluating the above integrals.

Let I_{cc} denote the moment of inertia about the line $x=x_0=\frac{4R}{3\pi}$ through the centroid. The parallel axis theorem shows that $I_{cc}=\left(\dfrac{\pi}{8}-\dfrac{8}{9\pi}\right)R^4$.

■

Example 5-18. Find the centroid and moments of inertia about the x and $y-$axes associated with the circular sector bounded by the rays $\theta = -\theta_0$ and $\theta = \theta_0$ and the circle $r = R$.



Solution The area inside the circular sector is given by $A = \theta_0 R^2$. Move to a general point (x, y) within the sector and construct an element of area $dA = r\,dr d\theta$. The x and y lever arms are given by $x = r\cos\theta$ and $y = r\sin\theta$. The first moment of this area element about the $y-$axis is given by

$$dM_y = x\,dA = (r\cos\theta) r\,dr d\theta$$

and a summation of these elements over the area of the sector gives

$$M_y = \int_{r=0}^{r=R}\int_{-\theta_0}^{\theta_0} \cos\theta\,d\theta\,r^2\,dr = \frac{2}{3}R^3\sin\theta_0$$

The $\bar{x}$ value for this area is given by $\bar{x} = \dfrac{M_y}{A} = \dfrac{2}{3}R\dfrac{\sin\theta_0}{\theta_0}$. By symmetry, the centroid lies on the ray $\theta = 0$ where $(\bar{x}, \bar{y}) = (\frac{2}{3}R\frac{\sin\theta_0}{\theta_0}, 0)$.

The elements for the second moments about the x and $y-$axes are given by

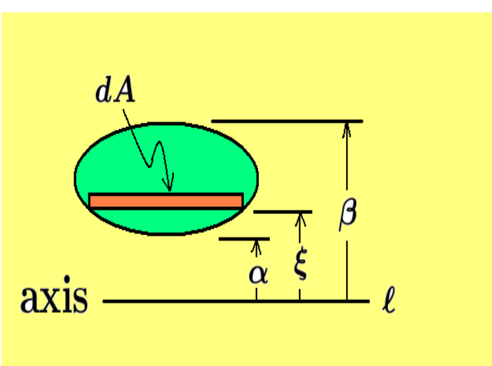
$$dI_{xx} = y^2\,dA = (r^2\sin^2\theta)\,r\,dr d\theta \qquad \text{and} \qquad dI_{yy} = x^2\,dA = (r^2\cos^2\theta)\,r\,dr d\theta$$

and summing these elements over the area of the sector gives the moments of inertia

$$I_{xx} = \int_0^R\int_{-\theta_0}^{\theta_0} r^3\sin^2\theta\,d\theta\,dr = (2\theta_0 - \sin 2\theta_0)\frac{R^4}{8}$$
$$I_{yy} = \int_0^r\int_{-\theta_0}^{\theta_0} r^3\cos^2\theta\,d\theta\,dr = (2\theta_0 + \sin 2\theta_0)\frac{R^4}{8}$$

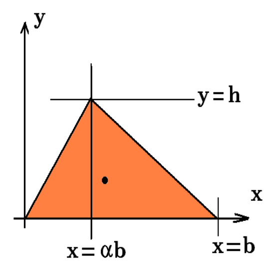
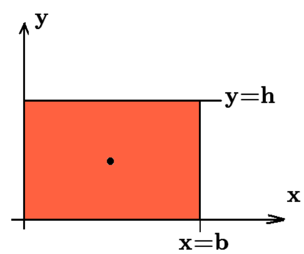
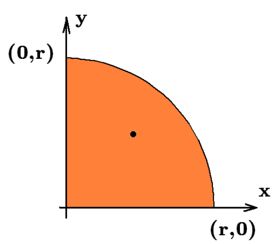
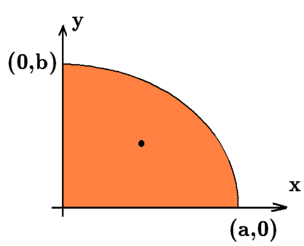
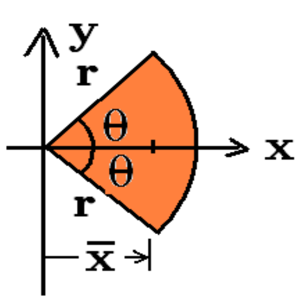
■

Moment of Inertia of an Area



The moment of inertia of a general plane area with respect to an axis can be calculated as follows. Construct elements of area dA all in the shape of a rectangle within the plane area and which are parallel to the axis not passing through the area. By definition, the element of moment of inertia associated with an element of area dA

Table 5-2 Moments of Inertia of Some Simple Shapes

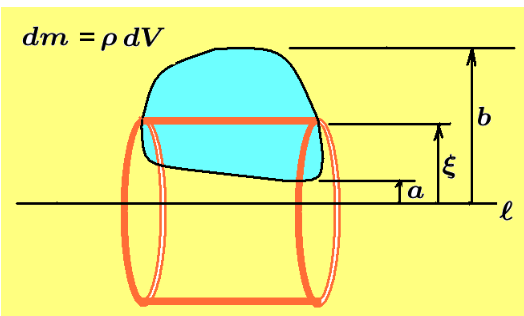
Shape		Area	I_{xx}	I_{yy}
Triangle		$\frac{1}{2}bh$	$\frac{1}{12}bh^3$	$\frac{1}{12}b^3h(1+\alpha+\alpha^2)$
Rectangle		bh	$\frac{1}{3}bh^3$	$\frac{1}{12}b^3h$
Quadrant of circle		$\frac{\pi}{4}r^2$	$\frac{1}{16}\pi r^4$	$\frac{1}{16}\pi r^4$
Quadrant of ellipse		$\frac{\pi}{4}ab$	$\frac{1}{16}\pi ab^3$	$\frac{1}{16}\pi a^3b$
Wedge		θr^2	$(2\theta_0 - \sin 2\theta_0)\frac{r^4}{8}$	$(2\theta_0 + \sin 2\theta_0)\frac{r^4}{8}$

is the lever arm squared times the element of area or $dI_{\ell\ell} = \xi^2\, dA$, where ξ represents the distance from the axis to an element of area dA. The total moment of inertia is then a summation over all rectangular elements. If α and β are values denoting the extreme distances of the plane area from the axis, then the moment of inertia of the plane area is determined by evaluating the integral

$$I_{\ell\ell} = \int_{\xi=\alpha}^{\xi=\beta} \xi^2\, dA \tag{5.53}$$

Here it is assumed that the element of area dA can be expressed in terms of the distance ξ.

Moment of Inertia of a Solid



In the study of mechanics one frequently encounters the necessity to calculate the moment of inertia of a solid which is generated by revolving a plane area about an axis. If a rectangular element of area is construct within the plane area and is rotated about an axis ℓ not passing through the area, then a cylindrical shell shaped volume element dV is generated. Multiplying this volume element by the density ρ of the solid creates an element of mass $dm = \rho\, dV$. The element of the moment of inertia is then given by

$$dI_{\ell\ell} = \xi^2\, dm = \xi^2\, \rho\, dV \tag{5.54}$$

where ξ is the distance of dm from the axis of rotation. If the extreme distances of the plane area from the axis of rotation have the values a and b, then the total moment of inertia about the rotation axis is

$$I_{\ell\ell} = \int_{\xi=a}^{\xi=b} \xi^2\, dm = \int_{\xi=a}^{\xi=b} \xi^2\, \rho\, dV \tag{5.55}$$

If the solid is homogeneous, then the density ρ is a constant so that the moment of inertia can be expressed

$$I_{\ell\ell} = \rho \int_{\xi=a}^{\xi=b} \xi^2\, dV \tag{5.56}$$

Moment of Inertia of Composite Shapes

To calculate the moment of inertia of a composite area about a selected axis

(i) Calculate the moment of inertia of each component about the selected axis.

(ii) Next one need only sum the moments of inertia calculated in step (i) to calculate the moment of inertia of the given composite area.

That is, **the moment of inertia of a composite area about an axis is equal to the sum of the moments of the component areas with respect to the same axis**. Note that if a component of the shape is removed, then this places a hole in the composite shape and in this case the moment of inertia of the component removed is then subtracted from the total sum.

Pressure

The **average density** ρ of a substance is defined as its mass m divided by its volume V or $\rho = \frac{m}{V}$, where $[\rho] = \text{kg/m}^3$, $[m] = \text{kg}$, $[V] = \text{m}^3$. The **relative density** of a substance is defined as the ratio of density of substance divided by the density of water. Pressure is **a scalar quantity** defined as the average force per unit of area and its **unit of measurement is the Pascal**, abbreviated (Pa), where $1\,\text{Pa} = 1\,\text{N}\,\text{m}^{-2}$.

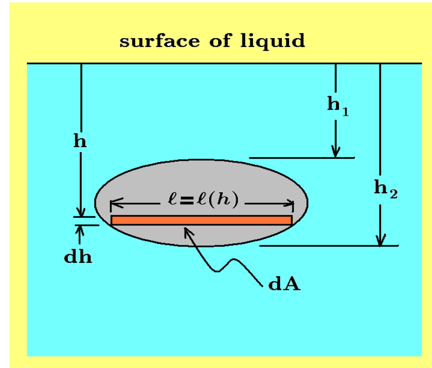
Liquid Pressure

Integration can be used to determine the forces acting on submerged objects. Pressure at a point is $p = \lim_{\Delta A \to 0} \dfrac{\Delta F}{\Delta A}$, and represents a derivative of the force with respect to area. An area submerged in water experiences **only a pressure normal to its surface** and there are no forces parallel to the area. This is known as **Pascal's law**. Knowing the pressure at a point, one can use integration to calculate the total force acting on a submerged object. The pressure p representing force per unit of area must be known when constructing water-towers, dams, locks, reservoirs, ships, submarines, under-water vessels as the total force acting on a submerged object must be known for certain design considerations.

Consider two points P_1 and P_2 beneath a fluid having a constant density ρ. If $\Delta h = |P_1 - P_2|$ is the distance between the points and ρ is the constant density of the liquid, then the change in pressure between the points P_1 and P_2 is given by

$$\Delta p = \rho\, g\, \Delta h \tag{5.57}$$

where h, $[h] = \text{m}$, is measured positive in the downward direction, ρ, $[\rho] = \text{kg/m}^3$, is the density and g, $[g] = \text{m/s}^2$ is the acceleration of gravity so that $[p] = \text{N/m}^2$. Defining $w = \rho\, g$, $[w] = \text{N/m}^3$ as the weight of the liquid per unit volume, one can write $\Delta p = w\,\Delta h$. Note the pressure increases with depth and depends only on the quantities vertical distance, density and acceleration of gravity. The pressure is then the same at all points of a submerged area lying on a horizontal line of constant depth within the liquid.



Consider a vertical plane area submerged in a liquid. Using the fact that the pressure is the same at all point lying on a horizontal line at constant depth, one can construct an element of area $dA = \ell(h)\,dh$ on the submerged area as illustrated, where $\ell = \ell(h)$ is the horizontal dimension of the element of area.

The element of force exerted by the liquid on one side of the submerged element of area is given by

$$dF = p\,dA = w\,h\,dA = w\,h\,\ell(h)\,dh$$

and so the total force acting on the submerged area is given by the summation of forces

$$F = \int_{h_1}^{h_2} w\,h\,\ell(h)\,dh \tag{5.58}$$

The representation for the total force given by equation (5.58) assumes that the element of area can be expressed in terms of the depth h. If one selects a different way of representing the position of the submerged object, say by constructing an x, y-axes somewhere, then the above quantities have to be modified accordingly.

Gas Pressure

The equation (5.57) is valid for the change in gas pressure between two points P_1 and P_2 for small volumes. However, for small volumes the gas pressure is very small and Δp remains small unless h is very large. One usually makes use of the fact that **the gas pressure is essentially constant at all points within a volume of reasonable size**. When dealing with volumes of a very large size, like the Earth's atmosphere, the equation (5.57) is no longer valid. Instead, one usually uses the fact that (i) the **pressure decreases** as the height h above the Earth increases and (ii) the density of the air **varies widely over the surface of the Earth**. Under these conditions one uses **the approximate relation** that the change in pressure with respect to height is proportional to $\rho\,g$ and one writes

$$\frac{dp}{dh} = -\rho\,g \tag{5.59}$$

The negative sign indicating that the pressure decreases with height. Note that the pressure has a wide range of values over the Earth's surface, varying with temperature, humidity, molar mass of dry air and sea level pressure. The **average sea level pressure** being 101.325 kPa or 760 mmHg. One can find various empirical formulas for variations of the density ρ determined by analyzing weather data.

Chemical Kinetics

In chemistry a chemical reaction describing how hydrogen (H_2) and oxygen (O_2) combine to form water is given by

$$2H_2 + O_2 \xrightarrow{k} 2H_2O$$

This reaction is a special case of a more general chemical reaction having the form

$$n_1A_1 + n_2A_2 + n_3A_3 + \cdots \underset{k_r}{\overset{k_f}{\rightleftharpoons}} m_1B_1 + m_2B_2 + m_3B_3 + \cdots \tag{5.60}$$

where $A_1, A_2, A_3, \ldots$ represent molecules of **the reacting substances**, called **reactants** and $B_1, B_2, B_3, \ldots$ represent molecules formed during the reaction, called **product elements of the reaction**. The coefficients $n_1, n_2, n_3, \ldots$ and $m_1, m_2, m_3, \ldots$ are either positive integers, zero or they have a fractional value. These values indicate the proportion of molecules involved in the reaction or proportions involved when the reactants combine. These coefficients are referred to as **stoichiometric coefficients**. The constants k_f and k_r are positive constants called the forward and reverse reaction rate coefficients. If $k_r = 0$, then the reaction goes in only one direction.

The stoichiometric representation of a reaction gives only **the net result of a reaction** and does not go into details about how the reaction is taking place. Other schemes for representing a reaction are used for more complicated reactions. One part of chemistry is the development of mathematical models which better describe the mechanisms of how elements and compounds react and involves the study of reaction dynamics of chemicals. This sometimes requires research involving extensive experimental and theoretical background work in order to completely understand all the bonding and subreactions which occur simultaneously during a given reaction. Simple chemical reactions can be described using our basic knowledge of calculus.

Rates of Reactions

A reaction is called **a simple reaction** if there are no intermediate reactions or processes taking place behind the scenes. For example, a simple reaction such as

$$n_1A_1 + n_2A_2 \xrightarrow{k} m_1B_1 + m_2B_2 \tag{5.61}$$

states that n_1 molecules of A_1 and n_2 molecules of A_2 combined to form m_1 molecules of B_1 and m_2 molecules of B_2. Let $[A_1], [A_2], [B_1], [B_2]$ denote respectively the concentrations of the molecules A_1, A_2, B_1, B_2 with the concentration measured in units of moles/liter. The stoichiometric reaction (5.61) states that as the concentrations of A_1 and A_2 decrease, then the concentrations of B_1 and B_2 increase. It is assumed that A_1 and A_2 decrease at the same rate so that

$$-\frac{1}{n_1}\frac{d[A_1]}{dt} = -\frac{1}{n_2}\frac{d[A_2]}{dt} \tag{5.62}$$

and the concentrations of B_1 and B_2 increase at the same rate so that

$$\frac{1}{m_1}\frac{d[B_1]}{dt} = \frac{1}{m_2}\frac{d[B_2]}{dt} \tag{5.63}$$

Equating the equations (5.62) and (5.63) gives

$$-\frac{1}{n_1}\frac{d[A_1]}{dt} = -\frac{1}{n_2}\frac{d[A_2]}{dt} = \frac{1}{m_1}\frac{d[B_1]}{dt} = \frac{1}{m_2}\frac{d[B_2]}{dt} \tag{5.64}$$

Here a standard rate of reaction is achieved by taking the rate of change of each substance and dividing by its stoichiometric coefficient. Also note that the minus signs are used to denote a decrease in concentration and a plus sign is used to denote an increase in concentration.

The Law of Mass Action

There are numerous and sometimes complicated rate laws for describing the chemical kinetics of a reaction. These complicated rate laws are avoided in presenting this introduction to chemical kinetics. For simple chemical reactions at a constant temperature which have the form of equation (5.60), let $x = x(t)$ denote the number of molecules per liter which have reacted after a time t. Many of these simple equations obey **the law of mass action** which states that the rate of change of $x = x(t)$ with respect to time t can be represented

$$\frac{dx}{dt} = k[A_1]^{n_1}[A_2]^{n_2}[A_3]^{n_3}\cdots \tag{5.65}$$

Each of the superscripts $n_1, n_2, n_3, \ldots$ have known values and k is a rate or velocity coefficient having units of $1/time$. The order of the chemical reaction is represented by the sum $n_1 + n_2 + n_3 + \cdots$ of these exponents. If the sum is one, the reaction is called a first-order reaction or unimolecular reaction. If the sum is two, the reaction is called a second-order reaction or bimolecular. If the sum is three, the reaction is called a third-order reaction or trimolecular, etc. Note that trimolecular and higher order reactions are rare.

An example of a unimolecular reaction is **a substance disintegrating** and this type of reaction can be represented

$$-\frac{d[A]}{dt} = k[A], \qquad [A] = [A](t) \tag{5.66}$$

and this equation is a way of stating that the rate of change of a decaying substance is proportional to the amount present. The proportionality constant k being called **the rate coefficient**. Separate the variables in equation (5.66) and write $\frac{d[A]}{[A]} = -k\,dt$ and then integrate both sides from 0 to t, assuming that at time $t = 0$, $[A](0) = [A]_0$ is the initial concentration. One can then write

$$\int_{[A]_0}^{[A]} \frac{d[A]}{[A]} = -k \int_0^t dt$$

and then these equations can be integrated to obtain

$$\ln [A] - \ln [A]_0 = -kt \qquad \Longrightarrow \qquad [A] = [A](t) = [A]_0\, e^{-kt} \tag{5.67}$$

where the reaction rate k has dimensions of $1/time$. Use equation (5.67) and plot $[A](t)$ versus t one finds the result is a straight line on semi-log paper. A second-order reaction or bimolecular reaction has the form

$$A_1 + A_2 \underset{k_r}{\overset{k_f}{\rightleftharpoons}} B_1 \tag{5.68}$$

and represents a reversible bimolecular reaction. Here k_f is the forward rate constant and k_r is the reverse rate constant. One can alternatively write the forward and reverse reactions as two separate equations. If $k_r = 0$, then there is no reverse reaction. Apply the law of mass action to the stoichiometric reaction (5.68) gives the differential equations

$$\begin{aligned}\frac{d[A_1]}{dt} =& k_r[B_1] - k_f[A_1][A_2]\\ \frac{d[A_2]}{dt} =& k_r[B_1] - k_f[A_1][A_2]\\ \frac{d[B_1]}{dt} =& -k_r[B_1] + k_f[A_1][A_2]\end{aligned} \tag{5.69}$$

Note that the rate coefficients k_r and k_f can have very large differences in magnitudes thus driving the reaction more in one direction than the other and for the reaction

(5.68) the rate coefficients k_f and k_r do not have the same units of measurements. To show this one should perform a dimensional analysis on each of the terms in the equations (5.69). **Each group of terms in equation (5.69) must have the same units of measurements** so that by examining the dimensions of each term in a group one can show the reaction rates k_f and k_r do not have the same dimensions. For example, if the concentrations are measured in units of $mol/liter$, then the terms on the left-hand side of the equations (5.69) all have units of $mol/liter\ per\ second$ and consequently each group of terms on the right-hand side of the equations (5.69) must also have this same unit of measurement. This requires that k_r have units of $1/second$ and k_f have units of $\dfrac{1}{\frac{mol}{liter}\cdot second}$. If different units of measurement are used one must perform a similar type of analysis of the dimensions associated with each group of terms. The requirement that each group of terms have the same dimensions is known as **requiring that the equations be homogeneous in their dimensions**. If an equation is not **dimensionally homogeneous**, then you know it is wrong.

In the equations (5.69) let $k_r = 0$ to obtain

$$\begin{aligned}\frac{d[A_1]}{dt} &= -\,k_f[A_1][A_2]\\ \frac{d[A_2]}{dt} &= -\,k_f[A_1][A_2]\\ \frac{d[B_1]}{dt} &= k_f[A_1][A_2]\end{aligned} \tag{5.70}$$

Let $y = y(t) = [B_1]$ denote the concentration of B_1 as a function of time t. If at time $t = 0$ the concentrations of A_1 and A_2 are denoted by $[A_1]_0$ and $[A_2]_0$, then after a time t one can express the amount of A_1 and A_2 by using the equations

$$[A_1] = [A_1](t) = [A_1]_0 - y \qquad \text{and} \qquad [A_2] = [A_2](t) = [A_2]_0 - y$$

Substituting these values into the last of the equations (5.70) gives the result

$$\frac{dy}{dt} = k_f([A_1]_0 - y)([A_2]_0 - y) \tag{5.71}$$

Let $\alpha_1 = [A_1]_0$ and $\alpha_2 = [A_2]_0$ and assume $\alpha_1 \neq \alpha_2$ so that equation (5.71) can be expressed in the form

$$\frac{dy}{(\alpha_1 - y)(\alpha_2 - y)} = k_f\,dt \tag{5.72}$$

where the variables have been separated. Use partial fractions and write

$$\frac{1}{(\alpha_1 - y)(\alpha_2 - y)} = \frac{A}{\alpha_1 - y} + \frac{B}{\alpha_2 - y}$$

and show $A = \frac{1}{\alpha_2 - \alpha_1}$ and $B = \frac{-1}{\alpha_2 - \alpha_1} = -A$. The equation (5.72) can then be expressed the following form

$$\frac{A\,dy}{\alpha_1 - y} - \frac{A\,dy}{\alpha_2 - y} = k_f\,dt, \qquad \alpha_1 \neq \alpha_2 \tag{5.73}$$

which is easily integrated to obtain

$$-A\ln|\alpha_1 - y| + A\ln|\alpha_2 - y| = k_f\,t + C$$

where C is a constant of integration. Solving for y gives

$$\begin{aligned} A\ln\left|\frac{\alpha_2 - y}{\alpha_1 - y}\right| =& k_f t + C \\ \ln\left|\frac{\alpha_2 - y}{\alpha_1 - y}\right| =& k_f(\alpha_2 - \alpha_1)t + C^*, \quad \text{where } C^* = C(\alpha_2 - \alpha_1) \\ \frac{\alpha_2 - y}{\alpha_1 - y} =& K e^{k_f(\alpha_2 - \alpha_1)t}, \text{ where } K = e^{C^*} \text{ is some new constant.} \end{aligned} \tag{5.74}$$

Evaluate equation (5.74) at time $t = 0$ with $y(0) = 0$ to show $K = \frac{\alpha_2}{\alpha_1}$. Substituting this value into the last equation in (5.74) and using algebra to solve for y one finds that

$$y = y(t) = \alpha_1\alpha_2 \frac{\left(1 - e^{k_f(\alpha_2 - \alpha_2)t}\right)}{\alpha_1 - \alpha_2 e^{k_f(\alpha_2 - \alpha_1)t}} \tag{5.75}$$

In the special case $\alpha_1 = \alpha_2$, the equation (5.72) takes on the form

$$\frac{dy}{(\alpha_1 - y)^2} = k_f\,dt \tag{5.76}$$

As an exercise integrate both sides of this equation and show

$$y = y(t) = \frac{\alpha_1^2 k_f t}{1 + \alpha_1 k_f t} \tag{5.77}$$

Differential Equations

Equations which contain derivatives which are of the form

$$L(y) = a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-2}(x)\frac{d^2 y}{dx^2} + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{5.78}$$

where $a_0, a_1, \ldots, a_n$ are constants or functions of x, are called **linear nth order homogeneous differential equations** and linear differential equations of the form

$$L(y) = a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-2}(x)\frac{d^2y}{dx^2} + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \tag{5.79}$$

are called **linear nth order nonhomogeneous differential equations**. The symbol L is a shorthand notation to denote the **linear differential operator**

$$L(\ ) = a_0(x)\frac{d^n(\ )}{dx^n} + a_1(x)\frac{d^{n-1}(\ )}{dx^{n-1}} + \cdots + a_{n-2}(x)\frac{d^2(\ )}{dx^2} + a_{n-1}(x)\frac{d(\ )}{dx} + a_n(x)(\ ) \tag{5.80}$$

An operator L is called a linear differential operator when it satisfies the conditions

$$\begin{aligned} L(y_1 + y_2) &= L(y_1) + L(y_2) \\ L(\alpha y) &= \alpha L(y) \end{aligned} \tag{5.81}$$

where α is a constant. The first condition is satisfied because a derivative of sum is the sum of the derivatives and the second condition is satisfied because the derivative of a constant times a function is the constant times the derivative of the function. Differential equations with ordinary derivatives, not having the forms of equations (5.78) or (5.79) , are called **nonlinear differential equations**. In general, linear differential equations are easier to solve than nonlinear differential equations.

A **solution of the differential equation** (5.78) is any continuous function $y = y(x)$ which can be differentiated n-times and one can show that when the function $y = y(x)$ and its derivatives are substituted into the equation (5.78) then an identity results. The function $y = y(x)$ is then said to have satisfied the conditions specified by the differential equation. The differential equation (5.78) has a nth derivative term and consequently it would require n-integrations to obtain the solution. The general solution will therefore contain n arbitrary constants. Sometimes it is possible to integrate the differential equation and determine the solution by integration methods. Sometimes the given differential equation has a special form where short cut methods have been developed for obtaining a solution.

The general procedure to solve the linear nth order nonhomogeneous equation (5.79) is to first solve the homogeneous differential equation (5.78) by finding n **linearly independent solutions** $\{y_1(x), y_2(x), \ldots, y_n(x)\}$, called a **fundamental set of solutions**, where each function $y_i(x)$ satisfies $L(y_i(x)) = 0$ for $i = 1, 2, \ldots, n$. The **general solution** of the linear homogeneous differential equation (5.78) is then **any linear**

combination of the functions in the fundamental set. The general solution of the linear homogeneous equation can be expressed

$$y_c = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \tag{5.82}$$

where $c_1, c_2, \ldots, c_n$ are constants and y_c is called a **complementary solution**. After determining the complementary solution one then tries to find **any function** y_p which satisfies $L(y_p) = F(x)$. The function y_p is then called a **particular solution of the nonhomogeneous linear differential equation** (5.79). The **general solution** to the linear nonhomogeneous differential equation (5.79) is written

$$y = y_c + y_p \tag{5.83}$$

which represents a sum of the complementary and particular solutions.

Spring-mass System

Consider a vertical spring which is suspended from a support as illustrated in figure 5-4(a). Consider what happens when a weight W is attached to a linear spring, and the spring stretches some distance s_0, and the weight remains at rest in **an equilibrium position** as illustrated in the figure 5-4(b). The weight is in equilibrium because the downward force W is offset by the upward spring restoring force and these forces must be equal and in opposite directions. If the weight is displaced from this equilibrium position and then released, it undergoes a vibratory motion with respect to a set of **reference axes constructed at the equilibrium position** as illustrated in figure 5-4(c).

In order to model the above problem, the following assumptions are made:

(a) No motion exists in the horizontal direction.
(b) A downward displacement is considered as positive.
(c) The spring is a **linear spring** and obeys **Hooke's**[11] **law** which states that the **restoring force of the spring is proportional to the spring displacement**.

Using Hooke's law the spring force holding the weight in equilibrium can be calculated. In figure 5-4(b), there is no motion because the weight W acting down is offset by the spring force acting upward. Let f_s denote the spring force illustrated in figure 5-4(d). Using Hooke's law, the spring force f_s is proportional to the displacement s and is written $f_s = Ks$, where K is the proportionality constant called

[11] Robert Hooke (1635–1703) English physicist.

the spring constant. The graph of f_s versus displacement s is therefore a straight line with slope K.

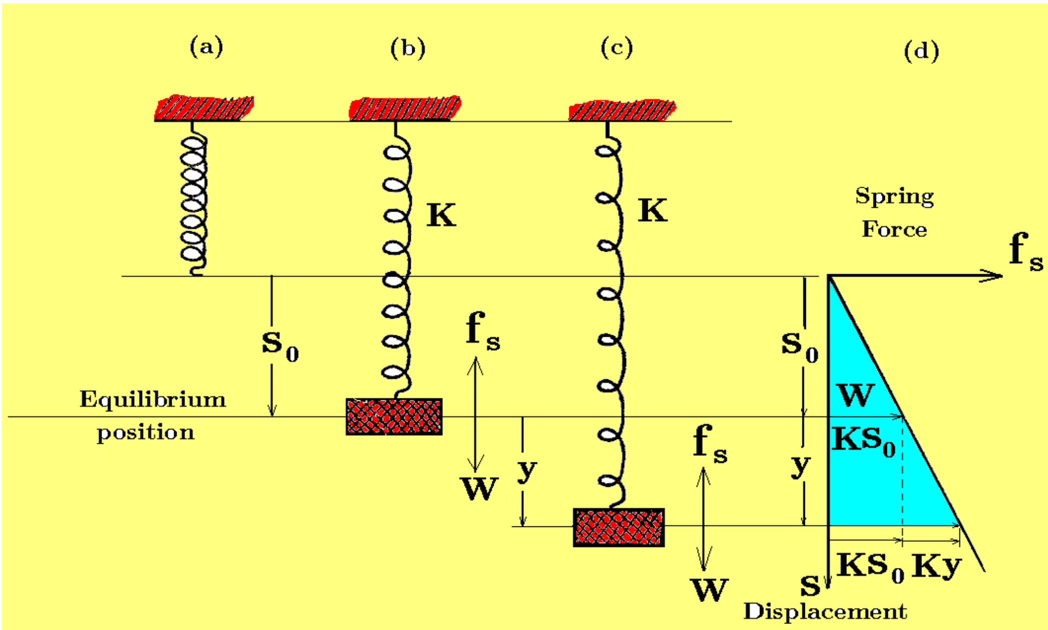


Figure 5-4. Spring-mass system.

Summation of the forces in equilibrium is represented in figure 5-4(b) which illustrates the spring force equal to the weight acting down or $f_s = Ks_0 = W$. This determines the spring constant K as

$$K = \frac{W}{s_0}. \tag{5.84}$$

In figure 5-4(c), the spring force acting on the weight is given by $f_s = K(s_0 + y)$ where y is **the displacement from the equilibrium position**. The vibratory motion can be describe by using Newton's second law that the sum of the forces acting on the mass must equal the mass times acceleration. The motion of the weight is thus modeled by summing the forces in the y direction and writing Newton's second law as

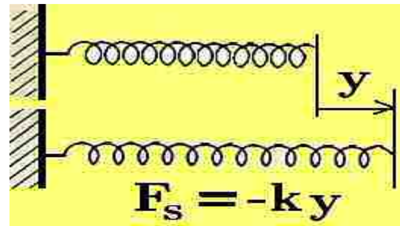
$$m\frac{d^2y}{dt^2} = W - f_s = W - Ks_0 - Ky = -Ky, \quad \text{or} \quad m\frac{d^2y}{dt^2} + Ky = 0 \tag{5.85}$$

or

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \qquad \omega^2 = K/m \tag{5.86}$$

Here the substitution $\omega^2 = K/m$ or $\omega = \sqrt{K/m}$ has been made to simply the representation of the differential equation describing the motion of the spring-mass system. The quantity ω is called the **natural frequency** of the undamped system.

Simple Harmonic Motion



Consider the spring illustrated which is stretched a distance y. **Hooke's law** states the for **a linear spring**, the restoring force F_s is proportional to the displacement y and one can write $F_s = -Ky$, where K is the proportionality constant called the spring constant. The negative sign indicating that the restoring force is in the opposite direction of the spring displacement.

The **elastic potential energy of the spring** is defined as follows. The work done in stretching a spring a distance y is given by

$$
\begin{aligned}
W =&(average\ force)(displacement)\\
W =&\left(\frac{1}{2}K\,y\right)(y) = \frac{1}{2}K\,y^2
\end{aligned}
$$

In stretching the spring using a force Ky, the spring exerts an opposite force $-Ky$ which does negative work. This negative work is called the **elastic potential energy of the spring** and it is denoted by

$$E_p = \frac{1}{2}Ky^2$$

Multiply equation (5.85) by $\frac{dy}{dt}\,dt$ to obtain

$$m\frac{dy}{dt}\frac{d^2y}{dt^2}\,dt + K\,y\,\frac{dy}{dt}\,dt = 0 \tag{5.87}$$

Note that the integration of each term in equation (5.87) is of the form $\int u\,du$ for an appropriate value of u. One can verify that an integration of equation (5.87) produces the result that the kinetic energy plus spring potential energy is a constant and represented

$$\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}Ky^2 = E \tag{5.88}$$

where E is a constant of integration. Observe that the terms in equation (5.88) represent

$$\begin{aligned} E_k =& \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 = \text{Kinetic energy of system} \\ E_p =& \frac{1}{2}K\,y^2 = \text{Spring potential energy} \\ E =& \text{Total energy of the system} \end{aligned}$$

Equation (5.88) can be integrated to obtain the displacement $y = y(t)$ as a function of time t. Write equation (5.88) as

$$\left(\frac{dy}{dt}\right)^2 = \frac{2E}{m} - \frac{K}{m}y^2 = \omega^2\left(A^2 - y^2\right), \qquad \text{where } A^2 = \frac{2E}{K} \text{ and } \omega^2 = \frac{K}{m} \tag{5.89}$$

Take the square root of both sides to obtain the differential equation

$$\frac{dy}{dt} = \omega\sqrt{A^2 - y^2}$$

This is a differential equation where the variables can be separated to obtain

$$\frac{dy}{\sqrt{A^2 - y^2}} = \omega\, dt$$

and then integrated by making the substitution

$$y = A\sin\theta, \qquad dy = A\cos\theta\, d\theta \quad \text{to obtain} \quad \frac{A\cos\theta d\theta}{A\cos\theta} = d\theta = \omega\, dt$$

where another integration produces

$$\theta = \omega\, t + \theta_0 \quad \text{or} \quad \theta = \sin^{-1}\left(\frac{y}{A}\right) = \omega\, t + \theta_0 \quad \text{or} \quad y = y(t) = A\sin(\omega t + \theta_0) \tag{5.90}$$

where θ_0 is a constant of integration. Any **periodic motion** $y = y(t)$ represented by either of the equations

$$y = A\sin(\omega t + \theta_0) \qquad \text{or} \qquad y = A\cos(\omega t + \theta_0) \tag{5.91}$$

is said to be a **simple harmonic motion** with amplitude A. The **period** P **associated with the oscillation** is the time taken to complete one oscillation. The period of the motion described by equation (5.90) or (5.91) is $P = \frac{2\pi}{\omega}$. The **frequency of the motion** is $f = \frac{1}{P} = \frac{\omega}{2\pi}$ and represents the number of oscillation performed in one second, where **1 unit cycle per second is called a Hertz.** The angle θ_0 in equation (5.91) is called the **phase angle or phase shift** associated with the oscillations.

Note that equations (5.91) **differ only by a phase constant**, since one can write

$$
\begin{aligned}
A\sin(\omega t+\theta_0) =& A\cos(\omega t+\theta_0-\pi/2) = A\cos(\omega t+\phi_0), \qquad \phi_0=\theta_0-\pi/2\\
A\cos(\omega t+\theta_0) =& A\sin(\omega t+\theta_0+\pi/2) = A\sin(\omega t+\psi_0), \qquad \psi_0=\theta_0+\pi/2
\end{aligned}
$$

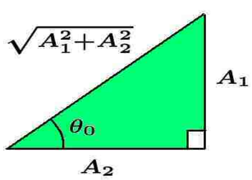
In general, given an equation of the form

$$y=y(t)=A_1\cos\omega t+A_2\sin\omega t$$

one can multiply both the numerator and denominator by $\sqrt{A_1^2+A_2^2}$ to obtain

$$y=y(t)=\sqrt{A_1^2+A_2^2}\left[\frac{A_1}{\sqrt{A_1^2+A_2^2}}\cos\omega t+\frac{A_2}{\sqrt{A_1^2+A_2^2}}\sin\omega t\right]$$

so that the oscillatory motion can be expressed in the following form.



$$A=\sqrt{A_1^2+A_2^2}$$
$$\sin\theta_0=\frac{A_1}{\sqrt{A_1^2+A_2^2}}$$
$$\cos\theta_0=\frac{A_2}{\sqrt{A_1^2+A_2^2}}$$

$$
\begin{aligned}
y =& y(t)=A\left[\sin\theta_0\cos\omega t+\cos\theta_0\sin\omega t\right]\\
y =& y(t)=A\sin(\omega t+\theta_0)
\end{aligned}
$$

Simple harmonic motion can be characterized by observing that the acceleration $\dfrac{d^2y}{dt^2}$ satisfies the conditions

(i) It is always proportional to its distance from a fixed point $\dfrac{d^2y}{dx^2}=-\omega^2 y$.

(ii) It is always directed toward the fixed point.

Damping Forces

Observe the sign of the spring force in equation (5.85). If $y>0$, the restoring force is in the negative direction. If $y<0$, the restoring force is in the positive direction. The directions of the forces are important because forces are vector quantities and must have both a magnitude and a direction. The direction of the forces is one check that the problem is correctly modeled.

If additional forces are added to the spring mass system, such as damping forces and external forces, then equation (5.85) must be modified to include these additional forces. In figure 5-5, assume a damper and an external force are attached to the spring as illustrated.

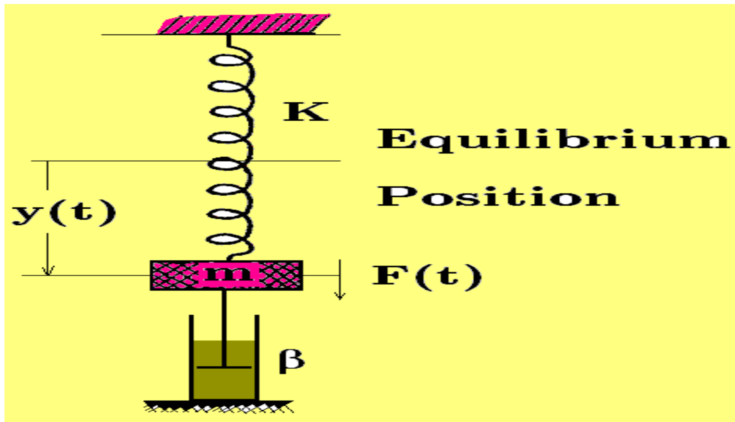


Figure 5-5. Spring mass system with additional forces.

If there is a damping force F_D which opposes the motion of the mass and the **magnitude of the damping force is proportional to the velocity**[12], then this can be represented by

$$F_D = -\beta \frac{dy}{dt} \tag{5.92}$$

where $\beta > 0$ is the proportionality constant called the **damping coefficient**. The sign of the damping force is determined by the sign of the derivative $\frac{dy}{dt}$. Note that if y is increasing and $\frac{dy}{dt} > 0$, the damping force is in the negative direction, whereas, if y is decreasing and $\frac{dy}{dt} < 0$, the damping force acts in the positive direction. In figure 5-5, the quantity $F(t)$ denotes an **external force** applied to drive the mass.

The use of Newton's second law of motion, together with the summation of forces, one can construct a mathematical model describing the motion of the spring mass system with damping and external force. The illustration in the figure 5-5 can be used as an aid to understanding the following equation

$$m\frac{d^2y}{dt^2} = -Ky - \beta\frac{dy}{dt} + F(t) \tag{5.93}$$

$$\text{or} \qquad m\ddot{y} + \beta\dot{y} + Ky = F(t) \qquad \dot{} = \frac{d}{dt}, \quad \ddot{} = \frac{d^2}{dt^2} \tag{5.94}$$

where the right-hand side of equation (5.93) represents a summation of the forces acting on the spring-mass system. Each term in equation (5.94) represents a **force**

[12] Various assumptions can be made to model other types of damping.

term and for the equation to be **dimensionally homogeneous**, every term must have dimensions of force. The quantity $m\ddot{y}$ is called the **inertial force**, $\beta\dot{y}$ is the **damping force**, Ky is the **spring force** and $F(t)$ is an **external force**. Here m, β and K are all positive constants with dimensions $[m] = \frac{[W]}{[g]} = \frac{\text{lbs}}{\text{ft/sec}^2}$, $[\beta] = \frac{\text{lbs}}{\text{ft/sec}}$, $[K] = \frac{\text{lbs}}{\text{ft}}$ with y and t having the dimensions $[y] = \text{ft}$ and $[t] = \text{seconds}$. It is left as an exercise to verify that the equation (5.94) is dimensionally homogeneous.[13]

To solve the differential equation (5.94) one first solves the homogeneous equation

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + Ky = 0 \tag{5.95}$$

by finding a set of two independent solutions $\{y_1(t), y_2(t)\}$ called a **fundamental set of solutions** to the homogeneous differential equation. The general solution to the homogeneous differential equation is then **any linear combination of the solutions from the fundamental set**. The general solution to equation (5.95) can be expressed

$$y_c = c_1y_1(t) + c_2y_2(t) \qquad \text{where } c_1,\ c_2 \text{ are arbitrary constants.} \tag{5.96}$$

This general solution is called the **complementary solution** and is usually denoted using the notation y_c. Any solution of the nonhomogeneous differential equation (5.94) is denoted using the notation y_p and is called **a particular solution.** The general solution to the differential equation (5.94) can then be expressed as $y = y_c + y_p$.

If the homogeneous differential equation has constant coefficients, one can assume an exponential solution $y = \exp(\gamma t) = e^{\gamma t}$, γ constant, to obtain the fundamental set of solutions.

Example 5-19. Solve the differential equation $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2e^{-3t}$

Solution Assume an **exponential solution** $y = e^{\gamma t}$ to the homogeneous differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0 \tag{5.97}$$

and substitute $y = e^{\gamma t}$, $\frac{dy}{dt} = \gamma e^{\gamma t}$, $\frac{d^2y}{dt^2} = \gamma^2 e^{\gamma t}$ into the homogeneous differential equation (5.97) to obtain **the algebraic equation**

$$\gamma^2 + 3\gamma + 2 = (\gamma + 2)(\gamma + 1) = 0 \tag{5.98}$$

[13] The word homogeneous is used quite frequently in the study of differential equations and its meaning depends upon the context in which it is used.

called the **characteristic equation**. The roots of this equation $\gamma = -2$ and $\gamma = -1$ are called the **characteristic roots**. Substituting these characteristic roots into the assumed solution produces the **fundamental set** $\{e^{-2t}, e^{-t}\}$. The **complementary solution** is then a **linear combination of the functions in the fundamental set**. This gives the complementary solution

$$y_c = c_1 e^{-2t} + c_2 e^{-t} \tag{5.99}$$

where c_1, c_2 are arbitrary constants.

The right-hand side of the given nonhomogeneous equation is an exponential function and since it is known that derivatives of exponential functions give exponential functions, one can assume that a **particular solution** y_p must have the form $y_p = Ae^{-3t}$ where A is a constant to be determined. The processes of examining the derivatives of the right-hand side of the nonhomogeneous equation and forming a linear combination of the basic terms associated with the right-hand side function and all its derivatives, is called **the method of undetermined coefficients** for obtaining a particular solution. Substituting the functions $y_p = Ae^{-3t}$, $\frac{dy_p}{dt} = -3Ae^{-3t}$, $\frac{d^2y_p}{dt^2} = 9Ae^{-3t}$ into the given nonhomogeneous differential equation one finds

$$9Ae^{-3t} - 9Ae^{-3t} + 2Ae^{-3t} = 2e^{-3t} \tag{5.100}$$

Simplify equation (5.100) and solving for the constant A one finds $A = 1$, so that $y_p = e^{-3t}$ is a particular solution. The general solution is then given by

$$y = y_c + y_p = c_1 e^{-2t} + c_2 e^{-t} + e^{-3t} \tag{5.101}$$

■

Example 5-20. (Representation of solution)

Obtain a general solution to the linear homogeneous differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \quad \omega \text{ is a constant} \tag{5.159}$$

which represents simple harmonic motion.

Solution When dealing with **homogeneous linear differential equations with constant coefficients one should assume an exponential solution** $y = e^{\gamma t}$ having derivatives $\dfrac{dy}{dt} = \gamma\, e^{\gamma t}$ and $\dfrac{d^2y}{dt^2} = \gamma^2\, e^{\gamma t}$. Substitute the assumed exponential solution into the given differential equation to obtain the characteristic equation

$$\gamma^2\, e^{\gamma t} + \omega^2\, e^{\gamma t} = 0 \implies \gamma^2 + \omega^2 = (\gamma - i\,\omega)(\gamma + i\,\omega) = 0 \tag{5.104}$$

The characteristic roots are the complex numbers $\gamma = i\,\omega$ and $\gamma = -i\,\omega$, where i is an imaginary unit satisfying $i^2 = -1$. These characteristic roots are substituted back into the assumed exponential solution to produce the fundamental set of solutions

$$\{\, e^{i\,\omega\, t},\ e^{-i\,\omega\, t}\,\} \tag{5.104}$$

Observe that any **linear combination of the solutions** from the fundamental set is also a solution of the differential equation (5.159), so that one can express the general solution to the homogeneous differential equation as[14]

$$y = c_1 e^{i\,\omega\, t} + c_2 e^{-i\,\omega\, t} \tag{5.105}$$

where c_1, c_2 are arbitrary constants. Use Euler's identity $e^{i\,\theta} = \cos\theta + i\,\sin\theta$ and consider the following special cases of equation (5.105).

(i) If $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$, the general solution becomes the real solution

$$y_1 = y_1(t) = \frac{1}{2}e^{i\,\omega\, t} + \frac{1}{2}e^{-i\,\omega\, t} = \cos\omega t$$

(ii) If $c_1 = \frac{1}{2i}$ and $c_2 = \frac{-1}{2i}$, the general solution becomes the real solution

$$y_2 = y_2(t) = \frac{1}{2i}e^{i\,\omega\, t} - \frac{1}{2i}e^{-i\,\omega\, t} = \sin\omega t$$

The functions $\cos\omega t$ and $\sin\omega t$ are real linearly independent solutions to the differential equation (5.159) and consequently one can state that the set of solutions

$$\{\,\cos\omega t,\ \sin\omega t\,\} \tag{5.106}$$

is a fundamental set of solutions to the equation (5.159), and

$$y = k_1\cos\omega t + k_2\sin\omega t \tag{5.107}$$

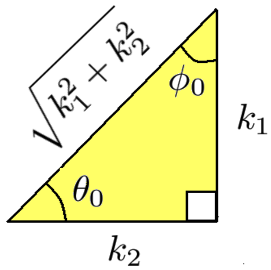
is a general solution[15] to the given differential equation (5.159), where k_1 and k_2 are arbitrary constants.

Multiply and divide equation (5.107) by $\sqrt{k_1^2 + k_2^2}$ to obtain

$$y = \sqrt{k_1^2 + k_2^2}\left(\frac{k_1}{\sqrt{k_1^2 + k_2^2}}\cos\omega t + \frac{k_2}{\sqrt{k_1^2 + k_2^2}}\sin\omega t\right) \tag{5.108}$$

[14] Electrical engineers prefer to use this form for the solution.

[15] Mechanical engineers prefer this form for the solution.



The substitutions

$$A = \sqrt{k_1^2+k_2^2}, \quad \sin\theta_0 = \frac{k_1}{\sqrt{k_1^2+k_2^2}}, \quad \cos\theta_0 = \frac{k_2}{\sqrt{k_1^2+k_2^2}}$$

allow one to express the general solution in the form

$$y = A\sin(\omega t + \theta_0)$$

The substitutions

$$A = \sqrt{k_1^2+k_2^2}, \quad \sin\phi_0 = \frac{k_2}{\sqrt{k_1^2+k_2^2}}, \quad \cos\phi_0 = \frac{k_1}{\sqrt{k_1^2+k_2^2}}$$

allows one to express the general solution in the form

$$y = A\cos(\omega t - \phi_0)$$

This example illustrates that one has many options available in representing the form for the general solution to a linear homogeneous differential equation with constant coefficients. The resulting form is closely associated with the selection of the two independent functions which make up the fundamental set of solutions.

■

Example 5-21. (Representation of solution)

Solve the linear homogeneous differential equation

$$\frac{d^2y}{dt^2} - \beta^2 y = 0, \qquad \beta \text{ is a constant} \tag{5.109}$$

Solution The given differential equation is a linear homogeneous second order differential equation with constant coefficients and so one can assume an exponential solution of the form $y = y(t) = e^{\gamma t}$ which has the derivatives $\frac{dy}{dt} = \gamma e^{\gamma t}$ and $\frac{d^2y}{dt^2} = \gamma^2 e^{\gamma t}$. Substitute the assumed solution and its derivatives into the above differential equation to obtain the characteristic equation

$$\gamma^2 e^{\gamma t} - \beta^2 e^{\gamma t} = 0 \qquad \Longrightarrow \qquad \gamma^2 - \beta^2 = (\gamma-\beta)(\gamma+\beta) = 0 \tag{5.110}$$

giving the characteristic roots $\gamma = \beta$ and $\gamma = -\beta$ from which one can construct the fundamental set of solutions

$$\{\, e^{\beta t},\ e^{-\beta t}\,\} \tag{5.111}$$

A general solution is then any linear combination of the functions in the fundamental set and so can be represented in the form

$$y = y(t) = c_1 e^{\beta t} + c_2 e^{-\beta t} \tag{5.112}$$

where c_1, c_2 are arbitrary constants. A special case of equation (5.112) occurs when $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$ and one finds the special solution

$$y_1 = y_1(t) = \frac{1}{2}e^{\beta t} + \frac{1}{2}e^{-\beta t} = \cosh \beta t \tag{5.113}$$

The special case where $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$ produces the solution

$$y_2 = y_2(t) = \frac{1}{2}e^{\beta t} - \frac{1}{2}e^{-\beta t} = \sinh \beta t \tag{5.114}$$

The functions $\cosh \beta t$ and $\sinh \beta t$ are linearly independent solutions to the differential equation (5.109) and therefore one can construct the fundamental set of solutions

$$\{ \cosh \beta t, \quad \sinh \beta t \} \tag{5.115}$$

and from this fundamental set one can construct the general solution in the form

$$y = y(t) = k_1 \cosh \beta t + k_2 \sinh \beta t \tag{5.116}$$

where k_1 and k_2 are arbitrary constants.

If one selects the constants k_1 and k_2 such that

$$k_1 = A \cosh \beta t_0 \qquad \text{and} \qquad k_2 = -A \sinh \beta t_0 \qquad t_0 \text{ is a constant}$$

then the general solution can be expressed in the form

$$y = y(t) = A\,(\cosh \beta t \cosh \beta t_0 - \sinh \beta t \sinh \beta t_0) = A \cosh \beta (t - t_0)$$

Alternatively, one can select the constants

$$k_1 = -A \sinh \beta t_0 \qquad \text{and} \qquad k_2 = A \cosh \beta t_0$$

and express the general solution in the alternative form

$$y = y(t) = A\,(\sinh \beta t \cosh \beta t_0 - \cosh \beta t \sinh \beta t_0) = A \sinh \beta (t - t_0)$$

This is another example, where the form selected for the fundamental set of solutions can lead to representing the general solution to the differential equation in a variety of forms. In selecting a particular form for representing the solution one should select a form where the representation of the solution and any required auxiliary conditions are easily handled.

Mechanical Resonance

In equation (5.94), let $F(t) = F_0 \cos \lambda t$, with λ is a constant, and then construct the general solution to equation (5.94) for this special case. To solve

$$L(y) = m\ddot{y} + \beta\dot{y} + Ky = F_0 \cos \lambda t \tag{5.117}$$

it is customary to first solve the homogeneous equation

$$L(y) = m\ddot{y} + \beta\dot{y} + Ky = 0. \tag{5.118}$$

This is an ordinary differential equation with constant coefficients and this type of equation can be solved by assuming an exponential solution $y = \exp(\gamma t) = e^{\gamma t}$. Substituting this assumed value for y into the differential equation (5.118) one obtains an equation for determining the constant(s) γ. This resulting equation is called **the characteristic equation** associated with the homogeneous differential equation and the roots of this equation are called **the characteristic roots.** One finds the characteristic equation

$$m\gamma^2 + \beta\gamma + K = 0$$

with characteristic roots

$$\gamma = \frac{-\beta \pm \sqrt{\beta^2 - 4mK}}{2m} = -\frac{\beta}{2m} \pm \sqrt{\left(\frac{\beta}{2m}\right)^2 - \frac{K}{m}}. \tag{5.119}$$

(i) If the characteristic roots are denoted by $\gamma_1,\ \gamma_2$ and these roots are distinct, then the set of solutions $\{e^{\gamma_1 t}, e^{\gamma_2 t}\}$ is called **a fundamental set of solutions** to the homogeneous differential equation and **the general solution** is denoted by the linear combination

$$y = c_1 e^{\gamma_1 t} + c_2 e^{\gamma_2 t}, \qquad c_1,\ c_2 \text{ are arbitrary constants} \tag{5.120}$$

(ii) If the characteristic roots are equal and $\gamma_1 = \gamma_2$, then one member of the fundamental set is $e^{\gamma_1 t}$. It has been found that **each time a characteristic root repeats itself, then one must multiply the first solution by** t. This rule gives the second member of the fundamental set as $te^{\gamma_1 t}$ The fundamental set is then given by $\{e^{\gamma_1 t}, te^{\gamma_1 t}\}$ and produces the general solution as a linear combination of the solutions in the fundamental set. The general solution can be written

$$y = c_1 e^{\gamma_1 t} + c_2 t e^{\gamma_1 t}, \qquad c_1,\ c_2 \text{ are arbitrary constants.} \tag{5.121}$$

(iii) If the characteristic roots are imaginary and of the form $\gamma_1 = \alpha + i\,\beta$ and $\gamma_2 = \alpha - i\,\beta$, then one can use the Euler formula $e^{ix} = \cos x + i \sin x$ to express the general solution in either of the forms

$$
\begin{aligned}
y =& c_1 e^{(\alpha+i\,\beta)t} + c_2 e^{(\alpha-i\,\beta)t}, \qquad \text{complex form for the solution} \\
y =& e^{\alpha t}\left(C_1 \cos\beta t + C_2 \sin\beta t\right), \qquad \text{real form for the solution}
\end{aligned}
\tag{5.122}
$$

where c_1, c_2, C_1, C_2 represent arbitrary constants.

