Statics and Mechanics of Materials

Chapter 4

Stress, Strain and Deformation: Axial Loading

Objectives:

- Learn and understand the concepts of internal forces, stresses, and strains
- Learn and understand the key concept of constitutive relationship of linear materials
- Know how to compute normal and shearing strains and stresses in mechanically and/or thermally loaded members (axial loading)
- Know how to compute strains and stresses of members belonging to indeterminate structures

Stress & Strain: Axial Loading

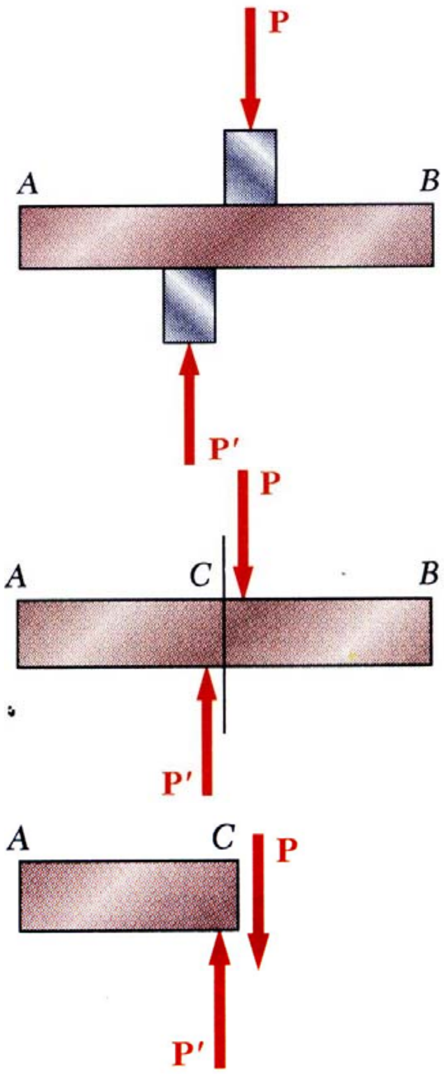
- Suitability of a structure or machine may depend on the deformations in the structure as well as the stresses induced under loading. Statics analyses alone are not sufficient.

- Considering structures as deformable allows determination of member forces and reactions which are statically indeterminate.

- Determination of the stress distribution within a member also requires consideration of deformations in the member.

- This chapter is concerned with deformation of a structural member under axial loading.

Shearing Stress



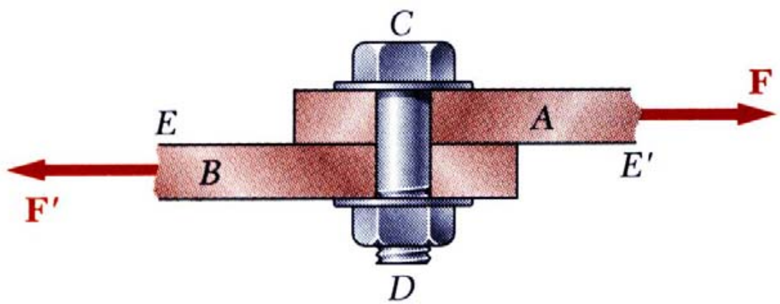
- Forces *P* and *P’* are applied transversely to the member *AB*.
- Corresponding internal forces act in the plane of section *C* and are called *shearing* forces.
- The resultant of the internal shear force distribution is defined as the *shear* of the section and is equal to the load *P*.
- The corresponding average shear stress is,

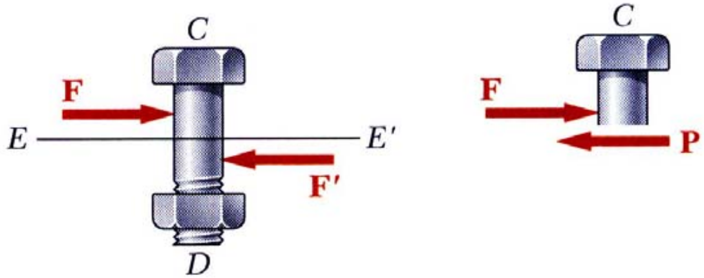
$$\tau_{\text{ave}} = \frac{P}{A}$$

- Shear stress distribution varies from zero at the member surfaces to maximum values that may be much larger than the average value.
- The shear stress distribution cannot be assumed to be uniform.