The general solution to the homogeneous differential equation (5.118) is called **the complementary solution.** In equation (5.119), the discriminant $(\beta/2m)^2 - K/m > 0$, determines the type of motion that results. The following cases are considered.

CASE I (Homogeneous Equation and Overdamping)

If $(\beta/2m)^2 - K/m > 0$, equation (5.119) has two distinct roots γ_1 and γ_2 where both γ_1 and γ_2 are negative, then the corresponding complementary solution of equation (5.118) has **transient terms** $e^{\gamma_1 t}$ and $e^{\gamma_2 t}$ and the general solution is of the form

$$
y_c = c_1 e^{\gamma_1 t} + c_2 e^{\gamma_2 t}, \quad \gamma_1 < 0,\ \gamma_2 < 0 \tag{5.123}
$$

where c_1, c_2 are arbitrary constants. This type of solution illustrates that if the damping constant β is too large, then no oscillatory motion can exist. In such a situation, the system is said to be **overdamped**.

CASE II (Homogeneous equation and underdamping)

For the condition $(\beta/2m)^2 - K/m < 0$, let $\omega_0^2 = K/m - (\beta/2m)^2$ and obtain from the characteristic equation (5.119) the two complex characteristic roots

$$
\gamma_1 = -\beta/2m + i\omega_0 \quad \text{and} \quad \gamma_2 = -\beta/2m - i\omega_0.
$$

These characteristic roots produce a complementary solution of the form

$$
y_c = e^{-\beta t/2m}\left(c_1 \sin\omega_0 t + c_2 \cos\omega_0 t\right)
$$

or

$$
y_c = \sqrt{c_1^2 + c_2^2}\, e^{-\beta t/2m} \cos(\omega_0 t - \phi) \tag{5.124}
$$

with c_1, c_2 arbitrary constants. Here ω_0 represents **the damped natural frequency of the system**. If β is small, ω_0 is **approximately the natural frequency of the undamped system** given by $\omega = \sqrt{K/m}$.

The solution equation (5.124) denotes **a damped oscillatory solution** which can be visualized by plotting the curves

$$y_1 = \sqrt{c_1^2 + c_2^2}\, e^{-\beta t/2m} \quad \text{and} \quad y_2 = -y_1$$

as **envelopes of the oscillation** $\cos(\omega_0 t - \phi)$ as is illustrated in figure 5-6. The term $-\beta/2m$ is called the damping constant or damping factor.

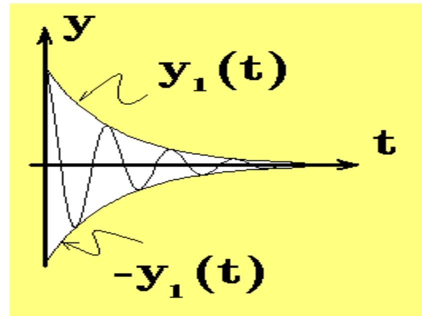


Figure 5-6. Damped oscillations.

CASE III (Homogeneous equation and critical damping) If $(\beta/2m)^2 - K/m = 0$, equation (5.119) has the repeated roots $\gamma_1 = \gamma_2 = -\beta/2m$ which produces the solution

$$y_c = (c_1 + c_2 t)e^{-\beta t/2m}. \tag{5.125}$$

By reducing the damping constant β one gets to a point where oscillations begin to occur. The motion is then said to be critically damped. The critical value for the damping constant β in this case is denoted by β_c and is determined by setting the discriminant equal to zero to obtain $\beta_c = 2m\omega$ where $\omega = \sqrt{K/m}$ is the natural frequency of the undamped system.

Particular Solution

Associated with the complementary solution from one of the cases I, II, or III, is **the particular solution** of the nonhomogeneous equation (5.117). The particular solution can be determined by **the method of undetermined coefficients**. Examine the function(s) on the right-hand side of the differential equation and all the derivatives associated with these function(s). Select the basic terms which keep occurring in the function and all of its derivatives and form a linear combination of these basic terms. For the equation (5.117) the basic terms which occur by continued differentiation of the right-hand side are the functions $\{\cos\lambda t,\ \sin\lambda t\}$ multiplied by some constant. One can then assume that **the particular solution** is of the form

$$y_p = A\cos\lambda t + B\sin\lambda t \tag{5.126}$$

with A **and** B **unknown constants to be determined**. Substituting this assumed particular solution into the differential equation (5.117) produces the equation

$$[(K - m\lambda^2)A + \beta\lambda B]\cos\lambda t + [-\beta\lambda A + (K - m\lambda^2)B]\sin\lambda t = F_0\cos\lambda t \tag{5.127}$$

Now equate the coefficients of like terms to obtain the system of equations

$$\begin{aligned}(K - m\lambda^2)A + \beta\lambda B &= F_0 \\ -\beta\lambda A + (K - m\lambda^2)B &= 0,\end{aligned} \tag{5.128}$$

which are equations used to determine the constants A and B. Solving equations (5.128) gives

$$A = \frac{(K - m\lambda^2)F_0}{\Delta} \quad \text{and} \quad B = \frac{\beta\lambda F_0}{\Delta} \tag{5.129}$$

where

$$\Delta = (K - m\lambda^2)^2 + \beta^2\lambda^2.$$

The particular solution[16] can then be expressed as

$$y_p = \frac{(K - m\lambda^2)F_0}{\Delta}\cos\lambda t + \frac{\beta\lambda F_0}{\Delta}\sin\lambda t = \frac{F_0}{\sqrt{\Delta}}\cos(\lambda t - \phi) \tag{5.130}$$

where ϕ is a phase angle defined by $\tan\phi = \beta\lambda/(K - m\lambda^2)$ for $m\lambda^2 \neq K$. The general solution to equation (5.119) can then be written $y = y_c + y_p$. In the general solution, the complementary solutions are transient solutions, and the particular solution represents the steady state oscillations. The amplitude of the steady state oscillations is given by

$$\text{Amp} = \frac{F_0}{\sqrt{\Delta}} = \frac{F_0}{\sqrt{(K - m\lambda^2)^2 + \beta^2\lambda^2}} = \frac{F_0}{m\sqrt{(\omega^2 - \lambda^2)^2 + 4\lambda^2\omega^2(\beta/\beta_c)^2}} \tag{5.131}$$

where ω is the natural frequency of the undamped system and $\beta_c = 2m\omega$ is the critical value of the damping. For $\beta = 0$ (no damping), the denominator in equation (5.131) becomes $m|\omega^2 - \lambda^2|$ and approaches zero as λ tends toward ω. Thus, with no damping, as the angular frequency λ of the forcing term approaches the natural frequency ω of the system, the denominator in equation (5.119) approaches zero, which in turn causes the amplitude of the oscillations to increase without bound. This is known as the phenomenon of resonance. For $\beta \neq 0$, there can still be a resonance-type behavior whereby the amplitude of the oscillations become large for some specific value of the forcing frequency λ.

Define the resonance frequency as the value of λ which produces the maximum amplitude of the oscillation, if an oscillation exists.

---

[16] Recall that $A\cos\lambda t + B\sin\lambda t = \sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\cos\lambda t + \frac{B}{\sqrt{A^2+B^2}}\sin\lambda t\right] = \sqrt{A^2+B^2}\cos(\lambda t - \phi)$

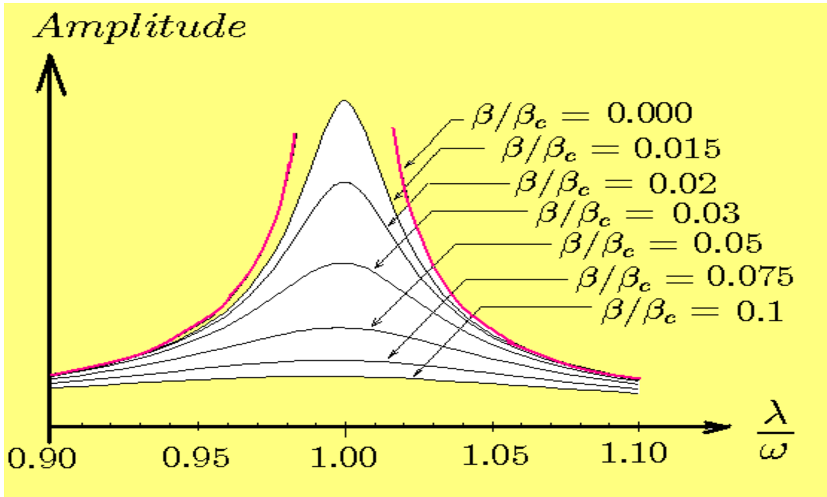


Figure 5-7. Amplitude versus frequency for a forced system.

This amplitude, given by equation (5.131), has a maximum value when the denominator is a minimum. Let

$$H = (\omega^2 - \lambda^2)^2 + 4\lambda^2\omega^2(\beta/\beta_c)^2$$

denote this denominator. The quantity H has a minimum value with respect to λ when the derivative of H with respect to λ is zero. Calculating this derivative gives

$$\frac{dH}{d\lambda} = 2(\omega^2 - \lambda^2)(-2\lambda) + 8\lambda\omega^2(\beta/\beta_c)^2 = 0$$

when

$$\lambda^2 = \omega^2 \left[1 - 2(\beta/\beta_c)^2\right]. \tag{5.132}$$

The phenomenon of resonance is illustrated graphically in figure 5-7 by plotting the amplitude, as given by equation (5.131), versus λ for various values of the ratio β/β_c.

In practical problems, it is important to be able to design vibratory structures to avoid resonance. For example, printing presses vibrating at the correct frequency can act as forcing functions to cause large vibrations and even collapse of the supporting floor. High winds can act as forcing functions to cause resonance oscillations of structures. Flutter of aircraft wings, if not controlled properly, can result in dynamic instability of an aircraft.

Resonance can also be a desired phenomenon such as in tuning an electrical circuit for a maximum response of a voltage of a specified frequency. In the study of electrical circuits where frequency is a variable, it is desirable to have frequency response characteristics for the circuit in a graphical form similar to figure 5-7.

Train yourself to look for curves which have shapes similar to those of the curves in figure 5-7. Chances are some kind of resonance phenomenon is taking place. Curves similar to the curves of figure 5-7 usually result from vibration models used to study a wide variety of subjects and are the design basis of a large number of measuring devices. The following is a brief list of subject areas where it is possible to find additional applications of the basic equations of vibratory phenomena and resonance. Examine topics listed under mechanical vibrations, earthquake modeling, atomic vibrations, atomic cross sections, vibrations of atoms in crystals, scattering of atoms, particles, and waves from crystal surfaces, sound waves, string instruments, tidal motions, lasers, electron spin resonance, nuclear magnetic resonance, and behavior of viscoelastic materials.

Torsional Vibrations

Torsional vibrations are similar in form to the spring mass system and the differential equation of the motion can be obtained from the example 5-15 presented earlier. From this example the relation

$$\sum \text{Torques} = M = I\alpha = I\frac{d^2\theta}{dt^2} \tag{5.133}$$

is employed from equation (5.52) where θ denotes the angular displacement, I is the moment of inertia of the body, and $\alpha = \ddot{\theta} = \dfrac{d^2\theta}{dt^2}$ is the angular acceleration. Consider a disk attached to a fixed rod as in figure 5-8.

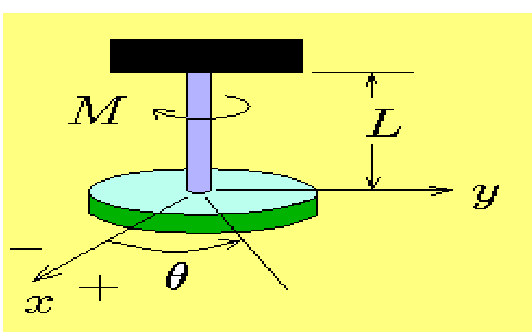


Figure 5-8. Torsional vibrations.

If the disk is rotated through an angle θ, there is a restoring moment M produced by the rod. Hooke's law states that the restoring moment is proportional to the angular displacement and

$$M = -K_T\theta \tag{5.134}$$

where K_T is called the spring constant of the shaft and is the proportionality constant associated with the angular displacement θ.

From the relation in equation (5.133) there results

$$M = -K_T\theta = I\frac{d^2\theta}{dt^2} \quad \text{or} \quad I\frac{d^2\theta}{dt^2} + K_T\theta = 0 \tag{5.135}$$

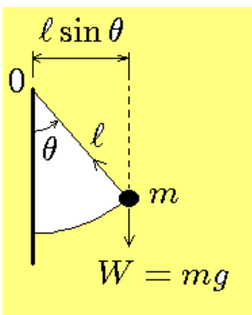
as the equation of motion describing the angular displacement θ.

By adding a linear damper and external force to equation (5.135), a more general equation results

$$I\frac{d^2\theta}{dt^2} + \beta\frac{d\theta}{dt} + K_T\theta = F(t). \tag{5.136}$$

From strength of materials, the constant K_T is given by the relation $K_T = GJ/L$, where G is called the shearing modulus of the rod material, J is the polar moment of inertia of the rod cross section, and L is the length of the shaft.

The simple pendulum



For the pendulum illustrated the forces about 0 are the weight of the mass and the radial force along the string. The radial force passes through the origin and so does not produce a moment about the origin. The moment of inertia of the mass m about 0 is given by $I = m\ell^2$ and the torque about 0 is given by $T = -(mg)(\ell\sin\theta)$and consequently the equation of motion can be expressed $T = -mg\ell\sin\theta = m\ell^2\dfrac{d^2\theta}{dt^2}$

Simplification reduces the equation of motion to the form

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0, \qquad \omega^2 = \frac{g}{\ell}$$

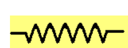
For small oscillations one can make the approximation $\sin\theta \approx \theta$ and write the equation for the oscillating pendulum in the form

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

which is the equation of a simple harmonic oscillator.

Electrical Circuits

The basic elements needed to study electrical circuits are as follows:

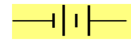
(a) Resistance R is denoted by the symbol 

The dimension of resistance is ohms[17] and written $[R] = \text{ohms}$.

[17] George Simon Ohm (1787–1854), German physicist.

(b) Inductance L is denoted by the symbol 
The dimension of inductance is henries[18] and written $[L] =$ henries.

(c) Capacitance C denoted by the symbol 
The dimension of capacitance is farads[19] and written $[C] =$ farads.

(d) Electromotive force (emf) E or V denoted by the symbols  or 
The dimensions of electromotive force is volts[20] and written $[E] = [V] =$ volts.

(e) Current I is a function of time, denoted $I = I(t)$, with dimensions of amperes[21] and written $[I] =$ amperes.

(f) Charge Q on the capacitance is a function of time and written $Q = Q(t)$, with dimensions $[Q] =$ coulombs.

The basic laws associated with electrical circuits are as follows: The current is the **time rate of change of charge**. This can be represented with the above notation as

$$I = \frac{dQ}{dt} \quad \text{with} \quad [I] = \text{amperes}, \quad [\frac{dQ}{dt}] = \text{coulombs/second} \tag{5.137}$$

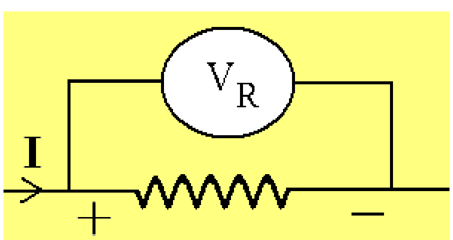


Figure 5-9.
Voltage drop V_R across a resistor.

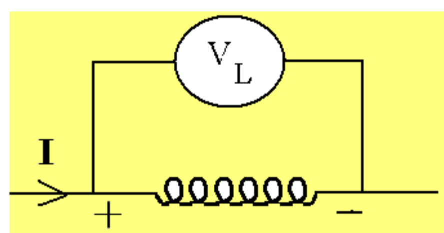


Figure 5-10.
Voltage drop V_L across inductor.

The **voltage drop V_R across a resistance**, see figure 5-9, is proportional to the current through the resistance. This is known as **Ohm's law**. The proportionality

[18] Joseph Henry (1797–1878), American physicist.
[19] Michael Faraday (1791– 1867) English physicist.
[20] Alessandro Volta (1745–1827) Italian scientist.
[21] André Marie Ampére (1775–1836) French physicist.

constant is called the resistance R. In symbols this can be represented as $V_R = RI$ where

$$[V_R] = \text{volts} = [R][I] = (\text{ohm})\,(\text{ampere}) \tag{5.138}$$

The **voltage drop V_L across an inductance**, see figure 5-10, is proportional to the time rate of change of current through the inductance. The proportionality constant is called the inductance L. In symbols this can be represented as

$$V_L = L\frac{dI}{dt} \quad \text{where} \quad [V_L] = \text{volts} = [L]\,[\frac{dI}{dt}] = (\text{henry})(\text{ampere/second}) \tag{5.139}$$

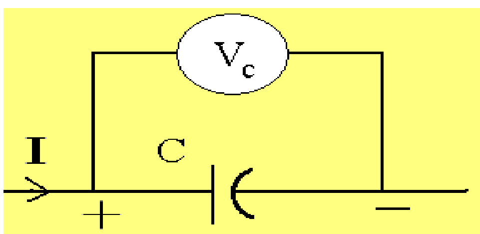


Figure 5-11.
Voltage drop V_C across capacitor.

The **voltage drop V_C across a capacitance**, see figure 5-11, is proportional to the charge Q of the capacitance. The proportionality constant is denoted $1/C$. In symbols this is represented as

$$V_C = \frac{Q}{C} \tag{5.140}$$

where $[V_C] = \text{volts} = \left[\frac{1}{C}\right][Q] = \text{coulombs/farad}$

The **Kirchoff laws** for an electric circuit are.

Kirchhoff's[22] first law :

The sum of the voltage drops around a closed circuit must equal zero.

Kirchhoff's second law:

The amount of current into a junction must equal the current leaving the junction.

The place in an electrical circuit where two or more circuit elements are joined together is called a **junction**. A closed circuit or loop occurs whenever a path constructed through connected elements within a circuit closes upon itself. Voltage drops are selected as positive, whereas voltage rises are selected as negative.

Example 5-22. For the RC-circuit illustrated in figure 5-12, set up the differential equation describing the rate of change of the charge Q on the capacitor. Make the assumption that $Q(0) = 0$.

[22] Gustav Robert Kirchhoff (1924–1887) German physicist.

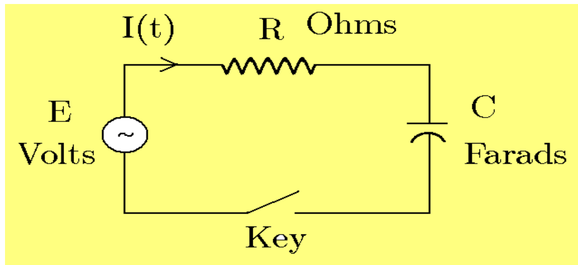


Figure 5-12. An RC-series circuit.

Solution For a path around the circuit illustrated in figure 5-12, the Kirchhoff's voltage law would be written

$$V_R + V_C - E = 0.$$

Let $I = I(t) = \frac{dQ}{dt}$ denote the current in the circuit at any time t. By Kirchhoff's first law:

$$\begin{array}{ccccc} \begin{pmatrix} Voltage\ drop \\ across\ R \end{pmatrix} & + & \begin{pmatrix} Voltage\ drop \\ across\ C \end{pmatrix} & = & \begin{pmatrix} Applied \\ emf \end{pmatrix} \\ V_R & + & V_C & = & E \\ RI & + & \frac{Q}{C} & = & E. \end{array}$$

This gives the differential equation

$$L(Q) = R\frac{dQ}{dt} + \frac{1}{C}Q = E \tag{5.141}$$

where R, C and E are constants. The solution of the homogeneous differential equation

$$R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

can be determined by separating the variables and integrating to obtain

$$\frac{dQ}{Q} = \frac{-1}{RC}\,dt \quad \text{and} \quad \int \frac{dQ}{Q} = \int \frac{-1}{RC}\,dt \implies \ln Q = \frac{-t}{RC} + \alpha$$

where α is a constant of integration. Solving for Q one finds is $Q_c = c_1 \exp(-t/RC)$ where $c_1 = e^{\alpha}$ is just some new constant. Since the right-hand side of the nonhomogeneous differential equation is a constant, one can assume a particular solution of the form $Q_p = c_2 E$ where c_2 is a constant to be determined. Substituting this assumed solution into the nonhomogeneous differential equation and solving for c_2 one finds $c_2 = C$ and so the particular solution can be written $Q_p = CE$. The general solution of equation (5.141) is represented by the sum $Q = Q_c + Q_p$ and the solution satisfying $Q(0) = 0$ is given by

$$Q = Q(t) = EC(1 - e^{-t/RC}) \tag{5.142}$$

The relation (5.142) is employed to determine the current I and voltages V_C and V_R as

$$\begin{aligned} I = I(t) = &\frac{dQ}{dt} = \frac{E}{R}\, e^{-t/RC} \\ V_C = &\frac{Q}{C} = E(1 - e^{-t/RC}) \\ V_R = &RI = E\, e^{-t/RC} \end{aligned} \tag{5.143}$$

In equations (5.142) and (5.143) the term $\exp(-t/RC)$ is called **a transient term** and the constant $\tau = RC$ is called **the time constant for the circuit**. In general, terms of the form $\exp(-t/\alpha)$ are transient terms, and such terms are short lived and quickly or slowly decay, depending upon the magnitude of the time constant $\tau = \alpha$. The following table illustrates values of $\exp(-t/\alpha)$ for t equal to various values of the time constant.

Time t	$\exp(-t/\alpha)$
α	0.3679
2α	0.1353
3α	0.0498
4α	0.0183
5α	0.0067

The values in the above table gives us valuable information concerning equations such as (5.142) and (5.143). The table shows that decaying exponential terms are essentially zero after five time constants. This is because the values of the exponential terms are less than 1 percent of their initial values.

Solutions to circuit problems are usually divided into two parts, called **transient terms** and **steady state terms**. Transient terms eventually decay and disappear and do not contribute to the solution after about 5 time constants. The steady state terms are the part of the solution which remains after the transient terms become negligible.

■

Example 5-23. For the parallel circuit illustrated in figure 5-13, apply Kirchhoff's first law to each of the three closed circuits.

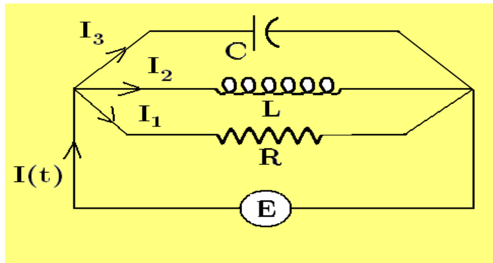


Figure 5-13. A parallel circuit.

Note that each closed circuit has the same voltage drop. This produces the following equations.

$$\begin{aligned} E &= RI_1 \\ E &= L\frac{dI_2}{dt} \\ E &= \frac{1}{C}\int I_3\,dt. \end{aligned} \tag{5.144}$$

Kirchhoff's second law applied to the given circuit tells us

$$I = I_1 + I_2 + I_3. \tag{5.145}$$

If the impressed current I is given, the above four equations can be reduced to one ordinary differential equation from which the impressed voltage E can be found. Write equation (5.145) in the form

$$I = \frac{E}{R} + \frac{1}{L}\int E\,dt + C\frac{dE}{dt}.$$

called a differential–integral equation. By differentiation of this equation there results an ordinary linear second-order differential equation

$$\frac{dI}{dt} = \frac{1}{R}\frac{dE}{dt} + \frac{E}{L} + C\frac{d^2E}{dt^2},$$

where E is the dependent variable to be determined.

Conversely, if E is given and I is unknown, then equations (5.144) give us I_1, I_2, and I_3, and equation (5.145) can be used to determine the current I.

■

Thermodynamics

Experiments on a fixed mass of gas has established the following gas laws which relate the pressure P, volume V and absolute temperature T.

Boyle's Law If T is held constant, then $pV = constant$.

Charles's Law If P is held constant, then $\dfrac{V}{T} = constant$.

Gay-Lussac Law If V is held constant, then $\dfrac{P}{T} = constant$.

These laws are summarized using the gas equation

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$$

where pressure P can be measured in units $[N/m^2]$, volume V can be measured in units $[m^3]$ and absolute temperature T is measured in units $[K]$.

The ideal gas absolute temperature is defined using Boyles law which states $PV \propto T$ which produces **the equation of state** for an ideal gas, which can be expressed in the form

$$\begin{aligned} PV =& nRT \\ [\frac{N}{m^2}][m^3] =& [mol]\left[\frac{J}{mol\,K}\right][K] \end{aligned}$$

where n is the number of moles of gas and $R = 8.314472\ [\frac{J}{mol\,K}]$ is the ideal gas constant or universal molar gas constant. Note that real gases may or may not obey the ideal gas law. For gases which are imperfect, there are many other proposed equations of state. Some of these proposed equations are valid over selected ranges and conditions and can be found under such names as *Van der Waals equation, Berthelot equation, Dieterici equation, Beattie-Bridgeman equation, Virial equation.*

The **zeroth law of thermodynamics** states that if two bodies are in thermal equilibrium with a third body, then the two bodies must be in thermal equilibrium with each other. The zeroth law is used to develop the concept of temperature. Here thermodynamic equilibrium infers that the system is (i) in chemical equilibrium and (ii) there are no pressure or temperature gradients which would cause the system to change with time. The **first law of thermodynamics** is an energy conservation principle which can be expressed $dQ = dU + dW$ where dQ is the heat supplied to a gas, dU is the change in internal energy of the gas and dW is the external work done. The **second law of thermodynamics** examines processes that can happen in an isolated system and states that the only processes which can occur are those for which the entropy either increases or remains constant. Here **entropy** S is related to the ability or inability of a systems energy to do work. The change in entropy is defined as $dS = dQ/T$ where dQ is the heat absorbed in an isothermal and reversible process and T denotes the absolute temperature.

Recall that the ability of gases to change when subjected to pressure and temperature variations can be described by the equation of state of an ideal gas

$$PV = nRT, \tag{5.146}$$

where P is the pressure [N/m^2], V is the volume [m^3], n is the amount of gas [moles], R is the universal gas constant [J/mol $\cdot$ K], and T is the temperature [K]. For an ideal gas, the gas constant R can also be expressed in terms of **the specific heat at constant pressure** C_p, [J/mol $\cdot$ K] and **the specific heat at constant volume** C_v, [J/mol $\cdot$ K] by **Mayer's equation** $R = C_p - C_v$. Equation (5.146) is illustrated in the pressure-volume diagram of figure 5-14.

The curves where T is a constant are called **isothermal curves** and are the hyperbolas labeled (b) and (c) illustrated in figure 5-14. These curves correspond to the temperature values T_1 and T_2. When a gas undergoes changes of state it can do so by an **isobaric process** (P is a constant) illustrated by line (a) in figure 5-14, an **isovolumetric process** (V is a constant) illustrated by the line (e) in figure 5-14, **an isothermal process** (T is a constant) illustrated by the hyperbolas with $T = T_1$ and $T = T_2$ in figure 5-14, or **an adiabatic process** (no heat is transferred) represented by the curve (d) in figure 5-14.

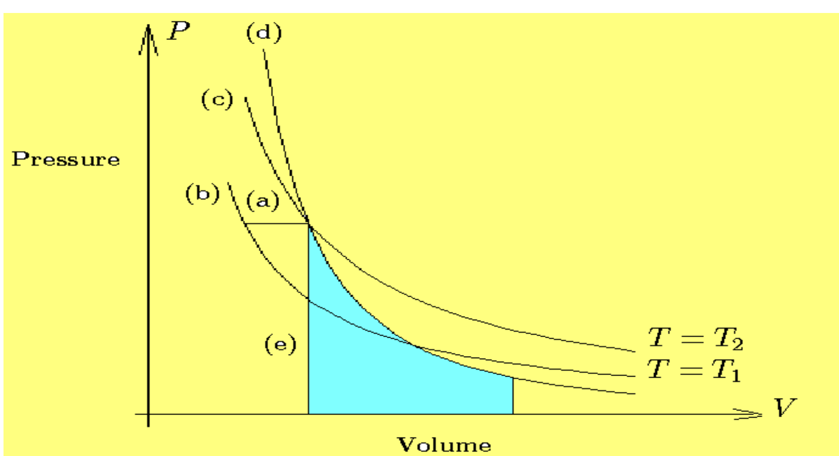


Figure 5-14. Pressure-Volume diagram.

The first law of thermodynamics states that when a gas undergoes a change, the equation $dU = dQ + dW$ must be satisfied, where dU is the change in internal energy, dQ is the change in heat supplied to the gas, and dW is the work done. An **adiabatic process** is one in which $dQ = 0$. For an adiabatic process, the first law of thermodynamics requires $dU = dW$. The work done dW is related to the volume change by the relation $dW = -PdV$, and the change in internal energy is related to the temperature change by the relation $dU = \mu C_p dT$. For an adiabatic process

$$\frac{dP}{P} + \gamma \frac{dV}{V} = 0 \tag{5.147}$$

Integrate the equations (5.147) and show the adiabatic curve (d) in figure 5-14 can be described by any of the equations

$$TV^{\gamma-1} = \text{Constant}, \quad TP^{\frac{1-\gamma}{\gamma}} = \text{Constant}, \quad or \quad PV^{\gamma} = \text{Constant},$$

where $\gamma = C_p/C_v$ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. Also note that during an adiabatic process $dQ = 0$ so that the work done by the system undergoing a change in volume is given by the integral of dW which is represented by the shaded area in the figure 5-14. This shaded area is represented by the integral

$$\text{work done} = \int_{v_1}^{v_2} P\, dV$$

Radioactive Decay

The periodic table of the chemical elements lists all 118 known chemical elements using the notation ${}^{\alpha}_{\eta}A$, where A represents a shorthand notation used to signify the name of an element, α is the atomic mass number or total number of protons and neutrons in the nucleus of the element and η is the atomic number or number of protons in the nucleus of the element. Isotopes of an element all have the same number of protons in the nucleus, but a different number of neutrons. For example, carbon is denoted ${}^{12}_{6}C$ and the elements ${}^{13}_{6}C$, ${}^{14}_{6}C$ are isotopes of carbon. Many of the isotopes experience a process known as **nuclear decay** or **radioactive decay**, where an isotope will emit some particles in a continuous way and lose some of its mass over time.

Let A denote the quantity of a radioactive substance, measured in grams, with the derivative $\frac{dA}{dt}$ denoting the rate of disintegration or amount of mass lost as a function of time t. In general, the amount of mass lost during radioactive decay is proportional to the amount present and so can be represented by the mathematical statement

$$\frac{dA}{dt} = -kA \tag{5.148}$$

where k is a proportionality constant and the minus sign indicates mass being lost. The proportionality constant k is referred to as the decay constant.

If $A = A_0$ at time $t = 0$ one can separate the variables in equation (5.148) and write

$$\frac{dA}{A} = -k dt \tag{5.149}$$

Integrate both sides of equation (5.149) and show

$$\int_{A_0}^{A} \frac{dA}{A} = -k \int_0^t dt \tag{5.150}$$

Here the limits of integration indicate that at time $t = 0$, $A = A_0$ and at time t, then $A = A(t)$. After integrating equation (5.150) one obtains

$$\ln A \Big|_{A_0}^{A} = -kt \Big|_0^t \implies \ln\left(\frac{A}{A_0}\right) = -kt \implies A = A_0 e^{-kt} \tag{5.151}$$

If, by experiment, it is found that p percent of A_0 disappears in T years, then $(1 - \frac{p}{100})\, A_0$ is the amount remaining after T years and so this information can be used to determine the decay constant k. At time T one has the equation

$$\left(1 - \frac{p}{100}\right) A_0 = A_0 e^{-kT} \tag{5.152}$$

which implies $\ln\left(1 - \dfrac{p}{100}\right) = -kT$ and so one can solve for the decay constant k and find

$$k = -\frac{1}{T} \ln\left(1 - \frac{p}{100}\right) \tag{5.153}$$

The **half-life of a radioactive material** is the time τ it takes for 50-percent of the material to disappear. Consequently, if $A = \frac{1}{2}A_0$ at time $t = \tau$, the equation (5.151) requires that

$$\frac{1}{2}A_0 = A_0 e^{-k\tau} \implies \ln\left(\frac{1}{2}\right) = -k\tau \implies \tau = \frac{1}{k}\ln(2) \tag{5.154}$$

The table below gives the half-life of some selected elements from the periodic table.

Using the results from equation (5.154) one can express the radioactive decay curve given by equation (5.151) in the form

$$\frac{A}{A_0} = e^{-(\ln 2)\, t/\tau} = \left(\frac{1}{2}\right)^{t/\tau} \tag{5.155}$$

The figure 5-15 is a sketch of $y = \dfrac{A}{A_0}$ versus $x = t/\tau$. Examine this figure and note the values of A for the values $t = \tau, 2\tau, 3\tau, 4\tau, \ldots$. One can then construct the table of values illustrated.

Element	Isotope	Half-Life
Silver (Ag)	^{105}Ag	41.29 days
	^{111}Ag	7.45 days
	^{107}Ag	44 seconds
Gold (Au)	^{197}Au	7.4 seconds
Iodine (I)	^{125}I	59 days
	^{123}I	13 hours
	^{131}I	8 days
Cesium (Ce)	^{137}Ce	30 years
Uranium (U)	^{238}U	$4.46\,(10)^9$ years
Thorium (Th)	^{232}Th	$14.05\,(10)^9$ years

The figure 5-15 illustrates that after a time of one half-life, then one-half of the material is gone. After another time span of one half-life, half of the remaining material is gone.

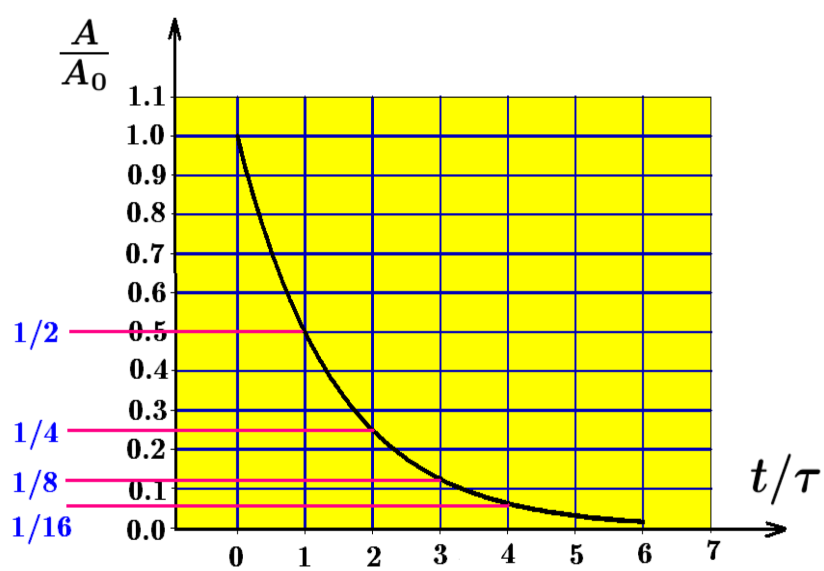


Figure 5-15. Radioactive decay curve $A = A_0 e^{-(\ln 2)\, t/\tau}$

t/τ	A/A_0
0	1
1	1/2
2	1/4
3	1/8
4	1/16
$\vdots$	$\vdots$

The reduction in the amount of material by one-half is illustrated by the scaling of the A and t axis of the radioactive decay curve given by equation (5.155). If one uses the axes A/A_0 and t/τ, then for each span of one half-life there is a decrease in the amount of material by one-half.

Different radioactive substances are used for scientific research in many disciplines. For example, archaeology uses ^{14}C for dating of ancient artifacts.

Carbon-12 is a stable element and its isotope carbon-14 is radioactive with a half-life of 5730 years. All living organisms contain both ^{12}C and ^{14}C in known ratios. However, after an organism dies, the carbon-12 amount remains the same but the carbon-14 begins to decay. By measuring the proportions of ^{12}C and ^{14}C in dead organisms one can estimate the elapsed time since death.

Geologist use ^{238}U, ^{206}Pb, ^{232}Th, and ^{208}Pb to determine the age of rocks. They measure the relative amounts of these radioactive substances and compare ratios of these amounts with rocks from an earlier age.

Radioactive substances are used for tracers and imaging in chemistry, biology and medicine.

Economics

Suppose that it cost $C = C(\xi)$ dollars to produce ξ number of units of a certain product. The function $C(x)$ is called the **cost function for production of x items**. Let $r = r(x)$ denote the **price received from the sale of 1 unit of the item** and let $P = P(x)$ denote the **profit from the sale of x items**. This profit can be represented

$$\begin{aligned} P = P(x) =& \text{(number of items sold)(selling price of 1 unit)} - \text{cost of production} \\ P = P(x) =& x \cdot r(x) - C(x) \end{aligned}$$

where the function $x \cdot r(x)$ is called the **revenue function**.

As a first approximation for the representations of $r(x)$ and $C(x)$ one can assume that they are linear functions of x and one can write

$$r = \alpha - \beta x \qquad \text{and} \qquad C(x) = a + bx$$

These assumptions have the following interpretations for α, β, a, b all constants.

(i) The minus sign in the representation for r indicates that an increase in the selling price will cause a decrease in sales. This can be seen by plotting the curve $x = \frac{\alpha}{\beta} - \frac{1}{\beta} r$ versus r which is a straight line with slope $-1/\beta$. This line tells one that as r increases (price increases), then the number of sales x decreases.

(ii) The constant α has to be large enough such that $\alpha - \beta x$ remains positive as you don't want to give away the product.

(iii) In the cost function, the constant a represents the overhead for the maintenance of the production facilities and the variable term bx represent the additional cost of production associated with producing x units. The units of measurements for each term must be in dollars so $[a] = \$$ and $[b] = \$/\text{unit}$ and $[x] =$ number of units. Hence, one can interpret b as the cost to produce 1 unit.

Using the above assumptions the profit from the sale of x items is given by

$$P = P(x) = x(\alpha - \beta x) - (a + bx)$$

and if a profit is to be made from the sale of just one item, then it is required that $\alpha > \beta + a + b$. The derivative of the profit with respect to x is

$$\frac{dP}{dx} = x(-\beta) + (\alpha - \beta x) - b$$

The profit is a maximum when $\dfrac{dP}{dx} = 0$ or $x = \dfrac{\alpha - b}{2\beta}$ is a critical point to be investigated. The second derivative gives $\dfrac{d^2P}{dx^2} = -2\beta < 0$ indicating that the critical point produces a maximum value. These results are interpreted

(i) $x = \dfrac{\alpha - b}{2\beta}$ items should be produced for a maximum profit.

(ii) The sale price for each item should be $r = \dfrac{\alpha + b}{2}$ dollars per unit.

In economics the term $R(x) = x \cdot r(x)$ is called the **revenue function** and its derivative $\frac{dR}{dx}$ is called the **marginal revenue**. The term $P(x)$ is called the **profit function** and its derivative $\frac{dP}{dx}$ is called the **marginal profit**. The term $C(x)$ is called the **cost function** and its derivative $\frac{dC}{dx}$ is called the **marginal cost function**.

By collecting data from production costs and sales over a period of time one can construct better approximations for the price function and cost function and other models similar to the above can be constructed and analyzed.

Population Models

Mathematical modeling is used to study the growth and/or decay of a population. The population under study can be human populations subjected to a spreading disease, insect populations which can affect crops, bacteria growth or cell growth in the study of the spread of a disease or cancer cell growth. Predator-prey models are used to study the advance and decline of populations based upon food supplies. The effect of a certain type of medicine on the spread of bacteria or virus growth is still another example of population changes which can be studied using mathematics.

One begins by making some assumptions and starting with a simple model which is easy to solve. By adding perturbations to the simple model it can be made more complex and applicable to the type of problem one is trying to model. This type of modeling has produced many extremely accurate results and the models predictive capability has given much incite into the study of population growth or decay.

For example, an over simplified population growth model for say predicting census changes is to let N denote the current population number and then assume that the rate of change of a population is proportional to the number present. The resulting model is represented

$$\frac{dN}{dt} = \alpha N$$

Here $\alpha > 0$ is a proportionality constant. The conditions that at time $t = 0$ the population is N_0 can be used as an initial condition that the model must satisfy. This model is simple and easy to solve. The variables can be separated and the result integrated giving

$$\int_{N_0}^{N} \frac{dN}{N} = \int_0^t \alpha\, dt \qquad \Longrightarrow \qquad \ln N \Big|_{N_0}^{N} = \alpha\, t \Big|_0^t \qquad \Longrightarrow \qquad N = N_0\, e^{\alpha t}$$

This result states that there is an exponential increase in the population with time. One immediate method to modify the model is to investigate what happens if α is allowed to change with time. If $\alpha = \alpha(t)$ then the above integrations become

$$\int_{N_0}^{N} \frac{dN}{N} = \int_0^t \alpha(t)\, dt \qquad \Longrightarrow \qquad \ln N \Big|_{N_0}^{N} = \int_0^t \alpha(t)\, dt \qquad \Longrightarrow \qquad N = N_0\, e^{\int_0^t \alpha(t)\, dt}$$

Whenever the exponent of e gets too cumbersome it is sometimes convenient to represent the solution in the form

$$N = N_0 \exp\left[\int_0^t \alpha(t)\, dt\right]$$

To try and make the census population model more accurate one can make assumptions that include the rate of births and rate of deaths associated with the current population. If one makes the assumptions that the birth rate is proportional to N, say βN and the death rate is proportional to N^2, say $\delta\, N^2$, where β and δ are positive constants. The population model then has the form

$$\frac{dN}{dt} = \beta N - \delta N^2 = (\beta - \delta N)\,N \tag{5.156}$$

which states the rate of change of the population with time is determine by the birth rate minus the death rate. Analyze this differential equation to see if it makes sense by

(i) Determining conditions for when $\frac{dN}{dt} > 0$ which would indicate the population is increasing.

(ii) Determining conditions for when $\frac{dN}{dt} < 0$ which would indicate the population is decreasing.

(iii) Determining conditions for when $\frac{dN}{dt} = 0$ which would indicate no change in the population.

Setting the equation (5.156) equal to zero, implies that $N = N(t)$ is a constant, since $\frac{dN}{dt} = 0$. One finds the constant solutions $N = N(t) = 0$ and $N = N(t) = \beta/\delta$, are constant solutions for all values of time t. These solutions are called steady-state solutions and they do not change with time.

In order for $\frac{dN}{dt} > 0$, one must require that $N > 0$ and $(\beta - \delta N) > 0$ or $\beta/\delta > N$. In order for $\frac{dN}{dt} < 0$, one must require that either $N < 0$ and $(\beta - \delta N) > 0$ or $N > 0$ and $(\beta - \delta N) < 0$ as these conditions would indicate the population was decreasing.

One can add additional assumptions such as (i) N is never zero and (ii) either $N_0 < N < \beta/\delta$ for $t > 0$ producing an increasing population or (iii) $N_0 > N > \beta/\delta$ producing a decreasing population. In either of the cases where $\frac{dN}{dt}$ is different from zero, one can separate the variables in equation (5.156) and write

$$\frac{dN}{(\beta - \delta N)N} = dt$$

An integration of this equation gives

$$\int_{N_0}^{N} \frac{dN}{(\beta - \delta N)N} = \int_0^t dt$$

To integrate the left-hand side of the above equation use partial fractions and show the above integral reduces to

$$\int_{N_0}^{N} \left[\frac{\delta/\beta}{\beta - \delta N} + \frac{1/\beta}{N}\right] dN = \int_0^t dt$$

Scaling the integral properly, one can integrate this equation to obtain

$$\left[\ln\left|\frac{N}{\beta - \delta N}\right|\right]_{N_0}^{N} = \beta\, t\Big|_0^t \qquad \Longrightarrow \qquad \ln\left|\frac{N}{\beta - \delta N}\right| - \ln\left|\frac{N_0}{\beta - \delta N_0}\right| = \beta\, t$$

Solving for N gives the solution

$$N = N(t) = \frac{\beta N_0 e^{\beta t}}{b - \delta\, N_0 + \delta\, N_0\, e^{\beta t}} = \frac{\beta N_0}{\delta N_0 + (\beta - \delta N_0)\, e^{-\beta t}} \tag{5.157}$$

The equation (5.156) is called **the logistic equation**. The solution of this equation is given by equation (5.157) which gives the limiting value $\lim\limits_{t\to\infty} N(t) = \beta/\delta$. A graphical representation of the logistic equation solutions are given in the figure 5-16.

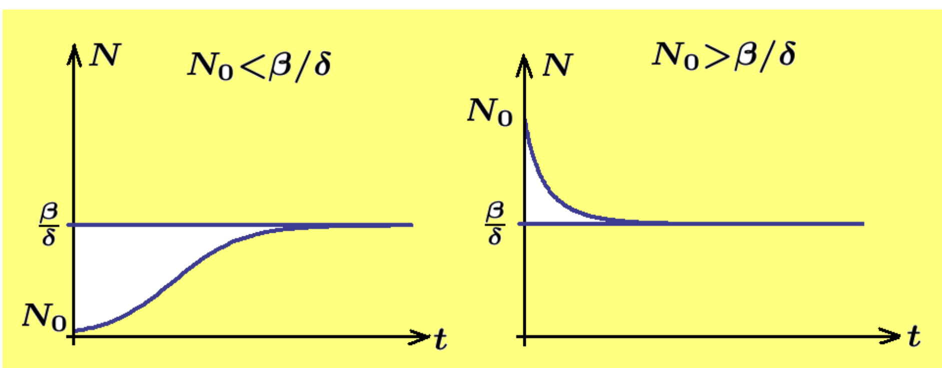


Figure 5-16. Solutions to the logistic equation.

There are many more population models which are much more complicated than the simple ones considered in this introduction.

Approximations

If the Greek letter epsilon ϵ is positive and very small, then this is expressed by writing $0 < |\epsilon| << 1$. For very small ϵ one can truncate certain Taylor series expansions to obtain the following formulas to approximate $f(x_0 + \epsilon)$. These approximate expansions are denoted using the symbol $\approx$ to represent approximation.

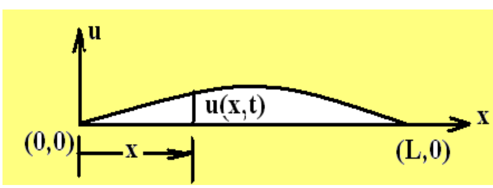
$$(1+\epsilon)^n \approx 1+n\epsilon \qquad \sin\epsilon \approx \epsilon - \frac{\epsilon^3}{3!} \qquad a^\epsilon \approx 1+\epsilon\ln a$$
$$\frac{1}{1+\epsilon} \approx 1-\epsilon \qquad \cos\epsilon \approx 1-\frac{\epsilon^2}{2!} \qquad e^\epsilon \approx 1+\epsilon$$
$$\frac{1}{\sqrt{1+\epsilon}} \approx 1-\frac{\epsilon}{2} \qquad \tan\epsilon \approx \epsilon+\frac{\epsilon^3}{3} \qquad \ln(x+\epsilon) \approx \ln x + \frac{\epsilon}{x}$$

It is left as an exercise to verify the above approximations.

Partial Differential Equations

Examples of partial differential equations can be found in just about all of the scientific disciplines. For example, partial differential equations are employed to describe such things as fluid motion, quantum mechanical interactions, diffusion processes, wave motion and electric and magnetic phenomena. The following are some examples of partial differential equations. The derivation of these well known partial differential equations are presented in more advanced courses.

The vibrating string



Let $u = u(x,t)$ denote the displacement of a string at position x and time t, where the string is stretched between the points $(0,0)$ and $(L,0)$. The assumption that T, the tension in the string, is much greater than the weight of the string, produces the equation describing the vibrations of the string. The partial differential equation describing the vibrations of the string is given by

$$\frac{\partial^2 u}{\partial t^2} = \omega^2 \frac{\partial^2 u}{\partial x^2}, \qquad u = u(x,t), \qquad \omega^2 = \frac{Tg}{\rho}$$

where T is the string tension, g the acceleration of gravity and ρ is the weight per unit length of string. This equation is called the **one-dimensional wave equation** and is subject to boundary conditions $u(0,t)=0$ and $u(L,t)=0$ and initial conditions

$$\begin{aligned} u(x,0) =& f(x) = \text{the initial shape of the string} \\ \text{and} \qquad \frac{\partial u(x,0)}{\partial t} =& g(x) = \text{the initial velocity of the string} \end{aligned}$$

The above quantities have the following dimensions $[x] = \text{cm}$, $[t] = \text{s}$, $[u] = \text{cm}$, $[\rho] = \text{gm/cm}$, $[T] = \text{dynes/cm}$ and $[g] = \text{cm/sec}^2$.

The above one-dimensional wave equation is a special case of a more general three-dimensional wave equation.

One-dimensional heat flow

The partial differential equation describing the **one-dimensional heat flow** in a rod along the x-axis is given by

$$k\frac{\partial^2 T}{\partial x^2} = c\rho\frac{\partial T}{\partial t}, \qquad T = T(x,t)$$

where $T = T(x,t)$ is the temperature at position x and time t in an insulated rod of length L, where k is called the thermal conductivity of the solid, ρ is the volume density of the solid and c is the specific heat of the solid. The above quantities have the following dimensions $[x] = \text{cm}$, $[t] = \text{s}$, $[T] = {}^\circ\text{C}$, $[c] = \text{cal/gm}\,{}^\circ\text{C}$, $[\rho] = \text{gm/cm}^3$ and $[k] = \text{cal/sec}\,\text{cm}^2\,{}^\circ\text{C/cm}$.

The one-dimensional heat equation is a special case of the more general three-dimensional diffusion equation.

Easy to Solve Partial Differential Equations

Partial differential equations of the form

$$\frac{\partial u}{\partial x} = f(x,y)$$

can be integrated partially with respect to x to obtain

$$\int \frac{\partial u}{\partial x}\,dx = \int f(x,y)\,dx \qquad \Longrightarrow \qquad u(x,y) = \int f(x,y)\,dx + \phi(y)$$

Here, y is held constant during the integration process and so the constant of integration can be any arbitrary function of y, represented here by $\phi(y)$.

Similarly, the partial differential equation

$$\frac{\partial u}{\partial y} = g(x,y)$$

can be integrated partially with respect to y to obtain

$$\int \frac{\partial u}{\partial y}\,dy = \int g(x,y)\,dy \qquad \Longrightarrow \qquad u(x,y) = \int g(x,y)\,dy + \psi(x)$$

Here x is held constant during the integration process and so any arbitrary function of x is considered as a constant of integration. This constant of integration is represented by $\psi(x)$.

Partial differential equations of the form

$$\frac{\partial^2 u}{\partial x\,\partial y} = h(x,y)$$

can be integrated with respect to y to obtain

$$\frac{\partial u}{\partial x} = \int h(x,y)\,dy + \psi(x)$$

where $\psi(x)$ is an arbitrary function of x representing the constant of integration during a partial integration with respect to y. One can then integrate with respect to x and obtain the solution in the form

$$u(x,y) = \int\int h(x,y)\,dy\,dx + \int \psi(x)\,dx + \phi(y)$$

where $\phi(y)$ is the constant of integration associated with a partial integration with respect to x. Note if $\psi(x)$ is arbitrary, then $\int \psi(x)\,dx$ is just some new arbitrary function of x.

If you use partial differentiation to differentiate each of the above solutions, holding the appropriate variables constant, you wind up with the integrand that you started with. These partial differentiations are left as an exercise.

Example 5-24. Determine by integration the solution $u = u(x,y)$ of the given partial differential equations. Remember, that when dealing with functions of more than one variable, you are going to be holding one of the variables constant during a partial differentiation or partial integration.

(i) The solution to the partial differential equation $\frac{\partial u}{\partial x} = 0$ is $u = u(x,y) = \phi(y)$ where $\phi(y)$ is an arbitrary function of y.

(ii) The solution to the partial differential equation $\frac{\partial u}{\partial y} = 0$ is $u = u(x,y) = \psi(x)$ where $\psi(x)$ is an arbitrary function of x.

(iii) The solution to the partial differential equation $\frac{\partial u}{\partial x} = x + y$ is given by $u = u(x,y) = \frac{x^2}{2} + xy + \phi(y)$ where $\phi(y)$ is an arbitrary function of y.

(iv) The solution to the partial differential equation $\frac{\partial u}{\partial y} = x + y$ is given by $u = u(x,y) = xy + \frac{y^2}{2} + \psi(x)$ where $\psi(x)$ is an arbitrary function of x.

Note that if a variable is held constant during a partial integration, then an arbitrary function of that variable can be considered as a constant of integration.

■

Example 5-25. Show that for f, g arbitrary functions which are continuous and differential, then the function $u = u(x,t) = f(x-ct) + g(x+ct)$ is a solution to the **one-dimensional wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u = u(x,t), \quad \text{c is a constant}$$

Solution Use the chain rule for differentiation and show

$$\begin{aligned}
\frac{\partial u}{\partial t} &= f'(x-ct)(-c) + g'(x+ct)(c) \\
\frac{\partial^2 u}{\partial t^2} &= f''(x-ct)(c^2) + g''(x+ct)(c^2) \\
\frac{\partial u}{\partial x} &= f'(x-ct) + g'(x+ct) \\
\frac{\partial^2 u}{\partial x^2} &= f''(x-ct) + g''(x+ct)
\end{aligned}$$

Substitute the derivatives in the one-dimensional wave equation and obtain the identity

$$c^2 f'' + c^2 g'' = c^2 f'' + c^2 g''$$

■

Maximum and Minimum for Functions of Two Variables

Finding the maximum and minimum values associated with a function $z = f(x,y)$, which is defined and continuous over a domain D, is similar to what has been done for functions of one variable. A function $z = f(x,y)$ is said to have a relative maximum value at a point $(x_0, y_0) \in D$ if $f(x,y) \le f(x_0, y_0)$ is satisfied for all points (x,y) in some δ-neighborhood of the point (x_0, y_0). Here a δ-neighborhood of the point (x_0, y_0) is defined at the set of points

$$N_\delta = \{\, (x,y) \,|\, (x-x_0)^2 + (y-y_0)^2 \le \delta^2 \,\} \tag{5.158}$$

A function $z = f(x,y)$ is said to have a relative minimum at a point (x_0, y_0) if the condition $f(x,y) \ge f(x_0, y_0)$ is satisfied for all (x,y) in some δ-neighborhood of the point (x_0, y_0). If (x_0, y_0) is a critical point to be tested for a relative maximum or minimum point, then one can reduce the test to a study of one-dimensional problems. Construct the three-dimensional curves

$$z = f(x_0, y) \qquad \text{and} \qquad z = f(x, y_0)$$

This is equivalent to letting the plane $x = x_0$ cut the surface $z = f(x, y)$ in the curve $z = f(x_0, y)$ and then letting the plane $y = y_0$ cut the surface $z = f(x, y)$ to produce the curve $z = f(x, y_0)$. If the curve $z = f(x_0, y)$ has a relative maximum or minimum value, then $\frac{\partial z}{\partial y} = 0$. If the curve $z = f(x, y_0)$ has a relative maximum or minimum value, then $\frac{\partial z}{\partial x} = 0$. Hence, a necessary condition that the point (x_0, y_0) have relative maximum or minimum value is for

$$\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = 0 \quad \text{simultaneously} \tag{5.159}$$

Note that if one of the functions $f(x_0, y)$ or $f(x, y_0)$ has a maximum at (x_0, y_0) and the other function has a minimum at the point (x_0, y_0), then the point (x_0, y_0) is called a saddle point. A surface with saddle point is illustrated in the following figure.

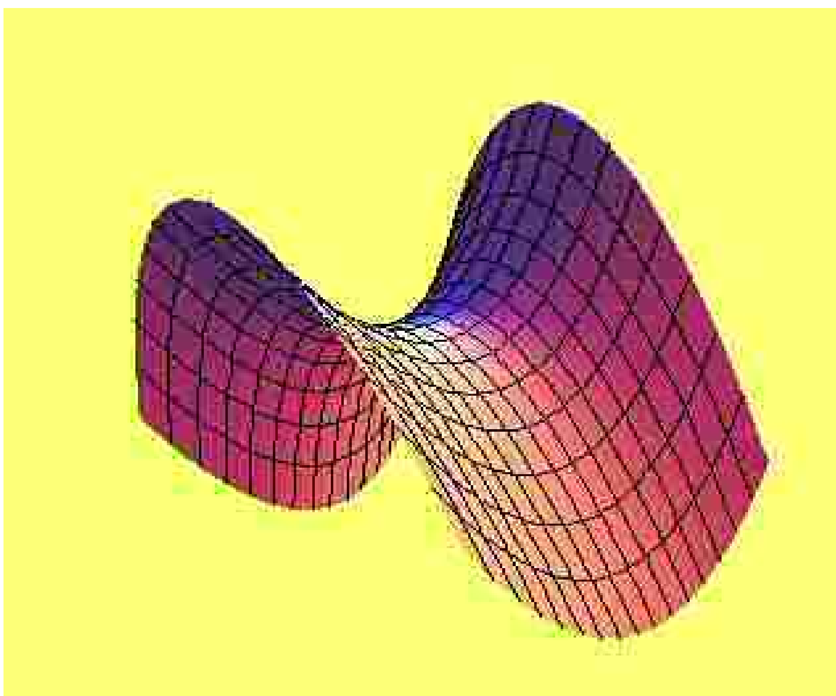


Figure 5-17. Surface in the shape of a saddle.

The study of maximum and minimum values are investigated in more detail in the next volume.

Exercises

▶ **5-1.** Consider a spherical balloon at the instant when the radius of the balloon is r_0 [cm]. If air is entering the balloon at the rate of α [cm^3/s], then at what rate is the radius of the balloon changing at this instant?

▶ **5-2.** Air expands adiabatically (no heat loss or gain) according to the gas law $pv^{1.4} =$ constant, where p is the pressure [dyne/cm^2] and v is the volume [cm^3].

(a) If the volume is increasing at a rate α [cm^3/s], then find the corresponding rate of change in pressure.

(b) If the pressure is decreasing at a rate β [dyne/cm^2 s] then find the corresponding rate of change in the volume.

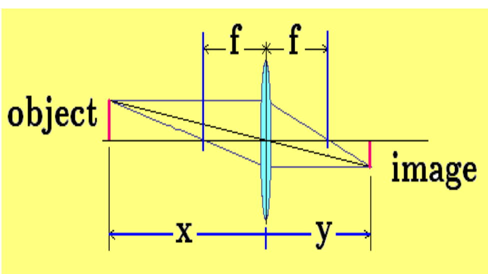
▶ **5-3.** A women who is 5.5 feet tall walks away from a street lamp, where the lamp is 10 feet above the ground. She walks at a rate of 4 ft/s

(a) At what rate is her shadow changing when she is 4 feet from the lamp post?

(b) Is the length of shadow increasing or decreasing as she walks away from the lamp?

(c) At what instant is the shadow 5.5 feet long?

▶ **5-4.** For a thin lens in air, let x denote the distance of the object from the lens and let y denote the distance of the image from the lens. The distances x and y are related by the thin lens formula $\frac{1}{x} + \frac{1}{y} = \frac{1}{f}$ where f is a constant representing the focal length of the lens.



(a) Show the thin lens formula can be written in the Newtonian form $S_1 S_2 = f^2$ where $S_1 = x - f > 0$ and $S_2 = y - f > 0$.

(b) If x changes at a rate $\frac{dx}{dt} = r_0$, then find a formula for the rate of change of y.

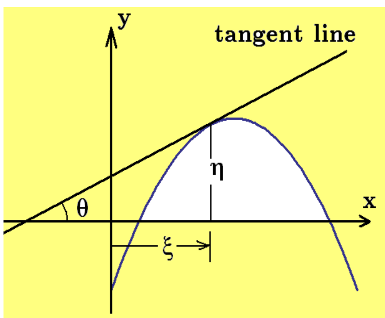
▶ **5-5.** The sides of an equilateral triangle increase at the rate of r_0 cm/hr. Find a formula for the rate of change of the area of an equilateral triangle when the length of a side is x_0 cm.

▶ **5-6.** A meteorologist, at a secret location, collects data and comes up with an atmospheric pressure formula $p = p_0\, e^{-\alpha_0 h}$ where $[p] = \text{lbs/ft}^2$, h has dimensions of feet and represents the altitude above sea-level. In the atmospheric pressure formula the quantities p_0 and α_0 are known constants.

(a) Find the dimensions of the constants p_0 and α_0.

(b) If the meteorologist gets into a balloon which rises at a rate of 10 ft/s, then find a formula representing the rate of change in the pressure when the altitude is h_0 feet.

▶ **5-7.**



Given the parabola $y - y_0 = -(x - x_0)^2$ where x_0, y_0 are known constants.

(a) At the point (ξ, η) on the curve a tangent line is constructed. Find the equation of the tangent line.

(b) The tangent line makes an angle θ with the x-axis as illustrated. If ξ changes at the rate of 1/2 cm/s, then at what rate does θ change?

▶ **5-8.** Charles[23] law, sometimes referred to as the law of volumes, states that at a constant pressure the volume V of a gas and gas temperature T satisfy the relation $\dfrac{V}{T} = C = \textit{constant}$, where V is the volume of the gas in cubic centimeters and T is the absolute temperature in degrees Kelvin. If at a certain instant when V has the volume V_0 and T has the temperature T_0, it is know that the volume of gas is changing at the rate $\dfrac{dV}{dt} = r_0$, then find how the temperature is changing.

▶ **5-9.** The Gay-Lussac[24] law states that if the mass and volume of an ideal gas are held constant, then the pressure of the gas varies directly with the gas absolute temperature. If P denotes pressure measured in Pascals and T is the absolute temperature in degrees Kelvin, then the Gay-Lussac law can be expressed $\dfrac{P}{T} = C = \textit{constant}$. If at some instant when P has the value P_0 and T has the value T_0, it is known that the temperature is change at the rate $\dfrac{dT}{dt} = r_0$, then find how the pressure is changing at this instant.

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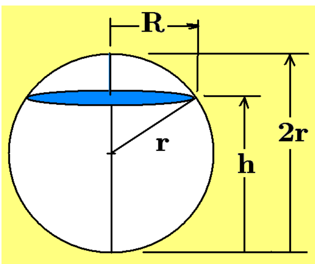
[23] Jacques Charles (1746-1823) French physicist and physical chemist as well as a balloonist.

[24] Joseph Louis Gay-Lussac (1778-1850) A French chemist who studied the expansion of gases.

▶ **5-10.** A rock is thrown off a cliff so that after a time t its height above the ground is $h = h(t) = 200 - 16t^2$.

(a) Find a formula representing the velocity of the rock.
(b) Find a formula representing the acceleration of the rock.
(c) What is the rocks velocity when it hits the ground?

▶ **5-11.**



A spherical water tank has a radius of $r = 12$ feet. Assume $h = h(t)$ is the depth of the water in the spherical tank. The empty space above the water level inside the tank defines a spherical cap whose volume is given by $V_{cap} = \frac{\pi}{3}(2r - h)^2(r + h)$

(a) Show the volume of water in the spherical tank is given by $V = V(h) = \frac{\pi}{3}(3rh^2 - h^3)$
(b) If water is entering the tank at 10 gallons per minute, then how fast is the water level rising when $h = 10$ feet? Hint: Use 1 gallon =0.1336 cubic feet.

▶ **5-12.** A spherical water tank has radius of $r = 12$ feet. Assume that $h = h(t)$ is the depth of the water in the spherical water tank and $R = R(t)$ is the radius of the top surface of the water. Find a relationship between $\frac{dh}{dt}$ and $\frac{dR}{dt}$.

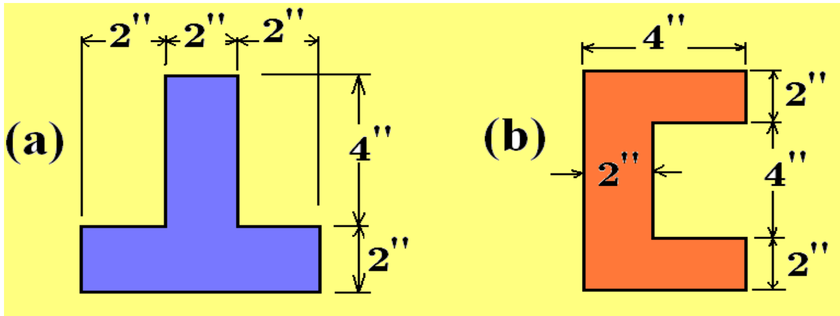
▶ **5-13.** A ball is shot from an air gun inclined at an angle θ with the horizontal. The height of the ball as a function of time is given by $y = y(t) = -16t^2 + 50\sqrt{3}t$ and the horizontal distance traveled is given by $x = x(t) = 50t$.
Answer the following questions.
(a) Find the maximum height of the ball.
(b) Find the time when the maximum height is achieved.
(c) Find the time when the ball hits the ground.
(d) Find the x position where the maximum height is achieved.
(e) Eliminate time t from $x = x(t)$ and $y = y(t)$ to obtain y as a function of x.

▶ **5-14.** Empirical data obtained by shooting bullets into maple wood blocks produces the formula

$$v = v(x) = K\sqrt{1 - 2x}, \qquad 0 < x < 1/2, \qquad (K \text{ is a constant })$$

for the speed [ft/s] of the bullet after it has penetrated the wood a distance x feet. Find the rate at which the speed of the bullet is decreasing after it enters the wood.

▶ **5-15.** Use the results from table 5-1 to find the centroid of the given composite shapes.



▶ **5-16.** Find the centroid of the area bounded by the curves

(a) The parabola $y = x^2$ and the line $y = y_0 > 0$

(b) The parabola $y = x^2$ and the lines $x = x_0 > 0$ and $y = 0$

(c) The parabola $y = x^2$ and the lines $x = x_0 > 0$, $x = x_1 > x_0 > 0$ and $y = 0$

▶ **5-17.** Find the centroid of the area bounded by the curves

(a) The parabola $x = 6y - y^2$ and the line $x = 0$

(b) The parabola $x = 6y - y^2$ and the line $x = 1$

(c) The parabola $x = 6y - y^2$ and the line $2y + x = 0$

▶ **5-18.** Find the centroid of the solid produced by rotation of the given area about the axis specified.

	Area defined by bounding curves	Axis of rotation
(a)	$by - hx = 0, \quad x = b, \quad y = 0$	x-axis
(b)	$by - hx = 0, \quad x = b, \quad y = 0$	y-axis
(c)	$by - hx = 0, \quad x = b, \quad y = 0$	The line $y = -2$
(d)	$y^2 = 4x, \quad x = 0, \quad y = y_0 > 0$	y-axis
(e)	$y^2 = 4x, \quad x = 0, \quad y = y_0 > 0$	x-axis
(f)	$y^2 = 4x, \quad x = 0, \quad y = y_0 > 0$	The line $x = -1$
b, h, y_0 are all positive constants.		

▶ **5-19.** The curve $y = 4 - x^2$, $0 \le x \le 2$, is revolved about the y-axis to form a solid of revolution. Find the centroid of this solid.

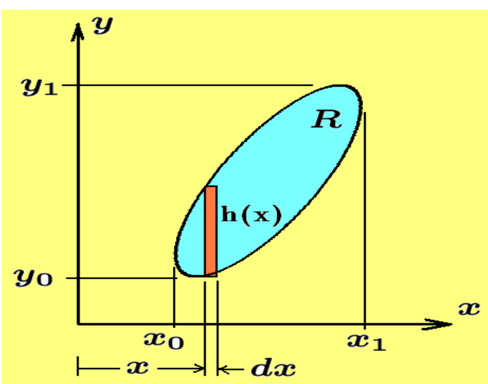
(a) Use disk shaped volume elements.

(b) Us cylindrical shell shaped volume elements.

▶ **5-20.** Newton's law of cooling states that the temperature T of a body cools with time t at a rate proportional to the temperature difference $T - T_{env}$ between the body temperature T and the temperature of the environment T_{env}. Newton's law of cooling can therefore be expressed by the differential equation $\frac{dT}{dt} = -k(T - T_{env})$, where $k > 0$ is a proportionality constant and the negative sign indicates the temperature is decreasing. (a) Use integration techniques to obtain the general solution to this differential equation. (b) If the body initially has a temperature $T = 100^\circ C$ and is cooling in an environment at $0^\circ C$, find $T = T(t)$. (c) If the body cools to $80^\circ C$ in 20 minutes, find the proportionality constant k. (d) Find the time it takes for the body to cool from $90^\circ C$ to $70^\circ C$. (e) Give units of measurement for each term in the solution to part (a).

▶ **5-21.** It has been found that under certain conditions, the number density N $(\#/cm^3)$ of a certain bacteria increases at a rate proportional to the amount N present. If at time $t = 0$, $N = N_0$ is the initial number of bacteria per cubic centimeter and if after 5 hours, the value of N has been found to increase to $3N_0$, then find the equation representing $N = N(t)$ as a function of time t. Give units of measurement for all terms in your equation.

▶ **5-22.** **Pappus's Theorem**



Pappus's[25] theorem states that if a region R is rotated about a line which does not pass through the region, then the volume of the solid of revolution equals the area of the region R multiplied by the distance traveled by the centroid of the region R. Assume the region R illustrated is rotated about the y-axis.

(a) Show the volume of the solid formed is $V = 2\pi \int_{x_0}^{x_1} x h(x)\, dx$

(b) Show the x-position of the centroid is $\bar{x} = \frac{1}{A} \int_{x_0}^{x_1} x h(x)\, dx$, where A is the area of the region R.

(c) Prove Pappus's theorem.

(d) Prove Pappus's theorem if the region R is rotated about the x-axis.

[25] Pappus of Alexandria (290-350) A Greek geometer.

▶ **5-23.** Use the Pappus theorem from the previous example

(a) to find the volume of a right circular cone obtained by revolving the line $y = \frac{r}{h}x$, $0 \le x \le h$ about the x-axis

(b) to find the volume of a sphere obtained by revolving the semi-circle $x^2 + y^2 = r^2$ $-r \le x \le r$, $y > 0$ about the x-axis.

▶ **5-24.** Solve the given differential equations using integration techniques.

$$\begin{array}{llllll} (a) & \dfrac{dy}{dx} = 0 & (d) & \dfrac{dy}{dx} = e^x & (g) & \dfrac{dy}{dx} = y^2 \\ (b) & \dfrac{dy}{dx} = 1 & (e) & \dfrac{dy}{dx} = y & (h) & \dfrac{dy}{dx} = \sin x \\ (c) & \dfrac{dy}{dx} = x & (f) & \dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} & (i) & \dfrac{dy}{dx} = \cos x \end{array}$$

▶ **5-25.** Solve each of the given differential equations by separating the variables and applying integration techniques.

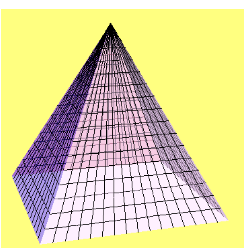
$$(a)\quad \frac{dy}{dx} = \frac{1+x}{1+y} \qquad (b)\quad \frac{dy}{dx} = \frac{1+x^2}{1+y^2} \qquad (c)\quad \frac{dy}{dx} = \frac{1+x^3}{1+y^3}$$

▶ **5-26.**

(a) Solve the differential equation $\dfrac{d^2y}{dt^2} + \omega^2 y = \cos \lambda t$, where ω and λ are constants.

(b) For what value λ does resonance occur?

▶ **5-27.**



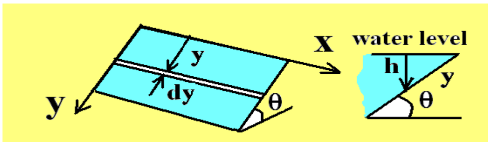
Use a plane to cut the regular pyramid with height h and square base having sides of length b and form an element of volume which can then be summed to determine the volume of the pyramid. (a) Find the volume of this pyramid. (b) Find the volume of a frustum of this pyramid.

▶ **5-28.** A piece of cardboard having length $\ell = \frac{1}{2}(15 + \sqrt{33})$ and width $w = \frac{1}{2}(15 - \sqrt{33})$ is to be made into a box by cutting squares of length x from each corner and then turning up the sides followed by reinforcing the sides with tape. Find the box that can be constructed which has the maximum volume.

▶ **5-29.** Find the centroid of the region bounded by the following curves.

(a) $y = x^2$ and $y = 2x$ $\qquad (b)$ $y = x^2$ and $y = mx$, $m > 0$ $\qquad (c)$ $y^2 = x$ and $y = 2(x-1)$

► **5-30.**



The face of a dam is a rectangular plate of width w and length ℓ. The plate is inclined at an angle θ so that the longer side is at the water level. Let $\rho = 62.4\,\text{lbs/ft}^3$ denote the water density and construct an x, y-axis lying on the plate with origin at one corner as illustrated in the sketch.

(i) Show the distance y along the plate, measured from the water surface, corresponds to a water depth $h = y \sin\theta$

(ii) Show a rectangular element of area dA on the plate is given by $dA = \ell\, dy$

(iii) At depth h, the pressure p acting on the element of area dA is given by $p = \rho h$.

(iv) Show the element of force dF acting on the element of area is $dF = p\, dA = p\,\ell\, dy$

(v) Find the total force acting normal to the face of the dam.

► **5-31.** Assume a body falls from rest from a height of 100 meters in air and the body experiences a drag force proportional to its velocity.

(i) Show that Newton's law of motion is represented

$$m\frac{dv}{dt} = mg - kv$$

where k is a proportionality constant.

(ii) Separate the variables and then integrate to determine the velocity as a function of time.

(iii) When does the body hit the ground? What is its velocity when is hits the ground.

(iv) Give units of measurement for all terms in the equations you used to obtain your answer.

► **5-32.** For each of the given differential equations assume an exponential solution $e^{\gamma x}$ and find

(a) The characteristic equation

(b) The characteristic roots

(c) A fundamental set of solutions

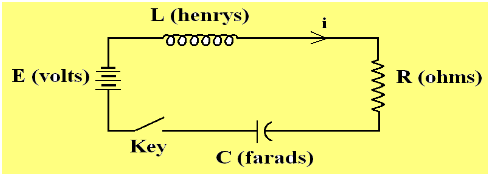
(d) The general solution

$$(a)\quad \frac{dy}{dx} - \alpha y = 0 \qquad\qquad (d)\quad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

$$(b)\quad \frac{d^2y}{dx^2} + \omega^2 y = 0 \qquad\qquad (e)\quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$$

$$(c)\quad \frac{d^2y}{dx^2} - \omega^2 y = 0 \qquad\qquad (f)\quad \frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$$

▶ **5-33.**



For the RLC circuit illustrated Kirchhoff's law produces the differential equation

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E, \qquad i = \frac{dq}{dt}$$

(a) Set $L = 0$ and $i = \dfrac{dq}{dt}$ and then solve the initial-value problem that $q = 0$ when $t = 0$ and show $q = q(t) = CE\left(1 - e^{-t/RC}\right)$ and then find the current i in the circuit.

(b) Set $E = 0$, $R = 0$ to find the discharge of a condenser through an inductance L. Assume the initial-values $q = q_0$ and $i = 0$ at time $t = 0$
Hint: One method is to let $i = \frac{dq}{dt}$ and write $\frac{d^2q}{dt^2} = \frac{di}{dt} = \frac{di}{dq}\frac{dq}{dt} = i\frac{di}{dq}$ and then separate variables.

(c) Set $C = 0$, $E = 0$ to find the decay of current in the circuit containing a resistance and inductance. Assume the initial-value $i = i_0$ at time $t = 0$ and show the current decays according to the law $i = i_0 e^{-(R/L)t}$. Find the current at the times $t = L/R, t = 2L/R, t = 3L/R, t = 4/R$ and $t = 5L/R$

▶ **5-34.** **A Paradox**

The curve $y = \dfrac{1}{x}$ for $1 \le x \le T$ is revolved about the x-axis to form a surface.
(a) Find the volume $V = V(T)$ bounded by the surface and the planes $x = 1$ and $x = T$. (b) Find the surface area $S = S(T)$ and show $S(T) > 2\pi \ln T$. (c) Show that in the limit as $T \to \infty$ that $V(T)$ is finite, but $S(T)$ becomes infinite.

The above results shows that you can take paint and fill up the infinite volume, but you can't paint the surface of this volume. Question: If you fill up the volume with paint and then pour it out, does this count as painting the outside surface?

▶ **5-35.** Assume $y_1 = y_1(x)$ is a solution of the differential equation

$$L(y) = \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

What condition must $u = u(x)$ satisfy, if $y_2(x) = u(x)y_1(x)$ is also a solution to the same differential equation?

(a) Determine the general solution to the differential equation $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$ if it is known that $y_1 = y_1(x) = e^{-x}$ is a solution of this equation.

(b) Determine the general solution to the differential equation $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$ if it is know that $y_1 = y_1(x) = x$ is a solution of this equation.

▶ **5-36.** Use partial integration to solve the given partial differential equations for the most general representation of the unknown function $u = u(x, y)$.

$$(a)\quad \frac{\partial u}{\partial x} = 6x^2 + y \qquad (d)\quad y\frac{\partial^2 u}{\partial x\,\partial y} = x + y \qquad (g)\quad \frac{\partial u}{\partial y} = x^2 + y$$

$$(b)\quad x\frac{\partial u}{\partial x} + u = y + x \qquad (e)\quad \frac{\partial^2 u}{\partial y^2} = x + y \qquad (h)\quad \frac{\partial u}{\partial x} = xy$$

$$(c)\quad y\frac{\partial u}{\partial y} + u = x + y \qquad (f)\quad \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 1 + y \qquad (i)\quad \frac{\partial^2 u}{\partial x\,\partial y} = xy$$

Hint for (f): After one integration, multiply through by e^x.

▶ **5-37.** Assume $f(x + iy) = u(x, y) + i\,v(x, y)$ is such that $u = u(x, y)$ and $v = v(x, y)$ are real continuous functions with partial derivatives of the first and second order which satisfy the **Cauchy-Riemann conditions** $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(a) Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (b) Show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

▶ **5-38.** Find the largest rectangle that can be inscribed inside a circle of radius r.

▶ **5-39.** Find the length of each of the given curves.

(a) $y = x^2$ from $x = 0$ to $x = 1$ (c) $y = \cosh x$ from $x = 0$ to $x = 1$

(b) $y^2 = 4x$ from $x = 0$ to $x = 1$ (d) $y = e^x$ from $x = 0$ to $x = 1$

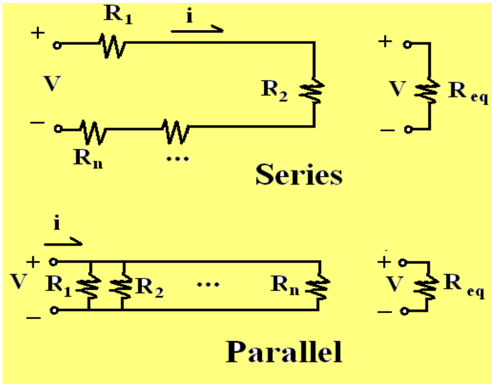
▶ **5-40.** A cylindrical can is to be constructed to hold πV_0 cubic meters of material. The cost of construction for the sides of the cylinder is c_0 dollars per square meter and the cost of constructing the top and bottom is $3c_0$ dollars per square meter. This is because the side of the cylinder can be considered as a rectangle with height h and width $2\pi r$, where h is the height of the cylinder and r is the radius of the cylinder. The top and bottom of the cylinder are circles and consequently more manufacturing techniques and waste of material occurs in their construction. Find the dimensions of the cylinder requiring minimum cost of construction. Hint: The volume of the cylinder is $V = \pi r^2 h = \pi V_0$.

▶ **5-41.** Evaluate the following limits

$$(a)\quad \lim_{h\to 0} \frac{u(x + h, y) - u(x, y)}{h} \qquad (b)\quad \lim_{k\to 0} \frac{u(x, y + k) - u(x, y)}{k}$$

assuming these limits exist.

▶ **5-42.**



(a) **(Resistors in Series)** Show that Ohm's law requires the voltage drop in moving around the series circuit requires $V = iR_1 + iR_2 + \cdots + iR_n$ and so one can replace the sum of the resistors by an equivalent resistance R_{eq} given by

$$R_{eq} = R_1 + R_2 + \cdots + R_n.$$

(b) **(Resistors in Parallel)** Use Kirchhoff's law and show the current i in the parallel circuit must satisfy $i = i_1 + i_2 + \cdots + i_n$ and that $V = i_1 R_1$, $V = i_2 R_2, \ldots, V = i_n R_n$.

(c) Show an equivalent resistance R_{eq} must satisfy $V = R_{eq} i$ and from this result show

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$$

▶ **5-43.** Assume an open container with vertical sides where the bottom of the container has the same shape as the top of the container. If water evaporates from this open container at a rate which is directly proportional to the exposed surface area, use calculus to show that the depth of water in the container changes at a constant rate and it doesn't matter what shape the top and bottom have as long as they are the same.

▶ **5-44.** Evaluate the integral $I = \int \tan^4 x\, dx$ for $0 < x < \frac{\pi}{2}$

(a) Use the substitution $z = \tan x$ and show $dx = \frac{dz}{1+z^2}$ so that the integral becomes

$$I = \int \frac{z^4}{1+z^2}\, dz = \int \frac{z^4 + z^2 - (z^2+1) + 1}{z^2+1}\, dz = \int \left(z^2 - 1 + \frac{1}{1+z^2} \right) dz$$

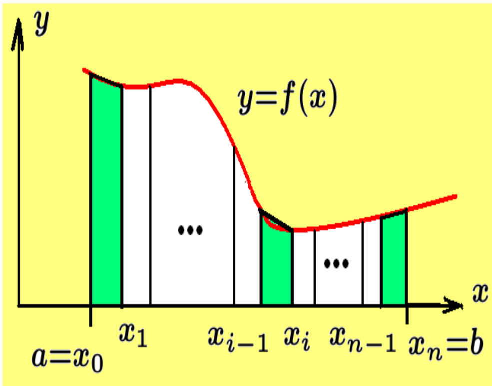
(b) Integrate the result from part (a) and then use back substitution to express the integral I in terms of x.

▶ **5-45.** Show that integrals of the type $I = \int f(\sin x, \cos x)\, dx$ where $f(u,v)$ is a rational function of u, v, can be simplified by making the substitution $z = \tan \dfrac{x}{2}$

(a) Show $\sin x = \dfrac{2z}{1+z^2}, \quad \cos x = \dfrac{1-z^2}{1+z^2}, \quad dx = \dfrac{2dz}{1+z^2}$ Hint: Show $\cos^2 \dfrac{x}{2} = \dfrac{1}{1 + \tan^2 \frac{x}{2}}$

(b) Evaluate the integral $I = \displaystyle\int \frac{1+\cos^2 x}{\cos^4 x}\, dx$

▶ **5-46. The Trapezoidal Rule**



Given a curve $y = f(x)$, $a \leq x \leq b$, one can partition the interval $[a, b]$ into n-parts by defining a step size $h = \Delta x = \frac{(b-a)}{n}$ and then labeling the points

$$a = x_0, \; x_1 = x_0 + h, \ldots, x_{n-1} = x_0 + (n-1)h, \; x_n = x_0 + nh$$

Next construct trapezoids, the ith trapezoid has the vertices $x_{i-1}, y_{i-1}, x_i, y_i$ as illustrated in the figure, where $y_{i-1} = f(x_{i-1})$ and $y_i = f(x_i)$.

(a) Show the area of the ith trapezoid is given by

$$A_i = \frac{1}{2}(y_{i-1} + y_i)h = \frac{h}{2}[f(x_{i-1}) + f(x_i)]$$

(b) Show the area of all the trapezoids is

$$A = \text{total area} = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}(y_{i-1} + y_i)h = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{i=1}^{n-1} y_i\right]$$

(c) **Computer Problem**

For the functions given over the interval specified compare the area under the curve using the definite integral $\int_a^b f(x)\,dx$, with the trapezoidal rule for approximating the area. Fill in the following table for values of $n = 10, 50$ and 100 associated with each function.

Area under curve $y = f(x)$ for $a \leq x \leq b$				
n=	Function $f(x)$	Interval $a \leq x \leq b$	Trapezoidal $\sum_{i=1}^{n} A_i$	Integration $\int_a^b f(x)\,dx$
	x	$0 \leq x \leq 2$		
	x^2	$0 \leq x \leq 2$		
	x^3	$0 \leq x \leq 2$		
	$\sin x$	$0 \leq x \leq \pi$		
	$\cos x$	$\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$		

(d) If the theoretical error of approximation using the trapezoidal rule is $E = \left|\frac{(b-a)^3}{12n^2} f''(\xi)\right|$, where $a < \xi < b$, compare your errors with the maximum theoretical error.

▶ **5-47.** For $a > 0$, verify the improper integrals

$$(a) \quad \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2} \qquad (b) \quad \int_0^\infty e^{-ax} \cos bx \, dx = \frac{1}{a^2+b^2}$$

▶ **5-48.** If the Gamma function is defined by the improper integral

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dx$$

(a) Use integration by parts to show $\Gamma(x+1) = x\Gamma(x)$

(b) Show for n an integer $\Gamma(n+1) = n!$

▶ **5-49.** Consider two particles starting at the origin at the same time and moving along the x-axis such that their positions at any time t are given by

$$s_1(t) = 2t^2 + t \quad \text{and} \quad s_2(t) = 11t - 3t^2$$

(a) At what time will the particles have the same position and what will be their velocities at this position?

(b) Find the particles positions when they have the same speed? What is this same speed?

(c) Describe the motion of each particle.

▶ **5-50.** Find the maximum and minimum values for the given functions

$$(a) \quad f(x,y) = x^2 + y^2 - 2x - 4y - 20 \qquad (b) \quad g(x,y) = 4x^2 + 9y^2 - 16x - 54y + 61$$

▶ **5-51.** Given a point $(x_0, y_0) \neq (0,0)$ lying in the first quadrant. Pick a point $x_1 > x_0$ on the x-axis and draw a line from $(x_1, 0)$ through (x_0, y_0) which intersects the y-axis. Find the shortest line from the x-axis, through the point (x_0, y_0) which intersects the y-axis Hint: If ℓ is the length of the line segment, then minimize ℓ^2.

▶ **5-52.** Find the maximum and minimum distances from the origin to points on the circle $(x-6)^2 + (y-8)^2 = 25$

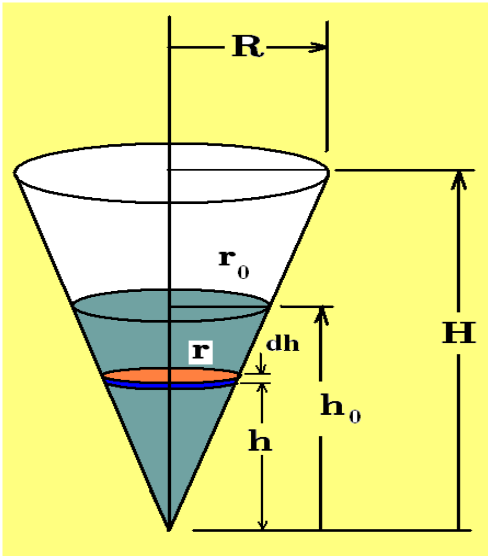
▶ **5-53.** A particle undergoes simple harmonic motion on the x-axis according to the law $x = x_0 + a\cos\omega t + b\sin\omega t$, where x_0, a, b, ω are given constant values.

(a) Find the center and amplitude of the motion.

(b) Find the period and frequency associated with the motion.

(c) Find the maximum acceleration.

▶ **5-54.**



A right circular conical water tank, as illustrated has a top radius R and height H. Assume water is in the tank at a depth h_0. Let $\omega = 62.5\, lbs/ft^3$ denote the density of water.
(a) Show weight of disk at height h produces force $dF = \omega \pi r^2\, dh$
(b) Show element of work done in lifting disk a distance $H - h$ is

$$dW = (\omega \pi r^2\, dh)(H - h)$$

(c) Show the work done in pumping the water out over the top of the tank is

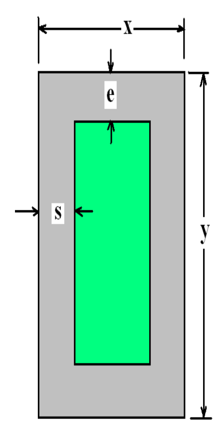
$$W = \left(\frac{R}{H}\right)^2 \pi\, \omega \int_0^{h_0} h^2 (H - h)\, dh$$

and then evaluate this integral.

▶ **5-55.** For p pressure and v volume, the integral $W = \int_{v_1}^{v_2} p\, dv$ occurs in the study of thermodynamics and represents work done by a gas.
(a) Evaluate this integral for an isothermal expansion where $pv = c = constant$.
(b) Evaluate this integral for an adiabatic expansion where $pv^\gamma = c = constant$, where $\gamma = 1.41$ is also a constant.

▶ **5-56.**

A botanical gardens is planning the construction of flower beds to display their hosta plants. The flower beds are to be rectangular and constructed inside a rectangular area having a known perimeter P. There is to be a walk surrounding each flower bed having dimensions of s-feet on each side and e-feet on each end. Design studies are to begin where the exact values of P,s and e are to be supplied for each flower bed. For a given value of P, s and e find the dimensions of the flower bed if the area of the flower bed is to be a maximum.



APPENDIX A
Units of Measurement

The following units, abbreviations and prefixes are from the Système International d'Unitès (designated SI in all Languages.)

Prefixes.

Abbreviations		
Prefix	Multiplication factor	Symbol
exa	10^{18}	W
peta	10^{15}	P
tera	10^{12}	T
giga	10^{9}	G
mega	10^{6}	M
kilo	10^{3}	K
hecto	10^{2}	h
deka	10	da
deci	10^{-1}	d
centi	10^{-2}	c
milli	10^{-3}	m
micro	10^{-6}	μ
nano	10^{-9}	n
pico	10^{-12}	p
femto	10^{-15}	f
atto	10^{-18}	a

Basic Units.

Basic units of measurement		
Unit	Name	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	degree Kelvin	$^\circ$K
Luminous intensity	candela	cd

Supplementary units		
Unit	Name	Symbol
Plane angle	radian	rad
Solid angle	steradian	sr

DERIVED UNITS		
Name	Units	Symbol
Area	square meter	m^2
Volume	cubic meter	m^3
Frequency	hertz	Hz (s^{-1})
Density	kilogram per cubic meter	$\mathrm{kg/m}^3$
Velocity	meter per second	m/s
Angular velocity	radian per second	rad/s
Acceleration	meter per second squared	$\mathrm{m/s}^2$
Angular acceleration	radian per second squared	$\mathrm{rad/s}^2$
Force	newton	N $(\mathrm{kg\cdot m/s}^2)$
Pressure	newton per square meter	$\mathrm{N/m}^2$
Kinematic viscosity	square meter per second	$\mathrm{m}^2\mathrm{/s}$
Dynamic viscosity	newton second per square meter	$\mathrm{N\cdot s/m}^2$
Work, energy, quantity of heat	joule	J $(\mathrm{N\cdot m})$
Power	watt	W (J/s)
Electric charge	coulomb	C $(\mathrm{A\cdot s})$
Voltage, Potential difference	volt	V (W/A)
Electromotive force	volt	V (W/A)
Electric force field	volt per meter	V/m
Electric resistance	ohm	Ω (V/A)
Electric capacitance	farad	F $(\mathrm{A\cdot s/V})$
Magnetic flux	weber	Wb $(\mathrm{V\cdot s})$
Inductance	henry	H $(\mathrm{V\cdot s/A})$
Magnetic flux density	tesla	T $(\mathrm{Wb/m}^2)$
Magnetic field strength	ampere per meter	A/m
Magnetomotive force	ampere	A

Physical Constants:

- $4\arctan 1 = \pi = 3.14159\,26535\,89793\,23846\,2643\ \ldots$
- $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.71828\,18284\,59045\,23536\,0287\ \ldots$
- Euler's constant $\gamma = 0.57721\,56649\,01532\,86060\,6512\ \ldots$
- $\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$ Euler's constant
- Speed of light in vacuum $= 2.997925(10)^8\ m\ s^{-1}$
- Electron charge $= 1.60210(10)^{-19}\ C$
- Avogadro's constant $= 6.0221415(10)^{23}\ mol^{-1}$
- Plank's constant $= 6.6256(10)^{-34}\ J\,s$
- Universal gas constant $= 8.3143\ J\,K^{-1}\,mol^{-1} = 8314.3\ J\,Kg^{-1}\,K^{-1}$
- Boltzmann constant $= 1.38054(10)^{-23}\ J\,K^{-1}$
- Stefan–Boltzmann constant $= 5.6697(10)^{-8}\ W\,m^{-2}\,K^{-4}$
- Gravitational constant $= 6.67(10)^{-11}\ N\,m^2 kg^{-2}$

APPENDIX B

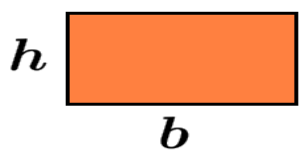
Background Material

Geometry

Rectangle

Area $=$ (base)(height) $= bh$

Perimeter $= 2b + 2h$

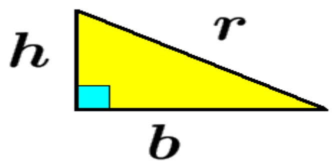


Right Triangle

Area $= \frac{1}{2}$(base)(height) $= \frac{1}{2}bh$

Perimeter $= b + h + r$

where $r^2 = b^2 + h^2$ is the Pythagorean theorem



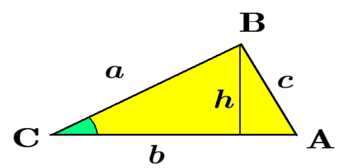
Triangle with sides a, b, c and angles A, B, C

Area $= \frac{1}{2}$(base)(height) $= \frac{1}{2}bh = \frac{1}{2}b(a\sin C)$

Perimeter $= a + b + c$

Law of Sines $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Law of Cosines $\quad c^2 = a^2 + b^2 - 2ab\cos C$

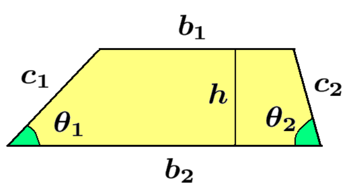


Trapezoid

Area $= \frac{1}{2}(b_1 + b_2)h$

Perimeter $= b_1 + b_2 + c_1 + c_2$

$c_1 = \dfrac{h}{\sin\theta_1} \qquad c_2 = \dfrac{h}{\sin\theta_2}$

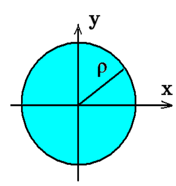


Circle

Area $= \pi\rho^2$

Perimeter $= 2\pi\rho$

Equation $\quad x^2 + y^2 = \rho^2$

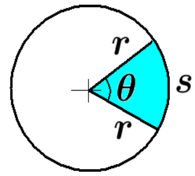


Sector of Circle

Area $= \frac{1}{2}r^2\theta$, $\quad\theta$ in radians

$s = \text{arclength} = r\theta$, $\quad\theta$ in radians

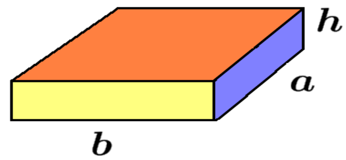
Perimeter $= 2r + s$



Rectangular Parallelepiped

V = Volume $= abh$

S = Surface area $= 2(ab + ah + bh)$



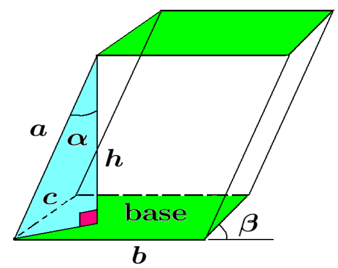
Parallelepiped

Composed of 6 parallelograms

V = Volume $= (Area\ of\ base)(height)$

A = Area of base $= bc\sin\beta$

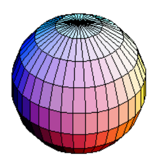
height $= h = a\cos\alpha$



Sphere of radius ρ

V = Volume $= \frac{4}{3}\pi\rho^3$

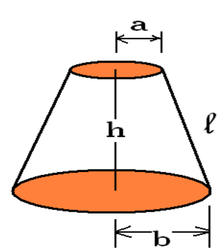
S = Surface area $= 4\pi\rho^2$



Frustum of right circular cone

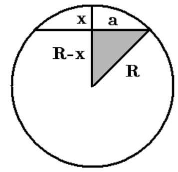
V = Volume $= \frac{\pi}{3}(a^2 + ab + b^2)h$

Lateral surface area $= \pi\ell(a + b)$



Chord Theorem for circle

$a^2 = x(2R - x)$

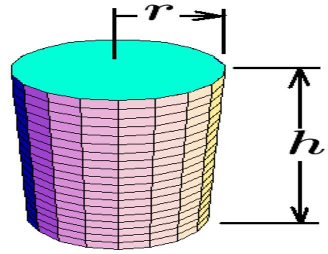


Right Circular Cylinder

$V =$ Volume $=$ (Area of base)(height) $= (\pi r^2)h$

Lateral surface area $= 2\pi r h$

Total surface area $= 2\pi r h + 2(\pi r^2)$

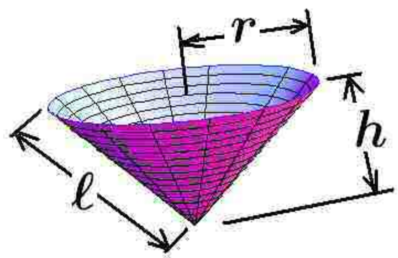


Right Circular Cone

$V =$ Volume $= \dfrac{1}{3}\pi r^2 h$

Lateral surface area $= \pi r \ell = \pi r \sqrt{h^2 + r^2}$

height $= h$, base radius r



Algebra

Products and Factors
$(x+a)(x+b) = x^2 + (a+b)x + ab$
$(x+a)^2 = x^2 + 2ax + a^2$
$(x-b)^2 = x^2 - 2bx + b^2$
$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ac+bc+ab)x + abc$
$x^2 - y^2 = (x-y)(x+y)$
$x^3 - y^3 = (x-y)(x^2+xy+y^2)$
$x^3 + y^3 = (x+y)(x^2-xy+y^2)$
$x^4 - y^4 = (x-y)(x+y)(x^2+y^2)$
If $ax^2+bx+c=0$, then $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

Binomial Expansion

For $n = 1, 2, 3, \ldots$ an integer, then

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n$$

where $n!$ is read n factorial and is defined

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \text{ and } 0! = 1 \text{ by definition.}$$

Binomial Coefficients

The binomial coefficients can also be defined by the expression

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{where } n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

where for $n = 1, 2, 3, \ldots$ is an integer. The binomial expansion has the alternative representation

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 \cdots + \binom{n}{n}y^n$$

Laws of Exponents

Let s and t denote real numbers and let m and n denote positive integers. For nonzero values of x and y

$$x^0 = 1, \quad x \neq 0 \qquad (x^s)^t = x^{st} \qquad x^{1/n} = \sqrt[n]{x}$$

$$x^s x^t = x^{s+t} \qquad (xy)^s = x^s y^s \qquad x^{m/n} = \sqrt[n]{x^m}$$

$$\frac{x^s}{x^t} = x^{s-t} \qquad x^{-s} = \frac{1}{x^s} \qquad \left(\frac{x}{y}\right)^{1/n} = \frac{x^{1/n}}{y^{1/n}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Laws of Logarithms

If $x = b^y$ and $b \neq 0$, then one can write $y = \log_b x$, where y is called the logarithm of x to the base b. For $P > 0$ and $Q > 0$, logarithms satisfy the following properties

$$\log_b(PQ) = \log_b P + \log_b Q$$

$$\log_b \frac{P}{Q} = \log_b P - \log_b Q$$

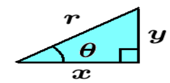
$$\log_b Q^P = P \log_b Q$$

Trigonometry

Pythagorean identities

Using the Pythagorean theorem $x^2 + y^2 = r^2$ associated with a right triangle with sides x, y and hypotenuse r, there results the following trigonometric identities, known as the Pythagorean identities.

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1, \qquad 1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2, \qquad \left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2,$$



$$\cos^2\theta + \sin^2\theta = 1, \qquad 1 + \tan^2\theta = \sec^2\theta, \qquad \cot^2\theta + 1 = \csc^2\theta,$$

Angle Addition and Difference Formulas

$$\sin(A+B) = \sin A\cos B + \cos A \sin B, \qquad \sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B, \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}, \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Double angle formulas

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\cot A}{\cot^2 A - 1}$$

Half angle formulas

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \qquad \begin{array}{c|c} + & + \\ \hline - & - \end{array}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \qquad \begin{array}{c|c} - & + \\ \hline - & + \end{array}$$

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A} \qquad \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

The sign depends upon the quadrant $A/2$ lies in.

Multiple angle formulas

$$\sin 3A = 3\sin A - 4\sin^3 A, \qquad \sin 4A = 4\sin A\cos A - 8\sin^3 A\cos A$$

$$\cos 3A = 4\cos^3 A - 3\cos A, \qquad \cos 4A = 8\cos^4 A - 8\cos^2 A + 1$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}, \qquad \tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$$

Multiple angle formulas

$$\begin{aligned}
\sin 5A =& 5\sin A - 20\sin^3 A + 16\sin^5 A\\
\cos 5A =& 16\cos^5 A - 20\cos^3 A + 5\cos A\\
\tan 5A =& \frac{\tan^5 A - 10\tan^3 A + 5\tan A}{1 - 10\tan^2 A + 5\tan^4 A}\\
\sin 6A =& 6\cos^5 A\sin A - 20\cos^3 A\sin^3 A + 6\cos A\sin^5 A\\
\cos 6A =& \cos^6 A - 15\cos^4 A\sin^2 A + 15\cos^2 A\sin^4 A - \sin^6 A\\
\tan 6A =& \frac{6\tan A - 20\tan^3 A + 6\tan^5 A}{1 - 15\tan^2 A + 15\tan^4 A - \tan^6 A}
\end{aligned}$$

Summation and difference formula

$$\begin{aligned}
\sin A + \sin B =& 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2}), & \sin A - \sin B =& 2\sin(\frac{A-B}{2})\cos(\frac{A+B}{2})\\
\cos A + \cos B =& 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2}), & \cos A - \cos B =& -2\sin(\frac{A-B}{2})\sin(\frac{A+B}{2})\\
\tan A + \tan B =& \frac{\sin(A+B)}{\cos A\cos B}, & \tan A - \tan B =& \frac{\sin(A-B)}{\cos A\cos B}
\end{aligned}$$

Product formula

$$\begin{aligned}
\sin A\sin B =& \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)\\
\cos A\cos B =& \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)\\
\sin A\cos B =& \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)
\end{aligned}$$

Additional relations

$$\begin{aligned}
\sin(A+B)\sin(A-B) =& \sin^2 A - \sin^2 B,\\
-\sin(A+B)\sin(A-B) =& \cos^2 A - \cos^2 B,\\
\cos(A+B)\cos(A-B) =& \cos^2 A - \sin^2 B,
\end{aligned}$$

$$\begin{aligned}
\frac{\sin A \pm \sin B}{\cos A + \cos B} =& \tan(\frac{A \pm B}{2})\\
\frac{\sin A \pm \sin B}{\cos A - \cos B} =& -\cot(\frac{A \mp B}{2})\\
\frac{\sin A + \sin B}{\sin A - \sin B} =& \frac{\tan(\frac{A+B}{2})}{\tan(\frac{A-B}{2})}
\end{aligned}$$

Powers of trigonometric functions

$$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A, \qquad \cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$$
$$\sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A, \qquad \cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$$
$$\sin^4 A = \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A, \qquad \cos^4 A = \frac{3}{8} + \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$$

Inverse Trigonometric Functions

$$\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x \qquad \sin^{-1}\frac{1}{x} = \csc^{-1} x$$
$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \qquad \cos^{-1}\frac{1}{x} = \sec^{-1} x$$
$$\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x \qquad \tan^{-1}\frac{1}{x} = \cot^{-1} x$$

Symmetry properties of trigonometric functions

$$\sin\theta = -\sin(-\theta) = \cos(\pi/2-\theta) = -\cos(\pi/2+\theta) = +\sin(\pi-\theta) = -\sin(\pi+\theta)$$
$$\cos\theta = +\cos(-\theta) = \sin(\pi/2-\theta) = +\sin(\pi/2+\theta) = -\cos(\pi-\theta) = -\cos(\pi+\theta)$$
$$\tan\theta = -\tan(-\theta) = \cot(\pi/2-\theta) = -\cot(\pi/2+\theta) = -\tan(\pi-\theta) = +\tan(\pi+\theta)$$
$$\cot\theta = -\cot(-\theta) = \tan(\pi/2-\theta) = -\tan(\pi/2+\theta) = -\cot(\pi-\theta) = +\cot(\pi+\theta)$$
$$\sec\theta = +\sec(-\theta) = \csc(\pi/2-\theta) = +\csc(\pi/2+\theta) = -\sec(\pi-\theta) = -\sec(\pi+\theta)$$
$$\csc\theta = -\csc(-\theta) = \sec(\pi/2-\theta) = +\sec(\pi/2+\theta) = +\csc(\pi-\theta) = -\csc(\pi+\theta)$$

Transformations

The following transformations are sometimes useful in simplifying expressions.

1. If $\tan\frac{u}{2} = A$, then

$$\sin u = \frac{2A}{1+A^2}, \qquad \cos u = \frac{1-A^2}{1+A^2}, \qquad \tan u = \frac{2A}{1-A^2}$$

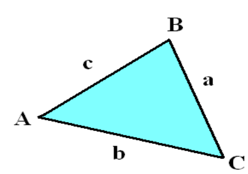
2. The transformation $\sin v = y$, requires $\cos v = \sqrt{1-y^2}$, and $\tan v = \dfrac{y}{\sqrt{1-y^2}}$

Law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



Special Numbers

Rational Numbers

All those numbers having the form p/q, where p and q are integers and q is understood to be different from zero, are called rational numbers.

Irrational Numbers

Those numbers that cannot be written as the ratio of two numbers are called irrational numbers.

The Number π

The Greek letter π (pronounced pi) is an irrational number and can be defined as the limiting sum[1] of the infinite series

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots + \frac{(-1)^n}{2n+1} + \cdots\right)$$

Using a computer one can verify that the numerical value of π to 50 decimal places is given by

$$\pi = 3.14159265358979323846264338327950288419716939937 51\ldots$$

The number π has the physical significance of representing the circumference C of a circle divided by its diameter D. The symbol π for the ratio C/D was introduced by William Jones (1675-1749), a Welsh mathematician. It became a standard notation for representing C/D after Euler also started using the symbol π for this ratio sometime around 1737.

The Number e

The limiting sum

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

is an irrational number which by agreement is called the number e. Using a computer this number, to 50 decimal places, has the numerical value

$$e = 2.71828182845904523536028747135266249775724709369996\ldots$$

The number e is referred to as the base of the natural logarithm and the function $f(x) = e^x$ is called the exponential function.

[1] Limits are very important in the study of calculus.

Greek Alphabet

Letter		Name
A	α	alpha
B	β	beta
Γ	γ	gamma
Δ	δ	delta
E	ϵ	epsilon
Z	ζ	zeta
H	η	eta
Θ	θ	theta
I	ι	iota
K	κ	kappa
Λ	λ	lambda
M	μ	mu

Letter		Name
N	ν	nu
Ξ	ξ	xi
O	o	omicron
Π	π	pi
P	ρ	rho
Σ	σ	sigma
T	τ	tau
Υ	υ	upsilon
Φ	ϕ	phi
X	χ	chi
Ψ	ψ	psi
Ω	ω	omega

Notation

By convention letters from the beginning of an alphabet, such as $a, b, c, \ldots$ or the Greek letters $\alpha, \beta, \gamma, \ldots$ are often used to denote quantities which have a constant value. Subscripted quantities such as $x_0, x_1, x_2, \ldots$ or $y_0, y_1, y_2, \ldots$ can also be used to represent constant quantities. A variable is a quantity which is allowed to change its value. The letters u, v, w, x, y, z or the Greek letters ξ, η, ζ are most often used to denote variable quantities.

Inequalities

The mathematical symbols $=$ (equals), $\neq$ (not equal), $<$ (less than), $<<$ (much less than), $\leq$ (less than or equal), $>$ (greater than), $>>$ (much greater than) $\geq$ (greater than or equal), and $|\,|$ (absolute value) occur frequently in mathematics to compare real numbers $a, b, c, \ldots$. The law of trichotomy states that if a and b are real numbers, then exactly one of the following must be true. Either a equals b, a is less than b or a is greater than b. These statements are expressed using the mathematical notations[2]

$$a = b, \qquad a < b, \qquad a > b$$

[2] In mathematical notation, the statement $b > a$, read "b is greater than a", can also be represented $a < b$ or "a is less than b"depending upon your way of looking at things.

Inequalities can be defined in terms of addition or subtraction. For example, one can define

$a < b$ if and only if $a - b < 0$

$a > b$ if and only if $a - b > 0$, or alternatively

$a > b$ if and only if there exists a positive number x such that $b + x = a$.

In dealing with inequalities be sure to observe the following properties associated with real numbers $a, b, c, \dots$

1. A constant can be added to both sides of an inequality without changing the inequality sign.

$$\text{If } a < b, \text{ then } a + c < b + c \text{ for all numbers c}$$

2. Both sides of an inequality can be multiplied or divided by a positive constant without changing the inequality sign.

$$\text{If } a < b \text{ and } c > 0, \text{ then } ac < bc \quad \text{or} \quad a/c < b/c$$

3. If both sides of an inequality are multiplied or divided by a negative quantity, then the inequality sign changes.

$$\text{If } b > a \text{ and } c < 0, \text{ then } bc < ac \quad \text{or} \quad b/c < a/c$$

4. The transitivity law

$$\text{If } a < b, \text{ and } b < c, \text{ then } a < c$$
$$\text{If } a = b \text{ and } b = c, \text{ then } a = c$$
$$\text{If } a > b, \text{ and } b > c, \text{ then } a > c$$

5. If $a > 0$ and $b > 0$, then $ab > 0$
6. If $a < 0$ and $b < 0$, then $ab > 0$ or $0 < ab$
7. If $a > 0$ and $b > 0$ with $a < b$, then $\sqrt{a} < \sqrt{b}$

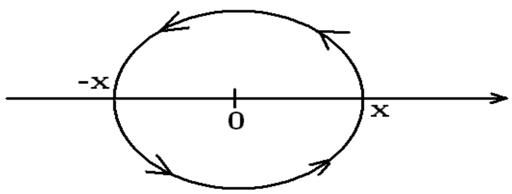
A negative times a negative is a positive

To prove that a real negative number multiplied by another real negative number gives a positive number start by assuming a and b are real numbers satisfying $a < 0$ and $b < 0$, then one can write

$$-a + a < -a \quad \text{or } 0 < -a \qquad \text{and} \qquad -b + b < -b \quad \text{or } 0 < -b$$

since equals can be added to both sides of an inequality without changing the inequality sign. Using the fact that both sides of an inequality can be multiplied by a positive number without changing the inequality sign, one can write

$$0 < (-a)(-b) \qquad \text{or} \qquad (-a)(-b) > 0$$



Another way to show a negative times a negative is a positive is as follows. Think of a number line with the number 0 separating the positive numbers and negative numbers. By agreement, if a number on this number line is multiplied by -1, then the number is to be rotated counterclockwise 180 degrees. If the positive number x is multiplied by -1, then it is rotated counterclockwise 180 degrees to produce the number $-x$. If the number $-x$ is multiplied by -1, then it is to be rotated 180 degrees counterclockwise to produce the positive number x. If $a > 0$ and $b > 0$, then the product $a(-b)$ scales the number $-b$ to produce the negative number $-ab$. If the number $-ab$ is multiplied by -1, which is equivalent to the product $(-a)(-b)$, one obtains by rotation the number $+ab$.

Absolute Value

The absolute value of a number x is defined

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

The symbol $\Longleftrightarrow$ is often used to represent equivalence of two equations. For example, if a and b are real numbers the statements

$$|x-a| \leq b \quad \Longleftrightarrow \quad -b \leq x-a \leq b \quad \Longleftrightarrow \quad a-b \leq x \leq a+b$$

are all equivalent statements involving restrictions on the real number x.

An important inequality known as the triangle inequality is written

$$|x+y| \leq |x| + |y| \tag{1.1}$$

where x and y are real numbers. To prove this inequality observe that $|x|$ satisfies $-|x| \leq x \leq |x|$ and also $-|y| \leq y \leq |y|$, so that by adding these results one obtains

$$-(|x|+|y|) \leq x+y \leq |x|+|y| \qquad \text{or} \qquad |x+y| \leq |x|+|y| \tag{1.2}$$

Related to the inequality (1.2) is the reverse triangle inequality

$$|x-y| \geq |x| - |y| \tag{1.3}$$

a proof of which is left as an exercise.

Cramer's Rule

The system of two equations in two unknowns

$$\begin{aligned} \alpha_1 x + \beta_1 y =& \gamma_1 \\ \alpha_2 x + \beta_2 y =& \gamma_2 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

has a unique solution if $\alpha_1\beta_2 - \alpha_2\beta_1$ is nonzero. The unique solution is given by

$$x = \frac{\begin{vmatrix} \gamma_1 & \beta_1 \\ \gamma_2 & \beta_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}} \qquad \text{where} \qquad \begin{matrix} & -\alpha_2\beta_1 \\ \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} & = \alpha_1\beta_2 - \alpha_2\beta_1 \\ & +\alpha_1\beta_2 \end{matrix}$$

is a single number called the determinant of the coefficients.

The system of three equations in three unknowns

$$\begin{aligned} \alpha_1 x + \beta_1 y + \gamma_1 z =& \delta_1 \\ \alpha_2 x + \beta_2 y + \gamma_2 z =& \delta_2 \\ \alpha_3 x + \beta_3 y + \gamma_3 x =& \delta_3 \end{aligned}$$

has a unique solution if the determinant of the coefficients

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \alpha_1\beta_2\gamma_3 + \beta_1\gamma_2\alpha_3 + \gamma_1\alpha_2\beta_3 - \alpha_3\beta_2\gamma_1 - \beta_3\gamma_2\alpha_1 - \gamma_3\alpha_2\beta_2$$

is nonzero. A mnemonic device to aid in calculating the determinant of the coefficients is to append the first two columns of the coefficients to the end of the array and then draw diagonals through the coefficients. Multiply the elements along an arrow and place a plus sign on the products associated with the down arrows and a minus sign associated with the products of the up arrows. This gives the figure

$$\left|\begin{matrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{matrix}\right| \begin{matrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{matrix} = \alpha_1\beta_2\gamma_3 + \beta_1\gamma_2\alpha_3 + \gamma_1\alpha_2\beta_3 - \alpha_3\beta_2\gamma_1 - \beta_3\gamma_2\alpha_1 - \gamma_3\alpha_2\beta_2$$

The solution of the three equations, three unknown system of equations is given by the determinant ratios

$$x = \frac{\begin{vmatrix} \delta_1 & \beta_1 & \gamma_1 \\ \delta_2 & \beta_2 & \gamma_2 \\ \delta_3 & \beta_3 & \gamma_3 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} \alpha_1 & \delta_1 & \gamma_1 \\ \alpha_2 & \delta_2 & \gamma_2 \\ \alpha_3 & \delta_3 & \gamma_3 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}}, \qquad z = \frac{\begin{vmatrix} \alpha_1 & \beta_1 & \delta_1 \\ \alpha_2 & \beta_2 & \delta_2 \\ \alpha_3 & \beta_3 & \delta_3 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}}$$

and is known as Cramer's rule for solving a system of equations.

Appendix C
Table of Integrals
Indefinite Integrals

General Integration Properties

1. If $\dfrac{dF(x)}{dx} = f(x)$, then $\displaystyle\int f(x)\,dx = F(x) + C$

2. If $\displaystyle\int f(x)\,dx = F(x) + C$, then the substitution $x = g(u)$ gives $\displaystyle\int f(g(u))\,g'(u)\,du = F(g(u)) + C$
For example, if $\displaystyle\int \frac{dx}{x^2+\beta^2} = \frac{1}{\beta}\tan^{-1}\frac{x}{\beta} + C$, then $\displaystyle\int \frac{du}{(u+\alpha)^2+\beta^2} = \frac{1}{\beta}\tan^{-1}\frac{u+\alpha}{\beta} + C$

3. Integration by parts. If $v_1(x) = \displaystyle\int v(x)\,dx$, then $\displaystyle\int u(x)v(x)\,dx = u(x)v_1(x) - \int u'(x)v_1(x)\,dx$

4. Repeated integration by parts or generalized integration by parts.
If $v_1(x) = \int v(x)\,dx, \;\; v_2(x) = \int v_1(x)\,dx, \ldots, v_n(x) = \int v_{n-1}(x)\,dx$, then

$$\int u(x)v(x)\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots + (-1)^{n-1}u^{n-1}v_n + (-1)^n \int u^{(n)}(x)v_n(x)\,dx$$

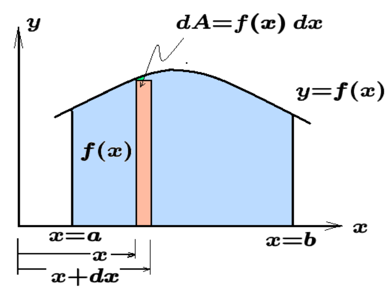
5. If $f^{-1}(x)$ is the inverse function of $f(x)$ and if $\displaystyle\int f(x)\,dx$ is known, then

$$\int f^{-1}(x)\,dx = zf(z) - \int f(z)\,dz, \qquad \text{where} \quad z = f^{-1}(x)$$

6. Fundamental theorem of calculus.
If the indefinite integral of $f(x)$ is known, say $\displaystyle\int f(x)\,dx = F(x) + C$, then the definite integral

$$\int_a^b dA = \int_a^b f(x)\,dx = F(x)]_a^b = F(b) - F(a)$$

represents the area bounded by the x-axis, the curve $y = f(x)$ and the lines $x = a$ and $x = b$.



7. Inequalities.
(i) If $f(x) \le g(x)$ for all $x \in (a,b)$, then $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$
(ii) If $|f(x) \le M|$ for all $x \in (a,b)$ and $\int_a^b f(x)\,dx$ exists, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b f(x)\,dx \le M(b-a)$$

8. $\displaystyle\int \frac{u'(x)\,dx}{u(x)} = \ln|u(x)| + C$

9. $\displaystyle\int (\alpha u(x)+\beta)^n u'(x)\,dx = \frac{(\alpha u(x)+\beta)^{n+1}}{\alpha(n+1)} + C$

10. $\displaystyle\int \frac{u'(x)v(x)-v'(x)u(x)}{v^2(x)}\,dx = \frac{u(x)}{v(x)} + C$

11. $\displaystyle\int \frac{u'(x)v(x)-u(x)v'(x)}{u(x)v(x)}\,dx = \ln|\frac{u(x)}{v(x)}| + C$

12. $\displaystyle\int \frac{u'(x)v(x)-u(x)v'(x)}{u^2(x)+v^2(x)}\,dx = \tan^{-1}\frac{u(x)}{v(x)} + C$

13. $\displaystyle\int \frac{u'(x)v(x)-u(x)v'(x)}{u^2(x)-v^2(x)}\,dx = \frac{1}{2}\ln|\frac{u(x)-v(x)}{u(x)+v(x)}| + C$

14. $\displaystyle\int \frac{u'(x)\,dx}{\sqrt{u^2(x)+\alpha}} = \ln|u(x)+\sqrt{u^2(x)+\alpha}| + C$

15. $\displaystyle\int \frac{u(x)\,dx}{(u(x)+\alpha)(u(x)+\beta)} = \begin{cases} \dfrac{\alpha}{\alpha-\beta}\displaystyle\int\frac{dx}{u(x)+\alpha} - \dfrac{\beta}{\alpha-\beta}\displaystyle\int\frac{dx}{u(x)+\beta}, & \alpha\neq\beta \\ \displaystyle\int\frac{dx}{u(x)+\alpha} - \alpha\int\frac{dx}{(u(x)+\alpha)^2}, & \beta=\alpha \end{cases}$

16. $\displaystyle\int \frac{u'(x)\,dx}{\alpha u^2(x)+\beta u(x)} = \frac{1}{\beta}\ln|\frac{u(x)}{\alpha u(x)+\beta}| + C$

17. $\displaystyle\int \frac{u'(x)\,dx}{u(x)\sqrt{u^2(x)-\alpha^2}} = \frac{1}{\alpha}\sec^{-1}\frac{u(x)}{\alpha} + C$

18. $\displaystyle\int \frac{u'(x)\,dx}{\alpha^2+\beta^2u^2(x)} = \frac{1}{\alpha\beta}\tan^{-1}\frac{\beta u(x)}{\alpha} + C$

19. $\displaystyle\int \frac{u'(x)\,dx}{\alpha^2u^2(x)-\beta^2} = \frac{1}{2\alpha\beta}\ln|\frac{\alpha u(x)-\beta}{\alpha u(x)+\beta}| + C$

20. $\displaystyle\int f(\sin x)\,dx = 2\int f\left(\frac{2u}{1+u^2}\right)\frac{du}{1+u^2}, \qquad u=\tan\frac{x}{2}$

21. $\displaystyle\int f(\sin x)\,dx = \int f(u)\frac{du}{\sqrt{1-u^2}}, \qquad u=\sin x$

22. $\displaystyle\int f(\cos x)\,dx = 2\int f\left(\frac{1-u^2}{1+u^2}\right)\frac{du}{1+u^2}, \qquad u=\tan\frac{x}{2}$

23. $\displaystyle\int f(\cos x)\,dx = -\int f(u)\frac{du}{\sqrt{1-u^2}}, \qquad u=\cos x$

24. $\displaystyle\int f(\sin x,\cos x)\,dx = \int f(u,\sqrt{1-u^2})\frac{du}{\sqrt{1-u^2}}, \qquad u=\sin x$

25. $\displaystyle\int f(\sin x,\cos x)\,dx = 2\int f\left(\frac{2u}{1+u^2},\frac{1-u^2}{1+u^2}\right)\frac{du}{1+u^2}, \qquad u=\tan\frac{x}{2}$

26. $\displaystyle\int f(x,\sqrt{\alpha+\beta x})\,dx = \frac{2}{\beta}\int f\left(\frac{u^2-\alpha}{\beta},u\right)u\,du, \qquad u^2=\alpha+\beta x$

27. $\displaystyle\int f(x,\sqrt{\alpha^2-x^2})\,dx = \alpha\int f(\alpha\sin u, a\cos u)\,\cos u\,du, \qquad x=\alpha\sin u$

General Integrals

28. $\int c\,u(x)\,dx = c\int u(x)\,dx$

29. $\int [u(x)+v(x)]\,dx = \int u(x)\,dx + \int v(x)\,dx$

30. $\int u(x)\,u'(x)\,dx = \frac{1}{2}\mid u(x)\mid^2 + C$

31. $\int [u(x)-v(x)]\,dx = \int u(x)\,dx - \int v(x)\,dx]$

32. $\int u^n(x)\,u'(x)\,dx = \frac{[u(x)]^{n+1}}{n+1} + C$

33. $\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx$

34. $\int F'[u(x)]\,u'(x)\,dx = F[u(x)] + C$

35. $\int \frac{u'(x)}{u(x)}\,dx = \ln\mid u(x)\mid + C$

36. $\int \frac{u'}{2\sqrt{u}}\,dx = \sqrt{u} + C$

37. $\int 1\,dx = x + C$

38. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

39. $\int \frac{1}{x}\,dx = \ln\mid x\mid + C$

40. $\int e^{au}\,u'\,dx = \frac{1}{a}e^{au} + C$

41. $\int a^u\,u'\,dx = \frac{1}{\ln a}a^u + C$

42. $\int \sin u\,u'\,dx = \cos u + C$

43. $\int \cos u\,u'\,dx = -\sin u + C$

44. $\int \tan u\,u'\,dx = \ln\mid \sec u\mid + C$

45. $\int \cot u\,u'\,dx = \ln\mid \sin u\mid + C$

46. $\int \sec u\,u'\,dx = \ln\mid \sec u + \tan u\mid + C$

47. $\int \csc u\,u'\,dx = \ln\mid \csc u - \cot u\mid + C$

48. $\int \sinh u\,u'\,dx = \cosh u + C$

49. $\int \cosh u\,u'\,dx = \sinh u + C$

50. $\int \tanh u\,u'\,dx = \ln\cosh u + C$

51. $\int \coth u\,u'\,dx = \ln\sinh u + C$

52. $\int \operatorname{sech} u\,u'\,dx = \sin^{-1}(\tanh u) + C$

53. $\int \operatorname{csch} u\,u'\,dx = \ln\tanh\frac{u}{2} + C$

54. $\int \sin^2 u\,u'\,dx = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

55. $\int \cos^2 u\,u'\,dx = \frac{u}{2} + \frac{1}{4}\sin 2u + C$

56. $\int \tan^2 u\,u'\,dx = \tan u - u + C$

57. $\int \cot^2 u\,u'\,dx = -\cot u - u + C$

58. $\int \sec^2 u\,u'\,dx = \tan u + C$

59. $\int \csc^2 u\,u'\,dx = -\cot u + C$

60. $\int \sinh^2 u\,u'\,dx = \frac{1}{4}\sinh 2u - \frac{1}{2}u + C$

61. $\int \cosh^2 u\,u'\,dx = \frac{1}{4}\sinh 2u + \frac{1}{2}u + C$

62. $\int \tanh^2 u\,u'\,dx = u - \tanh u + C$

63. $\int \coth^2 u\,u'\,dx = u - \coth u + C$

64. $\int \operatorname{sech}^2 u\,u'\,dx = \tanh u + C$

65. $\int \operatorname{csch}^2 u\,u'\,dx = -\coth u + C$

66. $\int \sec u\tan u\,u'\,dx = \sec u + C$

67. $\int \csc u\cot u\,u'\,dx = -\csc u + C$

68. $\int \operatorname{sech} u\tanh u\,u'\,dx = -\operatorname{sech} u + C$

69. $\int \operatorname{csch} u\coth u\,u'\,dx = -\operatorname{csch} u + C$

Integrals containing $X = a + bx$, $a \neq 0$ and $b \neq 0$

70. $\int X^n\,dx = \frac{X^{n+1}}{b(n+1)} + C, \quad n \neq -1$

71. $\int xX^n\,dx = \frac{X^{n+2}}{b^2(n+2)} - \frac{aX^{n+1}}{b^2(n+1)} + C, \quad n \neq -1, n \neq -2$

72. $\int X(x+c)^n\,dx = \frac{b}{n+2}(x+c)^{n+2} + \frac{a-bc}{n+1}(x+c)^{n+1} + C$

73. $\displaystyle\int x^2 X^n\,dx = \frac{1}{b^3}\left[\frac{X^{n+3}}{n+3} - \frac{2aX^{n+2}}{n+2} + \frac{a^2 X^{n+1}}{n+1}\right] + C$

74. $\displaystyle\int x^{n-1}X^m\,dx = \frac{1}{n+m}x^n X^m + \frac{am}{m+n}\int x^{n-1}X^{m-1}\,dx$

75. $\displaystyle\int \frac{X^m}{x^{n+1}}\,dx = -\frac{1}{na}\frac{X^{m+1}}{x^n} + \frac{m-n+1}{n}\frac{b}{a}\int \frac{X^m}{x^n}\,dx$

76. $\displaystyle\int \frac{dx}{X} = \frac{1}{b}\ln X + C$

77. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{b^2}\left(X - a\ln\mid X\mid\right) + C$

78. $\displaystyle\int \frac{x^2\,dx}{X} = \frac{1}{2b^3}\left(X^2 - 4aX + 2a^2\ln\mid X\mid\right) + C$

79. $\displaystyle\int \frac{dx}{xX} = \frac{1}{a}\ln\mid\frac{x}{X}\mid + C$

80. $\displaystyle\int \frac{dx}{x^3 X} = -\frac{a+2bx}{a^2 xX} + \frac{2b}{a^3}\ln\mid\frac{X}{x}\mid + C$

81. $\displaystyle\int \frac{dx}{X^2} = -\frac{1}{bX} + C$

82. $\displaystyle\int \frac{x\,dx}{X^2} = \frac{1}{b^2}\left[\ln\mid X\mid + \frac{a}{X}\right] + C$

83. $\displaystyle\int \frac{x^2\,dx}{X^2} = \frac{1}{b^3}\left[X - 2a\ln\mid X\mid - \frac{a^2}{X}\right] + C$

84. $\displaystyle\int \frac{dx}{xX^2} = \frac{1}{aX} - \frac{1}{a^2}\ln\mid\frac{X}{x}\mid + C$

85. $\displaystyle\int \frac{dx}{x^2 X^2} = -\frac{a+2bx}{a^2 xX} + \frac{2b}{a^3}\ln\mid\frac{X}{x}\mid + C$

86. $\displaystyle\int \frac{dx}{X^3} = -\frac{1}{2bX^2} + C$

87. $\displaystyle\int \frac{x\,dx}{X^3} = \frac{1}{b^2}\left[\frac{-1}{X} + \frac{a}{2X^2}\right] + C$

88. $\displaystyle\int \frac{x^2\,dx}{X^3} = \frac{1}{b^3}\left[\ln\mid X\mid + \frac{2a}{X} - \frac{a^2}{2X^2}\right] + C$

89. $\displaystyle\int \frac{dx}{xX^3} = \frac{1}{2aX^2} + \frac{1}{aX} - \ln\mid\frac{X}{x}\mid + C$

90. $$\int \frac{dx}{x^2X^3} = \frac{-b}{2a^2X} - \frac{2b}{a^3X} - \frac{1}{a^3x} + \frac{3b}{a^4}\ln\left|\,\frac{X}{x}\,\right|$$

91. $$\int \frac{x\,dx}{X^n} = \frac{1}{b^2}\left[\frac{-1}{(n-2)X^{n-2}} + \frac{a}{(n-1)X^{n-1}}\right] + C, \qquad n \neq 1,2$$

92. $$\int \frac{x^2\,dx}{X^n} = \frac{1}{b^3}\left[\frac{-1}{(n-3)X^{n-3}} + \frac{2a}{(n-2)X^{n-2}} - \frac{a^2}{(n-1)X^{n-1}}\right] + C, \qquad n \neq 1,2,3$$

93. $$\int \sqrt{X}\,dx = \frac{2}{3b}X^{3/2} + C$$

94. $$\int x\sqrt{X}\,dx = \frac{2}{15b^2}(3bx-2a)X^{3/2} + C$$

95. $$\int x^2\sqrt{X}\,dx = \frac{2}{105b^3}(8a^2 - 12abx + 15b^2x^2)X^{3/2} + C$$

96. $$\int \frac{\sqrt{X}}{x}\,dx = 2\sqrt{X} + a\int \frac{dx}{x\sqrt{X}}$$

97. $$\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}}$$

98. $$\int \frac{dx}{\sqrt{X}} = \frac{2}{b}\sqrt{X} + C$$

99. $$\int \frac{x\,dx}{\sqrt{X}} = \frac{2}{3b^2}(bx-2a)\sqrt{X} + C$$

100. $$\int \frac{x^2\,dx}{\sqrt{X}} = \frac{2}{15b^3}(8a^2 - 4abx + 3b^2x^2)\sqrt{X} + C$$

101. $$\int \frac{dx}{x\sqrt{X}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\,\dfrac{\sqrt{X}-\sqrt{a}}{\sqrt{X}+\sqrt{a}}\,\right| + C_1, & a > 0 \\ \dfrac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\dfrac{X}{-a}} + C_2, & a < 0 \end{cases}$$

102. $$\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}$$

103. $$\int x^n\sqrt{X}\,dx = \frac{2}{(2n+3)b}x^nX^{3/2} - \frac{2na}{(2n+3)b}\int x^{n-1}\sqrt{X}\,dx$$

104. $$\int \frac{\sqrt{X}}{x^n}\,dx = \frac{-1}{(n-1)a}\frac{X^{3/2}}{x^{n-1}} - \frac{(2n-5)b}{2(n-1)a}\int \frac{\sqrt{X}}{x^{n-1}}\,dx$$

105. $$\int x^{m-1}X^n\,dx = \frac{x^mX^n}{m+n} + \frac{an}{m+n}\int x^{m-1}X^{n-1}\,dx + C$$

106. $$\int \frac{X^n}{x^{m+1}}\,dx = -\frac{X^{n+1}}{ma\,x^m} + \frac{n-m+1}{m}\frac{b}{a}\int \frac{X^n}{x^m}\,dx$$

107. $\displaystyle\int \frac{X^n}{x}\,dx = \frac{X^n}{n} + a\int \frac{X^{n-1}}{x}\,dx$

Integrals containing $X = a + bx$ and $Y = \alpha + \beta x$, $\quad (b \neq 0,\ \beta \neq 0,\ \Delta = a\beta - \alpha b \neq 0\ )$

108. $\displaystyle\int \frac{dx}{XY} = \frac{1}{\Delta}\ln \mid \frac{Y}{X} \mid + C$

109. $\displaystyle\int \frac{x\,dx}{XY} = \frac{1}{\Delta}\left[\frac{a}{b}\ln \mid X \mid - \frac{\alpha}{\beta}\ln \mid Y \mid\right] + C$

110. $\displaystyle\int \frac{x^2\,dx}{XY} = \frac{x}{b\beta} = \frac{a^2}{b^2\Delta}\ln|X| + \frac{\alpha^2}{\beta^2\Delta}\ln|Y| + C$

111. $\displaystyle\int \frac{dx}{X^2Y} = \frac{1}{\Delta}\left(\frac{1}{X} + \frac{\beta}{\Delta}\ln|\frac{Y}{X}|\right) + C$

112. $\displaystyle\int \frac{x\,dx}{X^2Y} = -\frac{a}{b\Delta X} - \frac{\alpha}{\Delta^2}\ln|\frac{Y}{X}| + C$

113. $\displaystyle\int \frac{x^2\,dx}{X^2Y} = \frac{a^2}{b^2\Delta X} + \frac{1}{\Delta^2}\left[\frac{\alpha^2}{\beta}\ln|Y| + \frac{a(a\beta - 2\alpha b)}{b^2}\ln|X|\right] + C$

114. $\displaystyle\int \frac{X}{Y}\,dx = \frac{b}{\beta}x + \frac{\Delta}{\beta^2}\ln|\frac{Y}{X}| + C$

115. $\displaystyle\int \sqrt{XY}\,dx = \frac{\Delta + 2bY}{4b\beta}\sqrt{XY} - \frac{\Delta^2}{8b\beta}\int \frac{dx}{\sqrt{XY}}$

116. $\displaystyle\int \frac{dx}{X^nY^m} = \frac{-1}{(m-1)\Delta X^{n-1}Y^{m-1}} + \frac{(m+n-2)b}{(m-1)\Delta}\int \frac{dx}{X^nY^{m-1}}, \qquad m \neq 1$

117. $$\int \frac{dx}{Y\sqrt{X}} = \begin{cases} \dfrac{2}{\sqrt{-\Delta\beta}}\tan^{-1}\dfrac{\beta\sqrt{X}}{\sqrt{-\Delta\beta}}, + C_1 & \Delta\beta < 0 \\[2ex] \dfrac{1}{\sqrt{\Delta\beta}}\ln|\dfrac{\beta\sqrt{X} - \sqrt{\Delta\beta}}{\beta\sqrt{X} + \sqrt{\Delta\beta}}| + C_2, & \Delta\beta > 0 \end{cases}$$

118. $$\int \frac{dx}{\sqrt{XY}} = \begin{cases} \dfrac{2}{\sqrt{-b\beta}}\tan^{-1}\sqrt{\dfrac{-\beta X}{bY}} + C_1, & b\beta < 0,\ bY > 0 \\[2ex] \dfrac{2}{\sqrt{b\beta}}\tanh^{-1}\sqrt{\dfrac{\beta X}{bY}} + C_2, & b\beta > 0,\ bY > 0 \end{cases}$$

119. $\displaystyle\int \frac{x\,dx}{\sqrt{XY}} = \frac{1}{b\beta}\sqrt{XY} - \frac{(b\alpha + a\beta)}{2b\beta}\int \frac{dx}{\sqrt{XY}}$

120. $\displaystyle\int \frac{\sqrt{Y}}{\sqrt{X}}\,dx = \frac{1}{b}\sqrt{XY} - \frac{\Delta}{2b}\int \frac{dx}{\sqrt{XY}}$

121. $\displaystyle\int \frac{\sqrt{X}}{Y}\,dx = \frac{2}{\beta}\sqrt{X} + \frac{\Delta}{\beta}\int \frac{dx}{Y\sqrt{X}}$

Integrals containing terms of the form $a + bx^n$

122. $$\int \frac{dx}{a+bx^2} = \begin{cases} \dfrac{1}{\sqrt{ab}} \tan^{-1}\left(\sqrt{\dfrac{b}{a}}x\right) + C, & ab > 0 \\ \dfrac{1}{2\sqrt{-ab}} \ln\left|\dfrac{a+\sqrt{-ab}x}{a-\sqrt{-ab}x}\right| + C, & ab < 0 \end{cases}$$

123. $$\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\ln|x^2 + \frac{a}{b}| + C$$

124. $$\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+bx^2}$$

125. $$\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int \frac{dx}{a+bx^2}$$

126. $$\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a}\ln\left|\frac{x^2}{a+bx^2}\right| + C$$

127. $$\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}$$

128. $$\int \frac{dx}{(a+bx^2)^{n+1}} = \frac{1}{2na}\frac{x}{(a+bx^2)^n} + \frac{2n-1}{2na}\int \frac{dx}{(a+bx^2)^n}$$

129. $$\int \frac{dx}{\alpha^3+\beta^3x^3} = \frac{1}{6\alpha^2\beta}\left[2\sqrt{3}\tan^{-1}\left(\frac{2\beta x-\alpha}{\sqrt{3}\alpha}\right) + \ln\left|\frac{(\alpha+\beta x)^2}{\alpha^2-\alpha\beta x+\beta^2x^2}\right|\right] + C$$

130. $$\int \frac{x\,dx}{\alpha^3+\beta^3x^3} = \frac{1}{6\alpha\beta^2}\left[2\sqrt{3}\tan^{-1}\left(\frac{2\beta x-\alpha}{\sqrt{3}\alpha}\right) - \ln\left|\frac{(\alpha+\beta x)^2}{\alpha^2-\alpha\beta x+\beta^2x^2}\right|\right] + C$$

If $X = a + bx^n$, then

131. $$\int x^{m-1}X^p\,dx = \frac{x^mX^p}{m+pn} + \frac{apn}{m+pn}\int x^{m-1}X^{p-1}\,dx$$

132. $$\int x^{m-1}X^p\,dx = -\frac{x^mX^{p+1}}{an(p+1)} + \frac{m+pn+n}{an(p+1)}\int x^{m-1}X^{p+1}\,dx$$

133. $$\int x^{m-1}X^P\,dx = \frac{x^{m-n}X^{p+1}}{b(m+pn)} - \frac{(m-n)\,a}{b(m+pn)}\int x^{m-n-1}X^p\,dx$$

134. $$\int x^{m-1}X^p\,dx = \frac{x^mX^{p+1}}{am} - \frac{(m+pn+n)\,b}{am}\int x^{m+n-1}X^p\,dx$$

135. $$\int x^{m-1}X^p\,dx = \frac{x^{m-n}X^{p+1}}{bn(p+1)} - \frac{m-n}{bn(p+1)}\int x^{m-n-1}X^{p+1}\,dx$$

136. $$\int x^{m-1}X^p\,dx = \frac{x^mX^p}{m} - \frac{bpn}{m}\int x^{m+n-1}X^{p-1}\,dx$$

Integrals containing $X = 2ax - x^2, \quad a \neq 0$

137. $$\int \sqrt{X}\,dx = \frac{(x-a)}{2}\sqrt{X} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{|a|}\right) + C$$

138. $$\int \frac{dx}{\sqrt{X}} = \sin^{-1}\left(\frac{x-a}{|a|}\right) + C$$

139. $$\int x\sqrt{X}\,dx = \sin^{-1}\left(\frac{x-a}{|a|}\right) + C$$

140. $$\int \frac{x\,dx}{\sqrt{X}} = -\sqrt{X} + a\sin^{-1}\left(\frac{x-a}{|a|}\right) + C$$

141. $$\int \frac{dx}{X^{3/2}} = \frac{x-a}{a^2\sqrt{X}} + C$$

142. $$\int \frac{x\,dx}{X^{3/2}} = \frac{x}{a\sqrt{X}} + C$$

143. $$\int \frac{dx}{X} = \frac{1}{2a}\ln|\frac{x}{x-2a}| + C$$

144. $$\int \frac{x\,dx}{X} = -\ln|x-2a| + C$$

145. $$\int \frac{dx}{X^2} = -\frac{1}{4ax} - \frac{1}{4a^2(x-2a)} + \frac{1}{4a^2}\ln|\frac{x}{x-2a}| + C$$

146. $$\int \frac{x\,dx}{X^2} = -\frac{1}{2a(x-2a)} + \frac{1}{4a^2}\ln|\frac{x}{x-2a}| + C$$

147. $$\int x^n\sqrt{X}\,dx = -\frac{1}{n+2}x^{n-1}X^{3/2} + \frac{(2n+1)a}{n+2}\int x^{n-1}\sqrt{X}\,dx, \quad n \neq -2$$

148. $$\int \frac{\sqrt{X}\,dx}{x^n} = \frac{1}{(3-2n)a}\frac{X^{3/2}}{x^n} + \frac{n-3}{(2n-3)a}\int \frac{\sqrt{X}}{x^{n-1}}\,dx, \quad n \neq 3/2$$

Integrals containing $X = ax^2 + bx + c$ with $\Delta = 4ac - b^2, \quad \Delta \neq 0, \quad a \neq 0$

149. $$\int \frac{dx}{X} = \begin{cases} \frac{1}{\sqrt{-\Delta}}\ln\left(\frac{2ax+b-\sqrt{-\Delta}}{2ax+b+\sqrt{-\Delta}}\right) + C_1, & \Delta < 0 \\ \frac{2}{\sqrt{\Delta}}\tan^{-1}\frac{2ax+b}{\sqrt{\Delta}} + C_2, & \Delta > 0 \\ -\frac{1}{a(x+b/2a)} + C_3, & \Delta = 0 \end{cases}$$

150. $$\int \frac{x\,dx}{X} = \frac{1}{2c}\ln|X| - \frac{b}{2a}\int \frac{1}{X}\,dx$$

151. $$\int \frac{x^2\,dx}{X} = \frac{x}{a} - \frac{b}{2a^2}\ln|X| + \frac{2ac-\Delta}{2a^2}\int \frac{dx}{X}$$

152. $\displaystyle \int \frac{dx}{xX} = \frac{1}{2c}\ln|\frac{x^2}{X}| - \frac{b}{2c}\int \frac{dx}{X}$

153. $\displaystyle \int \frac{dx}{x^2X} = \frac{b}{2c^2}\ln|\frac{X}{x^2}| - \frac{1}{cx} + \frac{2ac-\Delta}{2c^2}\int \frac{dx}{X}$

154. $\displaystyle \int \frac{dx}{X^2} = \frac{bx+2c}{\Delta X} - \frac{b}{\Delta}\int \frac{dx}{X}$

155. $\displaystyle \int \frac{x\,dx}{X^2} = -\frac{bx+2c}{\Delta X} - \frac{b}{\Delta}\int \frac{dx}{X}$

156. $\displaystyle \int \frac{x^2\,dx}{X^2} = \frac{(2ac-\Delta)x+bc}{a\Delta X} + \frac{2c}{\Delta}\int \frac{dx}{X}$

157. $\displaystyle \int \frac{dx}{xX^2} = \frac{1}{2cX} - \frac{b}{2c}\int \frac{dx}{X^2} + \frac{1}{c}\int \frac{dx}{xX}$

158. $\displaystyle \int \frac{dx}{x^2X^2} = -\frac{1}{cxX} - \frac{3a}{c}\int \frac{dx}{X^2} - \frac{2b}{c}\int \frac{dx}{xX^2}$

159. $$\int \frac{dx}{\sqrt{X}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln|2\sqrt{aX}+2ax+b| + C_1, & a>0 \\ \dfrac{1}{\sqrt{a}}\sinh^{-1}\left(\dfrac{2ax+b}{\sqrt{\Delta}}\right) + C_2, & a>,\ \Delta>0 \\ -\dfrac{1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{2ax+b}{\sqrt{-\Delta}}\right) + C_3, & a<0,\ \Delta<0 \end{cases}$$

160. $\displaystyle \int \frac{x\,dx}{\sqrt{X}} = \frac{1}{a}\sqrt{X} - \frac{b}{2a}\int \frac{dx}{\sqrt{X}}$

161. $\displaystyle \int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2a} - \frac{3b}{4a^2}\right)\sqrt{X} + \frac{2b^2-\Delta}{8a^2}\int \frac{dx}{\sqrt{X}}$

162. $$\int \frac{dx}{x\sqrt{X}} = \begin{cases} -\dfrac{1}{\sqrt{c}}\ln|\dfrac{2\sqrt{cX}}{x} + \dfrac{2c}{x} + b| + C_1, & c>0 \\ -\dfrac{1}{\sqrt{c}}\sinh^{-1}\left(\dfrac{bx+2c}{x\sqrt{\Delta}}\right) + C_2, & c>0,\ \Delta>0 \\ \dfrac{1}{\sqrt{-c}}\sin^{-1}\left(\dfrac{bx+2c}{x\sqrt{-\Delta}}\right) + C_3, & c<0,\ \Delta<0 \end{cases}$$

163. $\displaystyle \int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{cx} - \frac{b}{2c}\int \frac{dx}{x\sqrt{X}}$

164. $\displaystyle \int \sqrt{X}\,dx = \frac{1}{4a}(2ax+b)\sqrt{X} + \frac{\Delta}{8a}\int \frac{dx}{\sqrt{X}}$

165. $\displaystyle \int x\sqrt{X}\,dx = \frac{1}{3a}X^{3/2} - \frac{b(2ax+b)}{8a^2}\sqrt{X} - \frac{b\Delta}{16a^2}\int \frac{dx}{\sqrt{X}}$

166. $\displaystyle \int x^2\sqrt{X}\,dx = \frac{6ax-5b}{24a^2}X^{3/2} + \frac{4b^2-\Delta}{16a^2}\int \sqrt{X}\,dx$

167. $$\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + c\int \frac{dx}{x\sqrt{X}}$$

168. $$\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + a\int \frac{dx}{\sqrt{X}} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}}$$

169. $$\int \frac{dx}{X^{3/2}} = \frac{2(2ax+b)}{\Delta\sqrt{X}} + C$$

170. $$\int \frac{x\,dx}{X^{3/2}} = \frac{-2(bx+2c)}{\Delta\sqrt{X}} + C$$

171. $$\int \frac{x^2\,dx}{X^{3/2}} = \frac{(b^2-\Delta)x+2bc}{a\Delta\sqrt{X}} + \frac{1}{a}\int \frac{dx}{\sqrt{X}}$$

172. $$\int \frac{dx}{xX^{3/2}} = \frac{1}{x\sqrt{X}} + \frac{1}{c}\int \frac{dx}{x\sqrt{X}} - \frac{b}{2c}\int \frac{dx}{X^{3/2}}$$

173. $$\int \frac{dx}{x^2X^{3/2}} = -\frac{ax^2+2bx+c}{c^2x\sqrt{X}} + \frac{b^2-2ac}{2c^2}\int \frac{dx}{X^{3/2}} - \frac{3b}{2c^2}\int \frac{dx}{x\sqrt{X}}$$

174. $$\int \frac{dx}{X\sqrt{X}} = \frac{2(2ax+b)}{\Delta\sqrt{X}} + C$$

175. $$\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2ax+b)}{3\Delta\sqrt{X}}\left(\frac{1}{X} + \frac{8a}{\Delta}\right) + C$$

176. $$\int X\sqrt{X}\,dx = \frac{(2ax+b)}{8a}\sqrt{X}\left(X + \frac{3\Delta}{8a}\right) + \frac{3\Delta^2}{128a^2}\int \frac{dx}{\sqrt{X}}$$

177. $$\int X^2\sqrt{X}\,dx = \frac{(2ax+b)}{8a}\sqrt{X}\left(X^2 + \frac{5\Delta}{16a}X + \frac{15\Delta^2}{128a^2}\right) + \frac{5\Delta^3}{1024a^3}\int \frac{dx}{\sqrt{X}}$$

178. $$\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2c)}{\Delta\sqrt{X}} + C$$

179. $$\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(b^2-\Delta)x+2bc}{a\Delta\sqrt{X}} + \frac{1}{a}\int \frac{dx}{\sqrt{X}}$$

180. $$\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5a} - \frac{b}{2a}\int X\sqrt{X}\,dx$$

181. $\int f(x, \sqrt{ax^2+bx+c})\,dx$ Try substitutions (i) $\sqrt{ax^2+bx+c} = \sqrt{a}(x+z)$

(ii) $\sqrt{ax^2+bx+c} = xz+\sqrt{c}$ and if $ax^2+bx+c = a(x-x_1)(x-x_2)$, then (iii) let $(x-x_2) = z^2(x-x_1)$

Integrals containing $X = x^2 + a^2$

182. $$\int \frac{dx}{X} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \quad \text{or} \quad \frac{1}{a}\cos^{-1}\frac{a}{\sqrt{x^2+a^2}} + C \quad \text{or} \quad \frac{1}{a}\sec^{-1}\frac{\sqrt{x^2+a^2}}{a} + C$$

183. $$\int \frac{x\,dx}{X} = \frac{1}{2}\ln X + C$$

184. $\displaystyle\int \frac{x^2\,dx}{X} = x - a\tan^{-1}\frac{x}{a} + C$

185. $\displaystyle\int \frac{x^3\,dx}{X} = \frac{x^2}{2} - \frac{a^2}{2}\ln|x^2+a^2| + C$

186. $\displaystyle\int \frac{dx}{xX} = \frac{1}{2a^2}\ln|\frac{x^2}{X}| + C$

187. $\displaystyle\int \frac{dx}{x^2X} = -\frac{1}{a^2x} - \frac{1}{a^3}\tan^{-1}\frac{x}{a} + C$

188. $\displaystyle\int \frac{dx}{x^3X} = -\frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

189. $\displaystyle\int \frac{dx}{X^2} = \frac{x}{2a^2X} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + C$

190. $\displaystyle\int \frac{x\,dx}{X^2} = -\frac{1}{2X} + C$

191. $\displaystyle\int \frac{x^2\,dx}{X^2} = -\frac{x}{2X} + \frac{1}{2a}\tan^{-1}\frac{x}{a} + C$

192. $\displaystyle\int \frac{x^3\,dx}{X^2} = \frac{a^2}{2X} + \frac{1}{2}\ln|X| + C$

193. $\displaystyle\int \frac{dx}{xX^2} = \frac{1}{2a^2X} + \frac{1}{2a^4}\ln|\frac{x}{X}| + C$

194. $\displaystyle\int \frac{dx}{x^2X^2} = -\frac{1}{a^4X} - \frac{x}{2a^4X} - \frac{3}{2a^5}\tan^{-1}\frac{x}{a} + C$

195. $\displaystyle\int \frac{dx}{x^3X^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4X} - \frac{1}{a^6}\ln|\frac{x^2}{X}| + C$

196. $\displaystyle\int \frac{dx}{X^3} = \frac{x}{4a^2X^2} + \frac{3x}{8a^4X} + \frac{3}{8a^5}\tan^{-1}\frac{x}{a} + C$

197. $\displaystyle\int \frac{dx}{X^n} = \frac{x}{2(n-1)a^2X^{n-1}} + \frac{2n-3}{(2(n-1)a^2}\int\frac{dx}{X^{n-1}}, \qquad n>1$

198. $\displaystyle\int \frac{x\,dx}{X^n} = -\frac{1}{2(n-1)X^{n-1}} + C$

199. $\displaystyle\int \frac{dx}{xX^n} = \frac{1}{2(n-1)a^2X^{n-1}} + \frac{1}{a^2}\int\frac{dx}{xX^{n-1}}$

Integrals containing the square root of $X = x^2 + a^2$

200. $\displaystyle\int \sqrt{X}\,dx = \frac{1}{2}xX + \frac{a^2}{2}\ln|x+\sqrt{X}| + C$

201. $\displaystyle\int x\sqrt{X}\,dx = \frac{1}{3}X^{3/2} + C$

202. $\displaystyle\int x^2\sqrt{X}\,dx = \frac{1}{4}xX^{3/2} - \frac{1}{8}a^2x\sqrt{X} - \frac{a^2}{8}\ln|x+\sqrt{X}| + C$

203. $\displaystyle\int x^3\sqrt{X}\,dx = \frac{1}{5}X^{5/2} - \frac{a^2}{3}X^{3/2} + C$

204. $\displaystyle\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\ln|\frac{a+\sqrt{X}}{x}| + C$

205. $\displaystyle\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \ln|x+\sqrt{X}| + C$

206. $\displaystyle\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} - \frac{1}{2a}\ln|\frac{a+\sqrt{X}}{x}| + C$

207. $\displaystyle\int \frac{dx}{\sqrt{X}} = \ln|x+\sqrt{X}| + C \quad \text{or} \quad \sinh^{-1}\frac{x}{a} + C$

208. $\displaystyle\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X} + C$

209. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} - \frac{a^2}{2}\ln|x+\sqrt{X}| + C$

210. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X}} = \frac{1}{3}X^{3/2} - a^2\sqrt{X} + C$

211. $\displaystyle\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\ln|\frac{a+\sqrt{X}}{x}| + C$

212. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x} + C$

213. $\displaystyle\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$

214. $\displaystyle\int X^{3/2}\,dx = \frac{1}{4}X^{3/2} + \frac{3}{8}a^2x\sqrt{X} + \frac{3}{8}a^4\ln|x+\sqrt{X}| + C$

215. $\displaystyle\int xX^{3/2}\,dx = \frac{1}{5}X^{5/2} + C$

216. $\displaystyle\int x^2X^{3/2}\,dx = \frac{1}{6}X^{5/2} - \frac{1}{24}a^2xX^{3/2} - \frac{1}{16}a^4x\sqrt{X} - \frac{1}{16}a^6\ln|x+\sqrt{X}| + C$

217. $\displaystyle\int x^3X^{3/2}\,dx = \frac{1}{7}X^{7/2} - \frac{1}{5}a^2X^{5/2} + C$

218. $\displaystyle\int \frac{X^{3/2}}{x}\,dx = \frac{1}{3}X^{3/2} + a^2\sqrt{X} - a^3 \ln|\frac{a+\sqrt{X}}{x}| + C$

219. $\displaystyle\int \frac{X^{3/2}}{x^2}\,dx = -\frac{X^{3/2}}{x} + \frac{3}{2}x\sqrt{X} + \frac{3}{2}a^2 \ln|x+\sqrt{X}| + C$

220. $\displaystyle\int \frac{X^{3/2}}{x^3}\,dx = -\frac{X^{3/2}}{2x^2} + \frac{3}{2}\sqrt{X} - \frac{3}{2}a \ln|\frac{a+\sqrt{x}}{x}| + C$

221. $\displaystyle\int \frac{dx}{X^{3/2}} = \frac{x}{a^2\sqrt{X}} + C$

222. $\displaystyle\int \frac{x\,dx}{X^{3/2}} = \frac{-1}{\sqrt{X}} + C$

223. $\displaystyle\int \frac{x^2\,dx}{X^{3/2}} = \frac{-x}{\sqrt{X}} + \ln|x+\sqrt{X}| + C$

224. $\displaystyle\int \frac{x^3\,dx}{X^{3/2}} = \sqrt{X} + \frac{a^2}{\sqrt{X}} + C$

225. $\displaystyle\int \frac{dx}{xX^{3/2}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$

226. $\displaystyle\int \frac{dx}{x^2X^{3/2}} = -\frac{\sqrt{X}}{a^4 x} - \frac{x}{a^4\sqrt{X}} + C$

227. $\displaystyle\int \frac{dx}{x^3X^{3/2}} = \frac{-1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} + \frac{3}{2a^5}\ln|\frac{a+\sqrt{X}}{x}| + C$

228. $\displaystyle\int f(x,\sqrt{X})\,dx = a\int f(a\tan u, a\sec u)\,\sec^2 u\,du, \qquad x = a\tan u$

Integrals containing $X = x^2 - a^2$ with $x^2 > a^2$

229. $\displaystyle\int \frac{dx}{X} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + C \quad \text{or} \quad -\frac{1}{a}\coth^{-1}\frac{x}{a} + C \quad \text{or} \quad -\frac{1}{a}\tanh^{-1}\frac{a}{x} + C$

230. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{2}\ln X + C$

231. $\displaystyle\int \frac{x^2\,dx}{X} = x + \frac{a}{2}\ln|\frac{x-a}{x+a}| + C$

232. $\displaystyle\int \frac{x^3\,dx}{X} = \frac{x^2}{2} + \frac{a^2}{2}\ln|X| + C$

233. $\displaystyle\int \frac{dx}{xX} = \frac{1}{2a^2}\ln|\frac{X}{x^2}| + C$

234. $\displaystyle\int \frac{dx}{x^2X} = \frac{1}{a^2x} + \frac{1}{2a^3}\ln|\frac{x-a}{x+a}| + C$

235. $\displaystyle\int \frac{dx}{x^3X} = \frac{1}{2a^2x} - \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

236. $\displaystyle\int \frac{dx}{X^2} = \frac{-x}{2a^2X} - \frac{1}{4a^3}\ln|\frac{x-a}{x+a}| + C$

237. $\displaystyle\int \frac{x\,dx}{X^2} = \frac{-1}{2X} + C$

238. $\displaystyle\int \frac{x^2\,dx}{X^2} = \frac{-x}{2X} + \frac{1}{4a}\ln|\frac{x-a}{x+a}| + C$

239. $\displaystyle\int \frac{x^3\,dx}{X^2} = \frac{-a^2}{2X} + \frac{1}{2}\ln|X| + C$

240. $\displaystyle\int \frac{dx}{xX^2} = \frac{-1}{2a^2X} + \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

241. $\displaystyle\int \frac{dx}{x^2X^2} = -\frac{1}{a^4x} - \frac{x}{2a^4X} - \frac{3}{4a^5}\ln|\frac{x-a}{x+a}| + C$

242. $\displaystyle\int \frac{dx}{x^3X^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4X} + \frac{1}{a^6}\ln|\frac{x^2}{X}| + C$

243. $\displaystyle\int \frac{dx}{X^n} = \frac{-x}{2(n-1)a^2X^{n-1}} - \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{X^{n-1}}, \qquad n>1$

244. $\displaystyle\int \frac{x\,dx}{X^n} = \frac{-1}{2(n-1)X^{n-1}} + C$

245. $\displaystyle\int \frac{dx}{xX^n} = \frac{-1}{2(n-1)a^2X^{n-1}} - \frac{1}{a^2}\int \frac{dx}{xX^{n-1}}$

Integrals containing the square root of $X = x^2 - a^2$ with $x^2 > a^2$

246. $\displaystyle\int \sqrt{X}\,dx = \frac{1}{2}x\sqrt{X} - \frac{a^2}{2}\ln|x+\sqrt{X}| + C$

247. $\displaystyle\int x\sqrt{X}\,dx = \frac{1}{3}X^{3/2} + C$

248. $\displaystyle\int x^2\sqrt{X}\,dx = \frac{1}{4}xX^{3/2} + \frac{1}{8}a^2x\sqrt{X} - \frac{a^4}{8}\ln|x+\sqrt{X}| + C$

249. $\displaystyle\int x^3\sqrt{X}\,dx = \frac{1}{5}X^{5/2} + \frac{1}{3}a^2X^{3/2} + C$

250. $\displaystyle\int \frac{X}{x}\,dx = \sqrt{X} - a\sec^{-1}|\frac{x}{a}| + C$

251. $\displaystyle\int \frac{X}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \ln|x + \sqrt{X}| + C$

252. $\displaystyle\int \frac{X}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\sec^{-1}|\frac{x}{a}| + C$

253. $\displaystyle\int \frac{dx}{\sqrt{X}} = \ln|x + \sqrt{X}| + C$

254. $\displaystyle\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X} + C$

255. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X}} = \frac{1}{2}x\sqrt{X} + \frac{a^2}{2}\ln|x + \sqrt{X}| + C$

256. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X}} = \frac{1}{3}X^{3/2} + a^2\sqrt{X} + C$

257. $\displaystyle\int \frac{dx}{x\sqrt{X}} = \frac{1}{a}\sec^{-1}|\frac{x}{a}| + C$

258. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = \frac{\sqrt{X}}{a^2x} + C$

259. $\displaystyle\int \frac{dx}{x^3\sqrt{X}} = \frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}|\frac{x}{a}| + C$

260. $\displaystyle\int X^{3/2}\,dx = \frac{x}{4}X^{3/2} - \frac{3}{8}a^2x\sqrt{X} + \frac{3}{8}a^4\ln|x + \sqrt{X}| + C$

261. $\displaystyle\int xX^{3/2}\,dx = \frac{1}{5}X^{5/2} + C$

262. $\displaystyle\int x^2X^{3/2}\,dx = \frac{1}{6}xX^{5/2} + \frac{1}{24}a^2xX^{3/2} - \frac{1}{16}a^4x\sqrt{X} + \frac{a^6}{16}\ln|x + \sqrt{X}| + C$

263. $\displaystyle\int x^3X^{3/2}\,dx = \frac{1}{7}X^{7/2} + \frac{1}{5}a^2X^{5/2} + C$

264. $\displaystyle\int \frac{X^{3/2}}{x}\,dx = \frac{1}{3}X^{3/2} - a^2\sqrt{X} + a^3\sec^{-1}|\frac{x}{a}| + C$

265. $\displaystyle\int \frac{X^{3/2}}{x^2}\,dx = -\frac{X^{3/2}}{x} + \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2\ln|x + \sqrt{X}| + C$

266. $\displaystyle\int \frac{X^{3/2}}{x^3}\,dx = -\frac{X^{3/2}}{2x^2} + \frac{3}{2}\sqrt{X} - \frac{3}{2}a\sec^{-1}|\frac{x}{a}| + C$

267. $\displaystyle\int \frac{dx}{X^{3/2}} = -\frac{x}{a^2\sqrt{X}} + C$

268. $\displaystyle\int \frac{x\,dx}{X^{3/2}} = \frac{-1}{\sqrt{X}} + C$

269. $\displaystyle\int \frac{x^2\,dx}{X^{3/2}} = -\frac{x}{\sqrt{X}} - \frac{a^2}{\sqrt{X}} + C$

270. $\displaystyle\int \frac{x^3\,dx}{X^{3/2}} = \sqrt{X} + \ln|x + \sqrt{X}| + C$

271. $\displaystyle\int \frac{dx}{xX^{3/2}} = \frac{-1}{a^2\sqrt{X}} - \frac{1}{a^3}\sec^{-1}|\frac{x}{a}| + C$

272. $\displaystyle\int \frac{dx}{x^2X^{3/2}} = -\frac{\sqrt{X}}{a^4x} - \frac{x}{a^4\sqrt{X}} + C$

273. $\displaystyle\int \frac{dx}{x^3X^{3/2}} = \frac{1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\sec^{-1}|\frac{x}{a}| + C$

Integrals containing $X = a^2 - x^2$ with $x^2 < a^2$

274. $\displaystyle\int \frac{dx}{X} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{or} \quad \frac{1}{a}\tanh^{-1}\frac{x}{a} + C$

275. $\displaystyle\int \frac{x\,dx}{X} = -\frac{1}{2}\ln X + C$

276. $\displaystyle\int \frac{x^2\,dx}{X} = -x + \frac{a}{2}\ln|\frac{a+x}{a-x}| + C$

277. $\displaystyle\int \frac{x^3\,dx}{X} = -\frac{1}{2}x^2 - \frac{a^2}{2}\ln|X| + C$

278. $\displaystyle\int \frac{d}{xX} = \frac{1}{2a^2}\ln|\frac{x^2}{X}| + C$

279. $\displaystyle\int \frac{dx}{x^2X} = -\frac{1}{a^2x} + \frac{1}{2a^3}\ln|\frac{a+x}{a-x}| + C$

280. $\displaystyle\int \frac{dx}{x^3X} = -\frac{1}{2a^2x^2} + \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

281. $\displaystyle\int \frac{dx}{X^2} = \frac{x}{2a^2X} + \frac{1}{4a^3}\ln|\frac{a+x}{a-x}| + C$

282. $\displaystyle\int \frac{x\,dx}{X^2} = \frac{1}{2X} + C$

283. $\displaystyle\int \frac{x^2\,dx}{X^2} = \frac{x}{2X} - \frac{1}{4a}\ln|\frac{a+x}{a-x}| + C$

284. $$\int \frac{x^3\,dx}{X^2} = \frac{a^2}{2X} + \frac{1}{2}\ln|X| + C$$

285. $$\int \frac{dx}{xX^2} = \frac{1}{2a^2X} + \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$$

286. $$\int \frac{dx}{x^2X^2} = -\frac{1}{a^4x} + \frac{x}{2a^4X} + \frac{3}{4a^5}\ln|\frac{a+x}{a-x}| + C$$

287. $$\int \frac{dx}{x^3X^2} = -\frac{1}{2a^4x^2} + \frac{1}{2a^4X} + \frac{1}{a^6}\ln|\frac{x^2}{X}| + C$$

288. $$\int \frac{dx}{X^n} = \frac{x}{2(n-1)a^2X^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{X^{n-1}}$$

289. $$\int \frac{x\,dx}{X^n} = \frac{1}{2(n-1)X^{n-1}} + C$$

290. $$\int \frac{dx}{xX^n} = \frac{1}{2(n-1)a^2X^{n-1}} + \frac{1}{a^2}\int \frac{dx}{xX^{n-1}}$$

Integrals containing the square root of $X = a^2 - x^2$ with $x^2 < a^2$

291. $$\int \sqrt{X}\,dx = \frac{1}{2}x\sqrt{X} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

292. $$\int x\sqrt{X}\,dx = -\frac{1}{3}X^{3/2} + C$$

293. $$\int x^2\sqrt{X}\,dx = -\frac{1}{4}xX^{3/2} + \frac{1}{8}a^2x\sqrt{X} + \frac{1}{8}a^4\sin^{-1}\frac{x}{a} + C$$

294. $$\int x^3\sqrt{X}\,dx = \frac{1}{5}X^{5/2} - \frac{1}{3}a^2X^{3/2} + C$$

295. $$\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\ln|\frac{a+\sqrt{X}}{x}| + C$$

296. $$\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} - \sin^{-1}\frac{x}{a} + C$$

297. $$\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\ln|\frac{a+\sqrt{X}}{x}| + C$$

298. $$\int \frac{dx}{\sqrt{X}} = \sin^{-1}\frac{x}{a} + C$$

299. $$\int \frac{x\,dx}{\sqrt{X}} = -\sqrt{X} + C$$

300. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X}} = -\frac{1}{2}x\sqrt{X} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

301. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X}} = \frac{1}{3}X^{3/2} - a^2\sqrt{X} + C$

302. $\displaystyle\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\ln|\frac{a+\sqrt{X}}{x}| + C$

303. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x} + C$

304. $\displaystyle\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} - \frac{1}{2a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$

305. $\displaystyle\int X^{3/2}\,dx = \frac{1}{4}xX^{3/2} + \frac{3}{8}a^2x\sqrt{X} + \frac{3}{8}a^4\sin^{-1}\frac{x}{a} + C$

306. $\displaystyle\int xX^{3/2}\,dx = -\frac{1}{5}X^{5/2} + C$

307. $\displaystyle\int x^2X^{3/2}\,dx = -\frac{1}{6}xX^{5/2} + \frac{1}{24}a^2xX^{3/2} + \frac{1}{16}a^4x\sqrt{X} + \frac{a^6}{16}\sin^{-1}\frac{x}{a} + C$

308. $\displaystyle\int x^3X^{3/2}\,dx = \frac{1}{7}X^{7/2} - \frac{1}{5}a^2X^{5/2} + C$

309. $\displaystyle\int \frac{X^{3/2}}{x}\,dx = \frac{1}{3}X^{3/2}a^2\sqrt{X} - a^3\ln|\frac{a+\sqrt{X}}{x}| + C$

310. $\displaystyle\int \frac{X^{3/2}}{x^2}\,dx = -\frac{X^{3/2}}{x} - \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2\sin^{-1}\frac{x}{a} + C$

311. $\displaystyle\int \frac{X^{3/2}}{x^3}\,dx = -\frac{X^{3/2}}{2x^2} - \frac{3}{2}\sqrt{X} + \frac{3}{2}a\ln|\frac{a+\sqrt{X}}{x}| + C$

312. $\displaystyle\int \frac{dx}{X^{3/2}} = \frac{x}{a^2\sqrt{X}} + C$

313. $\displaystyle\int \frac{x\,dx}{X^{3/2}} = \frac{1}{\sqrt{X}} + C$

314. $\displaystyle\int \frac{x^2\,dx}{X^{3/2}} = \frac{x}{\sqrt{X}} - \sin^{-1}\frac{x}{a} + C$

315. $\displaystyle\int \frac{x^3\,dx}{X^{3/2}} = \sqrt{X} + \frac{a^2}{\sqrt{X}} + C$

316. $\displaystyle\int \frac{dx}{xX^{3/2}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$

317. $\displaystyle\int \frac{dx}{x^2X^{3/2}} = -\frac{\sqrt{X}}{a^4x} + \frac{x}{a^4\sqrt{X}} + C$

318. $\displaystyle\int \frac{dx}{x^3X^{3/2}} = -\frac{1}{2a^2x^2\sqrt{X}} + \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\ln|\frac{a+\sqrt{X}}{x}| + C$

Integrals Containing $X = x^3 + a^3$

319. $\displaystyle\int \frac{dx}{X} = \frac{1}{6a^2}\ln|\frac{(x+a)^3}{X}| + \frac{1}{\sqrt{3}a^2}\tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

320. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{6a}\ln|\frac{X}{(x+a)^3}| + \frac{1}{\sqrt{3}a}\tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

321. $\displaystyle\int \frac{x^2\,dx}{X} = \frac{1}{2}\ln|X| + C$

322. $\displaystyle\int \frac{dx}{xX} = \frac{1}{3a^3}\ln|\frac{x^3}{X}| + C$

323. $\displaystyle\int \frac{dx}{x^2X} = -\frac{1}{a^2x} - \frac{1}{6a^4}\ln|\frac{X}{(x+a)^3}| - \frac{1}{\sqrt{3}a^4}\tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

324. $\displaystyle\int \frac{dx}{X^2} = \frac{x}{3a^3X} + \frac{1}{9a^5}\ln|\frac{(x+a)^3}{X}| + \frac{2}{3\sqrt{3}a^5}\tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

325. $\displaystyle\int \frac{x\,dx}{X^2} = \frac{x^2}{3a^3X} + \frac{1}{18a^4}\ln|\frac{X}{(x+a)^3}| + \frac{1}{3\sqrt{3}a^4}\tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

326. $\displaystyle\int \frac{x^2\,dx}{X^2} = -\frac{1}{3X} + C$

327. $\displaystyle\int \frac{dx}{xX^2} = \frac{1}{3a^2X} + \frac{1}{3a^6}\ln|\frac{x^3}{X}| + C$

328. $\displaystyle\int \frac{dx}{x^2X^2} = -\frac{1}{a^6x} - \frac{x^2}{3a^6X} - \frac{4}{3a^6}\int \frac{x\,dx}{X}$

329. $\displaystyle\int \frac{dx}{X^3} = \frac{1}{54a^3}\left[\frac{9a^5x}{X^2} + \frac{15a^2x}{X} + 10\sqrt{3}\tan^{-1}(\frac{2x-a}{\sqrt{3}a}) + 10\ln|x+a| - 5\ln|x^2-ax+a^2|\right] + C$

Integrals containing $X = x^4 + a^4$

330. $\displaystyle\int \frac{dx}{X} = \frac{1}{4\sqrt{2}a^3}\ln|\frac{X}{(x^2-\sqrt{2}ax+a^2)^2}| - \frac{1}{2\sqrt{2}a^3}\tan^{-1}\left(\frac{\sqrt{2}ax}{x^2-a^2}\right) + C$

331. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{2a^2}\tan^{-1}\left(\frac{x^2}{a^2}\right) + C$

332. $\displaystyle\int \frac{x^2\,dx}{X} = \frac{1}{4\sqrt{2}a}\ln|\frac{X}{(x^2+\sqrt{2}ax+a^2)^2}| - \frac{1}{2\sqrt{2}a}\tan^{-1}\left(\frac{\sqrt{2}ax}{x^2-a^2}\right) + C$

333. $\displaystyle\int \frac{x^3\,dx}{X} = \frac{1}{4}\ln|X| + C$

334. $\displaystyle\int \frac{dx}{xX} = \frac{1}{4a^4}\ln|\frac{x^4}{X}| + C$

335. $\displaystyle\int \frac{dx}{x^2X} = -\frac{1}{a^4x} - \frac{1}{\sqrt{2}4a^5}\ln|\frac{(x^2-\sqrt{2}ax+a^2)^2}{X}| + \frac{1}{2\sqrt{2}a^5}\tan^{-1}\left(\frac{\sqrt{2}ax}{x^2-a^2}\right) + C$

336. $\displaystyle\int \frac{dx}{x^3X} = -\frac{1}{2a^4x^2} - \frac{1}{2a^6}\tan^{-1}\left(\frac{x^2}{a^2}\right) + C$

Integrals containing $X = x^4 - a^4$

337. $$\int \frac{dx}{X} = \frac{1}{4a^3}\ln|\frac{x-a}{x+a}| - \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + C$$

338. $$\int \frac{x\,dx}{X} = \frac{1}{4a^2}\ln|\frac{x^2-a^2}{x^2+a^2}| + C$$

339. $$\int \frac{x^2\,dx}{X} = \frac{1}{4a}\ln|\frac{x-a}{x+a}| + \frac{1}{2a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

340. $$\int \frac{x^3\,dx}{X} = \frac{1}{4}\ln|X| + C$$

341. $$\int \frac{dx}{xX} = \frac{1}{4a^4}\ln|\frac{X}{x^4}| + C$$

342. $$\int \frac{dx}{x^2X} = \frac{1}{a^4x} + \frac{1}{4a^5}\ln|\frac{x-a}{x+a}| + \frac{1}{2a^5}\tan^{-1}\left(\frac{x}{a}\right) + C$$

343. $$\int \frac{dx}{x^3X} = \frac{1}{2a^4x^2} + \frac{1}{4a^6}\ln|\frac{x^2-a^2}{x^2+a^2}| + C$$

Miscelaneous algebraic integrals

344. $$\int \frac{dx}{b^2+(x+a)^2} = \frac{1}{b}\tan^{-1}\frac{x+a}{b} + C$$

345. $$\int \frac{dx}{b^2-(x+a)^2} = \frac{1}{b}\tanh^{-1}\frac{x+a}{b} + C$$

346. $$\int \frac{dx}{(x+a)^2-b^2} = -\frac{1}{b}\coth^{-1}\frac{x+a}{b} + C$$

347. $$\int \frac{dx}{\sqrt{x(a-x)}} = 2\sin^{-1}\sqrt{\frac{x}{a}} + C$$

348. $$\int \frac{dx}{\sqrt{x(a+x)}} = 2\sinh^{-1}\sqrt{\frac{x}{a}} + C$$

349. $$\int \frac{dx}{\sqrt{x(x-a)}} = 2\cosh^{-1}\sqrt{\frac{x}{a}} + C$$

350. $$\int \frac{dx}{(b+x)(a-x)} = 2\tan^{-1}\sqrt{\frac{b+x}{a-x}} + C, \qquad a > x$$

351. $$\int \frac{dx}{(x-b)(a-x)} = 2\tan^{-1}\sqrt{\frac{x-b}{a-x}} + C, \qquad a > x > b$$

352. $$\int \frac{dx}{(x+b)(x+a)} = \begin{cases} 2\tanh^{-1}\sqrt{\frac{x+b}{x+a}} + C_1, & a > b \\ 2\tanh^{-1}\sqrt{\frac{x+a}{x+b}} + C_2, & a < b \end{cases}$$

353. $\displaystyle\int \frac{dx}{x\sqrt{x^{2n}-a^{2n}}} = -\frac{1}{na^n}\sin^{-1}\left(\frac{a^n}{x^n}\right) + C$

354. $\displaystyle\int \sqrt{\frac{x+a}{x-a}}\,dx = \sqrt{x^2-a^2} + a\cosh^{-1}\frac{x}{a} + C$

355. $\displaystyle\int \sqrt{\frac{a+x}{a-x}}\,dx = a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + C$

356. $\displaystyle\int x\sqrt{\frac{a-x}{a+x}}\,dx = \frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right) + \frac{(x-2a)}{2}\sqrt{a^2-x^2} + C, \quad a > x$

357. $\displaystyle\int x\sqrt{\frac{a+x}{a-x}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x+2a}{2}\sqrt{a^2-x^2} + C$

358. $\displaystyle\int (x+a)\sqrt{\frac{x+b}{x-b}}\,dx = (x+a+b)\sqrt{x^2-b^2} + \frac{b}{2}(2a+b)\cosh^{-1}\frac{x}{b} + C$

359. $\displaystyle\int \frac{dx}{\sqrt{2ax+x^2}} = \ln|x+a+\sqrt{2ax+x^2}| + C$

360. $\displaystyle\int \sqrt{ax^2+c}\,dx = \begin{cases} \dfrac{1}{2}x\sqrt{ax^2+c} + \dfrac{c}{2\sqrt{a}}\ln|\sqrt{a}x + \sqrt{ax^2+c}| + c, & a > 0 \\ \frac{1}{2}x\sqrt{ax^2+c} + \frac{c}{2\sqrt{-a}}\sin^{-1}\left(\sqrt{\frac{-a}{c}}\,x\right) + C, & a < 0 \end{cases}$

361. $\displaystyle\int \sqrt{\frac{1+ax}{1-ax}}\,dx = \frac{1}{a}\sin^{-1}x - \frac{1}{a}\sqrt{1-x^2} + C$

362. $\displaystyle\int \frac{dx}{(ax+b)^2+(cx+d)^2} = \frac{1}{ad-bc}\tan^{-1}\left[\frac{(a^2+c^2)x+(ab+cd)}{ad-bc}\right] + C, \quad ad-bc \neq 0$

363. $\displaystyle\int \frac{dx}{(ax+b)^2-(cx+d)^2} = \frac{1}{2(bc-ad)}\ln\left|\frac{(a+c)x+(b+d)}{(a-c)x+(b-d)}\right| + C, \quad ad-bc \neq 0$

364. $\displaystyle\int \frac{x\,dx}{(ax^2+b)^2+(cx^2+d)^2} = \frac{1}{2(ad-bc)}\tan^{-1}\left[\frac{(a^2+c^2)x^2+(ab+cd)}{ad-bc}\right] + C, \quad ad-bc \neq 0$

365. $\displaystyle\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{1}{b}\tan^{-1}\frac{x}{b}\right) + C$

366. $\displaystyle\int \frac{(x^2+a^2)(x^2+b^2)}{(x^2+c^2)(x^2+d^2)}\,dx = x + \frac{1}{d^2-c^2}\left[\frac{(a^2-c^2)(b^2-c^2)}{c}\tan^{-1}\frac{x}{c} - \frac{(a^2-d^2)(b^2-d^2)}{d}\tan^{-1}\frac{x}{d}\right] + C$

367. $\displaystyle\int \frac{ax^2+b}{(cx^2+d)(ex^2+f)}\,dx = \frac{1}{\sqrt{cd}}\left(\frac{ad-bc}{ed-fc}\right)\tan^{-1}\left(\sqrt{\frac{c}{d}}\,x\right) + \frac{1}{\sqrt{ef}}\left(\frac{af-be}{fc-ed}\right)\tan^{-1}\left(\sqrt{\frac{e}{f}}\,x\right) + C$

368. $\displaystyle\int \frac{x\,dx}{(ax^2+bx+c)^2+(ax^2-bx+c)^2} = \frac{1}{4b\sqrt{b^2+4ac}}\ln\left|\frac{2a^2x^2+2ac+b^2-b\sqrt{b^2+4ac}}{2a^2x^2+2ac+b^2+b\sqrt{b^2+4ac}}\right| + C, \quad b^2+4ac > 0$

369. $$\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{1}{b}\tan^{-1}\frac{x}{b}\right) + C$$

370. $$\int \frac{(x^2+\alpha^2)(x^2+\beta^2)}{(x^2+\gamma^2)(x^2+\delta^2)}\,dx = x + \frac{1}{\delta^2-\gamma^2}\left[\frac{(\alpha^2-\gamma^2)(\beta^2-\gamma^2)}{\gamma}\tan^{-1}\frac{x}{\gamma} - \frac{(\alpha^2-\delta^2)(\beta^2-\delta^2)}{\delta}\tan^{-1}\frac{x}{\delta}\right] + C$$

371. $$\int \frac{\alpha x^2+\beta}{(\gamma x^2+\delta)(\epsilon x^2+\zeta)}\,dx = \frac{1}{\sqrt{\gamma\delta}}\frac{\alpha\delta-\beta\gamma}{\epsilon\delta-\zeta\gamma}\tan^{-1}\left(\sqrt{\frac{\gamma}{\delta}}\,x\right) + \frac{1}{\sqrt{\epsilon\zeta}}\frac{\alpha\zeta-\beta\epsilon}{\zeta\gamma-\epsilon\delta}\tan^{-1}\left(\sqrt{\frac{\epsilon}{\zeta}}\,x\right) + C$$

372. $$\int \frac{dx}{\sqrt{(x+a)(x+b)}} = \cosh^{-1}\left(\frac{2x+a+b}{a-b}\right) + C, \qquad a \neq b$$

373. $$\int \frac{dx}{\sqrt{(x-b)(a-x)}} = 2\tan^{-1}\sqrt{\frac{x-b}{a-x}} + C$$

374. $$\int \frac{dx}{(\alpha x+\beta)^2+(\gamma x+\delta)^2} = \frac{1}{\alpha\delta-\beta\gamma}\tan^{-1}\left[\frac{(\alpha^2+\gamma^2)x+(\alpha\beta+\gamma\delta)}{\alpha\delta-\beta\gamma}\right] + C$$

375. $$\int \frac{x\,dx}{(a^2+b^2-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{2ab}\sin^{-1}\left[\frac{(a^2+b^2)x^2-(a^4+b^4)}{(a^2-b^2)(a^2+b^2-x^2)}\right] + C$$

376. $$\int \frac{(x+b)\,dx}{(x^2+a^2)\sqrt{x^2+c^2}} = \frac{1}{\sqrt{a^2-c^2}}\sin^{-1}\sqrt{\frac{x^2+c^2}{x^2+a^2}} + \frac{b}{a\sqrt{a^2-c^2}}\cosh^{-1}\left[\frac{a}{c}\sqrt{\frac{x^2+c^2}{x^2+a^2}}\right] + C$$

377. $$\int \frac{px+q}{ax^2+bx+c}\,dx = \frac{p}{2a}\ln|ax^2+bx+c| + \left(q-\frac{pb}{2a}\right)\int\frac{dx}{ax^2+bx+c}$$

378. $$\int \frac{(\sqrt{a}-\sqrt{x})^2}{(a^2+ax+x^2)\sqrt{x}}\,dx = \frac{2\sqrt{3}}{\sqrt{a}}\tan^{-1}\frac{2\sqrt{x}+\sqrt{a}}{\sqrt{3a}} - \frac{2}{\sqrt{3a}}\tan^{-1}\frac{2\sqrt{x}-\sqrt{a}}{\sqrt{3a}} + C$$

379. $$\int (a+x)\sqrt{a^2+x^2}\,dx = \frac{1}{6}(2x^2+3ax+2a^2)\sqrt{a^2+x^2} + \frac{1}{2}a^2\sinh^{-1}\frac{x}{a} + C$$

380. $$\int \frac{x^2+a^2}{x^4+a^2x^2+a^4}\,dx = \frac{1}{a\sqrt{3}}\tan^{-1}\frac{ax\sqrt{3}}{a^2-x^2} + C$$

381. $$\int \frac{x^2-a^2}{x^4+a^2x^2+a^4}\,dx = \frac{1}{2a^3}\ln\frac{x^2-ax+a^2}{x^2+ax+a^2} + C$$

Integrals containing $\sin ax$

382. $$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

383. $$\int x\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$$

384. $$\int x^2\sin ax\,dx = \frac{2}{a^2}x\sin ax + \left(\frac{2}{a^3}-\frac{x^2}{a}\right)\cos ax + C$$

385. $$\int x^3 \sin ax\, dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax + \left(\frac{6x}{a} - \frac{x^3}{a}\right)\cos ax + C$$

386. $$\int x^n \sin ax\, dx = -\frac{1}{a}x^n \cos ax + \frac{n}{a^2}x^{n-1}\sin ax - \frac{n(n-1)}{a^2}\int x^{n-2}\sin ax\, dx$$

387. $$\int \frac{\sin ax}{x}\, dx = ax - \frac{a^3x^3}{3\cdot 3!} + \frac{a^5x^5}{5\cdot 5!} - \frac{a^7x^7}{7\cdot 7!} + \cdots + \frac{(-1)^n x^{2n+1}x^{2n+1}}{(2n+1)\cdot(2n+1)!} + \cdots$$

388. $$\int \frac{\sin ax}{x^2}\, dx = -\frac{1}{a}\sin ax + a\int \frac{\cos ax}{x}\, dx$$

389. $$\int \frac{\sin ax}{x^3}\, dx = -\frac{a}{2x}\cos ax - \frac{1}{2x^2}\sin ax - \frac{a^2}{2}\int \frac{\sin ax}{x}\, dx$$

390. $$\int \frac{\sin ax}{x^n}\, dx = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}\, dx$$

391. $$\int \frac{dx}{\sin ax} = \frac{1}{a}\ln|\csc as - \cot ax| + C$$

392. $$\int \frac{x\, dx}{\sin ax} = \frac{1}{a^2}\left[ax + \frac{a^3x^3}{18} + \frac{7a^5x^5}{1800} + \cdots + \frac{2(2^{2n-1}-1)\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!} + \cdots\right] + C$$

where $\mathfrak{B}_n$ is the nth Bernoulli number $\mathfrak{B}_1 = 1/6, \mathfrak{B}_2 = 1/30, \ldots$ Note scaling and shifting

393. $$\int \frac{dx}{x\sin ax} = -\frac{1}{ax} + \frac{ax}{6} + \frac{7a^3x^3}{1080} + \cdots + \frac{2(2^{2n-1}-1)\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n-1)(2n)!} + \cdots + C$$

394. $$\int \sin^2 ax\, dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$$

395. $$\int x\sin^2 ax\, dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2} + C$$

396. $$\int x^2\sin^2 ax\, dx = \frac{1}{6a} - \frac{1}{4a^2}\cos 2ax + \frac{1}{24a^3}(3 - 6a^2x^2)\sin 2ax + C$$

397. $$\int \sin^3 ax\, dx = -\frac{\cos ax}{a} + \frac{\cos^2 ax}{3a} + C$$

398. $$\int x\sin^3 ax\, dx = \frac{1}{12a}x\cos 3ax - \frac{1}{36a^2}\sin 3ax - \frac{3}{4a}x\cos ax + \frac{3}{4a^2}\sin ax + C$$

399. $$\int \sin^4 ax\, dx = \frac{3}{8}x - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a} + C$$

400. $$\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax + C$$

401. $$\int \frac{x\,dx}{\sin^2 ax} = -\frac{x}{a}\cot ax + \frac{1}{a^2}\ln|\sin ax| + C$$

402. $$\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a\sin^2 ax} + \frac{1}{2a}\ln|\tan\frac{ax}{2}| + C$$

403. $$\int \frac{dx}{\sin^n ax} = \frac{-\cos ax}{(n-1)a\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}$$

404. $$\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$$

405. $$\int \frac{dx}{a-\sin ax} = \frac{2}{a\sqrt{a^2-1}}\tan^{-1}\left[\frac{a\tan(ax/2)-1}{\sqrt{a^2-1}}\right] + C, \quad a > 1$$

406. $$\int \frac{x\,dx}{1-\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln|\sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)| + C$$

407. $$\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$$

408. $$\int \frac{dx}{a+\sin ax} = \frac{2}{a\sqrt{a^2-1}}\tan^{-1}\left[\frac{1+a\tan(ax/2)}{\sqrt{a^2-1}}\right] + C, \quad a > 1$$

409. $$\int \frac{x\,dx}{1+\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln|\sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)| + C$$

410. $$\int \frac{dx}{1+\sin^2 x} = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + C$$

411. $$\int \frac{dx}{1-\sin^2 x} = \tan x + C$$

412. $$\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$$

413. $$\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$$

414. $$\int \frac{dx}{\alpha+\beta\sin ax} = \begin{cases} \dfrac{2}{a\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\alpha\tan\dfrac{ax}{2}+\beta\right) + C, & \alpha^2 > \beta^2 \\ \dfrac{1}{a\sqrt{\beta^2-\alpha^2}}\ln\left|\dfrac{\alpha\tan\frac{ax}{2}+\beta-\sqrt{\beta^2-\alpha^2}}{\alpha\tan\frac{ax}{2}+\beta+\sqrt{\beta^2-\alpha^2}}\right| + C, & \alpha^2 < \beta^2 \\ \dfrac{1}{a\alpha}\tan\left(\dfrac{ax}{2} \pm \dfrac{\pi}{4}\right) + C, & \beta = \pm\alpha \end{cases}$$

415. $$\int \frac{dx}{\alpha^2+\beta^2\sin^2 ax} = \frac{1}{a\alpha\sqrt{\beta^2+\alpha^2}}\tan^{-1}\left(\frac{\sqrt{\beta^2+\alpha^2}}{\alpha}\tan ax\right) + C$$

416. $$\int \frac{dx}{\alpha^2-\beta^2\sin^2 ax} = \begin{cases} \dfrac{1}{a\alpha\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\dfrac{\sqrt{\alpha^2-\beta^2}}{\alpha}\tan ax\right)+C, & \alpha^2>\beta^2 \\ \dfrac{1}{2a\alpha\sqrt{\beta^2-\alpha^2}}\ln\left|\dfrac{\sqrt{\beta^2-\alpha^2}\tan ax+\alpha}{\sqrt{\beta^2-\alpha^2}\tan ax-\alpha}\right|+C, & \alpha^2<\beta^2 \end{cases}$$

417. $$\int \sin^n ax\,dx = -\frac{1}{an}\sin^{n-1} ax\cos ax + \frac{n-1}{n}\int \sin^{n-2} ax\,dx$$

418. $$\int \frac{dx}{\sin^n ax} = \frac{-\cos ax}{(n-1)a\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}$$

419. $$\int x^n \sin ax\,dx = -\frac{1}{a}x^n\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx$$

420. $$\int \frac{\alpha+\beta\sin ax}{1\pm\sin ax}\,dx = \beta x + \frac{\alpha\mp\beta}{a}\tan\left(\frac{\pi}{4}\mp\frac{ax}{2}\right)+C$$

421. $$\int \frac{\alpha+\beta\sin ax}{a+b\sin ax}\,dx = \frac{\beta}{b}x + \frac{\alpha b-a\beta}{b}\int \frac{dx}{a+b\sin ax}$$

422. $$\int \frac{dx}{\alpha+\frac{\beta}{\sin ax}} = \frac{x}{\alpha} - \frac{\beta}{\alpha}\int \frac{dx}{\beta+\alpha\sin ax}$$

Integrals containing cos ax

423. $$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

424. $$\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$$

425. $$\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a}-\frac{2}{a^3}\right)\sin ax + C$$

426. $$\int x^n\cos ax\,dx = \frac{1}{a}x^n\sin ax + \frac{n}{a^2}x^{n-1}\cos ax - \frac{n(n-1)}{a^2}\int x^{n-2}\cos ax\,dx$$

427. $$\int \frac{\cos ax}{x}\,dx = \ln|x| - \frac{a^2x^2}{2\cdot 2!} + \frac{a^4x^4}{4\cdot 4!} - \frac{a^6x^6}{6\cdot 6!} + \cdots + \frac{(-1)^n a^{2n}x^{2n}}{(2n)\cdot(2n)!} + \cdots + C$$

428. $$\int \frac{\cos ax\,dx}{x^n} = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int \frac{\sin ax}{x^{n-1}}\,dx$$

429. $$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln|\sec ax+\tan ax| + C$$

430. $$\int \frac{x\,dx}{\cos ax} = \frac{1}{a^2}\left[\frac{a^2x^2}{2} + \frac{a^4x^4}{4\cdot 2!} + \frac{5a^6x^6}{6\cdot 4!} + \cdots + \frac{\mathfrak{E}_n a^{2n+2}x^{2n+2}}{(2n+2)\cdot(2n)!} + \cdots\right] + C$$

431. $$\int \frac{dx}{x\cos ax} = \ln|x| + \frac{a^2x^2}{4} + \frac{5a^4x^4}{96} + \cdots + \frac{\mathfrak{E}_n a^{2n}x^{2n}}{2n(2n)!} + \cdots + C$$

where $\mathfrak{E}_n$ is the nth Euler number $\mathfrak{E}_1=1, \mathfrak{E}_2=5, \mathfrak{E}_3=61, \ldots$ Note scaling and shifting

432. $\displaystyle\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$

433. $\displaystyle\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$

434. $\displaystyle\int \sqrt{1-\cos ax}\,dx = -2\sqrt{2}\cos\frac{ax}{2} + C$

435. $\displaystyle\int \sqrt{1+\cos ax}\,dx = 2\sqrt{2}\sin\frac{ax}{2} + C$

436. $\displaystyle\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$

437. $\displaystyle\int x\cos^2 ax\,dx = \frac{x^2}{4} + \frac{1}{4a}x\sin 2ax + \frac{1}{8a^2}\cos 2ax + C$

438. $\displaystyle\int \cos^3 ax\,dx = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a} + C$

439. $\displaystyle\int \cos^4 ax\,dx = \frac{3}{8}x + \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax + C$

440. $\displaystyle\int \frac{dx}{\cos^2 ax} = \frac{1}{a}\tan ax + C$

441. $\displaystyle\int \frac{x\,dx}{\cos^2 ax} = \frac{x}{a}\tan ax + \frac{1}{a^2}\ln|\cos ax| + C$

442. $\displaystyle\int \frac{dx}{\cos^3 ax} = \frac{1}{2a}\frac{\sin ax}{\cos^2 ax} + \frac{1}{2a}\ln|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)| + C$

443. $\displaystyle\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$

444. $\displaystyle\int \frac{x\,dx}{1-\cos ax} = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\ln|\sin\frac{ax}{2}| + C$

445. $\displaystyle\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$

446. $\displaystyle\int \frac{x\,dx}{1+\cos ax} = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\ln|\cos\frac{ax}{2}| + C$

447. $\displaystyle\int \frac{dx}{1+\cos^2 ax} = -\frac{1}{\sqrt{2}a}\tan^{-1}(\sqrt{2}\cot ax) + C$

448. $\displaystyle\int \frac{dx}{1-\cos^2 ax} = -\frac{1}{a}\cot ax + C$

449. $$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2} + C$$

450. $$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^2\frac{ax}{2} + C$$

451. $$\int \frac{dx}{\alpha+\beta\cos ax} = \begin{cases} \dfrac{2}{a\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\sqrt{\dfrac{\alpha-\beta}{\alpha+\beta}}\tan\dfrac{ax}{2}\right) + C, & \alpha^2 > \beta^2 \\ \dfrac{1}{a\sqrt{\beta^2-\alpha^2}}\ln\left|\dfrac{\sqrt{\beta+\alpha}+\sqrt{\beta-\alpha}\tan\frac{ax}{2}}{\sqrt{\beta+\alpha}-\sqrt{\beta-\alpha}\tan\frac{ax}{2}}\right| + C, & \alpha^2 < \beta^2 \end{cases}$$

452. $$\int \frac{dx}{\alpha+\frac{\beta}{\cos ax}} = \frac{x}{\alpha} - \frac{\beta}{\alpha}\int\frac{dx}{\beta+\alpha\cos ax}$$

453. $$\int \frac{dx}{(\alpha+\beta\cos ax)^2} = \frac{\alpha\sin ax}{a(\beta^2-\alpha^2)(\alpha+\beta\cos ax)} - \frac{\alpha}{\beta^2-\alpha^2}\int\frac{dx}{\alpha+\beta\cos ax}, \qquad \alpha \neq \beta$$

454. $$\int \frac{dx}{\alpha^2+\beta^2\cos^2 ax} = \frac{1}{a\alpha\sqrt{\alpha^2+\beta^2}}\tan^{-1}\left(\frac{\alpha\tan ax}{\sqrt{\alpha^2+\beta^2}}\right) + C$$

455. $$\int \frac{dx}{\alpha^2-\beta^2\cos^2 ax} = \begin{cases} \dfrac{1}{a\alpha\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\dfrac{\alpha\tan ax}{\sqrt{\alpha^2-\beta^2}}\right) + C, & \alpha^2 > \beta^2 \\ \dfrac{1}{2a\alpha\sqrt{\beta^2-\alpha^2}}\ln\left|\dfrac{\alpha\tan ax-\sqrt{\beta^2-\alpha^2}}{\alpha\tan ax+\sqrt{\beta^2-\alpha^2}}\right| + C, & \alpha^2 < \beta^2 \end{cases}$$

456. $$\int \frac{dx}{\cos^n ax} = \frac{\sec^{(n-2)} ax\tan ax}{(n-1)a} + \frac{n-2}{n-1}\int\sec^{n-2} ax\,dx + C$$

Integrals containing both sine and cosine functions

457. $$\int \sin ax\cos ax\,dx = \frac{1}{2a}\sin^2 ax + C$$

458. $$\int \frac{dx}{\sin ax\cos ax} = -\frac{1}{a}\ln|\cot ax| + C$$

459. $$\int \sin ax\cos bx\,dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} + C, \qquad a \neq b$$

460. $$\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C$$

461. $$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C$$

462. $$\int \sin^n ax\cos ax\,dx = \frac{sin^{n+1}ax}{(n+1)a} + C$$

463. $$\int \cos^n ax\sin ax\,dx = -\frac{\cos^{n+1} ax}{(n+1)a} + C$$

464. $\displaystyle\int \frac{\sin ax\, dx}{\cos ax} = \frac{1}{a}\ln|\sec ax| + C$

465. $\displaystyle\int \frac{\cos ax\, dx}{\sin ax} = \frac{1}{a}\ln|\sin ax| + C$

466. $\displaystyle\int \frac{x\sin ax\, dx}{\cos ax} = \frac{1}{a^2}\left[\frac{a^3x^3}{3} + \frac{a^5x^5}{5} + \frac{2a^7x^7}{105} + \cdots + \frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!}\right] + C$

467. $\displaystyle\int \frac{x\cos ax\, dx}{\sin ax} = \frac{1}{a^2}\left[ax - \frac{a^3x^3}{9} - \frac{a^5x^5}{225} - \cdots - \frac{2^{2n}\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!} - \cdots\right] + C$

468. $\displaystyle\int \frac{\cos ax\, dx}{x\sin ax} = -\frac{1}{ax} - \frac{ax}{2} - \frac{a^3x^3}{135} - \cdots - \frac{2^{2n}\mathfrak{B}_n a^{2n-1}x^{2n-1}}{(2n-1)(2n)!} - \cdots + C$

469. $\displaystyle\int \frac{\sin ax}{x\cos ax}\, dx = ax + \frac{a^3x^3}{9} + \frac{2a^5x^5}{75} + \cdots + \frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n-1}x^{2n-1}}{(2n-1)(2n)!} + \cdots + C$

470. $\displaystyle\int \frac{\sin^2 ax}{\cos^2 ax}\, dx = \frac{1}{a}\tan ax - x + C$

471. $\displaystyle\int \frac{\cos^2 ax}{\sin^2 ax}\, dx = -\frac{1}{a}\cot ax - x + C$

472. $\displaystyle\int \frac{x\sin^2 ax}{\cos^2 ax}\, dx = \frac{1}{a}x\tan ax + \frac{1}{a^2}\ln|\cos ax| - \frac{1}{2}x^2 + C$

473. $\displaystyle\int \frac{x\cos^2 ax}{\sin^2 ax}\, dx = -\frac{1}{a}x\cot ax + \frac{1}{a^2}\ln|\sin ax| - \frac{1}{2}x^2 + C$

474. $\displaystyle\int \frac{\cos ax}{\sin ax}\, dx = \frac{1}{a}\ln|\sin ax| + C$

475. $\displaystyle\int \frac{\sin^3 ax}{\cos^3 ax}\, dx = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\ln|\cos ax| + C$

476. $\displaystyle\int \frac{\cos^3 ax}{\sin^3 ax}\, dx = -\frac{1}{2a}\cot^2 ax - \frac{1}{a}\ln|\sin ax| + C$

477. $\displaystyle\int \sin(ax+b)\sin(ax+\beta)\, dx = \frac{x}{2}\cos(b-\beta) - \frac{1}{4a}\sin(2ax+b+\beta) + C$

478. $\displaystyle\int \sin(ax+b)\cos(ax+\beta)\, dx = \frac{x}{2}\sin(b-\beta) - \frac{1}{4a}\cos(2ax+b+\beta) + C$

479. $\displaystyle\int \cos(ax+b)\cos(ax+\beta)\, dx = \frac{x}{2}\cos(b-\beta) + \frac{1}{4a}\sin(2ax+b+\beta) + C$

480. $\displaystyle\int \sin^2 ax\cos^2 bx\, dx = \begin{cases} \dfrac{x}{4} - \dfrac{\sin 2ax}{8a} + \dfrac{\sin 2bx}{8b} - \dfrac{\sin 2(a-b)x}{16(a-b)} - \dfrac{\sin 2(a+b)x}{16(a+b)} + C, & b \neq a \\ \dfrac{x}{8} - \dfrac{\sin 4ax}{32a} + C, & b = a \end{cases}$

481. $\displaystyle\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a}\ln|\tan ax| + C$

482. $\displaystyle\int \frac{dx}{\sin^2 ax \cos ax} = \frac{1}{a}\ln|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)| - \frac{1}{a\sin ax} + C$

483. $\displaystyle\int \frac{dx}{\sin ax \cos^2 ax} = \frac{1}{a}\ln|\tan\frac{ax}{2}| + \frac{1}{a\cos ax} + C$

484. $\displaystyle\int \frac{dx}{\sin^2 ax \cos^2 ax} = -\frac{2\cos 2ax}{a} + C$

485. $\displaystyle\int \frac{\sin^2 ax}{\cos ax}\,dx = -\frac{\sin ax}{a} + \frac{1}{a}\ln|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)| + C$

486. $\displaystyle\int \frac{\cos^2 ax}{\sin ax}\,dx = \frac{\cos ax}{a} + \frac{1}{a}\ln|\tan\frac{ax}{2}| + C$

487. $\displaystyle\int \frac{dx}{\cos ax\,(1+\sin ax)} = \frac{1}{2a(1+\sin ax)}\left[-1 + (1+\sin ax)\ln\left|\frac{\cos\frac{ax}{2}+\sin\frac{ax}{2}}{\cos\frac{ax}{2}-\sin\frac{ax}{2}}\right|\right] + C$

488. $\displaystyle\int \frac{dx}{\sin ax\,(1+\cos ax)} = \frac{1}{4a}\sec^2\frac{ax}{2} + \frac{1}{2a}\ln|\tan\frac{ax}{2}| + C$

489. $\displaystyle\int \frac{dx}{\sin ax\,(\alpha+\beta\sin ax)} = \frac{1}{a\alpha}\ln|\tan\frac{ax}{2}| - \frac{\beta}{\alpha}\int\frac{dx}{\alpha+\beta\sin ax}$

490. $\displaystyle\int \frac{dx}{\cos ax\,(\alpha+\beta\sin ax)} = \frac{1}{\alpha^2-\beta^2}\left[\frac{\alpha}{a}\ln|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)| - \frac{\beta}{\alpha}\ln\left|\frac{\alpha+\beta\sin ax}{\cos ax}\right|\right] + C, \qquad \beta\neq\alpha$

491. $\displaystyle\int \frac{dx}{\sin ax\,(\alpha+\beta\cos ax)} = \frac{1}{\alpha^2-\beta^2}\left[\frac{\alpha}{a}\ln|\tan\frac{ax}{2}| + \frac{\beta}{a}\ln|\frac{\alpha+\beta\cos ax}{\sin ax}|\right] + C, \qquad \beta\neq\alpha$

492. $\displaystyle\int \frac{dx}{\cos ax\,(\alpha+\beta\cos ax)} = \frac{1}{a\alpha}\ln|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)| - \frac{\beta}{\alpha}\int\frac{dx}{\alpha+\beta\cos ax}$

493. $$\int \frac{dx}{\alpha+\beta\cos ax+\gamma\sin ax} = \begin{cases} \dfrac{2}{a\sqrt{-R}}\tan^{-1}\left(\dfrac{\gamma+(\alpha-\beta)\tan\frac{ax}{2}}{\sqrt{-R}}\right) + C, & \begin{array}{c}\alpha^2>\beta^2+\gamma^2\\ R=\beta^2+\gamma^2-\alpha^2\end{array} \\ \dfrac{1}{a\sqrt{R}}\ln\left|\dfrac{\gamma-\sqrt{R}+(\alpha-\beta)\tan\frac{ax}{2}}{\gamma+\sqrt{R}+(\alpha-\beta)\tan\frac{ax}{2}}\right| + C, & \alpha^2<\beta^2+\gamma^2 \\ \dfrac{1}{a\beta}\ln\left|\beta+\gamma\tan\dfrac{ax}{2}\right| + C, & \alpha=\beta \\ \dfrac{1}{a\beta}\ln\left|\dfrac{cos\frac{ax}{2}+\sin\frac{ax}{2}}{(\beta+\gamma)\cos\frac{ax}{2}+(\gamma-\beta)\sin\frac{ax}{2}}\right| + C, & \alpha=\gamma \\ \dfrac{1}{a\gamma}\ln|1+\tan\dfrac{ax}{2}| + C, & \alpha=\beta=\gamma \end{cases}$$

494. $\displaystyle\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{\sqrt{2}a}\ln|\tan\left(\frac{ax}{2}\pm\frac{\pi}{8}\right)| + C$

495. $$\int \frac{\sin ax\, dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \ln|\sin ax \pm \cos ax| + C$$

496. $$\int \frac{\cos ax\, dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a}\ln|\sin axx \pm \cos ax| + C$$

497. $$\int \frac{\sin ax\, dx}{\alpha + \beta \sin ax} = \frac{1}{a\beta}\ln|\alpha + \beta \sin ax| + C$$

498. $$\int \frac{\cos ax\, dx}{\alpha + \beta \sin ax} = \frac{1}{a\beta}\ln|\alpha + \beta \sin ax| + C$$

499. $$\int \frac{\sin ax \cos ax\, dx}{\alpha^2 \cos^2 ax + \beta^2 ax} = \frac{1}{2a(\beta^2 - \alpha^2)}\ln|\alpha^2 \cos^2 ax + \beta^2 \sin^2 ax| + C, \qquad \beta \neq \alpha$$

500. $$\int \frac{dx}{\alpha^2 \sin^2 ax + \beta^2 \cos^2 ax} = \frac{1}{a\alpha\beta}\tan^{-1}\left(\frac{\alpha}{\beta}\tan ax\right) + C$$

501. $$\int \frac{dx}{\alpha^2 \sin^2 ax - \beta^2 \cos^2 ax} = \frac{1}{2a\alpha\beta}\ln\left|\frac{\alpha \tan ax - \beta}{\alpha \tan ax + \beta}\right| + C$$

502. $$\int \frac{\sin^n ax}{\cos^{(n+2} ax}\, dx = \frac{\tan^{n+1} ax}{(n+1)a} + C$$

503. $$\int \frac{\cos^n ax}{\sin^{(n+2)} ax}\, dx = -\frac{\cot^{(n+1)} ax}{(n+1)a} + C$$

504. $$\int \frac{dx}{\alpha + \beta \frac{\sin ax}{\cos ax}} = \frac{\alpha x}{\alpha^2 + \beta^2} + \frac{\beta}{a(\alpha^2 + \beta^2)}\ln|\beta \sin ax + \alpha \cos ax| + C$$

505. $$\int \frac{dx}{\alpha + \beta \frac{\cos ax}{\sin ax}} = \frac{\alpha x}{\alpha^2 + \beta^2} - \frac{\beta}{a(\alpha^2 + \beta^2)}\ln|\alpha \sin ax + \beta \cos ax| + C$$

506. $$\int \frac{\cos^n ax}{\sin^n ax}\, dx = -\frac{\cot^{(n-1)} ax}{(n-1)a} - \int \cot^{(n-2)} ax\, dx$$

507. $$\int \frac{\sin^n ax}{\cos^n ax}\, dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \frac{\sin^{n-2} ax}{\cos^{n-2} ax}\, dx$$

508. $$\int \frac{\sin ax}{\cos^{(n+1)} ax}\, dx = \frac{1}{na}\sec^n ax + C$$

509. $$\int \frac{\alpha \sin x + \beta \cos x}{\gamma \sin x + \delta \cos x}\, dx = \frac{[(\alpha\gamma + \beta\delta)x + (\beta\gamma - \alpha\gamma)\ln|\gamma \sin x + \delta \cos x|]}{\gamma^2 + \delta^2} + C$$

510. $$\int \frac{\alpha + \beta \sin x}{a + b \cos x}\, dx = \begin{cases} \dfrac{2\alpha}{\sqrt{a^2 - b^2}}\tan^{-1}\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2} - \dfrac{\beta}{b}\ln|a + b\cos x| + C, & a > b \\ \dfrac{2\alpha}{\sqrt{b^2 - a^2}}\tanh^{-1}\sqrt{\dfrac{b-a}{b+a}}\tan\dfrac{x}{2} - \dfrac{\beta}{b}\ln|a + b\cos x| + C, & a < b \end{cases}$$

511. $$\int \frac{dx}{a^2 - b^2\cos^2 x} = \begin{cases} \dfrac{1}{a\sqrt{a^2-b^2}} \tan^{-1}\left(\dfrac{a}{\sqrt{a^2-b^2}}\tan x\right) + C, & a > b \\ \frac{-1}{a\sqrt{b^2-a^2}} \tanh^{-1}\left(\frac{a}{\sqrt{b^2-a^2}}\tan x\right) + C, & b > a \end{cases}$$

512. $$\int \frac{dx}{(a\cos x + b\sin x)^2} = \frac{1}{a^2+b^2}\tan\left(x - \tan^{-1}\frac{b}{a}\right) + C$$

513. $$\int \frac{\sin x\,dx}{\sqrt{a\cos^2 x + 2b\cos x + c}} = \begin{cases} \dfrac{-1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{\sqrt{-a(a\cos^2 x + 2b\cos x + c)}}{\sqrt{b^2-ac}}\right) + C, \ b^2 > ac, \ a < 0 \\ \dfrac{-1}{\sqrt{a}}\sinh^{-1}\left(\dfrac{\sqrt{a(a\cos^2 x + 2b\cos x + c)}}{\sqrt{b^2-ac}}\right) + C, \ b^2 > ac, \ a > 0 \\ \dfrac{-1}{\sqrt{a}}\cosh^{-1}\left(\dfrac{\sqrt{a(a\cos^2 x + 2b\cos x + c)}}{\sqrt{ac-b^2}}\right) + C, \ b^2 < ac, \ a > 0 \end{cases}$$

514. $$\int \frac{\cos x\,dx}{\sqrt{a\sin^2 x + 2b\sin x + c}} = \begin{cases} \dfrac{1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{\sqrt{-a(a\sin^2 x + 2b\sin x + c)}}{\sqrt{b^2-ac}}\right) + C, \ b^2 > ac, \ a < 0 \\ \dfrac{1}{\sqrt{a}}\sinh^{-1}\left(\dfrac{\sqrt{a(a\sin^2 x + 2b\sin x + c)}}{\sqrt{b^2-ac}}\right) + C, \ b^2 > ac, \ a > 0 \\ \dfrac{1}{\sqrt{a}}\cosh^{-1}\left(\dfrac{\sqrt{a(a\sin^2 x + 2b\sin x + c)}}{\sqrt{ac-b^2}}\right) + C, \ b^2 < ac, \ a > 0 \end{cases}$$

Integrals containing $\tan ax$, $\cot ax$, $\sec ax$, $\csc ax$

Write integrals in terms of $\sin ax$ and $\cos ax$ and see previous listings.

Integrals containing inverse trigonmetric functions

515. $$\int \sin^{-1}\frac{x}{a}\,dx = x\sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2} + C$$

516. $$\int \cos^{-1}\frac{x}{a}\,dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + C$$

517. $$\int \tan^{-1}\frac{x}{a}\,dx = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln|x^2+a^2| + C$$

518. $$\int \cot^{-1}\frac{x}{a}\,dx = x\cot^{-1}\frac{x}{a} + \frac{a}{2}\ln|x^2+a^2| + C$$

519. $$\int \sec^{-1}\frac{x}{a}\,dx = \begin{cases} x\sec^{-1}\dfrac{x}{a} - a\ln|x + \sqrt{x^2-a^2}| + C, & 0 < \sec^{-1}\frac{x}{a} < \pi/2 \\ x\sec^{-1}\dfrac{x}{a} + a\ln|x + \sqrt{x^2-a^2} + C, & \pi/2 < \sec^{-1}\frac{x}{a} < \pi \end{cases}$$

520. $$\int \csc^{-1}\frac{x}{a}\,dx = \begin{cases} x\csc^{-1}\dfrac{x}{a} + a\ln|x + \sqrt{x^2-a^2}| + C, & 0 < \csc^{-1}\frac{x}{a} < \pi/2 \\ x\csc^{-1}\dfrac{x}{a} - a\ln|x + \sqrt{x^2-a^2}| + C, & -\pi/2 < \csc^{-1}\frac{x}{a} < 0 \end{cases}$$

521. $$\int x\sin^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\sin^{-1}\frac{x}{a} + \frac{1}{4}x\sqrt{a^2-x^2} + C$$

522. $$\int x\cos^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\cos^{-1}\frac{x}{a} - \frac{1}{4}x\sqrt{a^2-x^2} + C$$

523. $$\int x\tan^{-1}\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln|x^2+a^2| + C$$

524. $$\int x\cot^{-1}\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\cot^{-1}\frac{x}{a} + \frac{a}{2}x + C$$

525. $$\int x\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{2}x^2\sec^{-1}\frac{x}{a} - \frac{a}{2}\sqrt{x^2-a^2} + C, & 0 < \sec^{-1}\frac{x}{a} < \pi/2 \\ \frac{1}{2}x^2\sec^{-1}\frac{x}{a} + \frac{a}{2}\sqrt{x^2-a^2} + C, & \pi/2 < \sec^{-1}\frac{x}{a} < \pi \end{cases}$$

526. $$\int x\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{2}x^2\csc^{-1}\frac{x}{a} + \frac{a}{2}\sqrt{x^2-a^2} + C, & 0 < \csc^{-1}\frac{x}{a} < \pi/2 \\ \frac{1}{2}x^2\csc^{-1}\frac{x}{a} - \frac{a}{2}\sqrt{x^2-a^2} + C, & -\pi/2 < \csc^{-1}\frac{x}{a} < 0 \end{cases}$$

527. $$\int x^2\sin^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\sin^{-1}\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2} + C$$

528. $$\int x^2\cos^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\cos^{-1}\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2} + C$$

529. $$\int x^2\tan^{-1}\frac{x}{a}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{a} - \frac{a}{6}x^2 + \frac{a^3}{6}\ln|x^2+a^2| + C$$

530. $$\int x^2\cot^{-1}\frac{x}{a}\,dx = \frac{1}{3}\cot^{-1}\frac{x}{a} + \frac{a}{6}x^2 - \frac{a^3}{6}\ln|a^2+x^2| + C$$

531. $$\int x^2\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{3}x^3\sec^{-1}\frac{x}{a} - \frac{a}{6}x\sqrt{x^2-a^2} - \frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}| + C, & 0 < \sec^{-1}\frac{x}{a} < \pi/2 \\ \frac{1}{3}x^3\sec^{-1}\frac{x}{a} + \frac{a}{6}x\sqrt{x^2-a^2} + \frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}| + c, & \pi/2 < \sec^{-1}\frac{x}{a} < \pi \end{cases}$$

532. $$\int x^2\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{3}x^3\csc^{-1}\frac{x}{a} + \frac{a}{6}x\sqrt{x^2-a^2} + \frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}| + C, & 0 < \csc^{-1}\frac{x}{a} < \pi/2 \\ \frac{1}{3}x^3\csc^{-1}\frac{x}{a} - \frac{a}{6}x\sqrt{x^2-a^2} - \frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}| + C, & -\pi/2 < \csc^{-1}\frac{x}{a} < 0 \end{cases}$$

533. $$\int \frac{1}{x}\sin^{-1}\frac{x}{a}\,dx = \frac{x}{a} + \frac{1}{2\cdot 3\cdot 3}\left(\frac{x}{a}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\left(\frac{x}{a}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots + C$$

534. $$\int \frac{1}{x}\cos^{-1}\frac{x}{a}\,dx = \frac{\pi}{2}\ln|x| + -\int \frac{1}{x}\sin^{-1}\frac{x}{a}\,dx$$

535. $$\int \frac{1}{x}\tan^{-1}\frac{x}{a}\,dx = \frac{x}{a} - \frac{1}{3^2}\left(\frac{x}{a}\right)^3 + \frac{1}{5^2}\left(\frac{x}{a}\right)^5 - \frac{1}{7^2}\left(\frac{x}{a}\right)^7 + \cdots + C$$

536. $$\int \frac{1}{x}\cot^{-1}\frac{x}{a}\,dx = \frac{\pi}{2}\ln|x| - \int \frac{1}{x}\tan^{-1}\frac{x}{a}\,dx$$

537. $$\int \frac{1}{x}\sec^{-1}\frac{x}{a}\,dx = \frac{\pi}{2}\ln|x| + \frac{a}{x} + \frac{1}{2\cdot 3\cdot 3}\left(\frac{x}{a}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\left(\frac{x}{a}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\left(\frac{x}{a}\right)^7 + \cdots + C$$

538. $$\int \frac{1}{x}\csc^{-1}\frac{x}{a}\,dx = -\left(\frac{a}{x}+\frac{1}{2\cdot 3\cdot 3}\left(\frac{x}{a}\right)^3+\frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\left(\frac{x}{a}\right)^5+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\left(\frac{x}{a}\right)^7+\cdots\right)+C$$

539. $$\int \frac{1}{x^2}\sin^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\sin^{-1}\frac{x}{a}-\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{a}\right|+C$$

540. $$\int \frac{1}{x^2}\cos^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\cos^{-1}\frac{x}{a}+\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{a}\right|+C$$

541. $$\int \frac{1}{x^2}\tan^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\tan^{-1}\frac{x}{a}-\frac{1}{2a}\ln\left|\frac{x^2+a^2}{a^2}\right|+C$$

542. $$\int \frac{1}{x^2}\cot^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\cot^{-1}\frac{x}{a}+\frac{1}{2a}\int \frac{1}{x}\tan^{-1}\frac{x}{a}\,dx$$

543. $$\int \frac{1}{x^2}\sec^{-1}\frac{x}{a}\,dx = \begin{cases} -\dfrac{1}{x}\sec^{-1}\dfrac{x}{a}+\dfrac{1}{ax}\sqrt{x^2-a^2}+C, & 0<\sec^{-1}\frac{x}{a}<\pi/2 \\ -\dfrac{1}{x}\sec^{-1}\dfrac{x}{a}-\dfrac{1}{ax}\sqrt{x^2-a^2}+C, & \pi/2<\sec^{-1}\frac{x}{a}<\pi \end{cases}$$

544. $$\int \frac{1}{x^2}\csc^{-1}\frac{x}{a}\,dx = \begin{cases} -\dfrac{1}{x}\csc^{-1}\dfrac{x}{a}-\dfrac{1}{ax}\sqrt{x^2-a^2}+C, & 0<\csc^{-1}\frac{x}{a}<\pi/2 \\ -\dfrac{1}{x}\csc^{-1}\dfrac{x}{a}+\dfrac{1}{ax}\sqrt{x^2-a^2}+C, & -\pi/2<\csc^{-1}\frac{x}{a}<0 \end{cases}$$

545. $$\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx = (a+x)\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{ax}+C$$

546. $$\int \cos^{-1}\sqrt{\frac{x}{a+x}}\,dx = (2a+x)\tan^{-1}\sqrt{\frac{x}{2a}}-\sqrt{2ax}+C$$

Integrals containing the exponential function

547. $$\int e^{ax}\,dx = \frac{1}{a}e^{ax}+C$$

548. $$\int xe^{ax}\,dx = \left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax}+C$$

549. $$\int x^2e^{ax}\,dx = \left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right)e^{ax}+C$$

550. $$\int x^ne^{ax}\,dx = \frac{1}{a}x^ne^{ax}-\frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

551. $$\int \frac{1}{x}e^{ax}\,dx = \ln|x|+\frac{ax}{1\cdot 1!}+\frac{(ax)^2}{2\cdot 2!}+\frac{(ax)^3}{3\cdot 3!}+\cdots+C$$

552. $$\int \frac{1}{x^n}e^{ax}\,dx = -\frac{1}{(n-1)x^{n-1}}e^{ax}+\frac{a}{n-1}\int \frac{1}{x^{n-1}}e^{ax}\,dx$$

553. $$\int \frac{e^{ax}}{\alpha+\beta e^{ax}}\,dx = \frac{1}{a\beta}\ln|\alpha+\beta e^{ax}|+C$$

554. $\displaystyle\int e^{ax}\sin bx\,dx = \left(\frac{a\sin bx - b\cos bx}{a^2+b^2}\right)e^{ax} + C$

555. $\displaystyle\int e^{ax}\cos bx\,dx = \left(\frac{a\cos bx + b\sin bx}{a^2+b^2}\right)e^{ax} + C$

556. $\displaystyle\int e^{ax}\sin^n bx\,dx = \left(\frac{a\sin bx - nb\cos bx}{a^2+n^2b^2}\right)e^{ax}\sin^{n-1} bx + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\sin^{n-2} bx\,dx$

557. $\displaystyle\int e^{ax}\cos^n bx\,dx = \left(\frac{a\cos bx + nb\sin bx}{a^2+n^2b^2}\right)e^{ax}\cos^{n-1} bx + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\cos^{n-2} bx\,dx$

Another way to express the above integrals is to define

$C_n = \displaystyle\int e^{ax}\cos^n bx\,dx$ and $S_n = \displaystyle\int e^{ax}\sin^n bx\,dx$, then one can write the reduction formulas

$$\begin{aligned}
C_n =& \frac{a\cos bx + nb\sin bx}{a^2+n^2b^2}e^{ax}\cos^{n-1} bx + \frac{n(n-1)b^2}{a^2+n^2b^2}C_{n-2}\\
S_n =& \frac{a\sin bx - nb\cos bx}{a^2+n^2b^2}e^{ax}\sin^{n-1} bx + \frac{n(n-1)b^2}{a^2+n^2b^2}S_{n-2}
\end{aligned}$$

558. $\displaystyle\int xe^{ax}\sin bx\,dx = \left(\frac{[2ab - b(a^2+b^2)x]\cos bx + [a(a^2+b^2)x - a^2 + b^2]\sin bx}{(a^2+b^2)^2}\right)e^{ax} + C$

559. $\displaystyle\int xe^{ax}\cos bx\,dx = \left(\frac{[a(a^2+b^2)x - a^2 + b^2]\cos bx + [b(a^2+b^2)x - 2ab]\sin bx}{(a^2+b^2)^2}\right)e^{ax} + C$

560. $\displaystyle\int e^{ax}\ln x\,dx = \frac{1}{a}e^{ax}\ln x - \frac{1}{a}\int \frac{1}{x}e^{ax}\,dx$

561. $\displaystyle\int e^{ax}\sinh bx\,dx = \left[\frac{a\sinh bx - b\cosh bx}{(a-b)(a+b)}\right]e^{ax} + C, \qquad a \neq b$

562. $\displaystyle\int e^{ax}\sinh ax\,dx = \frac{1}{4a}e^{2ax} - \frac{x}{2} + C$

563. $\displaystyle\int e^{ax}\cosh bx\,dx = \left[\frac{a\cosh bx - b\sinh bx}{(a-b)(a+b)}\right]e^{ax} + C, \qquad a \neq b$

564. $\displaystyle\int e^{ax}\cosh ax\,dx = \frac{1}{4a}e^{2ax} + \frac{x}{2} + C$

565. $\displaystyle\int \frac{dx}{\alpha + \beta e^{ax}} = \frac{x}{\alpha} - \frac{1}{a\alpha}\ln|\alpha + \beta e^{ax}| + C$

566. $\displaystyle\int \frac{dx}{(\alpha + \beta e^{ax})^2} = \frac{x}{\alpha^2} + \frac{1}{a\alpha(\alpha+\beta e^{ax})} - \frac{1}{a\alpha^2}\ln|\alpha + \beta e^{ax}| + C$

567. $\displaystyle\int \frac{dx}{\alpha e^{ax} + \beta e^{-ax}} = \begin{cases} \dfrac{1}{a\sqrt{\alpha\beta}}\tan^{-1}\left(\sqrt{\dfrac{\alpha}{\beta}}e^{ax}\right) + C, & \alpha\beta > 0\\ \dfrac{1}{2a\sqrt{-\alpha\beta}}\ln\left|\dfrac{e^{ax} - \sqrt{-\beta/\alpha}}{e^{ax} + \sqrt{-\beta/\alpha}}\right| + C, & \alpha\beta < 0 \end{cases}$

568. $\displaystyle\int e^{ax}\sin^2 bx\,dx = \left(\frac{a^2+4b^2-a^2\cos(2bx)-2ab\sin(2bx)}{2a(a^2+4b^2}\right)e^{ax}+C$

569. $\displaystyle\int e^{ax}\cos^2 bx\,dx = \left(\frac{a^2+4b^2+a^2\cos(2bx)+2ab\sin(2bx)}{2a(a^2+4b^2}\right)e^{ax}+C$

Integrals containing the logarithmic function

570. $\displaystyle\int \ln x\,dx = x\ln|x|+C$

571. $\displaystyle\int x\ln x\,dx = \frac{1}{2}x^2\ln|x|-\frac{1}{4}x^2+C$

572. $\displaystyle\int x^n\ln x\,dx = \frac{1}{(n+1)^2}x^{n+1}+\frac{1}{n+1}x^{n+1}\ln|x|+C, \qquad n\neq -1$

573. $\displaystyle\int \frac{1}{x}\ln x\,dx = \frac{1}{2}(\ln|x|)^2+C$

574. $\displaystyle\int \frac{dx}{x\ln x} = \ln|\ln|x||+C$

575. $\displaystyle\int \frac{1}{x^2}\ln x\,dx = -\frac{1}{x}-\frac{1}{x}\ln|x|+C$

576. $\displaystyle\int (\ln|x|)^2\,dx = x(\ln|x|)^2-2x\ln|x|+2x+C$

577. $\displaystyle\int \frac{1}{x}(\ln|x|)^n\,dx = \frac{1}{n+1}(\ln|x|)^{n+1}+C, \qquad n\neq -1$

578. $\displaystyle\int (\ln|x|)^n\,dx = x(\ln|x|)^n-n\int(\ln|x|)^{n-1}\,dx$

579. $\displaystyle\int \ln|x^2+a^2|\,dx = x\ln|x^2+a^2|-2x+2a\tan^{-1}\frac{x}{a}+C$

580. $\displaystyle\int \ln|x^2-a^2|\,dx = x\ln|x^2-a^2|-2x+a\ln|\frac{x+a}{x-a}|+C$

581. $\displaystyle\int (ax+b)\ln(\beta x+\gamma)\,dx = \frac{\beta^2(ax+b)^2-(b\beta-a\gamma)^2}{2a\beta^2}\ln(\beta x+\gamma)-\frac{a}{4\beta^2}(\beta x+\gamma)^2-\frac{1}{\beta}(b\beta-a\gamma)x+C$

582. $\displaystyle\int (\ln ax)^2\,dx = x(\ln ax)^2-2x\ln ax+2x+C$

Integrals containing the hyperbolic function sinh ax

583. $\displaystyle\int \sinh ax\,dx = \frac{1}{a}\cosh ax+C$

584. $\displaystyle\int x\sinh ax\,dx = \frac{1}{a}x\cosh ax - \frac{1}{a^2}\sinh ax + C$

585. $\displaystyle\int x^2\sinh ax\,dx = \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\cosh ax - \frac{2x}{a^2}\sinh ax + C$

586. $\displaystyle\int x^n\sinh ax\,dx = \frac{1}{a}x^n\cosh ax - \frac{n}{a}\int x^{n-1}\cosh ax\,dx$

587. $\displaystyle\int \frac{1}{x}\sinh ax\,dx = ax + \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{4\cdot 5!} + \cdots + C$

588. $\displaystyle\int \frac{1}{x^2}\sinh ax\,dx = -\frac{1}{x}\sinh ax + a\int \frac{1}{x}\cosh ax\,dx$

589. $\displaystyle\int \frac{1}{x^n}\sinh ax\,dx = -\frac{\sinh ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{1}{x^{n-1}}\cosh ax\,dx$

590. $\displaystyle\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln|\tanh\frac{ax}{2}| + C$

591. $\displaystyle\int \frac{x\,dx}{\sinh ax} = \frac{1}{a^2}\left[ax - \frac{(ax)^3}{18} + frac7(ax)^5 1800 + \cdots + (-1)^n\frac{2(2^{2n}-1)\mathfrak{B}_n\,a^{2n+1}x^{2n+1}}{(2n+1)!} + \cdots\right] + C$

592. $\displaystyle\int \sinh^2 ax\,dx = \frac{1}{2a}x\sinh 2ax - \frac{1}{2}x + C$

593. $\displaystyle\int \sinh^n ax\,dx = \frac{1}{na}\sinh^{n-1}ax\cosh ax - \frac{n-1}{n}\int \sinh^{n-2}ax\,dx$

594. $\displaystyle\int x\sinh^2 ax\,dx = \frac{1}{4a}x\sinh 2ax - \frac{1}{8a^2}\cosh 2ax - \frac{1}{4}x^2 + C$

595. $\displaystyle\int \frac{dx}{\sinh^2 ax} = -\frac{1}{a}\coth ax + C$

596. $\displaystyle\int \frac{dx}{\sinh^3 ax} = -\frac{1}{2a}\operatorname{csch} ax\coth ax - \frac{1}{2a}\ln|\tanh\frac{ax}{2}| + C$

597. $\displaystyle\int \frac{x\,dx}{\sinh^2 ax} = -\frac{1}{a}x\coth ax + \frac{1}{a^2}\ln|\sinh ax| + C$

598. $\displaystyle\int \sinh ax\sinh bx\,dx = \frac{1}{2(a+b)}\sinh(a+b)x - \frac{1}{2(a-b)}\sinh(a-b)x + C$

599. $\displaystyle\int \sinh ax\sin bx\,dx = \frac{1}{a^2+b^2}\left[a\cosh ax\sin bx - b\sinh ax\cos bx\right] + C$

600. $\displaystyle\int \sinh ax\cos bx\,dx = \frac{1}{a^2+b^2}\left[a\cosh ax\cos bx + b\sinh ax\sin bx\right] + C$

601. $$\int \frac{dx}{\alpha+\beta\sinh ax} = \frac{1}{\sqrt{\alpha^2+\beta^2}}\ln\left|\frac{\beta e^{ax}+\alpha-\sqrt{\alpha^2+\beta^2}}{\beta e^{ax}+\alpha+\sqrt{\alpha^2+\beta^2}}\right|+C$$

602. $$\int \frac{dx}{(\alpha+\beta\sinh ax)^2} = \frac{-\beta}{a(\alpha^2+\beta^2)}\frac{\cosh ax}{\alpha+\beta\sinh ax}+\frac{\alpha}{\alpha^2+\beta^2}\int\frac{dx}{\alpha+\beta\sinh ax}$$

603. $$\int \frac{dx}{\alpha^2+\beta^2\sinh^2 ax} = \begin{cases} \dfrac{1}{a\alpha\sqrt{\beta^2-\alpha^2}}\tan^{-1}\left(\dfrac{\sqrt{\beta^2-\alpha^2}\tanh ax}{\alpha}\right)+C, & \beta^2>\alpha^2 \\ \dfrac{1}{2a\alpha\sqrt{\alpha^2-\beta^2}}\ln\left|\dfrac{\alpha+\sqrt{\alpha^2-\beta^2}\tanh ax}{\alpha-\sqrt{\alpha^2-\beta^2}\tanh ax}\right|+C, & \beta^2<\alpha^2 \end{cases}$$

604. $$\int \frac{dx}{\alpha^2-\beta^2\sinh^2 ax} = \frac{1}{2a\alpha\sqrt{\alpha^2+\beta^2}}\ln\left|\frac{\alpha+\sqrt{\alpha^2+\beta^2}\tanh ax}{\alpha-\sqrt{\alpha^2+\beta^2}\tanh ax}\right|+C$$

Integrals containing the hyperbolic function cosh *ax*

605. $$\int \cosh ax\,dx = \frac{1}{a}\sinh ax+C$$

606. $$\int x\cosh ax\,dx = \frac{1}{a}x\sinh ax-\frac{1}{a^2}\cosh ax+C$$

607. $$\int x^2\cosh ax\,dx = -\frac{2}{a^2}x\cosh ax+\left(\frac{x^2}{a}+\frac{2}{a^3}\right)\sinh ax+C$$

608. $$\int x^n\cosh ax\,dx = \frac{1}{a}x^n\sinh ax-\frac{n}{a}\int x^{n-1}\sinh ax\,dx$$

609. $$\int \frac{1}{x}\cosh ax\,dx = \ln|x|+\frac{(ax)^2}{2\cdot 2!}+\frac{(ax)^4}{4\cdot 4!}+\frac{(ax)^6}{6\cdot 6!}+\cdots+C$$

610. $$\int \frac{1}{x^2}\cosh ax\,dx = -\frac{1}{x}\cosh ax+a\int\frac{1}{x}\sinh ax\,dx$$

611. $$\int \frac{1}{x^n}\cosh ax\,dx = -\frac{1}{n-1}\frac{\cosh ax}{x^{n-1}}+\frac{a}{n-1}\int\frac{\sinh ax}{x^{n-1}}\,dx, \qquad n>1$$

612. $$\int \frac{dx}{\cosh ax} = \frac{2}{a}\tan^{-1}e^{ax}+C$$

613. $$\int \frac{x\,dx}{\cosh ax} = \frac{1}{a^2}\left[\frac{a^2x^2}{2}-\frac{a^4x^4}{8}+\frac{5a^6x^6}{144}+\cdots+(-1)^n\frac{\mathfrak{E}_n a^{2n+2}x^{2n+2}}{(2n+2)\cdot(2n)!}+\cdots\right]+C$$

614. $$\int \cosh^2 ax\,dx = \frac{1}{2}x+\frac{1}{2}\sinh ax\cosh ax+C$$

615. $$\int \cosh^n ax\,dx = \frac{1}{na}\cosh^{n-1}ax\sinh ax+\frac{n-1}{n}\int\cosh^{n-2}ax\,dx$$

616. $$\int x\cosh^2 ax\,dx = \frac{1}{4}x^2+\frac{1}{4a}x\sinh 2ax-\frac{1}{8a^2}\cosh 2ax+C$$

617. $$\int \frac{dx}{\cosh^2 ax} = \frac{1}{a}\tanh ax + C$$

618. $$\int \frac{x\,dx}{\cosh^2 ax} = \frac{1}{a} x \tanh ax - \frac{1}{a^2}\ln|\cosh ax| + C$$

619. $$\int \frac{dx}{\cosh^n ax} = \frac{1}{(n-1)a}\frac{x\sinh ax}{\cosh^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cosh^{n-2} ax, \quad n>1}$$

620. $$\int \cosh ax \cosh bx\,dx = \frac{1}{2(a-b)}\sinh(a-b)x + \frac{1}{2(a+b)}\sinh(a+b)x + C$$

621. $$\int \cosh ax \sin bx\,dx = \frac{1}{a^2+b^2}\left[a\sinh ax \sin bx - b\cosh ax \cos bx\right] + C$$

622. $$\int \cosh ax \cos bx\,dx = \frac{1}{a^2+b^2}\left[a\sinh ax \cos bx + b\cosh ax \sin bx\right] + C$$

623. $$\int \frac{dx}{\alpha + \beta\cosh ax} = \begin{cases} \dfrac{2}{\sqrt{\beta^2-\alpha^2}}\tan^{-1}\dfrac{\beta e^{ax}+\alpha}{\sqrt{\beta^2-\alpha^2}} + C, & \beta^2 > \alpha^2 \\ \dfrac{1}{a\sqrt{\alpha^2-\beta^2}}\ln\left|\dfrac{\beta e^{ax}+\alpha-\sqrt{\alpha^2-\beta^2}}{\beta e^{ax}+\alpha+\sqrt{\alpha^2-\beta^2}}\right| + C, & \beta^2 < \alpha^2 \end{cases}$$

624. $$\int \frac{dx}{1+\cosh ax} = \frac{1}{a}\tanh ax + C$$

625. $$\int \frac{x\,dx}{1+\cosh ax} = \frac{x}{a}\tanh\frac{ax}{2} - \frac{2}{a^2}\ln|\cosh\frac{ax}{2}| + C$$

626. $$\int \frac{dx}{-1+\cosh ax} = -\frac{1}{a}\coth\frac{ax}{2} + C$$

627. $$\int \frac{dx}{(\alpha+\beta\cosh ax)^2} = \frac{\beta\sinh ax}{a(\beta^2-\alpha^2)(\alpha+\beta\cosh ax)} - \frac{\alpha}{\beta^2-\alpha^2}\int\frac{dx}{\alpha+\beta\cosh ax}$$

628. $$\int \frac{dx}{\alpha^2-\beta^2\cosh^2 ax} = \begin{cases} \dfrac{1}{2a\alpha\sqrt{\alpha^2-\beta^2}}\ln\left|\dfrac{\alpha\tanh ax+\sqrt{\alpha^2-\beta^2}}{\alpha\tanh ax-\sqrt{\alpha^2-\beta^2}}\right| + C, & \alpha^2 > \beta^2 \\ \dfrac{-1}{a\alpha\sqrt{\beta^2-\alpha^2}}\tan^{-1}\dfrac{\alpha\tanh ax}{\sqrt{\beta^2-\alpha^2}} + C, & \alpha^2 < \beta^2 \end{cases}$$

629. $$\int \frac{dx}{\alpha^2+\beta^2\cosh^2 ax} = \frac{1}{a\alpha\sqrt{\alpha^2+\beta^2}}\tanh^{-1}\left(\frac{\alpha\tanh ax}{\sqrt{\alpha^2+\beta^2}}\right) + C$$

Integrals containing the hyperbolic functions $\sinh ax$ and $\cosh ax$

630. $$\int \sinh ax\cosh ax\,dx = \frac{1}{2a}\sinh^2 ax + C$$

631. $$\int \sinh ax\cosh bx\,dx = \frac{1}{2(a+b)}\cosh(a+b)x + \frac{1}{2(a-b)}\cosh(a-b)x + C$$

632. $$\int \sinh^2 ax \cosh^2 ax\, dx = \frac{1}{32a}\sinh 4ax - \frac{1}{8}x + C$$

633. $$\int \sinh^n ax \cosh ax\, dx = \frac{1}{(n+1)a}\sinh^{n+1} ax + C, \qquad n \neq -1$$

634. $$\int \cosh^n ax \sinh ax\, dx = \frac{1}{(n+1)a}\cosh^{n+1} ax + C, \qquad n \neq -1$$

635. $$\int \frac{\sinh ax}{\cosh ax}\, dx = \frac{1}{a}\ln|\cosh ax| + C$$

636. $$\int \frac{\cosh ax}{\sinh ax}\, dx = \frac{1}{a}\ln|\sinh ax| + C$$

637. $$\int \frac{dx}{\sinh ax \cosh ax} = \frac{1}{a}\ln|\tanh ax| + C$$

638. $$\int \frac{x \sinh ax}{\cosh ax}\, dx = \frac{1}{a^2}\left[\frac{a^3x^3}{3} - \frac{a^5x^5}{15} + \cdots + (-1)^n \frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!} + \cdots\right] + C$$

639. $$\int \frac{x \cosh ax}{\sinh ax}\, dx = \frac{1}{a^2}\left[ax + \frac{a^3x^3}{9} - \frac{a^5x^5}{225} + \cdots + (-1)^{n-1} \frac{2^{2n}\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!} + \cdots\right] + C$$

640. $$\int \frac{\sinh^2 ax}{\cosh^2 ax}\, dx = x - \frac{1}{a}\tanh ax + C$$

641. $$\int \frac{\cosh^2 ax}{\sinh^2 ax}\, dx = x - \frac{1}{a}\coth ax + C$$

642. $$\int \frac{x\sinh^2 ax}{\cosh^2 ax}\, dx = \frac{1}{2}x^2 - \frac{1}{a}x\tanh ax + \frac{1}{a^2}\ln|\cosh ax| + C$$

643. $$\int \frac{x\cosh^2 ax}{\sinh^2 ax}\, dx = \frac{1}{2}x^2 - \frac{1}{a}x\coth ax + \frac{1}{a^2}\ln|\sinh ax| + C$$

644. $$\int \frac{\sinh ax}{x\cosh ax}\, dx = ax - \frac{a^3x^3}{9} + \cdots + (-1)^{n-1}\frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n-1}x^{2n-1}}{(2n-1)(2n)!} + \cdots + C$$

645. $$\int \frac{\cosh ax}{x\sinh ax}\, dx = -\frac{1}{ax} + \frac{ax}{3} - \frac{a^3x^3}{135} + \cdots + (-1)^{n}\frac{2^{2n}\mathfrak{B}_n a^{2n-1}x^{2n-1}}{(2n-1)(2n)!} + \cdots + C$$

646. $$\int \frac{\sinh^3 ax}{\cosh^3 ax}\, dx = \frac{1}{a}\ln|\cosh ax| - \frac{1}{2a}\tanh^2 ax + C$$

647. $$\int \frac{\cosh^3 ax}{\sinh^3 ax}\, dx = \frac{1}{a}\ln|\sinh ax| - \frac{1}{2a}\coth^2 ax + C$$

648. $$\int \frac{dx}{\sinh ax\cosh^2 ax} = \frac{1}{a}\operatorname{sech} ax + \frac{1}{a}\ln\tanh\frac{ax}{2}| + C$$

649. $$\int \frac{dx}{\sinh^2 ax \cosh ax} = -\frac{1}{a}\tan^{-1}(\sinh ax) - \frac{1}{a}\operatorname{csch} ax + C$$

650. $$\int \frac{dx}{\sinh^2 ax \cosh^2 ax} = -\frac{2}{a}\coth ax + C$$

651. $$\int \frac{\sinh^2 ax}{\cosh ax}\,dx = \frac{1}{a}\sinh ax - \frac{1}{a}\tan^{-1}(\sinh ax) + C$$

652. $$\int \frac{\cosh^2 ax}{\sinh ax}\,dx = \frac{1}{a}\cosh ax + \frac{1}{a}\ln|\tanh\frac{ax}{2}| + C$$

653. $$\int \frac{dx}{\cosh ax\,(1+\sinh ax)} = \frac{1}{2a}\ln\left|\frac{1+\sinh ax}{\cosh ax}\right| + \frac{1}{a}\tan^{-1} e^{ax} + C$$

654. $$\int \frac{dx}{\sinh ax\,(\cosh ax + 1)} = \frac{1}{2a}\ln|\tanh\frac{ax}{2}| + \frac{1}{2a(\cosh ax + 1)} + C$$

655. $$\int \frac{dx}{\sinh ax\,(\cosh ax - 1)} = -\frac{1}{2a}\ln|\tanh\frac{ax}{2}| - \frac{1}{2a(\cosh ax - 1)} + C$$

656. $$\int \frac{dx}{\alpha + \beta\frac{\sinh ax}{\cosh ax}} = \frac{\alpha x}{\alpha^2-\beta^2} - \frac{\beta}{a(\alpha^2-\beta^2)}\ln|\beta\sinh ax + \alpha\cosh ax| + C$$

657. $$\int \frac{dx}{\alpha + \beta\frac{\cosh ax}{\sinh ax}} = \frac{\alpha x}{\alpha^2-\beta^2} + \frac{\beta}{a(\alpha^2-\beta^2)}\ln|\alpha\sinh ax + \beta\cosh ax| + C$$

658. $$\int \frac{dx}{b\cosh ax + c\sinh ax} = \begin{cases} \dfrac{1}{a\sqrt{b^2-c^2}}\sec^{-1}\left[\dfrac{b\cosh ax + c\sinh ax}{\sqrt{b^2-c^2}}\right] + C, & b^2 > c^2 \\ \dfrac{-1}{a\sqrt{c^2-b^2}}\operatorname{csch}^{-1}\left[\dfrac{b\cosh ax + c\sinh ax}{\sqrt{c^2-b^2}}\right] + C, & b^2 < c^2 \end{cases}$$

Integrals containing the hyperbolic functions $\tanh ax$, $\coth ax$, $\operatorname{sech} ax$, $\operatorname{csch} ax$

Express integrals in terms of $\sinh ax$ and $\cosh ax$ and see previous listings.

Integrals containing inverse hyperbolic functions

659. $$\int \sinh^{-1}\frac{x}{a}\,dx = x\sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2} + C$$

660. $$\int \cosh^{-1}\frac{x}{a}\,dx = \begin{cases} x\cosh^{-1}(x/a) - \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ x\cosh^{-1}(x/a) + \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

661. $$\int \tanh^{-1}\frac{x}{a}\,dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\ln|a^2-x^2| + C$$

662. $$\int \coth^{-1}\frac{x}{a}\,dx = x\coth^{-1}\frac{x}{a} + \frac{a}{2}\ln|x^2-a^2| + C$$

663. $$\int \operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} x\operatorname{sech}^{-1}\dfrac{x}{a} + a\sin^{-1}\dfrac{x}{a} + C, & \operatorname{sech}^{-1}(x/a) > 0 \\ x\operatorname{sech}^{-1}\dfrac{x}{a} - a\sin^{-1}\dfrac{x}{a} + C, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$$

664. $\displaystyle\int \operatorname{csch}^{-1}\frac{x}{a}\,dx = x\operatorname{csch}^{-1}\frac{x}{a} \pm a\sinh^{-1}\frac{x}{a}, \quad + \text{ for } x>0 \text{ and } - \text{ for } x<0$

665. $\displaystyle\int x\sinh^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2}+\frac{a^2}{4}\right)\sinh^{-1}\frac{x}{a} - \frac{1}{4}xx\sqrt{x^2+a^2}+C$

666. $\displaystyle\int x\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{4}(2x^2-a^2)\cosh^{-1}\frac{x}{a} - \frac{1}{4}x\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)>0 \\ \frac{1}{4}(2x^2-a^2)\cosh^{-1}\frac{x}{a} + \frac{1}{4}x\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)<0 \end{cases}$

667. $\displaystyle\int x\tanh^{-1}\frac{x}{a}\,dx = \frac{ax}{2}+\frac{1}{2}(x^2-a^2)\tanh^{-1}\frac{x}{a}+C$

668. $\displaystyle\int x\coth^{-1}\frac{x}{a}\,dx = \frac{ax}{2}+\frac{1}{2}(x^2-a^2)\coth^{-1}\frac{x}{a}+C$

669. $\displaystyle\int x\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{2}x^2\operatorname{sech}^{-1}\frac{x}{a} - \frac{1}{2}a\sqrt{a^2-x^2}, & \operatorname{sech}^{-1}(x/a)>0 \\ \frac{1}{2}x\operatorname{sech}^{-1}\frac{x}{a} + \frac{1}{2}a\sqrt{a^2-x^2}+C, & \operatorname{sech}^{-1}(x/a)<0 \end{cases}$

670. $\displaystyle\int x\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{1}{2}x^2\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{2}\sqrt{x^2+a^2}+C, \quad + \text{ for } x>0 \text{ and } - \text{ for } x<0$

671. $\displaystyle\int x^2\sinh^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\sinh^{-1}\frac{x}{a}+\frac{1}{9}(2a^2-x^2)\sqrt{x^2+a^2}+C$

672. $\displaystyle\int x^2\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{3}x^3\cosh^{-1}\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)>0 \\ \frac{1}{3}x^3\cosh^{-1}\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)<0 \end{cases}$

673. $\displaystyle\int x^2\tanh^{-1}\frac{x}{a}\,dx = \frac{a}{6}x^2+\frac{1}{3}x^3\tanh^{-1}\frac{x}{a}+\frac{1}{6}a^3\ln|a^2-x^2|+C$

674. $\displaystyle\int x^2\coth^{-1}\frac{x}{a}\,dx = \frac{a}{6}x^2+\frac{1}{3}x^3\coth^{-1}\frac{x}{a}+\frac{1}{6}a^3\ln|x^2-a^2|+C$

675. $\displaystyle\int x^2\operatorname{sech}^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\operatorname{sech}^{-1}\frac{x}{a} - \frac{1}{3}\int\frac{x^3\,dx}{\sqrt{x^2+a^2}}$

676. $\displaystyle\int x^2\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{3}\int\frac{x^2\,dx}{\sqrt{x^2+a^2}}$

677. $\displaystyle\int x^n\sinh^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\sinh^{-1}\frac{x}{a} - \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{x^2-a^2}}$

678. $\displaystyle\int x^n\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{n+1}x^{n+1}\cosh^{-1}\frac{x}{a} - \frac{1}{n+1}\int\frac{x^{n+1}}{\sqrt{x^2-a^2}}, & \cosh^{-1}(x/a)>0 \\ \frac{1}{n+1}x^{n+1}\cosh^{-1}\frac{x}{a} + \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{x^2-a^2}}, & \cosh^{-1}(x/a)<0 \end{cases}$

679. $\displaystyle\int x^n\tanh^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\tanh^{-1}\frac{x}{a} - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2-x^2}$

680. $\displaystyle\int x^n \coth^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\coth^{-1}\frac{x}{a} - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{a^2-x^2}$

681. $\displaystyle\int x^n \operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{1}{n+1}x^{n+1}\operatorname{sech}^{-1}\dfrac{x}{a} + \dfrac{a}{n+1}\displaystyle\int \frac{x^n\,dx}{\sqrt{a^2-x^2}}, & \operatorname{sech}^{-1}(x/a) > 0 \\ \dfrac{1}{n+1}x^{n+1}\operatorname{sech}^{-1}\dfrac{x}{a} - \dfrac{a}{n+1}\displaystyle\int \frac{x^n\,dx}{\sqrt{a^2-x^2}}, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$

682. $\displaystyle\int x^n \operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{n+1}\int \frac{x^n\,dx}{\sqrt{x^2+a^2}}, \quad + \text{ for } x>0,\ - \text{ for } x<0$

683. $\displaystyle\int \frac{1}{x}\sinh^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x}{a} - \dfrac{(x/a)^3}{2\cdot 3\cdot 3} + \dfrac{1\cdot 3(x/a)^5}{2\cdot 4\cdot 4\cdot 5} - \dfrac{1\cdot 3\cdot 5(x/a)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots + C, & |x| > a \\ \dfrac{1}{2}\left(\ln|\dfrac{2x}{a}|\right)^2 - \dfrac{(a/x)^2}{2\cdot 2\cdot 2} + \dfrac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} - \dfrac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} + \cdots + C, & x > a \\ -\dfrac{1}{2}\left(\ln|\dfrac{-2x}{a}|\right)^2 + \dfrac{(a/x)^2}{2\cdot 2\cdot 2} - \dfrac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} + \dfrac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} + \cdots + C, & x < -a \end{cases}$

684. $\displaystyle\int \frac{1}{x}\cosh^{-1}\frac{x}{a}\,dx = \pm\left[\frac{1}{2}\left(\ln|\frac{2x}{a}|\right)^2 + \frac{(a/x)^2}{2\cdot 2\cdot 2} + \frac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} + \frac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} + \cdots\right] + C$

$+$ for $\cosh^{-1}(x/a) > 0$, $-$ for $\cosh^{-1}(x/a) < 0$

685. $\displaystyle\int \frac{1}{x}\tanh^{-1}\frac{x}{a}\,dx = \frac{x}{a} + \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} + \cdots + C$

686. $\displaystyle\int \frac{1}{x}\coth^{-1}\frac{x}{a}\,dx = \frac{ax}{2} + \frac{1}{2}(x^2-a^2)\coth^{-1}\frac{x}{a} + C$

687. $\displaystyle\int \frac{1}{x}\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} -\dfrac{1}{2}\ln|\dfrac{a}{x}|\ \ln|\dfrac{4a}{x}| - \dfrac{(x/a)^2}{2\cdot 2\cdot 2} - \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} - \cdots + C, & \operatorname{sech}^{-1}(x/a) > 0 \\ \dfrac{1}{2}\ln|\dfrac{a}{x}|\ \ln|\dfrac{4a}{x}| + \dfrac{(x/a)^2}{2\cdot 2\cdot 2} + \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} + \cdots, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$

688. $\displaystyle\int \frac{1}{x}\operatorname{csch}^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{1}{2}\ln|\dfrac{x}{a}|\ \ln|\dfrac{4a}{x}| + \dfrac{(x/a)^2}{2\cdot 2\cdot 2} - \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} + \cdots + C, & 0 < x < a \\ \dfrac{1}{2}\ln|\dfrac{-x}{a}|\ \ln|\dfrac{-x}{4a}| - \dfrac{(x/a)^2}{2\cdot 2\cdot 2} + \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} - \cdots, & -a < x < 0 \\ -\dfrac{a}{x} + \dfrac{(a/x)^3}{2\cdot 3\cdot 3} - \dfrac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \cdots + C, & |x| > a \end{cases}$

Integrals evaluated by reduction formula

689. If $S_n = \displaystyle\int \sin^n x\,dx$, then $S_n = -\dfrac{1}{n}\sin^{n-1}x\cos x + \dfrac{n-1}{n}S_{n-2}$

690. If $C_n = \displaystyle\int \cos^n x\,dx$, then $C_n = \dfrac{1}{n}\sin x\cos^{n-1}x + \dfrac{n-1}{n}C_{n-2}$

691. If $I_n = \displaystyle\int \frac{\sin^n ax}{\cos ax}\,dx$, then $I_n = \dfrac{-1}{(n-1)\,a}\sin^{n-1}ax + I_{n-2}$

692. If $I_n = \displaystyle\int \frac{\cos^n ax}{\sin ax}\,dx$, then $I_n = \dfrac{1}{(n-1)\,a}\cos^{n-1}ax + I_{n-2}$

693. If $S_m = \int x^m \sin nx\,dx$ and $C_m = \int x^m \cos nx\,dx$, then

$$S_m = \frac{-1}{n}x^m \cos nx + \frac{m}{n}C_{m-1} \qquad \text{and} \qquad C_m = \frac{1}{n}x^m \sin nx - \frac{m}{n}S_{m-1}$$

694. If $I_1 = \int \tan x\,dx$, and $I_n = \int \tan^n x\,dx$, then $I_n = \dfrac{1}{n-1}\tan^{n-1} x - I_{n-2}, \quad n = 2, 3, 4, \dots$

695. If $I_n = \int \dfrac{\sin^n ax}{\cos ax}\,dx$, then $I_n = -\dfrac{\sin^{n-1} ax}{(n-1)a} + I_{n-2}$

696. If $I_n = \int \dfrac{\cos^n ax}{\sin ax}\,dx$, then $I_n = \dfrac{\cos^{n-1} ax}{(n-1)a} + I_{n-2}$

697. If $I_{n,m} = \int \sin^n x \cos^m x\,dx$, then

$$\begin{aligned}
I_{n,m} =& \frac{-1}{n+m}\sin^{n-1} x \cos^{m+1} x + \frac{n-1}{n+m}I_{n-2,m}\\
I_{n,m} =& \frac{1}{n+1}\sin^{n+1} x \cos^{m+1} x + \frac{n+m+2}{n+1}I_{n+2,m}\\
I_{n,m} =& \frac{1}{n+m}\sin^{n+1} x \cos^{m-1} x + \frac{m-1}{n+m}I_{n,m+2}\\
I_{n,m} =& \frac{-1}{m+1}\sin^{n+1} x \cos^{m+1} x + \frac{n+m+2}{m+1}I_{n,m+2}\\
I_{n,m} =& \frac{-1}{m+1}\sin^{n-1} x \cos^{m+1} x + \frac{n-1}{m+1}I_{n-2,m+2}\\
I_{n,m} =& \frac{1}{n+1}\sin^{n+1} x \cos^{m-1} x + \frac{m-1}{n+1}I_{n+2,m-2}
\end{aligned}$$

698. If $S_n = \int e^{ax}\sin^n bx\,dx$ and $C_n = \int e^{ax}\cos^n bx\,dx$, then

$$\begin{aligned}
C_n =& e^{ax}\cos^{n-1} bx\left[\frac{a\cos bx + nb\sin bx}{a^2+n^2b^2}\right] + \frac{n(n-1)\,b^2}{a^2+n^2b^2}C_{n-2}\\
S_n =& e^{ax}\sin^{n-1} ax\left[\frac{a\sin bx - nb\cos nx}{a^2+n^2b^2}\right] + \frac{n(n-1)\,b^2}{a^2+n^2b^2}S_{n-2}
\end{aligned}$$

699. If $I_n = \int x^m(\ln x)^n\,dx$, then $I_n = \dfrac{1}{m+1}x^{m+1}(\ln x)^n - \dfrac{n}{m+1}I_{n-1}$

Integrals involving Bessel functions

700. $\int J_1(x)\,dx = -J_0(x) + C$

701. $\int xJ_1(x)\,dx = -xJ_0(x) + \int J_0(x)\,dx$

702. $\int x^n J_1(x)\,dx = -x^n J_0(x) + n\int x^{n-1}J_0(x)\,dx$

703. $\displaystyle\int \frac{J_1(x)}{x}\,dx = -J_1(x) + \int J_0(x)\,dx$

704. $\displaystyle\int x^\nu J_{\nu-1}(x)\,dx = x^\nu J_\nu(x) + C$

705. $\displaystyle\int x^{-\nu} J_{\nu+1}(x)\,dx = x^{-\nu} J_\nu(x) + C$

706. $\displaystyle\int \frac{J_1(x)}{x^n}\,dx = \frac{-1}{n}\frac{J_1(x)}{x^{n-1}} + \frac{1}{n}\int \frac{J_0(x)}{x^{n-1}}\,dx$

707. $\displaystyle\int xJ_0(x)\,dx = xJ_1(x) + C$

708. $\displaystyle\int x^2 J_0(x)\,dx = x^2 J_1(x) + xJ_0(x) - \int J_0(x)\,dx$

709. $\displaystyle\int x^n J_0(x)\,dx = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2\int x^{n-2}J_0(x)\,dx$

710. $\displaystyle\int \frac{J_0(x)}{x^n}\,dx = \frac{J_1(x)}{(n-1)^2x^{n-2}} - \frac{J_0(x)}{(n-1)x^{n-1}} - \frac{1}{(n-1)^2}\int \frac{J_0(x)}{x^{n-2}}\,dx$

711. $\displaystyle\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2J_n(x)$

712. $\displaystyle\int xJ_n(\alpha x)J_n(\beta x)\,dx = \frac{x}{\beta^2-\alpha^2}\left[\alpha J_n'(\alpha x)J_n(\beta x) - \beta J_n'(\beta x)J_n(\alpha x)\right] + C$

713. If $\displaystyle I_{m,n} = \int x^m J_n(x)\,dx$, $\quad m \geq -n$, then

$$I_{m,n} = -x^m J_{n-1}(x) + (m+n-1)\,I_{m-1,n-1}$$

714. If $\displaystyle I_{n,0} = \int x^n J_0(x)\,dx$, then $I_{n,0} = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 I_{n-2,0}$ Note that $\displaystyle I_{1,0} = \int xJ_0(x)\,dx = xJ_1(x) + C$ and $\displaystyle I_{0,1} = \int J_1(x)\,dx = -J_0(x) + C$ Note also that the integral $\displaystyle I_{0,0} = \int J_0(x)\,dx$ cannot be given in closed form.

Definite integrals

General integration properties

1. If $\dfrac{dF(x)}{dx} = f(x)$, then $\displaystyle\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$

2.

$$\int_0^\infty f(x)\,dx = \lim_{b\to\infty}\int_0^b f(x)\,dx, \qquad \int_{-\infty}^\infty f(x)\,dx = \lim_{\substack{b\to\infty\\ a\to-\infty}}\int_a^b f(x)\,dx$$

3. If $f(x)$ has a singular point at $x=b$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\int_a^{b-\epsilon} f(x)\,dx$

4. If $f(x)$ has a singular point at $x=a$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\int_{a+\epsilon}^{b} f(x)\,dx$

5. If $f(x)$ has a singular point at $x=c$, $a<c<b$, then $\displaystyle\int_a^b f(x)\,dx = \int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx$

6.

$$\begin{aligned}\int_a^b cf(x)\,dx &= c\int_a^b f(x)\,dx, \quad c \text{ constant}\\ \int_a^a f(x)\,dx &= 0,\\ \int_0^b f(x)\,dx &= \int_0^b f(b-x)\,dx\end{aligned}$$

$$\begin{aligned}\int_a^b f(x)\,dx &= -\int_b^a f(x)\,dx,\\ \int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\end{aligned}$$

7. Mean value theorems

$$\int_a^b f(x)\,dx = f(c)(b-a), \quad a\le c\le b$$

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx, \quad g(x)\ge 0,\ a\le c\le b$$

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^\xi g(x)\,dx \qquad \int_a^b f(x)g(x)\,dx = f(b)\int_\eta^b g(x)\,dx$$

$$a<\xi<b \qquad\qquad a<\eta<b$$

The last mean value theorem requires that $f(x)$ be monotone increasing and nonnegative throughout the interval (a,b)

8. Numerical integration

Divide the interval (a,b) into n equal parts by defining a step size $h=\frac{b-a}{n}$.

Two numerical integration schemes are

(a) Trapezoidal rule with global error $-\frac{(b-a)}{12}h^2 f''(\xi)$ for $a<\xi<b$.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots 2f(x_{n-1}) + f(x_n)\right]$$

(b) Simpson's 1/3 rule with global error $-\frac{(b-a)}{90}h^4 f^{(iv)}(\xi)$ for $a<\xi<b$.

$$\int_a^b f(x)\,dx = \frac{2h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

9. If $f(x)$ is periodic with period L, then $f(x+L) = f(x)$ for all x and $\int_0^{nL} f(x)\,dx = n\int_0^L f(x)\,dx$, for integer values of n.

10.

$$\underbrace{\int_0^x dx \int_0^x dx \cdots \int_0^x}_{n \text{ integration signs}} dx\, f(x) = \frac{1}{(n-1)!}\int_0^x (x-u)^{n-1} f(u)\,du$$

Integrals containing algebraic terms

11. $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad m>0, n>0$

12. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}}\left[\Gamma(\frac{1}{4})\right]^2$

13. $\displaystyle\int_0^1 \frac{dx}{(1-x^{2n})^{n/2}} = \frac{\pi}{2n\sin\frac{\pi}{2n}}$

14. $\displaystyle\int_0^1 \frac{1}{\beta-\alpha x}\frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{\beta(\beta-\alpha)}}$

15. $\displaystyle\int_0^1 \frac{x^p - x^{-p}}{x^q - x^{-q}}\frac{dx}{x} = \frac{\pi}{2q}\tan\frac{p\pi}{2q}, \qquad |p|<q$

16. $\displaystyle\int_0^1 \frac{x^p + x^{-p}}{x^q + x^{-q}}\frac{dx}{x} = \frac{\pi}{2q}\sec\frac{p\pi}{2q}, \qquad |p|<q$

17. $\displaystyle\int_0^1 \frac{x^{p-1} - x^{1-p}}{1-x^2}\,dx = \frac{\pi}{2}\cot\frac{p\pi}{2}, \qquad 0<p<2$

18. $\displaystyle\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \frac{\pi}{2}$

19. $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx = \frac{\pi}{4}a^2$

20. $\displaystyle\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$

21. $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin\alpha\pi}, \qquad 0<\alpha<1$

22. $\displaystyle\int_0^1 \frac{x^{\alpha-1}+x^{-\alpha}}{1+x}\,dx = \frac{\pi}{\sin\alpha\pi}, \qquad 0<\alpha<1$

23. $\displaystyle\int_0^\infty \frac{x^m\,dx}{1+x^2} = \frac{\pi}{2}\sec\frac{m\pi}{2}$

24. $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1-x^2}\,dx = \frac{\pi}{2}\cot\frac{\alpha\pi}{2}$

25. $\displaystyle\int_0^\infty \frac{dx}{1-x^n} = \frac{\pi}{n}\cot\frac{\pi}{n}$

26. $\displaystyle\int_0^\infty \frac{dx}{(a^2x^2+c^2)(x^2+b^2)} = \frac{\pi}{2bc}\frac{1}{c+ab}$

27. $\displaystyle\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2}\frac{1}{ab\,(a+b)}$

28. $\displaystyle\int_0^\infty \frac{dx}{(a^2-x^2)(x^2+p^2)} = \frac{\pi}{2p}\frac{1}{a^2+p^2}$

29. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(a^2-x^2)(x^2+p^2)} = \frac{\pi}{2}\frac{p}{a^2+p^2}$

30. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2(a+b)(b+c)(c+a)}$

31. $\displaystyle\int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} = \frac{\pi}{\sqrt{2}}$

32. $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x)(1+x^2)} = \frac{\pi}{4}$

Integrals containing trigonometric terms

33. $\displaystyle\int_0^1 \frac{\sin^{-1}x}{x}\,dx = \frac{\pi}{2}\ln 2$

34. $\displaystyle\int_0^{\pi/2} \frac{\tan^{-1}(\frac{b}{a}\tan\theta)\,d\theta}{\tan\theta} = \frac{\pi}{2}\ln|1+\frac{b}{a}|$

35. $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx = \frac{\pi}{4}$

36. $\displaystyle\int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$

37. $\displaystyle\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{\cos^{-1}(b/a)}{\sqrt{a^2-b^2}}$

38. $\displaystyle\int_0^{\pi/2} \sin^{2m-1}x\cos^{2n-1}x\,dx = B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad m>0, n>0$

39. $\displaystyle\int_0^{\pi/2} \sin^p x\cos^q x\,dx = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q}{2}+1)}$

40. $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\tan^m x} = \frac{\pi}{4}$

41. $\displaystyle\int_0^{\pi} \cos p\theta\cos q\theta\,d\theta = \begin{cases} 0, & p\neq q \\ \frac{\pi}{2}, & p=q \end{cases}$

42. $$\int_0^\pi \sin p\theta \sin q\theta \, d\theta = \begin{cases} 0, & p \neq q \\ \dfrac{\pi}{2}, & p = q \end{cases}$$

43. $$\int_0^\pi \sin p\theta \cos q\theta \, d\theta = \begin{cases} 0, & p+q \text{ even} \\ \dfrac{2p}{p^2 - q^2}, & p+q \text{ odd} \end{cases}$$

44. $$\int_0^\pi \frac{x\,dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2-1}}$$

45. $$\int_0^\pi \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}$$

46. $$\int_0^\pi \frac{\sin\theta\,d\theta}{1 - 2a\cos\theta + a^2} = \frac{2}{a}\tanh^{-1} a$$

47. $$\int_0^\pi \frac{\sin 2\theta\,d\theta}{1 - 2a\cos\theta + a^2} = \frac{2}{a^2}(1 + a^2)\tanh^{-1} a - \frac{2}{a}$$

48. $$\int_0^\pi \frac{x\sin x\,dx}{1 - 2a\cos x + a^2} = \begin{cases} \dfrac{\pi}{a}\ln(1+a), & |a| < 1 \\ \pi\ln\left(1 + \dfrac{1}{a}\right), & |a| > 1 \end{cases}$$

49. $$\int_0^\pi \frac{\cos p\theta\,d\theta}{1 - 2a\cos\theta + a^2} = \begin{cases} \dfrac{\pi a^p}{1 - a^2}, & a^2 < 1 \\ \dfrac{\pi a^{-p}}{a^2 - 1}, & a^2 > 1 \end{cases}$$

50. $$\int_0^\pi \frac{\cos p\theta\,d\theta}{(1 - 2a\cos\theta + a^2)^2} = \begin{cases} \dfrac{\pi a^p}{(1 - a^2)^3}[(p+1) - (p-1)a^2], & a^2 < 1 \\ \dfrac{\pi a^{-p}}{(a^2 - 1)^3}[(1-p) + (1+p)a^2], & a^2 > 1 \end{cases}$$

51. $$\int_0^\pi \frac{\cos p\theta\,d\theta}{(1 - 2a\cos\theta + a^2)^3} = \begin{cases} \dfrac{\pi a^p}{2(1 - a^2)^5}\left[(p+2)(p+1) + 2(p+2)(p-2)a^2 + (p-2)(p-1)a^4\right], & a^2 < 1 \\ \dfrac{\pi a^{-p}}{2(a^2 - 1)^5}\left[(1-p)(2-p) + 2(2-p)(2+p)a^2 + (2+p)(1+p)a^4\right], & a^2 > 1 \end{cases}$$

52. $$\int_0^{2\pi} \frac{dx}{(a + b\sin x)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

53. $$\int_0^{2\pi} \frac{dx}{a + b\sin x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

54. $$\int_0^{2\pi} \frac{dx}{a + b\cos x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

55. $$\int_0^{2\pi} \frac{dx}{(a + b\sin x)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

56. $$\int_0^{2\pi} \frac{dx}{(a + b\cos x)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

57. $$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n, \quad m, n \text{ integers} \\ \frac{L}{2}, & m = n \end{cases}$$

58. $\displaystyle\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0$ for all integer m, n values

59. $\displaystyle\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \neq 0 \\ L, & m = n = 0 \end{cases}$

60. $\displaystyle\int_0^\infty \frac{x^m\,dx}{1+2x\cos\beta + x^2} = \frac{\pi}{\sin m\pi}\frac{\sin m\beta}{\sin\beta}$

61. $\displaystyle\int_0^\infty \frac{\sin\alpha x}{x}\,dx = \begin{cases} \pi/2, & \alpha > 0 \\ 0, & \alpha = 0 \\ -\pi/2, & \alpha < 0 \end{cases}$

62. $\displaystyle\int_0^\infty \frac{\sin\alpha x\sin\beta x}{x}\,dx = \begin{cases} 0, & \alpha > \beta > 0 \\ \pi/2, & 0 < \alpha < \beta \\ \pi/4, & \alpha = \beta > 0 \end{cases}$

63. $\displaystyle\int_0^\infty \frac{\sin\alpha x\sin\beta x}{x^2}\,dx = \begin{cases} \frac{\pi\alpha}{2}, & 0 < \alpha \leq \beta \\ \frac{\pi\beta}{2}, & \alpha \geq \beta > 0 \end{cases}$

64. $\displaystyle\int_0^\infty \frac{\sin^2\alpha x}{x^2}\,dx = \frac{\pi\alpha}{2}$

65. $\displaystyle\int_0^\infty \frac{1-\cos\alpha x}{x^2}\,dx = \frac{\pi\alpha}{2}$

66. $\displaystyle\int_0^\infty \frac{\cos\alpha x}{x^2+a^2}\,dx = \frac{\pi}{2a}e^{-\alpha a}$

67. $\displaystyle\int_0^\infty \frac{x\sin\alpha x}{x(x^2+a^2)}\,dx = \frac{\pi}{2}e^{-\alpha a}$

68. $\displaystyle\int_0^\infty \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}$

69. $\displaystyle\int_0^\infty \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}$

70. $\displaystyle\int_0^\infty \frac{\tan x}{x}\,dx = \frac{\pi}{2}$

71. $\displaystyle\int_0^\infty \frac{\sin\alpha x}{x(x^2+a^2)}\,dx = \frac{\pi}{2a^2}(1-e^{-\alpha a})$

72. $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$

73. $\displaystyle\int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$

74. $\displaystyle\int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$

75. $\displaystyle\int_0^\infty \sin ax^2 \cos 2bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} - \sin\frac{b^2}{a}\right)$

76. $\displaystyle\int_0^\infty \cos ax^2 \cos 2bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} + \sin\frac{b^2}{a}\right)$

77. $\displaystyle\int_0^\infty \frac{dx}{x^4 + 2a^2x^2\cos 2\beta + a^4} = \frac{\pi}{4a^3\cos\beta}$

78. $\displaystyle\int_0^\infty \cos\left(x^2 + \frac{a^2}{x^2}\right) dx = \frac{\sqrt{\pi}}{2}\cos(\frac{\pi}{4} + 2a)$

79. $\displaystyle\int_0^\infty \sin\left(x^2 + \frac{a^2}{x^2}\right) dx = \frac{\sqrt{\pi}}{2}\sin(\frac{\pi}{4} + 2a)$

80. $\displaystyle\int_0^\infty \frac{\tan bx\, dx}{x(p^2+x^2)} = \frac{\pi}{2p^2}\tanh bp$

81. $\displaystyle\int_0^\infty \frac{x\tan bx\, dx}{p^2+x^2} = \frac{\pi}{2} - \frac{\pi}{2}\tanh bp$

82. $\displaystyle\int_0^\infty \frac{x\cot bx\, dx}{p^2+x^2} = \frac{\pi}{2}\coth bp$

83. $\displaystyle\int_0^\infty \frac{\sin ax}{\sin bx}\frac{dx}{(p^2+x^2)} = \frac{\pi}{2p}\frac{\sinh ap}{\sinh bp}, \qquad a < b$

84. $\displaystyle\int_0^\infty \frac{\cos ax}{\cos bx}\frac{dx}{(p^2+x^2)} = \frac{\pi}{2p}\frac{\cosh ap}{\cosh bp}, \qquad a < b$

85. $\displaystyle\int_0^\infty \frac{\sin ax}{\cos bx}\frac{dx}{(p^2+x^2)} = \frac{\pi}{2p^2}\frac{\sinh ap}{\cosh bp}, \qquad a < b$

86. $\displaystyle\int_0^\infty \frac{\sin ax}{\cos bx}\frac{x\, dx}{(x^2+p^2)} = -\frac{\pi}{2}\frac{\sinh ap}{\cosh bp}, \qquad a < b$

87. $\displaystyle\int_0^\infty \frac{\cos ax}{\sin bx}\frac{x\, dx}{(p^2+x^2)} = \frac{\pi}{2}\frac{\cosh ap}{\sinh bp}, \qquad a < b$

Integrals containing exponential and logarithmic terms

88. $\displaystyle\int_0^1 \frac{\ln\frac{1}{x}}{1+x}\, dx = \frac{\pi^2}{12}$

89. $\displaystyle\int_0^1 \frac{\ln\frac{1}{x}}{(1-x)}\, dx = \frac{\pi^2}{6}$

90. $\displaystyle\int_0^1 \frac{\left(\ln\frac{1}{x}\right)^3}{1-x}\, dx = \frac{\pi^4}{15}$

91. $\displaystyle\int_0^1 \frac{\ln(1+x)}{x}\, dx = \frac{\pi^2}{12}$

92. $$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}$$

93. $$\int_0^1 (ax^2+bx+c)\frac{\ln\frac{1}{x}}{1-x}\,dx = (a+b+c)\frac{\pi^2}{6}-(a+b)-\frac{a}{4}$$

94. $$\int_0^1 \frac{\ln\frac{1}{x}}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\ln 2$$

95. $$\int_0^1 \frac{1-x^{p-1}}{(1-x)(1-x^p)}(\ln\frac{1}{x})^{2n-1}\,dx = \frac{1}{4n}(1-\frac{1}{p^{2n}})(2\pi)^{2n}\,\mathfrak{B}_{2n-1}$$

96. $$\int_0^1 \frac{x^m-x^n}{\ln x}\,dx = \ln\left|\frac{1+m}{1+n}\right|$$

97. $$\int_0^1 x^p(\ln x)^n\,dx = \begin{cases} (-1)^n\dfrac{n!}{(p+1)^{n+1}}, & \text{n an integer}\\ (-1)^n\dfrac{\Gamma(n+1)}{(p+1)^{n+1}}, & \text{n noninteger}\end{cases}$$

98. $$\int_0^{\pi/4} \ln(1+\tan x)\,dx = \frac{\pi}{8}\ln 2$$

99. $$\int_0^{\pi/2} \ln\sin\theta\,d\theta = \frac{\pi}{2}\ln(\frac{1}{2})$$

100. $$\int_0^{\pi} \ln(a+b\cos x)\,dx = \pi\ln\left|\frac{a+\sqrt{a^2+b^2}}{2}\right|$$

101. $$\int_0^{2\pi} \ln(a+b\cos x)\,dx = 2\pi\ln|a+\sqrt{a^2-b^2}|$$

102. $$\int_0^{2\pi} \ln(a+b\sin x)\,dx = 2i\ln|a+\sqrt{a^2-b^2}|$$

103. $$\int_0^{\infty} e^{-ax}\,dx = \frac{1}{a}$$

104. $$\int_0^{\infty} x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}}$$

105. $$\int_0^{\infty} e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma(\frac{1}{2})$$

106. $$\int_0^{\infty} x^n e^{-a^2x^2}\,dx = \frac{\Gamma(\frac{m+1}{2})}{2a^{m+1}}$$

107. $$\int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$$

108. $$\int_0^{\infty} e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

109. $$\int_0^\infty e^{-ax}\frac{\sin bx}{x}\,dx = \tan^{-1}\frac{b}{a}$$

110. $$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\frac{b}{a}$$

111. $$\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}e^{-b^2/4a^2}$$

112. $$\int_0^\infty e^{-(ax^2+b/x^2)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}$$

113. $$\int_0^\infty x^{2n}e^{-\beta x^2}\,dx = \frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{2^{n+1}\beta^n}\sqrt{\frac{\pi}{\beta}}$$

114. $$\int_0^\infty e^{-k\left(\frac{x^2}{a^2}+\frac{b^2}{x^2}\right)}\,dx = \frac{\sqrt{\pi}}{2}\frac{a}{\sqrt{k}}e^{-2kb/a}$$

115. $$\int_0^\infty \frac{\sin rx\,dx}{x(x^4+2a^2x^2\cos 2\beta+a^4)} = \frac{\pi}{2a^4}\left[1-\frac{\sin(ar\sin\beta+2\beta)}{\sin 2\beta}\,e^{-\beta r\cos\beta}\right]$$

116. $$\int_0^\infty \frac{\cos rx\,dx}{x^4+2a^2x^2\cos 2\beta+a^4} = \frac{\pi}{2a^3}\frac{\sin(\beta+ar\sin\beta)}{\sin 2\beta}\,e^{-ar\cos\beta}$$

117. $$\int_0^\infty \frac{\sin rx\,dx}{x(x^6+a^6)} = \frac{\pi}{6a^6}\left[3-e^{-ar}-2e^{-ar/2}\cos\frac{ar\sqrt{3}}{2}\right]$$

118. $$\int_0^\infty \frac{\cos rx\,dx}{x^6+a^6} = \frac{\pi}{6a^5}\left[e^{-ar}-2e^{-ar/2}\cos(\frac{ar\sqrt{3}}{2}+\frac{2\pi}{3})\right]$$

119. $$\int_0^\infty \frac{\sin\pi x\,dx}{x(1-x^2)} = \pi$$

120. $$\int_0^\infty \frac{e^{-qx}-e^{-px}}{x}\cos bx\,dx = \frac{1}{2}\ln\left|\frac{p^2+b^2}{q^2+b^2}\right|$$

121. $$\int_0^\infty \frac{e^{-qx}-e^{-px}}{x}\sin bx\,dx = \tan^{-1}\frac{p}{b}-\tan^{-1}\frac{q}{b}$$

122. $$\int_0^\infty e^{-ax}\frac{\sin px-\sin qx}{x}\,dx = \tan^{-1}\frac{p}{a}-\tan^{-1}\frac{q}{b}$$

123. $$\int_0^\infty e^{-ax}\frac{\cos px-\cos qx}{x}\,dx = \frac{1}{2}\ln\left|\frac{a^2+a^2}{a^2+p^2}\right|$$

124. $$\int_0^\infty xe^{-x^2}\sin ax\,dx = \frac{a\sqrt{\pi}}{4}e^{-a^2/4}$$

125. $$\int_0^\infty x^2e^{-x^2}\cos ax\,dx = \frac{\sqrt{\pi}}{4}\left(1-\frac{a^2}{2}\right)e^{-a^2/4}$$

126. $\displaystyle\int_0^\infty x^3 e^{-x^2} \sin ax\, dx = \frac{\sqrt{\pi}}{8}\left(3a - \frac{a^3}{2}\right) e^{-a^2/4}$

127. $\displaystyle\int_0^\infty x^4 e^{-x^2} \cos ax\, dx = \frac{\sqrt{\pi}}{8}\left(3 - 3a^2 + \frac{a^4}{4}\right) e^{-a^2/4}$

128. $\displaystyle\int_0^\infty \left(\frac{\ln x}{x-1}\right)^3 dx = \pi^2$

129. $\displaystyle\int_{-\infty}^\infty \frac{x \sin rx\, dx}{(x-b)^2 + a^2} = \frac{\pi}{a}(a\cos br + b \sin br)\, e^{-ar}$

130. $\displaystyle\int_{-\infty}^\infty \frac{\sin rx\, dx}{x[(x-b)^2 + a^2]} = \frac{\pi}{a(a^2+b^2)}\left[a - (\cos br - b\sin br)\, e^{-ar}\right]$

131. $\displaystyle\int_{-\infty}^\infty \frac{\cos rx\, dx}{(x-b)^2 + a^2} = \frac{\pi}{a} e^{-ar} \cos br$

132. $\displaystyle\int_{-\infty}^\infty \frac{\sin rx\, dx}{(x-b)^2 + a^2} = \frac{\pi}{a} e^{-ar} \sin br$

133. $\displaystyle\int_{-\infty}^\infty e^{-x^2} \cos 2nx\, dx = \sqrt{\pi}\, e^{-n^2}$

134. $\displaystyle\int_0^\infty \frac{x^{p-1} \ln x}{1+x}\, dx = \frac{-\pi^2}{\sin p\pi} \cot p\pi, \qquad 0 < p < 1$

135. $\displaystyle\int_0^\infty e^{-x} \ln x\, dx = -\gamma$

136. $\displaystyle\int_0^\infty e^{-x^2} \ln x\, dx = -\frac{\sqrt{\pi}}{4}(\gamma + 2\ln 2)$

137. $\displaystyle\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right) dx = \frac{\pi^2}{4}$

138. $\displaystyle\int_0^\infty \frac{x\, dx}{e^x - 1} = \frac{\pi^2}{6}$

139. $\displaystyle\int_0^\infty \frac{x\, dx}{e^x + 1} = \frac{\pi^2}{12}$

Integrals containing hyperbolic terms

140. $\displaystyle\int_0^1 \frac{\sinh(m \ln x)}{\sinh(\ln x)}\, dx = \frac{\pi}{2} \tan \frac{m\pi}{2}, \qquad |m| < 1$

141. $\displaystyle\int_0^\infty \frac{\sin ax}{\sinh bx}\, dx = \frac{\pi}{2b} \tanh(\frac{\pi a}{2b})$

142. $\displaystyle\int_0^\infty \frac{\cos ax}{\cosh bx}\, dx = \frac{\pi}{2b} \text{sech}\,(\frac{\pi a}{2b})$

143. $\displaystyle\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2}$

144. $\displaystyle\int_0^\infty \frac{\sinh px}{\sinh qx}\,dx = \frac{\pi}{2q}\tan(\frac{\pi p}{2q}), \qquad |p| < q$

145. $\displaystyle\int_0^\infty \frac{\cosh ax - \cosh bx}{\sinh \pi x}\,dx = \ln\left|\frac{\cos\frac{b}{2}}{\cos\frac{a}{2}}\right|, \qquad -\pi < b < a < \pi$

146. $\displaystyle\int_0^\infty \frac{\sinh px}{\sinh qx}\cos mx\,dx = \frac{\pi}{2q}\,\frac{\sin\frac{\pi p}{q}}{\cos\frac{\pi p}{q} + \cosh\frac{\pi m}{q}}, \qquad q > 0, p^2 < q^2$

147. $\displaystyle\int_0^\infty \frac{\sinh px}{\cosh qx}\sin mx\,dx = \frac{\pi}{q}\,\frac{\sin\frac{p\pi}{2q}\sinh\frac{m\pi}{2q}}{\cos\frac{p\pi}{q} + \cosh\frac{m\pi}{q}}$

148. $\displaystyle\int_0^\infty \frac{\cosh px}{\cosh qx}\cos mx\,dx = \frac{\pi}{q}\,\frac{\cos\frac{p\pi}{2q}\cosh\frac{m\pi}{2q}}{\cos\frac{p\pi}{q} + \cosh\frac{m\pi}{q}}$

Miscellaneous Integrals

149. $\displaystyle\int_0^x \xi^{\lambda-1}[1-\xi^\mu]^\nu\,d\xi = \frac{x^\lambda}{\lambda}F(-\nu, \frac{\lambda}{\mu}; \frac{\lambda}{\mu}+1; x^\mu)$ See hypergeometric function

150. $\displaystyle\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi = \pi\,J_n(x)$

151. $\displaystyle\int_{-a}^a (a+x)^{m-1}(a-x)^{n-1}\,dx = (2a)^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

152. If $f'(x)$ is continuous and $\displaystyle\int_1^\infty \frac{f(x)-f(\infty)}{x}\,dx$ converges, then

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = [f(0)-f(\infty)]\,\ln\frac{b}{a}$$

153. If $f(x) = f(-x)$ so that $f(x)$ is an even function, then

$$\int_0^\infty f\left(x - \frac{1}{x}\right)\,dx = \int_0^\infty f(x)\,dx$$

154. Elliptic integral of the first kind

$$\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\theta,k), \qquad 0 < k < 1$$

155. Elliptic integral of the second kind

$$\int_0^\theta \sqrt{1-k^2\sin^2\theta}\,d\theta = E(\theta,k)$$

156. Elliptic integral of the third kind

$$\int_0^\theta \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \Pi(\theta,k,n)$$

Appendix D
Solutions to Selected Problems

Chapter 1

▶ **1-1.**

(a)

$A \cup B = \{\, x \,|\, -2 < x < 7 \,\}$

$A \cap B = \{\, x \,|\, 2 \le x \le 4 \,\}$

(b)

$A \cup B = \{\, x \,|\, -2 < x \le 7 \,\}$

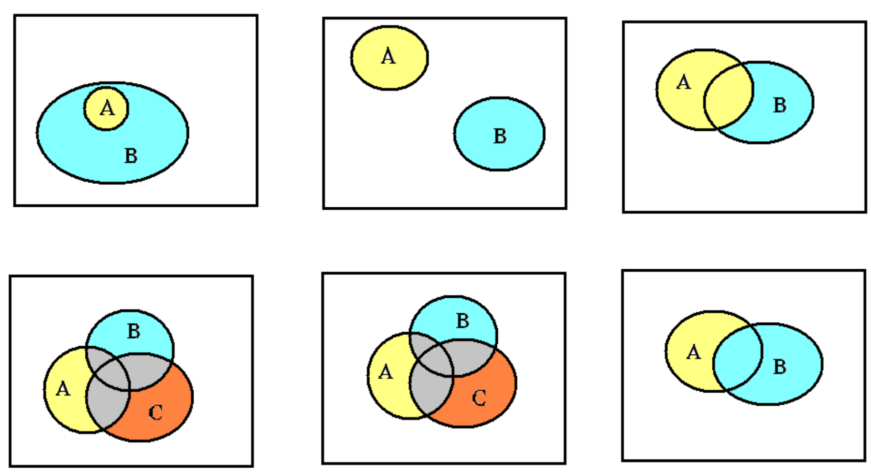
$A \cap B = \emptyset$

(c)

$A \cup B = \{\, x \,|\, x < 4 \,\}$

$A \cap B = \{\, x \,|\, -5 < x < 3 \}$

▶ **1-2.**



▶ **1-5.** (a) $A \cup (A \cap B) = (A \cap U) \cup (A \cap B)$ and use distributive law to write

$$(A \cap U) \cup (A \cap B) = A \cap (U \cup B) = A \cap U = A$$

▶ **1-8.**

(a) S_a bounded above $\ell.u.b. = 4$, S_a bounded below $g.\ell.b = -4$

(b) S_b bounded above $\ell.u.b. = 3$, S_b is not bounded below

(c) S_c bounded above $\ell.u.b. = 25$, S_c bounded below $g.\ell.b = 0$

(d) S_d is not bounded above, S_c bounded below $g.\ell.b. = 27$

▶ **1-9.**

$$(a) \quad y - 4 = -2(x-2)$$
$$(c) \quad y - 4 = -\frac{2}{3}(x-2)$$

▶ **1-10.**

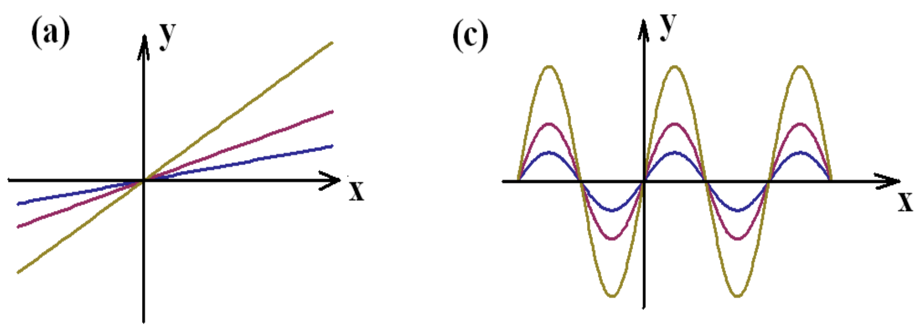
(a) $\quad y = -(3/4)x + 7/4, \qquad m = -3/4, \qquad b = 7/4$

(c) $\quad$ Polar form of $3x + 4y = 7$ is $r\cos(\theta - \beta) = d$, where $d = 7/5$ and $\tan\beta = 4/3$

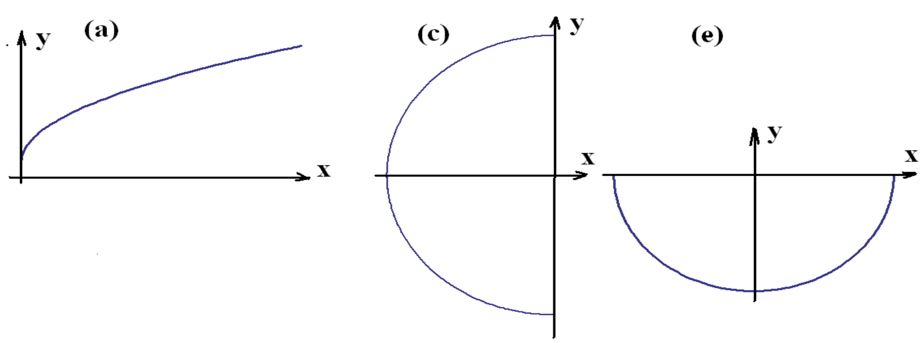
▶ **1-12.**

$$(b) \quad x \neq a, \qquad x \neq b, \qquad x \neq c$$
$$(c) \quad r \geq 0$$
$$(d) \quad x^3 + 1 > 0 \quad \Longrightarrow \quad x > -1$$

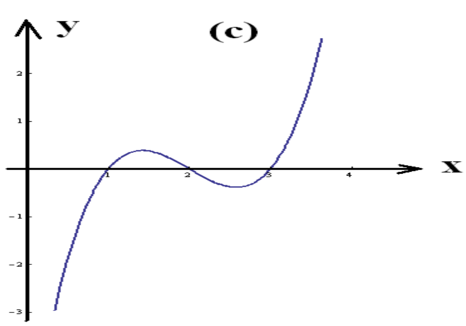
▶ **1-13.**



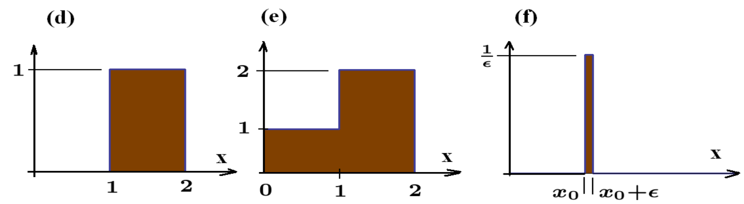
▶ **1-14.**



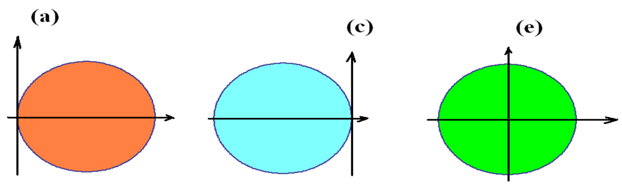
▶ **1-15.**



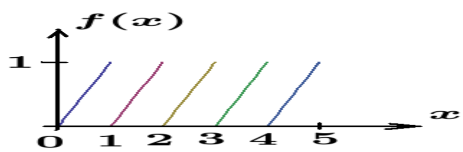
▶ **1-16.**



▶ **1-18.**



▶ **1-21.**

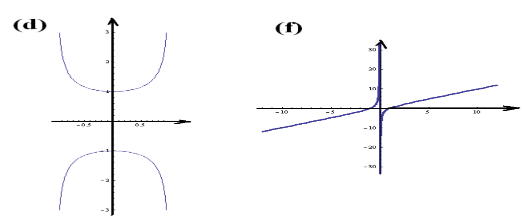


▶ **1-22.**

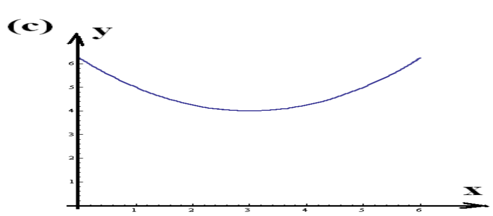
$$(e) \quad \frac{(x+h)^2 - x^2}{h} \qquad (g) \quad f(g(x)) = g^2(x) = (3-2x)^2$$

▶ **1-24.** (d) $f^{-1}(x) = x^3 - 4$

▶ **1-26.**



▶ **1-27.**



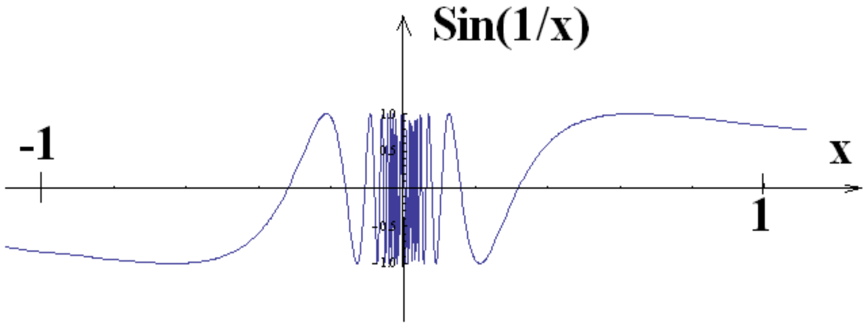
▶ **1-31.** (a) e^{α} (b) e^{β}

▶ **1-34.** (c) $y = -x$, (d) $y = \frac{4}{3}x$, (e) $y = \frac{3}{4}x$, (f) $y = \frac{-1}{2}x$, (g) $y = x$, (h) $y = \frac{4}{3}x$

▶ **1-37.** (b) $y - 4 = \frac{-1}{2}(x - 3)$

▶ **1-38.** (c) Multiply numerator and denominator by $\sqrt{x+h} + \sqrt{x}$ and simplify.
(d) $2x$
(f) $\frac{1}{4}$

▶ **1-39.** (c) Limit does not exist.



(e) Limit does not exist.
All other limits have finite values.

▶ **1-41.** (a) 4
(c) Multiply numerator and denominator by $(1 + \cos h)$
(e) 1

▶ **1-42.** $\frac{1}{2\sqrt{x}}$

▶ **1-43.** (a) $\sin x$ oscillates between -1 and +1.
(b) $\sin(\frac{1}{x})$ oscillates between -1 and +1.
(c) Put in form $\frac{\sin\theta}{\theta}$

▶ **1-44.** (d) $3y - 4x = 1$

▶ **1-45.** (b) Show $(1+p)^n > 1+np$ and $\dfrac{1}{(1+p)^n} < \dfrac{1}{1+np}$
(d) Show $(1+p)^n > 1+np$ and $1+np \to \infty$ as $n \to \infty$

▶ **1-47.** (c) Let the point $(3/2, 9/4)$ approach the point $(1,1)$, then secant line approaches the tangent line.
(d) $y-1 = 2(x-1)$

▶ **1-48.** (d) Complete the square $y^2-6y+9+12x-3-9=0$ and then simplify to obtain $(y-3)^2 = -12(x-1)$ From this equation show the focus is at (-2,3), the vertex is at (1,3), the directrix is the line $x=4$ ant the latus rectum is 12.

▶ **1-49.** (d) Complete the square on the x and y terms to obtain

$$25(y^2-6y+9)+16(x^2-4x+4) = 689-4(25)-4(16) = 400$$

which simplifies to

$$\frac{(y-3)^2}{4^2} + \frac{(x-2)^2}{5^2} = 1$$

representing an ellipse. Here the foci are at (5,3) and (-1,3), the directrices are at $x = 31/3$ and $x = -19/3$, the latus rectum is 32/5, the eccentricity is 3/5 and the center is (2,3).

▶ **1-50.** (d) Complete the square $(x^2-2x+1)-4(y^2-8y+16) = 67+1-64 = 4$ which simplifies to

$$\frac{(x-1)^2}{2^2} - \frac{(y-4)^2}{1^2} = 1$$

which is a hyperbola centered at (1,4). The foci are at $(1-2\sqrt{5}, 4)$ and $(1+2\sqrt{5}, 4)$, the verticies are at (-1,0) and (3,0, the directrices are at $x = 1-\sqrt{5}$ and $x = 1+\sqrt{5}$, the eccentricity is $\frac{2\sqrt{5}}{5}$ and the asymptotic lines are $y-4 = \frac{-1}{2}(x-1)$ and $y-4 = \frac{1}{2}(x-1)$.

▶ **1-51.** If $y_{line} = y_{parabola}$, then $x+b = 2\sqrt{x}$ so that x must satisfy $x^2+2xb+b^2 = 4x$ or $x^2+(2b-4)x+b^2 = 0$. This is quadratic equation with roots

$$x = \frac{4-2b \pm \sqrt{(2b-4)^2-4b^2}}{2}$$

The discriminant is $\sqrt{(2b-4)^2-4b^2} = \sqrt{16(1-b)}$ which tells one that

$b=1$, one point of intersection

$b<1$, two points of intersection

$b>1$, no points of intersection

▶ **1-52.** (a) If $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ are defined by equations (1.97), then

$$\bar{a} + \bar{c} = a(\cos^2\theta + \sin^2\theta) + c(\sin^2\theta + \cos^2\theta) = a + c$$

▶ **1-53.** Rotate axes $\pi/4$ radians and show that for $x = \bar{x}\frac{1}{\sqrt{2}} - \bar{y}\frac{1}{\sqrt{2}}$ and $y = \bar{x}\frac{1}{\sqrt{2}} + \bar{y}\frac{1}{\sqrt{2}}$, then

$$xy = \frac{\bar{x}^2}{2} - \frac{\bar{y}^2}{2} = a^2$$

▶ **1-54.** $(x+1)^2 = y^2$

▶ **1-57.** $F = \frac{9}{5}C + 32$

▶ **1-58.** (a) line $x = 3$ (c) circle $(x-2)^2 + y^2 = 2^2$

▶ **1-59.** (a) parabola opens to right $\epsilon < \theta < 2\pi - \epsilon$ (c) ellipse centered at (2,0) intersecting x-axis at (-2,0) and (6,0).

▶ **1-60.** (a) parabola opens upward $\frac{\pi}{2} + \epsilon < \theta < \frac{5\pi}{2} - \epsilon$ (c) ellipse centered at $(1,0)$ intersecting x-axis at (3,0) and (-1,0).

▶ **1-62.** $y = 3$ and $x = -2$

Chapter 2

► **2-1.** (b) $y - 4 = 4(x - 2)$ (c) $y - 4 = -4(x + 2)$

► **2-2.** (d) $\dfrac{dy}{dx} = -\dfrac{b + 2ax}{(ax^2 + bx + c)^2}$

(h) $\dfrac{dy}{dx} = 3(2x + 1)^2(2)(3x^2 - x)^2 + (2x + 1)^3 2(3x^2 - x)(6x - 1)$

which can also be expressed in the form

$\dfrac{dy}{dx} = 2x(3x - 1)(1 + 2x)^2(-1 + x + 21x^2)$

► **2-3.** (d) $\dfrac{dy}{dx} = \dfrac{-1}{2x^{3/2}}$ (j) $\dfrac{dy}{dx} = 1 + \ln x$ (l) $\dfrac{dy}{dx} = \dfrac{1}{n}(a + x)^{\frac{1}{n}-1}$

► **2-4.** (b) $\dfrac{dy}{dx} = -\dfrac{\sqrt{x}}{(1 + x)^2} + \dfrac{1}{2\sqrt{x}(1 + x)}$ (f) $\dfrac{dy}{dx} = \dfrac{x(3x^2 - 2x)}{2\sqrt{x^3 - x^2}} + \sqrt{x^3 - x^2}$

(k) $\dfrac{dy}{dx} = \dfrac{-x}{(1 + x^2)^{3/2}\sqrt{1 - \frac{1}{1+x^2}}}$

► **2-5.** (g) $\dfrac{dy}{d\theta} = 6\cos(3\theta)\sin(3\theta) = 3\sin(6\theta)$

(k) $\dfrac{dy}{dx} = -\dfrac{a - 2bx}{\sqrt{1 - (ax - bx^2)^2}}$

(l) $\dfrac{dy}{dx} = x^x(1 + \ln x) + x^{1/x}\left(\dfrac{1}{x^2} - \dfrac{\ln x}{x^2}\right)$

► **2-6.** (j) $y' = \dfrac{-2x}{(1 + x^2)^2\sqrt{\frac{1-x^2}{1+x^2}}}$

(k) $y' = \dfrac{(b + 2cx)\sec[\ln(a + bx + cx^2)]\tan[\ln(a + bx + cx^2)]}{a + bx + cx^2}$

► **2-7.** (e) $y' = \dfrac{-x}{y^2 - 2}$ (j) $y' = \dfrac{3x^2 - y + 6xy + 3y^2}{x - 3x^2 - 6xy - 3y^2}$ (l) $y' = \dfrac{yx^{y-1}}{1 - x^4\ln x}$

► **2-8.** (d) $y' = \dfrac{2x}{\sqrt{1 - x^4}}$ (g) $y' = \dfrac{1 + x}{\sqrt{1 - x^2}} + \sin^{-1} x$ (j) $y' = (\cos(3x))^x(\ln|\cos 3x| - 3x\tan(3x))$

► **2-9.** (d) $y' = (3 + x)^x\left(\dfrac{x}{3 + x} + \ln(3 + x)\right)$ (i) $y' = x^x(1 + \ln x)$

(l) $y' = \dfrac{6x\cos(4x)\sin^2(4x)}{\sqrt{\sin^3(4x)}} + \sqrt{\sin^3(4x)}$

► **2-10.** (b) $y' = \dfrac{-2ax}{\sqrt{1 - a^2x^4}}$ (h) $y' = \dfrac{3}{\sqrt{3x - 1}\sqrt{1 + 3x}} = \dfrac{3}{\sqrt{9x^2 - 1}}$

(l) $y' = \dfrac{3x^2}{\sqrt{1 - x^6}}$

► **2-11.** $\displaystyle\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$, let $x = x_0 + \Delta x$

► **2-13.** (d) $y' = \dfrac{3x\cos x}{2\sqrt{4+3\sin x}} + \sqrt{4+3\sin x}$

$$y'' = \frac{-9x\cos^2 x}{4(4+3\sin x)^{3/2}} + \frac{3\cos x}{\sqrt{4+3\sin x}} - \frac{3x\sin x}{2\sqrt{4+3\sin x}}$$

(e) $y' = \dfrac{ab-x^2}{(a-x)^2(b-x)^2}$

$$y'' = -\frac{2(a^2b+ab(b-3x)+x^2)}{(a-x)^3(x-b)^3}$$

► **2-14.**

$$\frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dt} \implies \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\text{Note that } \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt} = \frac{d^2y}{dx^2}\frac{dx}{dt}$$

so differentiating the above with respect to t gives

$$\frac{dy}{dx}\frac{d^2x}{dt^2} + \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 = \frac{d^2y}{dt^2} \implies \frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2} - \frac{dy}{dx}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}$$

(b)

$$x = 4\cos t \quad y = 4\sin t \text{ with } \frac{dx}{dt} = -4\sin t \quad \frac{dy}{dt} = 4\cos t \text{ and}$$

$$\frac{d^2x}{dt^2} = -4\cos t \quad \frac{d^2y}{dt^2} = -4\sin t$$

$$\text{so that } \frac{dy}{dx} = \frac{4\cos t}{-4\sin t} = -\frac{x}{y} \text{ and } \frac{d^2y}{dx^2} = \frac{-4\sin t - (-x/y)(-4\sin t)}{(-4\sin t)^2} = -\frac{y^2+x^2}{y^3}$$

Another method. $x^2+y^2 = 16$ so that $2x + 2yy' = 0 \implies y' = -x/y$ and $2 + 2yy'' + 2(y')^2 = 0 \implies y'' = -\dfrac{x^2+y^2}{y^3}$

► **2-15.** (e) $y' = x + 2x\ln(3x)$ (h) $y' = \dfrac{1+2x}{x+x^2}\cos(x^2) - 2x\ln(x+x^2)\sin(x^2)$

► **2-16.** (e) $y' = \dfrac{1}{(1+x^2)^{3/2}} - \dfrac{3x^2}{(1+x^2)^{5/2}}$ (h) $y' = 2x\cos(x^2)\ln(x^3) + \dfrac{3\sin(x^2)}{x}$

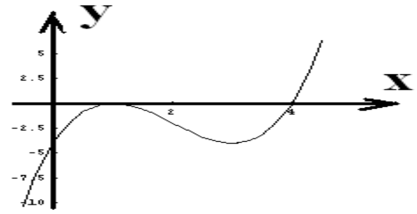
► **2-17.** (e)

$$y' = -6\cos(3x)\sin(3x)$$

$$y'' = -18\cos^2(3x) + 18\sin^2(3x)$$

▶ **2-18.** (b) $y' = \cos x$ so that $m = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and therefore
$(y - \dfrac{1}{\sqrt{2}}) = \dfrac{1}{\sqrt{2}}(x - \dfrac{\pi}{4})$

▶ **2-20.** (d)



▶ **2-21.**

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$\text{but} \qquad f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\text{therefor,} \qquad f''(x) = \lim_{h\to 0} \frac{\frac{f(x+2h)-f(x+h)}{h} - \left(\frac{f(x+h)-f(x)}{h}\right)}{h^2}$$

$$f''(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

▶ **2-22.** (b) $y' = \frac{x^2-1}{(x^2+1)^2}$ critical points at $x = \pm 1$

(e) local maximums at $x = (2n+1)\pi/2$
local minimums at $x = (2n+3)\pi/2$ where n is an integer.

▶ **2-23.** (d) Critical points at $x = \pm 1$ and curve symmetric about origin.

▶ **2-26.** Let $x + y = \ell$ with x used for square and y used for triangle, then area of square is $A_s = (x/4)(x/4) = x^2/16$ and the area of the triangle is $A_t = \dfrac{1}{2}\sin\dfrac{\pi}{3}(y/3)^2$. The sum of these areas can be expressed

$$A = \frac{x^2}{16} + \frac{\sqrt{3}}{36}y^2$$

$$A = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(\ell - x)^2$$

Show $x = 4\sqrt{3}$ and $y = 9$ when A is a minimum.

▶ **2-28.** (c) If $f'(x) = x(x-1)^2(x-3)^2$, then

$f''(x) = 3x(x-1)^2(x-3)^2 + 2x(x-1)(x-3)^3 + (x-1)^2(x-3)^3 = (x-1)(x-3)^2(6x^2-13x+3)$

$f'(x) = 0$ at $x = 0$, $x = 1$ and $x = 3$, these are the critical values.

At $x = 0$, $f''(x) = (-1)^2(-3)^3 < 0 \implies f(x)$ is local maximum.

At $x = 1$, $f''(1) = 0$, second derivative test fails, so use the first derivative test using the values $x = 1/2, x = 1$ and $x = 3/2$.

$$\begin{aligned} &\text{At } x = 1/2,\ f'(1/2) < 0 \text{ negative slope}\\ &\text{At } x = 1,\ f'(1) = 0 \text{ zero slope}\\ &\text{At } x = 3/2,\ f'(3/2) < 0 \text{ negative slope} \end{aligned}$$

Hence, the point where $x = 1$ corresponds to a point of inflection.

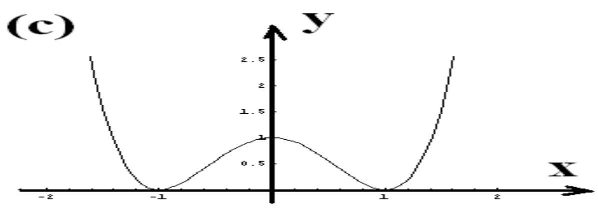
At $x = 3$, $f''(3) = 0$ second derivative test again fails, so use first derivative test with $x = 2.5, x = 3$ and $x = 3.5$. One finds

$$\begin{aligned} f'(2.5) &< 0 \text{ negative slope}\\ f'(3) &= 0 \text{ zero slope}\\ f'(3.5) &> 0 \text{ positive slope} \end{aligned}$$

so that $x = 3$ corresponds to a local minimum.

▶ **2-29.** (b) e (d) $1/2$ (g) $\dfrac{\ln a}{\ln b}$

▶ **2-30.**



▶ **2-33.** (c)

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{-x^2}{(x^2+y^2)^{3/2}} + \frac{1}{\sqrt{x^2+y^2}}\\ \frac{\partial^2 u}{\partial x\,\partial y} &= \frac{\partial^2 u}{\partial y\,\partial x} = \frac{-xy}{(x^2+y^2)^{3/2}}\\ \frac{\partial^2 u}{\partial y^2} &= \frac{-y^2}{\sqrt{x^2+y^2}^{3/2}} + \frac{1}{\sqrt{x^2+y^2}} \end{aligned}$$

► **2-35.**

$$\begin{aligned}(a)\quad D\,e^{\alpha x} =&\alpha e^{\alpha x}\\ D^2\,e^{\alpha x} =&\alpha^2 e^{\alpha x}\\ \vdots\quad&\quad\vdots\\ D^n\,e^{\alpha x} =&\alpha^n e^{\alpha x}\end{aligned}$$

$$\begin{aligned}(b)\quad D(a^x) =&D(e^{x\ln a}) = \ln a\,a^x\\ D^2(a^x) =&(\ln a)^2\,a^x\\ \vdots\quad&\quad\vdots\\ D^n(a^x) =&(\ln a)^n\,a^x\end{aligned}$$

$$\begin{aligned}(c)\quad D(\ln x) =&\frac{1}{x}\\ D^2(\ln x) =&\frac{-1}{x^2}\\ D^3(\ln x) =&\frac{(-1)(-2)}{x^3}\\ \vdots\quad&\quad\vdots\\ D^n(\ln x) =&\frac{(-1)(-2)(-3)\cdots(-n+1)}{x^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}\end{aligned}$$

$$\begin{aligned}(d)\quad D(x^n) =&nx^{n-1}\\ D^2(x^n) =&n(n-1)x^{n-2}\\ D^3(x^n) =&n(n-1)(n-2)x^{n-3}\\ \vdots\quad&\quad\vdots\\ D^m(x^n) =&n(n-1)(n-2)\cdots(n-(m-1))x^{n-m},\qquad m<n\end{aligned}$$

or one can write $D^m(x^n) = \dfrac{n!}{(n-m)!}x^{n-m}, \quad m<n$

$$\begin{aligned}(e)\quad D(\sin x) =& \cos x = \sin(x+\frac{\pi}{2})\\ D^2(\sin x) =& -\sin x = \sin(x+2\,\frac{\pi}{2})\\ D^3(\sin x) =& -\cos x = \sin(x+3\,\frac{\pi}{2})\\ \vdots\quad&\quad\vdots\\ D^n(\sin x) =& \sin(x+n\,\frac{\pi}{2})\end{aligned}$$

► **2-35.**

$$
\begin{aligned}
(f) \qquad D(\cos x) &= -\sin x = \cos(x + \frac{\pi}{2}) \\
D^2(\cos x) &= -\cos x = \cos(x + 2\,\frac{\pi}{2}) \\
D^3(\cos x) &= \sin x = \cos(x + 3\,\frac{\pi}{2}) \\
&\vdots \qquad \vdots \\
D^n(\cos x) &= \cos(x + n\,\frac{\pi}{2})
\end{aligned}
$$

(g) Express problem so that the results from parts (e) and (f) can be employed. Use $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$, then

$$
\begin{aligned}
D(\sin^3 x) =& \frac{3}{4} D(\sin x) - \frac{3}{4}\sin(3x + \frac{\pi}{2}) \\
D^2(\sin^3 x) =& \frac{3}{4} D^2(\sin x) - \frac{3^2}{4}\sin(3x + 2\,\frac{\pi}{2}) \\
\vdots \qquad & \vdots \\
D^n(\sin^3 x) =& \frac{3}{4} D^n(\sin x) - \frac{3^n}{4}\sin(3x + n\,\frac{\pi}{2}) \qquad \text{or} \\
D^n(\sin^3 x) =& \frac{3}{4}\sin(x + n\,\frac{\pi}{2}) - \frac{3^n}{4}\sin(3x + n\,\frac{\pi}{2})
\end{aligned}
$$

► **2-40.** $v = t^3 - 6t^2 + 11t - t$ and $a = 3t^2 - 12t + 11$
Hint: Show $v = \frac{ds}{dt} = (t-1)(t-2)(t-3)$

► **2-42.** $\frac{d}{dx}|u| = \frac{d}{dx}\sqrt{u^2} = \frac{1}{2}(u^2)^{-1/2}\,2u\frac{du}{dx} = \frac{u}{|u|}\frac{du}{dx}$

► **2-44.**

$$
\begin{aligned}
&(a) \qquad \ln y = \ln\alpha + x\ln\beta \implies Y = mx + b \quad \text{where } Y = \ln y,\ m = \ln\beta,\ b = \ln\alpha \\
&(b) \qquad \ln y = \ln\alpha + \beta\ln x \implies Y = \beta X + b \quad \text{where } Y = \ln y,\ X = \ln x,\ b = \ln\alpha
\end{aligned}
$$

▶ **2-45.**

$$\text{Let } y = y(x) = s^2 = (x-x_0)^2 + \left(\frac{(-c-ax)}{b} - y_0\right)^2$$

$$\frac{dy}{dx} = 2\left[ax^2 + b^2(x-x_0) + a(c+by_0)\right] = 0 \quad \text{at the point where}$$

$$(a^2+b^2)x - b^2x_0 + ac + aby_0 = 0$$

$$\text{or} \quad x = x^* = \frac{b^2x_0 - ac - aby_0}{a^2+b^2}$$

Show that $y_{min} = y(x^*) = s_min^2 = \dfrac{(c+ax_0+by_0)^2}{a^2+b^2}$ and that

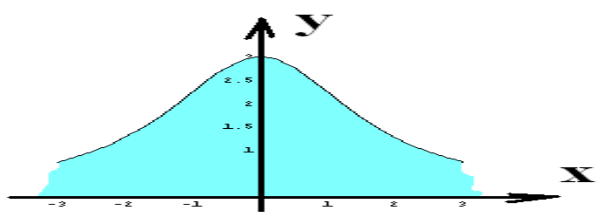
$$S_{min} = d = \frac{|c+ax_0+by_0|}{\sqrt{a^2+b^2}}$$

▶ **2-46.** Chain rule differentiation requires $\dfrac{dy}{dx}\dfrac{dx}{d\theta} = \dfrac{dy}{d\theta}$ or $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

▶ **2-50.** Cone with maximum volume has base radius $r = \frac{2\sqrt{2}}{3}R$ and height $h = \frac{4}{3}R$.

▶ **2-53.** A square with side $\sqrt{2}R$.

▶ **2-56.**



▶ **2-60.** (a) $\dfrac{m}{n}\alpha^{m-n}$

▶ **2-65.** By the law of cosines $\ell^2 = r^2 + s^2 - 2rs\cos\omega t$. Differentiate this relation and show

$$\frac{ds}{dt} = -\omega r\sin\omega t - \frac{\omega r^2\cos\omega t\sin\omega t}{s - r\cos\omega t}$$

From the law of cosines show $s - r\cos\omega t = \sqrt{\ell^2 - r^2\sin^2\omega t}$

Chapter 3

► **3-1.** (a) $\frac{4}{3}x^3+2x^2+x+C$ or $\frac{1}{6}(2x+1)^3+C_1$

(b) $-\csc(t)+C$

(c) $-\frac{1}{2}\cos^2 x+C_1$ or $\frac{1}{2}\sin^2 x+C_2$

► **3-2.** (a) $\frac{1}{10}\sin^5(2t)+\sin^4(2t)+4\sin^3(2t)+8\sin^2(2t)+8\sin(2t)+C_1$ or $\frac{1}{10}[2+\sin(2t)]^5+C_2$

(c) $\frac{4^{x^2}}{4\ln 2}+C$

(e) $-\frac{1}{3}\cos(3x+1)+C$

► **3-3.** (a) $\frac{1}{3}\tan(3x+4)+C$ (c) $\frac{1}{3}\sec(3x+4)+C$ (e) $\frac{1}{6}\ln|\sin(3x^2)|+C$

► **3-4.** (a) $\frac{1}{2}\sin^{-1}(x^2)+C$ (c) $-\sqrt{1-x^2}$ (e) $\frac{1}{2}\sinh(x^2)+C$

► **3-5.** (a) $-\frac{1}{3\sinh(3x+1)}+C$ (c) $-\frac{1}{3}\coth(3x+1)+C$ (e) $\frac{1}{3}\sqrt{(3x+1)^2-1}$

► **3-6.** (a) $-\frac{1}{x}+5x+\ln x+C$ (c) $2c\sqrt{t}+at+\frac{2}{3}bt^{3/2}+C$ (e) $\frac{1}{3b}(a+bu^2)\sqrt{a+bu^2}$

► **3-7.** (a) $\frac{26}{3}$ Area under parabola

(b) 2 Area under sine curve

(c) $\frac{1}{2}BH$ Area of triangle

► **3-8.** (a) Let $u=3x+1=\tan\theta$ with $du=3\,dx=\sec^2\theta\,d\theta$

then $\displaystyle \frac{1}{3}\int\csc\theta\,d\theta=-\frac{1}{3}\ln|\csc\theta+\cot\theta|=-\frac{1}{3}\ln\left|\frac{\sqrt{1+u^2}}{u}+\frac{1}{u}\right|$

$\displaystyle =-\frac{1}{3}\ln|1+\sqrt{1+u^2}|+\frac{1}{3}\ln u+C$ where $u=3x+1$

► **3-9.** Two functions with the same derivative differ by some constant value.

► **3-10.** Use table III with appropriate scaling.

► **3-11.** (a) $\ln|x-3|+\ln|x-2|+\ln|x-1|+C$

(c) $\tan^{-1}x+\ln(x+1)+\ln(x^2+1)+C$

(e) $\displaystyle x+\frac{3}{2}x^2+x^3+\frac{1}{2}x^4-\frac{1}{1+x+x^2}+\ln|x-1|+C$

▶ **3-12.** $(a)\ dy = x\,dx \quad \int_1^y dy = \int_1^x x\,dx \implies y-3 = \frac{1}{2}(x^2-1)$ or $y = \frac{1}{2}x^2 + \frac{5}{2}$

$(c)\ dy = \sin(3x)\,dx \quad \int_1^y dy = \frac{1}{3}\int_0^x \sin(3x)\,3dx \implies y-1 = \frac{1}{3}[1-\cos(3x)]$

or $y = \frac{4}{3} - \frac{1}{3}\cos(3x)$

$(e)\ dy = \sin^2(3x)\,dx \quad \int_1^y dy = \frac{1}{3}\int_0^x \sin^2(3x)\,3\,dx \implies y-1 = \frac{1}{12}[6x - \sin(6x)]$

or $y = 1 - \frac{1}{2}x - \frac{1}{12}\sin(6x)$

▶ **3-13.** $(b)\ x - \tan^{-1}x + C \quad (d)\ x - \frac{1}{x+1} - 2\ln(x+1) + C \quad (f)\ \frac{1}{a}\ln x - \frac{1}{a}\ln(x-a) + C$

▶ **3-14.** $(b)\ \frac{1}{(b-a)(x-a)} + \frac{1}{(a-b)^2}[\ln(x-b) - \ln(x-a)] + C$

$(d)\ \frac{\ln(x^2-b^2)}{2(b^2-a^2)} + \frac{\ln(x-a)}{a^2-b^2} - \frac{a}{b(b^2-a^2)}\tan^{-1}\left(\frac{x}{b}\right) + C$

$(f)\ \frac{-1}{a^2+b^2}\left[\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) + \frac{1}{a}\tan^{-1}\left(\frac{x}{b}\right)\right] + C$

▶ **3-15.** $(b)\ 1 \quad (d)\ \frac{1}{4} + \frac{e^2}{4} \quad (f)\ 100\sqrt{15}$

▶ **3-16.** $(b)\ (x^2-2x+2)e^x + C \quad (d)\ \frac{1}{2}(\sin x - \cos x)e^x + C \quad (f)\ \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

▶ **3-17.** $(a) \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C \quad (c)\ \frac{x}{2} - \frac{x^2}{4} - \frac{1}{2}\ln(x+1) + \frac{x^2}{2}\ln(x+1) + C \quad (e)\ e-2$

▶ **3-18.** $(a)\ \int_0^1 f(x)\,dx$

▶ **3-19.** $A = \int_{-1}^5 [(4y-1) - (y^2-6)]\,dy, \implies A = 36$

▶ **3-21.** (a) Use symmetry to find

area ABH = area BCD $= \int_0^4 (\sqrt{y+4} - \sqrt{4-y})\,dy = \frac{32}{3}(\sqrt{2}-1)$

▶ **3-23.** $(d)\ -\frac{1}{3}(a^2-x^2)^{3/2} + C \quad (e)\ 2x - 2x\ln x + x(\ln x)^2 + C \quad (f)\ \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} + C$

▶ **3-24.** $(a)\ \ln[t+2+\sqrt{t^2+4t-3}] + C \quad (d)\ \frac{1}{9}\tan^{-1}x + C \quad (e)\ \frac{1}{18}[\ln(x-1) - \ln(x+1)] + C$

▶ **3-25.** $(a)\ 2\ln(1+\sqrt{x}) + C \quad (f)\ -a\sqrt{b^2-(x+a)^2} + (b-a^2)\tan^{-1}\left[\frac{(a+x)\sqrt{b^2-(a+x)^2}}{b^2-a^2-2ax-x^2}\right] + C$

▶ **3-26.** $(c)\ 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$ $\quad(f)\ \dfrac{1}{2}\left(\sqrt{x-x^2} + (2x-1)\sin^{-1}\sqrt{x}\right) + C$

▶ **3-27.** $(c)\ (x^2+1)\,e^x$ $\quad(d)\ \sqrt{1-x^2}\left(\dfrac{5}{8}x - \dfrac{1}{4}x^2\right) + \dfrac{3}{8}\sin^{-1}x + C$

▶ **3-28.** Surface $S = \dfrac{64}{3}\pi$ and volume $V = 5\pi^2$

▶ **3-29.** $(b)\ \alpha = 1$

▶ **3-32.** (b)arc length$s = 16$

▶ **3-35.** (i) Surface area $S = \dfrac{28\sqrt{2}}{3}\pi$ (ii) volume $V = 12\pi$

▶ **3-36.** (i) Surface area $S = \dfrac{\pi}{27}\left[(145)^{3/2} - (10)^{3/2}\right]$ (ii) Volume $V = \dfrac{762}{7}\pi$

▶ **3-37.** $(a)\ \dfrac{2}{\pi}$ $\quad(b)\ \dfrac{2}{\pi}$ $\quad(c)\ \dfrac{4}{3\pi}$

▶ **3-38.** $(d)\ 2$ $\quad(f)\ \dfrac{1}{2}\ln 2$

▶ **3-39.** The area of three sides need to be calculated. Call these surface areas S_1, S_2 and S_3. Show $S = S_1 + S_2 + S_3 = 11\sqrt{2}\pi + 9\sqrt{2}\pi + 20\pi$

▶ **3-44.** $(a)\ A = \displaystyle\int_0^{2\pi} r\,d\theta$ $\quad(b)\ A = 2\pi\displaystyle\int_0^r x\,dx$ $\quad(c)\ A = \displaystyle\int_0^{2\pi}\int_0^r r\,dr d\theta$

▶ **3-45.** The figure illustrated in the problem 3-45, without the axes, is known as the symbol the Pythagoreans use to represent their society.

(a) Divide area into four symmetric parts and show the area of one of these parts is $A_1 = r_0^2\left(\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{8}\right)$ and the total area is $A = 4A_1 = r_0^2\left(\dfrac{2}{3}\pi - \dfrac{\sqrt{3}}{2}\right)$

(b) Volume is given by $V = \dfrac{5}{12}\pi\, r_0^3$

▶ **3-47.** $(a)\ 80$ $\quad(b)\ 8$ $\quad(c)\ \dfrac{1}{4}[\,a^2c^2 - b^2c^2 - a^2d^2 + b^2d^2]$

▶ **3-50.** $c = \frac{1}{2}(a+b)$

▶ **3-52.** The function $f(x)$ is periodic so that

$$f(x) = f(x+T) = f(x+2T) = \cdots = f(x+(m-1)T) = f(x+T) = \cdots$$

Write

$$I = \int_0^{nT} = \int_0^T f(x)\,dx + \int_T^{2T} f(x)\,dx + \cdots + \int_{(m-1)T}^{mT} f(x)\,dx + \cdots + \int_{(n-1)T}^{T} f(x)\,dx$$

or $I = \sum_{m=1}^n \int_{(m-1)T}^{mT} f(x)\,dx$ Let $x = u + (m-1)T$ with $dx = du$ and note that when $x = (m-1)T$, then $u = 0$ and when $x = mT$, then $u = T$, so that

$$I = \sum_{m=1}^n \int_0^T f(u+(m-1)T)\,du = \sum_{m=1}^n \int_0^T f(u)\,du = n\int_0^T f(u)\,du$$

▶ **3-54.**

(a) let $u = e^{2x},\ du = 2e^{2x}\,dx$ then interal has value $\dfrac{1}{2}\dfrac{(u+3)^{m+1}}{m+1} + C$

(b) let $u = e^x,\ du = e^x\,dx$ and show integral is $\displaystyle\int \frac{u^4+u^3}{u^2+1}\,du$

Show $\dfrac{u^4+u^3}{u^2+1} = -1 + u + u^2 + \dfrac{1-u}{1+u^2}$ and integral is

$-u + \dfrac{u^2}{2} + \dfrac{u^3}{3} + \tan^{-1} u - \dfrac{1}{2}\ln(1+u^2) + C$

(c) let $u = e^x,\ du = e^x\,dx$ with integral $\dfrac{1}{3}(u+1)^3 + C$

▶ **3-55.**

$$(a) \quad -\frac{a}{2x^2} - \frac{1}{x} + C$$

$$(b) \quad -\frac{ab}{2x^2} - \frac{(a+b)}{x} + \ln x + C$$

$$(c) \quad -\frac{abc}{x} - \frac{(bc+ab+ac)}{x} + x + (a+b+c)\ln x + C$$

▶ **3-56.**

$$(a) \quad -x + \ln(1+x) + x\ln(1+x) + C$$

$$(b) \quad Hint: \frac{x^4+1}{x-1} = 1 + x + x^2 + x^3 + \frac{2}{x-1}$$

$$\frac{1}{12}\left[12x + 6x^2 + 4x^3 + 3x^4 + 24\ln(x-1)\right] + C$$

$$(c) \quad \frac{(m+2)a + b(m+1) - c}{m^2+3m+2}(x+c)^{m+1} + C$$

▶ **3-57.** (a) $\dfrac{1}{s}$ (b) e^x (c) $\dfrac{1}{s^2}$

▶ **3-58.** Integrate term by term and show

$$\begin{aligned} J_0(x) =& 1 - \frac{x^2}{2^2 1!1!} + \frac{x^4}{2^4 2!2!} + \cdots (-1)^m \frac{x^{2m}}{2^{2m} m!m!} + \cdots \\ J_1(x) =& \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} + \cdots + (-1)^{m-1} \frac{x^{2m-1}}{2^{2m-1}(m-1)!m!} + \cdots \end{aligned}$$

▶ **3-59.** $I_n = x^n e^x - nI_{n-1}, \qquad I_4 = (x^4 - 4x^3 + 12x^2 - 24x + 24)\, e^x$

▶ **3-60.** Let $u = a - x$, with $du = -dx$, then when $x = 0, u = a$ and when $x = a, u = 0$ so that

$$\int_0^a g(a-x)\,dx = \int_a^0 g(u)(-du) = \int_0^a g(u)\,du = \int_0^a g(x)\,dx$$

▶ **3-65.** Write $\displaystyle\int_0^{2T} f(x)\,dx = \int_0^T f(x)\,dx + \int_T^{2T} f(x)\,dx$ and in the second integral let $x = 2T - u$ with $dx = -du$, so that when $x = T, u = T$ and when $x = 2T, u = 0$, then use $f(2T - x) = f(x)$ and write

$$\int_0^{2T} f(x)\,dx = \int_0^T f(\xi)\,d\xi + \int_T^0 f(2T-u)(-du) = \int_0^T f(\xi)\,d\xi + \int_0^T f(u)\,du = 2\int_0^T f(x)\,dx$$

Note—Definite integrals have dummy variables of integration.

▶ **3-66.** $\displaystyle V = \int_0^h A(y)\,dy$

▶ **3-67.**

$$(e) \quad \int_0^\beta \frac{dx}{(\beta - x)^p} = \begin{cases} \frac{\beta^{1-p}}{1-p} & \text{if } p < 1 \\ \text{Doesn't exist} & \text{if } p \ge 1 \end{cases}$$

$$(f) \quad \int_0^\infty \frac{dx}{(\beta + x)^p} = \begin{cases} \frac{\beta^{1-p}}{p-1} & \text{if } p > 1 \\ \text{Doesn't exist} & \text{if } p \le 1 \end{cases}$$

▶ **3-73.** Material removed is cylinder with spherical end caps of height $h = r - r_0$ where $r_0 = r\sqrt{1 - \alpha^2/4}$. Diameter of drill is αr implies radius of cylinder is $\alpha r/2$.

$$\begin{aligned} V_0 =& \text{volume sphere before drilling} = \frac{4}{3}\pi r^3 \\ V_1 =& \text{volume of cylinder without spherical caps} = \pi(\frac{\alpha r}{2})^2(2r_0) \\ V_2 =& \text{volume of 2 spherical caps} = 2\left(\frac{\pi}{3}h^2(3r - h)\right) \\ V_3 =& \text{volume removed by drill} = V_1 + V_2 \\ V_4 =& \text{Volume remaining after drilling} = V_0 - V_3 = V_0 - (V_1 + V_2) \end{aligned}$$

Chapter 4

► **4-1.** $(a)\quad 3/4 \qquad (c)\quad 1 \qquad (e)\quad 0$

► **4-2.** $(a)\quad -1/2 \qquad (d)\quad 0 \qquad (f)\quad 0$

► **4-3.** $S=\sum_{n=0}^{m} r^n = 1+r+r^2+\cdots+r^m, \qquad S=\dfrac{r^{m+1}-1}{r-1}$

$$
\begin{array}{ll}
(d) & 4\dfrac{3^{10}-1}{2}=118,096 \\
(e) & \dfrac{1-(.02)^{100}}{1-.02}=1.02041 \\
(f) & (2+\sqrt{2})\dfrac{\left(\frac{3-\sqrt{2}}{2+\sqrt{2}}\right)^{10}-1}{\frac{3-\sqrt{2}}{2+\sqrt{2}}-1}=6.37237
\end{array}
$$

► **4-4.** For S_3 use $r=\dfrac{7-4\sqrt{3}}{2-\sqrt{3}}$ and show $S_3=\dfrac{r}{1-r}=\dfrac{1}{2}(\sqrt{3}-1)$

► **4-5.** $S_3=0.83333\ldots$ and $S_4=0.16666\ldots$

► **4-6.** (b) Use ratio test and show convergence for $|x|<2$

► **4-7.** (ii) $E(X)=\sum_{k=1}^{\infty} kp_k = 3$

► **4-8.** $S_2=\sum_{i=1}^{\infty}\left(r_1^i+r_2^i\right) \qquad r_1=a^2/b \qquad r_2=1/b$

Show for a,b real and $b>a^2>1$, then series converges to $\dfrac{a^2}{b-a^2}+\dfrac{1}{b-1}$

Note if $S=1+r+r^2+\cdots$ converges, then

$$\frac{dS}{dr}=1+2r+3r^2+4r^3+\cdots=\sum_{k=0}^{\infty}kr^{k-1}=\frac{d}{dr}(1-r)^{-1}=\frac{1}{(1-r)^2}$$

Consequently, one can show $S_4=\dfrac{a}{1-r}+\dfrac{b}{(1-r)^2}$

► **4-9.** (b) $\displaystyle\sum_{n=3}^{\infty}\frac{1}{(n-2)(n-1)}=\sum_{n=3}^{\infty}\left(\frac{1}{n-2}-\frac{1}{n-1}\right)$

$S=\left(\dfrac{1}{1}-\dfrac{1}{2}\right)+\left(\dfrac{1}{2}-\dfrac{1}{3}\right)+\left(\dfrac{1}{3}-\dfrac{1}{4}\right)+\cdots$ which converges to $S=1$.

▶ **4-10.** (d) $u_n = \dfrac{n}{(n+1)(n+2)(n+3)} = \dfrac{1/2}{n+1} - \dfrac{1}{n+2} + \dfrac{1/2}{n+3}$ so that the Nth partial sum can be written

$$U_N = \sum_{n=1}^{N} u_n = \frac{5N+N^2}{12(6+5N+N^2)}$$

Divide numerator and denominator by N^2 and show $\lim\limits_{N\to\infty} U_N = \dfrac{1}{12}$

▶ **4-11.** (d) If $f(x) = \frac{1}{x\ln x}$, then $\int_M^T f(x)\,dx = \int_M^T \frac{1}{x\ln x}\,dx$

Let $u = \ln x$ with $du = \frac{1}{x}\,dx$ and show $\int_M^T f(x)\,dx = \ln[\ln x]_M^T = \ln[\ln T] - \ln[\ln M]$ Show this result increases without bound as $T \to \infty$ so the integral diverges and series diverges.

▶ **4-12.** Let $u_n = \dfrac{1}{n^p}$, where $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is the p-series which converges $p > 1$ and diverges $p \le 1$. Now select $v_n = n^p f(n)$ and follow results for modification of a series.

▶ **4-13.** (d) Since $3n^2+2n+1 > n^2$, then $\dfrac{1}{3n^2+2n+1} < \dfrac{1}{n^2}$ and we know the p-series, with $p = 2$ converges.

▶ **4-14.** (iv) $\displaystyle\sum_{n=1}^{4} (-1)^{n+1}\frac{1}{n^4} = 0.945939$ with error $E < \dfrac{1}{5^4} = 0.0016$

$\displaystyle\sum_{n=1}^{8} (-1)^{n+1}\frac{1}{n^4} = 0.94694$ with error $E| < \dfrac{1}{9^4} = 0.000152416$

▶ **4-15.** (A) converges slower than (B)

▶ **4-17.** $x_n = x_{n-1} - \dfrac{x_n^2 - 3x_n + 1}{2x_n - 3}$ $\quad x_0 = 2, \ldots, x_4 = 2.61803$

Exact roots are $x = \dfrac{3 \pm \sqrt{5}}{2}$ Convergence to desired root depends upon position of initial guess.

▶ **4-18.** Converges $|x| < 1$ and diverges for $|x| > 1$

▶ **4-20.** Assume $\displaystyle\sum_{n=1}^{\infty} u_n$ converges by ratio test so that $\lim\limits_{n\to\infty} \left|\dfrac{u_{n+1}}{u_n}\right| = p < 1$. Let $v_n = \frac{1}{u_n}$ and examine $\displaystyle\sum_{n=1}^{\infty} v_n$ using the ratio test to show series diverges.

▶ **4-21.** This is a telescoping series with

$$S_n = \sum_{k=0}^{n-1}(U_{k+1} - U_k) = (U_1 - U_0) + (U_2 - U_1) + (U_3 - U_2) + \cdots + (U_n - U_{n-1})$$

so that $S_n = U_n - U_0 = \frac{1}{3}n(n+1)(n+2)$

Note that this result can be generalized. If one is given a U_k and calculates the difference $\Delta U_k = U_{k+1} - U_k$, then one can write $\sum_{k=1}^{N} \Delta U_k = \sum_{k=1}^{N}(U_{k+1} - U_k) = U_{N+1} - U_1$

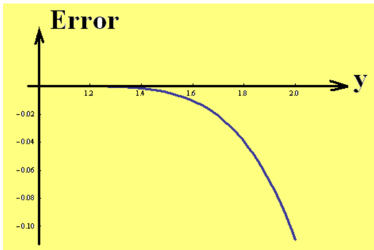
What would $\sum_{k=\ell}^{m} \Delta U_k$ produce for the answer?

▶ **4-22.**

$$x - 1 = -\ln(y) = -(y-1) + \frac{1}{2}(y-1)^2 - \frac{1}{3}(y-1)^3 + \frac{1}{4}(y-1)^4 - \frac{1}{5}(y-1)^5 + \cdots$$

$$Error = E = E(y) = -\ln(y) - \left[-(y-1) + \frac{1}{2}(y-1)^2 - \frac{1}{3}(y-1)^3 + \frac{1}{4}(y-1)^4\right]$$

Over the interval $1 \le y \le 2$ the error curve is as illustrated



▶ **4-23.** (b) (c) (e) all converge

▶ **4-24.** (a) $\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$ hence converges.

▶ **4-25.** (c) $\sin(\pi/3) = \frac{\sqrt{3}}{2}$ gives geometric series with sum $3 + 2\sqrt{3}$

▶ **4-26.** Use the nth term test $\to 0$ and note (e) is a form of the harmonic series.

▶ **4-27.** (f) divergent

▶ **4-28.** (f) $\frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdots n}{n(n-1)(n-2)\cdot 1} > n$ divergent

▶ **4-29.** (f) $\sqrt{n} < n$ and $\frac{\sqrt{n}}{n^2+1} < \frac{n}{n^2+1}$. The series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ can be compared with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ to show absolute convergence.

► **4-30.** Use ratio test

► **4-33.** Write $(a+b)^r = [a(1+b/a)]^r = a^r(1+b/a)^r$ and let $x = b/a$ and then examine the series $f(x) = (1+x)^r$. Here

$$\begin{aligned}
f'(x) =& r(1+x)^{r-1}\\
f''(x) =& r(r-1)(1+x)^{r-2}\\
\vdots\ & \vdots\\
f^{(n)}(x) =& r(r-1)(r-2)\cdots(r-(n-1)(1+x)^{r-n}
\end{aligned}$$

This series has the power series expansion

$$f(x) = (1+x)^r = 1 + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdot + R_n$$

Using the Lagrange form of the remainder

$$R_n = \frac{x^n}{n!}f^{(n)}(\theta x) = \binom{r}{n}x^n(1+\theta x)^{r-n}$$

where $0 < \theta < 1$. For $0 \le x < 1$ and $n > r$, then one can show $(1+\theta x)^{r-n} < 1$ and $\binom{r}{n}x^n \to 0$ as $n \to \infty$ The Lagrange form for the remainder doesn't aid in the analysis of the region $-1 < x \le 0$ and so one can use the Cauchy form of the remainder to analyze the remainder in this region.

The Cauchy form of the remainder is

$$R_n = \frac{r(r-1)(r-2)\cdots(r-n+1)}{(n-1)!}\frac{(1-\theta)^{n-1}x^n}{(}1+\theta x)^{n-m}, \qquad 0 < \theta < 1$$

If $|x| < 1$, then $\frac{1-\theta x}{1+\theta x} < 1$ so that

$$\frac{(1-\theta)^{n-1}}{(1+\theta x)^{n-1}\cdot \frac{1}{1+\theta x}^{1-m}} < \frac{1}{(1+\theta x)^{1-m}}$$

For $-1 < x \le 0$ the term $\frac{1}{(1+\theta x)^{1-m}} = K$ is some constant independent of the index n and the term

$$\frac{r(r-1)\cdots(r-n+1)}{(n-1)!} = r\left[\frac{(r-1)!}{(n-1)!(r-n)!}\right] = r\binom{r-1}{n-1}$$

Therefore,

$$|R_n| < K|r|\left|\binom{r-1}{n-1}\right||x|^n \to 0 \quad \text{as } n \to \infty$$

So the binomial series converges as $|R_n| \to 0$ for $|x| < 1$

► **4-36.** $y_1 = \alpha_0 + \frac{1}{\alpha_1 + \frac{1}{x}} = \alpha_0 + \frac{x}{\alpha_1 x + 1}$ Here $Q_1 = \alpha_1 x + 1$ and

$$\frac{dy_1}{dx} = \frac{(\alpha_1 x + 1)(1) - x(\alpha_1)}{(\alpha_1 x + 1)^2} = \frac{1}{[Q_1(x)]^2}$$

► **4-41.** For n, m integers and $n \neq m$

$$(\sin nx, \sin mx) = \int_0^\pi \sin nx \sin mx \, dx = \frac{\sin[(m-n)x]}{2(m-n)} - \frac{\sin[(m+n)x]}{2(m+n)}\bigg|_0^\pi = 0, \qquad m \neq n$$

$$\| \sin nx \|^2 = (\sin nx, \sin nx) = \int_0^\pi \sin^2(nx)\, dx = \frac{x}{2} - \frac{\sin 2nx}{4n}\bigg|_0^\pi = \frac{\pi}{2}$$

or $\| \sin nx \| = \sqrt{\frac{\pi}{2}}$. The set of functions $\left\{ \sqrt{\frac{2}{\pi}} \sin nx \right\}$ is therefore an orthonormal set with norm-squared equal to unity.

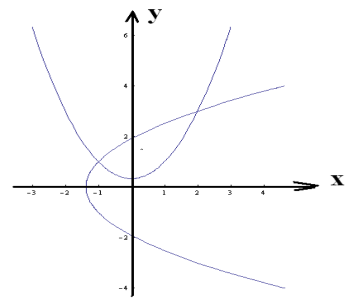
► **4-42.** (c)

$$\begin{aligned}
(1, \cos\frac{n\pi x}{L}) &= \int_{-L}^{L} \cos\frac{n\pi x}{L}\, dx = 0 \\
(1, \sin\frac{n\pi x}{L}) &= \int_{-L}^{L} \sin\frac{n\pi x}{L}\, dx = 0 \\
(\cos\frac{n\pi x}{L}, \cos\frac{m\pi x}{L}) &= \int_{-L}^{L} \cos\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = 0, \qquad m \neq n \\
(\cos\frac{n\pi x}{L}, \cos\frac{m\pi x}{L}) &= \int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = 0 \\
(\sin\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}) &= \int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = 0 \qquad m \neq n \\
(1, 1) &= \| 1 \|^2 = 2L \\
(\sin\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}) &= \| \sin\frac{n\pi x}{L} \|^2 = L \\
(\cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}) &= \| \cos\frac{n\pi x}{L} \|^2 = L
\end{aligned}$$

► **4-43.** (e) (i) If $f(x)$ is even, $b_n = 0$ and $a_0 = \frac{1}{L}\int_0^L f(x)\, dx, \quad a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\, dx$

(ii) If $f(x)$ is odd, $a_0 = a_n = 0$ and $b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$

► **4-44.** Roots are $(2,3)$ and $(-1,1)$



► **4-48.** (a) $C = \dfrac{C_0 e^{-k\tau}}{1 - e^{-k\tau}}$ (c) $k\tau = -\ln\left[\dfrac{\dfrac{C_s}{C_0} - 1}{\dfrac{C_s}{C_0}}\right]$

► **4-53.** $y = \sqrt{\sin x + y} \implies y^2 = \sin x + y$ Differentiate this relation to obtain given answer.

► **4-55.** If $e^{x\cos x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, then its derivative is

$$e^{x\cos x}[\cos x - x\sin x] = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Show

$$\cos x - x\sin x = 1 - \frac{3}{2}x^2 + \frac{5}{24}x^4 - \frac{7}{720}x^6 + \frac{1}{4480}x^8 + \cdots$$

and substitute the top line equation and third line equation into the second line equation to obtain

$$(a_0 + a_1 x + a_2 x^2 + \cdots)(1 - \frac{3}{2}x^2 + \frac{5}{24}x^4 + \cdots) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Expanding the left-hand side of this equation gives

$$a_0 + a_1 x + (a_2 - \frac{3}{2}a_0)x^2 + (a_3 - \frac{3}{2}a_1)x^3 + \cdots = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Equate like powers of x to find a relation between the coefficients. Use top line equation to show at $x = 0$ that $a_0 = 1$ and from equating like powers of x one finds

$$a_1 = 1, \quad a_2 = 1/2, \quad a_3 = -1/3, \quad a_4 = -11/24, \quad \text{etc}$$

Chapter 5

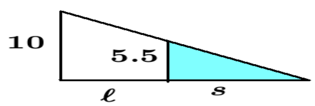
▶ **5-1.** $V = \dfrac{4}{3}\pi r^3, \qquad \dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}, \qquad \alpha = 4\pi r_0^2 \dfrac{dr}{dt} \implies \dfrac{dr}{dt} = \dfrac{\alpha}{4\pi r_0^2}$

▶ **5-2.** $pv^{1.4} = c, \quad \dfrac{dp}{dt} v^{1.4} + p(1.4)v^{0.4}\dfrac{dv}{dt} = 0$ or $\dfrac{dp}{dt} = -1.4 \dfrac{p}{v}\dfrac{dv}{dt}$

(a) $\frac{dp}{dt} = -1.4\frac{p}{v}\alpha$

(b) $\frac{dv}{dt} = \frac{\beta}{1.4}\frac{v}{p}$

▶ **5-3.**



$\dfrac{\ell + s}{10} = \dfrac{s}{5.5}$, if $\frac{d\ell}{dt} = 4\,\text{ft/s}$, then show $\dfrac{ds}{dt} = \dfrac{22}{4.5}$ ft/s.

▶ **5-4.** (a) Let $S_1 = x - f > 0$ and $S_2 = y - f > 0$, then $\dfrac{1}{x - f + f} + \dfrac{1}{y - f + f} = \dfrac{1}{f}$ or $\dfrac{1}{S_1 + f} + \dfrac{1}{S_2 + f} = \dfrac{1}{f}$ Simplify this expression and show $f^2 = S_1 S_2$

(b) Differentiate the lens law and show $\dfrac{dy}{dt} = -\dfrac{y^2}{x^2}\, r_0$

▶ **5-5.** Area of equilateral triangle with side x is $A = \dfrac{\sqrt{3}}{8}x^2$ so that $\dfrac{dA}{dt} = \dfrac{2\sqrt{3}}{8}x\dfrac{dx}{dt}$ Now substitute $x = x_0$ and $\frac{dx}{dt} = r_0$

▶ **5-6.** $p = p_0 e^{-\alpha_0 h}$

(a) $[p_0] = lbs/ft^2, \qquad [\alpha_0] = 1/ft$

(b) $\dfrac{dh}{dt} = 10\,ft/s \qquad \dfrac{dp}{dt} = p_0 e^{-\alpha_0 h}(-\alpha_0 \dfrac{dh}{dt})$ Substitute for $\frac{dh}{dt}$ and find the pressure decreases with height.

▶ **5-7.** $y - y_0 = \alpha(x - x_0)^2$ has the derivative $\dfrac{dy}{dx} = 2\alpha(x - x_0)$. When $x = \xi$, the slope of the tangent is $m = 2\alpha(\xi - x_0)$ and the equation of the tangent line is $y - \eta = 2\alpha(\xi - x_0)(x - \xi)$. Here $\tan\theta = m = 2\alpha(\xi - x_0)$ and if $\frac{d\xi}{dt} = 1/2\,cm/s$, then $\sec^2\theta \dfrac{d\theta}{dt} = 2\alpha\dfrac{d\xi}{dt}$ or $\dfrac{d\theta}{dt} = \dfrac{2\alpha}{1 + \tan^2\theta}\dfrac{d\xi}{dt}$ or $\dfrac{d\theta}{dt} = \dfrac{2\alpha}{1 + 4\alpha^2(\xi - x_0)^2}\dfrac{1}{2}$

▶ **5-8.** $\dfrac{dT}{dt} = \dfrac{r_0}{c_0}$ where $c_0 = V_0/T_0$.

▶ **5-9.** $\dfrac{dP}{dt} = c_0 r_0$ where $c_0 = P_0/T_0$.

▶ **5-10.**

$$\begin{aligned} h =& 200 - 16t^2 \ \ ft \\ v = \frac{dh}{dt} =& -32\,t \ \ ft/s \\ a = \frac{dv}{dt} = \frac{d^2h}{dt^2} =& -32 \ \ ft/s^2 \end{aligned}$$

When $h = 0$, then $t = \sqrt{2}10/4$ and $v = -80\sqrt{2} \ \ ft/s$

▶ **5-11.**

(a) $V = \frac{4}{3}\pi r^3 - \frac{pi}{3}(2r-h)^2(r+h) = \frac{\pi}{3}(3rh^2 - h^3)$ and $\frac{dV}{dt} = \frac{\pi}{3}(3r\,2h\frac{dh}{dt} - 3h^2\,\frac{dh}{dt})$, $\quad r$is a constant

(b) $\frac{dh}{dt} = 30/(\pi\,420)$

▶ **5-12.** $(h-r)^2 + R^2 = r^2$ Differentiate this relation and show

$$\begin{aligned} R\frac{dR}{dt} =& -(h-r)\frac{dh}{dt} \ \ r < h < 2r \\ R\frac{dR}{dt} =& (r-h)\frac{dh}{dt} \ \ 0 < h < r \end{aligned}$$

▶ **5-13.** $(a)\ 1125/16\,ft \quad (b)\ 50\sqrt{3}/2 \quad (c)\ 50\sqrt{3}/16 \quad (d)\ 625\,\sqrt{3}/8 \quad (e)\ parabola$

▶ **5-14.** $\frac{dv}{dx} = \frac{-K}{\sqrt{1-2x}}$

▶ **5-15.** (a) $M_y = 60 \quad M_x = 44 \quad \bar{x} = \frac{M_y}{A} = \frac{60}{20} = 3, \quad \bar{y} = \frac{M_x}{A} = \frac{44}{20} = \frac{11}{5}$

▶ **5-16.** (b) $\bar{x} = \frac{M_y}{A} = \frac{3}{4}x_0 \quad \bar{y} = \frac{M_x}{A} = \frac{3}{10}x_0^2$

▶ **5-17.** (b) $\bar{x} = \frac{21}{5} \qquad \bar{y} = 3$

▶ **5-18.** (c) $\bar{x} = \frac{b}{4}(\frac{3h+16}{h+6})$

▶ **5-19.** $\bar{y} = 4/3$

▶ **5-20.** (a) $T - T_{env} = T_0 e^{-kt}, \quad T_0\ constant$ (b) $T = 100e^{-kt}$ (c) $k = -\frac{1}{20}\ln(4/5)$ (d) $\Delta t = 20\ln(9/7)/ln(5/4)$

▶ **5-21.** $N = N_0 e^{\frac{t}{5}\ln 3}$

▶ **5-22.**

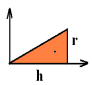
(a) $dV = 2\pi x h(x)\, dx \qquad V = \int_{x_0}^{x_1} 2\pi x h(x)\, dx$

(b) $dA = h(x)\, dx \qquad A = \int_{x_0}^{x_1} h(x)\, dx$

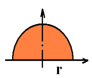
(c) $dM = x\, dA = x h(x)\, dx \qquad M = \int_{x_0}^{x_1} x h(x)\, dx \qquad \bar{x} = \frac{1}{A}\int_{x_0}^{x_1} x h(x)\, dx$

(d) (Area)(distance traveled by centroid)=$A \cdot 2\pi\bar{x}$=(volume)=$\int_{x_0}^{x_1} 2\pi x h(x)\, dx$
$A \cdot 2\pi \frac{1}{A}\int_{x_0}^{x_1} x h(x)\, dx = 2\pi \int_{x_0}^{x_1} x h(x)\, dx$ reduces to an identity.

▶ **5-23.**



(a) $\bar{y} = \frac{1}{3} r \;\; \bar{x} = \frac{2}{3} h \;\; A = \frac{1}{2} h r \qquad$ Volume=V=$2\pi\bar{y}A = \frac{\pi}{3} r^2 h$



(b) $\bar{y} = \frac{4}{3\pi} r, \;\; A = \frac{\pi}{2} r^2 \qquad$ Volume=$2\pi\bar{y}A = \frac{4}{3}\pi r^3$

▶ **5-24.** (e) $\frac{dy}{y} = dx, \qquad \int \frac{dy}{y} = \int dx, \qquad \ln y = x + C$ or $y = y_0 e^x, \qquad y_0 = e^C$

(f)

$$\int \frac{d^2y}{dx^2}\, dx = \int \frac{dy}{dx}\, dx \text{ gives } \frac{dy}{dx} = y + C$$

Now separate the variables and write

$$\frac{dy}{y+C} = dx \quad \text{with} \quad \int \frac{dy}{y+C} = \int dx \text{ giving } \ln(y+C) = x + C_2$$

which can also be expressed in the form $y + C = y_0 e^x$ where $y_0 = e^{C_2}$ is a new constant.

▶ **5-25.** (b) $\int (1+y^2)\, dy = \int (1+x^2)\, dx \quad$ gives $\quad y + \frac{y^3}{3} = x + \frac{x^3}{3} + C$

▶ **5-26.** Assume solution $y = e^{\gamma t}$ to the homogeneous equation $y'' + \omega^2 y = 0$ and show $\gamma = i\,\omega$ and $\gamma = -i\,\omega$ are characteristic roots which lead to the real fundamental set $\{\cos\omega t, \sin\omega t\}$ and complementary solution $y_c = c_1 \cos\omega t + c_2 \sin\omega t$.

Examine the right-hand side and its derivative to find basic terms needed for assumed solution. The terms $\cos\lambda t$ and $\sin\lambda t$ (multiplied by some constants) are the only basic terms and so one can assume a particular solution $y_p = A\cos\lambda t + B\sin\lambda t$ where A and B are constants to be determined. Substitute this solution and its derivatives into the nonhomogeneous equation and equate coefficients of like terms and show A, B must be selected to satisfy

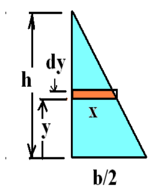
$$-A\lambda^2 + \omega^2 A = 1$$
$$-B\lambda^2 + \omega^2 B = 0$$

giving $B = 0$ and $A = \frac{1}{\omega^2 - \lambda^2}$. This gives the particular solution $y_p = \frac{1}{\omega^2-\lambda^2}\cos\lambda t$. The general solution is therefore

$$y = y_c + y_p = c_1 \cos\omega t + c_2 \sin\omega t + \frac{1}{\omega^2 - \lambda^2}\cos\lambda t$$

Resonance occurs as $\lambda \to \omega$.

▶ **5-27.**

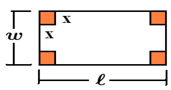


Use similar triangles and write $\dfrac{h}{b/2} = \dfrac{h-y}{x}$ or $x = \dfrac{h-y}{h}\dfrac{b}{2}$

This gives element of volume $dV = 4x^2\,dy = 4\left(\dfrac{h-y}{h}\right)^2 \dfrac{b^2}{4}\,dy$

Sum these elements from 0 to h and show volume of pyramid is $V = \frac{1}{3}b^2 h$.

▶ **5-28.**



The box has volume $V = x(w-2x)(\ell-2x) = 4x^3 - 2(w+\ell)x^2 + wlx$

Here $\dfrac{dV}{dx} = 12x^2 - 4(w+\ell)x + w\ell = 12x^2 - 60x + 48 = (2x-8)(6x-6)$

The second derivative is $\dfrac{d^2V}{dx^2} = 24x - 60$. When $x = 1$, $\frac{d^2V}{dx^2} < 0$ hence maximum box achieved. If $x = 4$, solution is meaningless as the volume becomes negative.

▶ **5-29.**

$$(a) \quad Area = \frac{1}{6} \quad M_x = \frac{1}{15} \quad M_y = \frac{1}{12}$$

$$(b) \quad Area = \frac{m^3}{6} \quad M_x = \frac{m^5}{15} \quad M_y = \frac{m^4}{12}$$

▶ **5-30.** $F = \rho\ell \sin\theta \dfrac{\omega^2}{2}$

▶ **5-32.** (d) $y'' + 3y_2'y = 0$ assume solution $y = e^{\gamma t}$ get characteristic equation $(\gamma+2)(\gamma+1) = 0$ with characteristic roots $\gamma = -2$ and $\gamma = -1$. This gives the fundamental set $\{e^{-2x}, e^{-x}\}$ and general solution $y = c_1 e^{-2x} + c_2 e^{-x}$, with c_1, c_2 arbitray constants.

▶ **5-33.** (b) $L\dfrac{di}{dt} + \dfrac{q}{C} = 0$ or $L\dfrac{di}{dq}\dfrac{dq}{dt} + \dfrac{q}{C} = 0 \implies i\dfrac{di}{dq} = -\dfrac{1}{LC}q$ Separate the variables and write $i\,di = -\frac{1}{LC}q\,dq$ and then integrate to obtain

$$\frac{i^2}{2} = -\frac{1}{LC}\frac{q^2}{2} + \frac{K}{2}$$

Here $K/2$ is selected as the constant of integration to help simplify the algebra. If $i = 0,\ q = q_0$ at $t = 0$, then $K = \frac{1}{LC}q_0^2$. This gives

$$\frac{dq}{dt} = \frac{1}{\sqrt{LC}}\sqrt{q_0^0 - q^2} \implies \frac{dq}{\sqrt{q_0^2 - q^2}} = \frac{1}{\sqrt{LC}}\,dt$$

Integrate this equation and show

$$q = q_0 \cos\left(\frac{t}{\sqrt{LC}}\right) \quad \text{and} \quad i = \frac{dq}{dt} = -\frac{q_0}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right)$$

▶ **5-35.** Hypothesis, $y_1 = y_1(x)$ satisfies the given differential equation so that $y'' + P(x)y' + Q(x)y = 0$. If $y_2 = uy_1 = u(x)y_1(x)$, then by differentiation

$$\begin{aligned} y_2' =& uy_1' + u'y_1 \\ y_2'' =& uy_1'' + 2u'y_1' + u''y_1 \end{aligned}$$

We desire to select $u = u(x)$ such that $y_2 = y_2(x)$ is also a function which satisfies the differential equation. If $y_2'' + P(x)y_2' + Q(x)y_2 = 0$, the $u = u(x)$ must be selected such that

$$uy_1'' + 2u'y_1' + P(x)[uy_1' + u'y_1] + Q(x)[uy_1] = 0$$

Rearrange terms and show

$$y_2'' + P(x)y_2' + Q(x)y_2 = u\,\underbrace{[y_1'' + P(x)y_1' + Q(x)y_1]}_{\text{zero by hypothesis}} + u'[2y_1' + P(x)y_1 + u''y_1 = 0$$

Therefore $u = u(x)$ must be selected to satisfy

$$u'[2y_1' + P(x)y_1] + u''y_1 = 0$$

Make the substitution $u' = v$ and $u'' = \frac{dv}{dx}$ and show

$$v = \frac{1}{y_1^2} e^{-\int P(x)\,dx}$$

then another integration gives u.

▶ **5-36.** (e) Integrate with respect to y and show

$$\frac{\partial u}{\partial y} = xy + \frac{y^2}{2} + f(x), \quad f(x) \text{ arbitrary function}$$

Integrate again with respect to y holding x constant to obtain

$$u = \frac{xy^2}{2} + \frac{y^3}{6} + yf(x) + g(x) \text{ where } g(x) \text{ is another arbitrary function}$$

(f) Integrate with respect to x and show

$$\frac{\partial u}{\partial x} + u = x + xy + f(y) \quad f(y) \text{ arbitrary function}$$

Now multiply by e^x and show $\frac{\partial u}{\partial x}e^x + ue^x = \frac{\partial}{\partial x}(ue^x) = xe^x + xye^x + f(y)e^x$
Integrate with respect to x and show

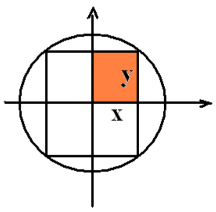
$$ue^x = e^x(x-1) + ye^x(x-1) + f(y)e^x + g(y), \quad g(y) \text{ arbitrary}$$

▶ **5-37.** (a) If $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, then differentiate the first equation with respect to x and differentiate the second equation with respect to y and show

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y\,\partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x\,\partial y}$$

then by addition of these equations one obtains $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

▶ **5-38.**



Area$= A = (2x)(2y) = 4xy = 4x\sqrt{r^2 - x^2}$ since $x^2 + y^2 = r^2$

$$\frac{dA}{dx} = \frac{-4x^2}{\sqrt{r^2 - x^2}} + 4\sqrt{r^2 - x^2}$$

Show the derivative is zero when $x = r/\sqrt{2}$ and $y = r/\sqrt{2}$ and a square is the maximum inscribed rectangle. To show it is a maximum examine $\frac{dA}{dx}$ for $x < r/\sqrt{2}$ and $\frac{dA}{dx}$ for $x > r/\sqrt{2}$

▶ **5-39.** $y = e^x$ for $0 \le x \le 1$ use $ds^2 = dx^2 + dy^2$ and write $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ so that $s = \int_0^1 \sqrt{1 + e^{2x}}\,dx$ To evaluate this integral make the substitution $u = e^x$ with $du = e^x\,dx$ and show $s = \int_1^e \sqrt{1 + u^2}\,\frac{du}{u}$ Make another substitution $w^2 = 1 + u^2$ with $2w\,dw = 2u\,du$ and show

$$s = \int_{\sqrt{2}}^{\sqrt{1+e^2}} w \cdot \frac{w\,dw}{\sqrt{w^2 - 1}}\frac{1}{\sqrt{w^2 - 1}}$$

$$s = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{w^2 - 1 + 1}{w^2 - 1}\,dw$$

$$s = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left[dw - \frac{dw}{1 - w^2}\right] = w\Big]_{\sqrt{2}}^{\sqrt{1+e^2}} - \tanh^{-1} w\Big]_{\sqrt{2}}^{\sqrt{1+e^2}}$$

$$s = \sqrt{1 + e^2} - \sqrt{2} + \tanh^{-1}\sqrt{2} - \tanh^{-1}(\sqrt{1 + e^2})$$

▶ **5-40.** The side area is $A_s = 2\pi rh$ and the ends (top and bottom) have area $A_e = 2\pi r^2$ where $\pi r^2 h = \pi V_0$ is to be satisfied. The cost of construction is

$$C = c_0(2\pi rh) + 3c_0(2\pi r^2)$$

Substitute into this equation $h = \frac{V_0}{r^2}$ to express $C = C(r)$ as a function of r. Show $\frac{dC}{dr} = 0$ when $r = \sqrt[3]{V_0}/6$ and $h = 6^{2/3}\sqrt[3]{V_0}$. Show curve for $C = C(r)$ is concave up at this point and hence a minimum is achieved.

► **5-41.** (a) $\frac{\partial u}{\partial x}$ (b) $\frac{\partial u}{\partial y}$

► **5-43.** $V = AH$ so that if $\frac{dV}{dt} = kA$ and A is a constant, then $\frac{dV}{dt} = A\frac{dh}{dt} = kA$ which implies $\dfrac{dh}{dt} = k$ is a constant. Here k is the proportionality constant.

► **5-44.** $I = x - \dfrac{4}{3}\tan x + \dfrac{1}{3}\sec^2 x \tan x + C$

► **5-45.** (b) $I = \frac{5}{3}\tan x + \frac{1}{3}\sec^2 x \tan x + C$

► **5-46.** For $n = 50$

$f(x)$	$\sum_{i=1}^{50}$	$\int_a^b f(x)\,dx$
x	2	2
x^2	2.6672	8/3
x^3	4.0016	4
$\sin x$	1.99934	2
$\cos x$	-1.99934	2

► **5-48.** $\Gamma(x) = \displaystyle\int_0^\infty e^{-t}t^{x-1}\,dx$, Integrate by parts with $U = t^{x-1}$ and $dV = e^{-t}\,dt$ to obtain $\Gamma(x) = (x-1)\Gamma(x-1)$ Now replace x by $x+1$ to obtain $\Gamma(x+1) = x\Gamma(x)$

$$\begin{aligned}
\Gamma(1) =& \int_0^\infty e^{-t}\,dt = 1 \\
\Gamma(2) =& 1\Gamma(1) = 1 \\
\Gamma(3) =& 2\Gamma(2) = 2 \cdot 1 = 2! \\
\Gamma(4) =& 3\Gamma(3) = 3 \cdot 2! = 3! \\
\vdots & \qquad \vdots \\
\Gamma(n+1) =& n\Gamma(n) = n \cdot (n-1)! = n!
\end{aligned}$$

► **5-49.** (a) $s_1(t) = s_2(t)$ when $2t^2 + t = 11t - 3t^2$ or $t = 0$ and $t = 2$.

$$\frac{ds_1}{dt} = v_1 = 4t + 1 \qquad \frac{ds_2}{dt} = v_2 = 11 - 6t$$

At $t = 0$, $v_1 = 1$ and $v_2 = 11$. At $t = 2$, $v_1 = 9$ and $v_2 = -1$.

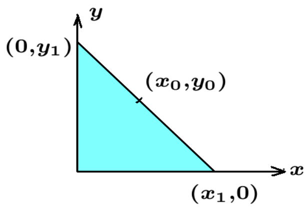
(b) $v_1 = v_2$ when $t = 1$ at positions $s_1(1) = 3$ and $s_2(1) = 8$.

(c) $v_1 = \frac{ds_1}{dt} > 0$ steadily increases, while $v_2 = \frac{ds_2}{dt} > 0$ for $t < 11/6$, s_2 increases and $\frac{ds_2}{dt} < 0$ for $t > 11/6$, then s_2 decreases.

▶ **5-50.** (a) $\dfrac{\partial f}{\partial x} = 2x - 2,$ and $\dfrac{\partial f}{\partial x} = 0$ when $x = 1$.

$\dfrac{\partial f}{\partial y} = 2y - 4$ and $\dfrac{\partial f}{\partial y} = 0$ when $y = 2$. $f(1,2) = -25$ is a minimum value, since for all (x, y) is a neighborhood of $(1,2)$ we have $f(x,y) > f(1,2)$.

▶ **5-51.**



$m = slope = \dfrac{-y_0}{x_1 - x_0}$ and equation of line is

$y - y_0 = \dfrac{(-y_0)}{(x_1 - x_0)}(x - x_0)$, when $x = 0$, $y_1 = y_0 + \frac{x_0 y_0}{x_1 - x_0}$.

Therefore

$$\ell^2 = x_1^2 + y_1^2 = x_1^2 + \left(y_0 + \frac{x_0 y_0}{x_1 - x_0}\right)^2, \quad x_0, y_0 \; fixed$$

Show $\frac{\partial \ell}{\partial x_1} = 0$, when $x_1 = x_0 + x_0^{1/3} y_0^{2/3}$ This is the value of x_1 which will produce the shortest line.

▶ **5-52.** $x = 3,\ y = 4$ and $x = 9,\ y = 12$

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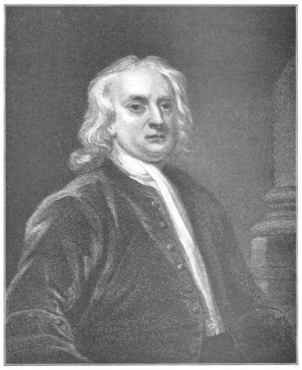
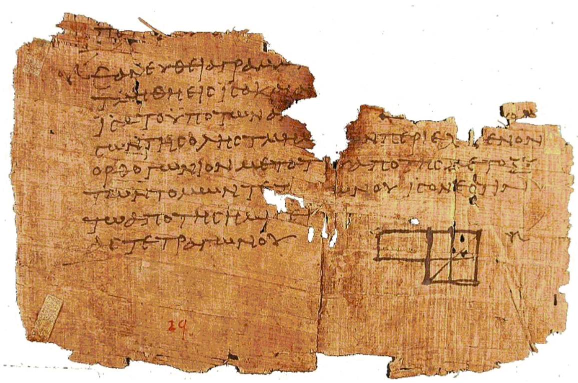


Isaac Newton (1642-1727)

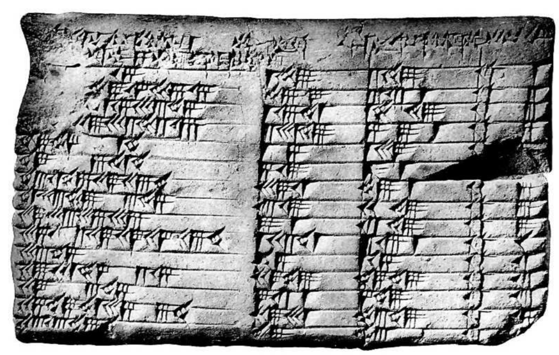
Gottfried Wilhelm Leibnitz (1646-1716)

Rhind mathematical papyrus written around 1650 BCE can be found at the British Museum.

Translation available on internet.



Papyrus containing part of Euclid's Elements is located at the University of Pennsylvania. Euclid's Elements consists of 13 books on geometry written around 300 BCE.



Plimpton 322 is a mathematical clay tablet from Babylon around 1800 BCE. For more details read, Eleanor Robson, "Words and pictures: new light on Plimpton 322", American Mathematical Monthly,109(2):105-120. Tablet can be found at Columbia University.