Shearing Stress Examples

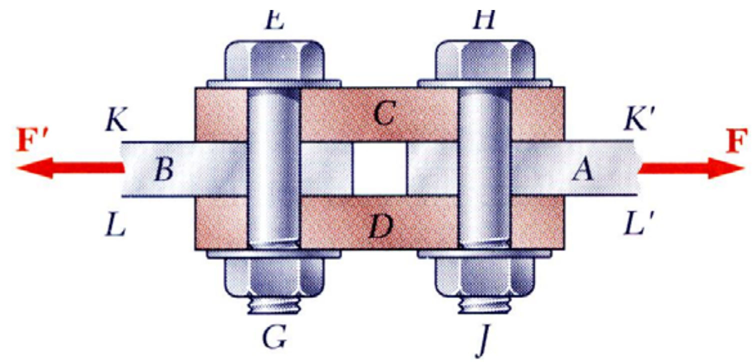
Single Shear

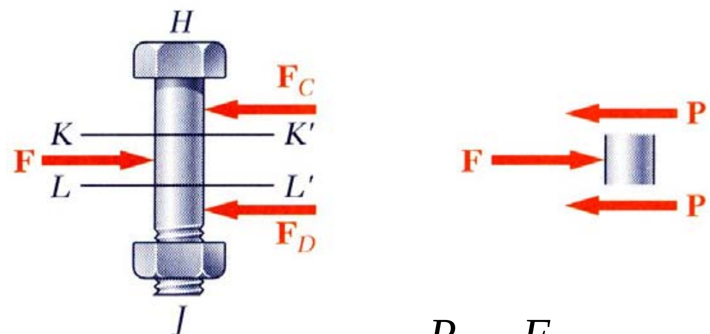




$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{A}$$

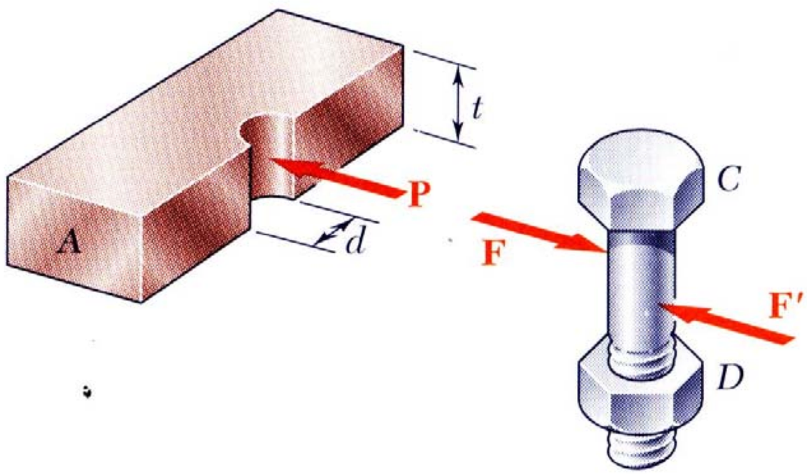
Double Shear

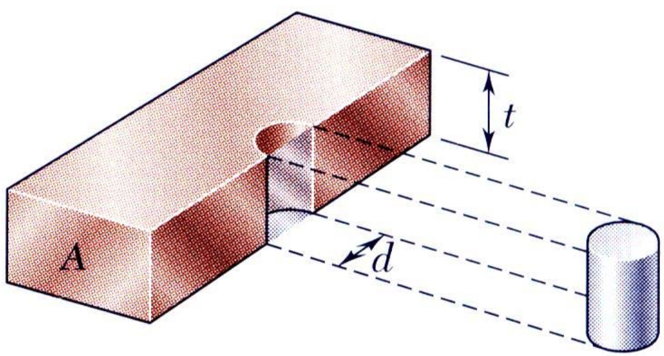




$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{2A}$$

Bearing Stress in Connections





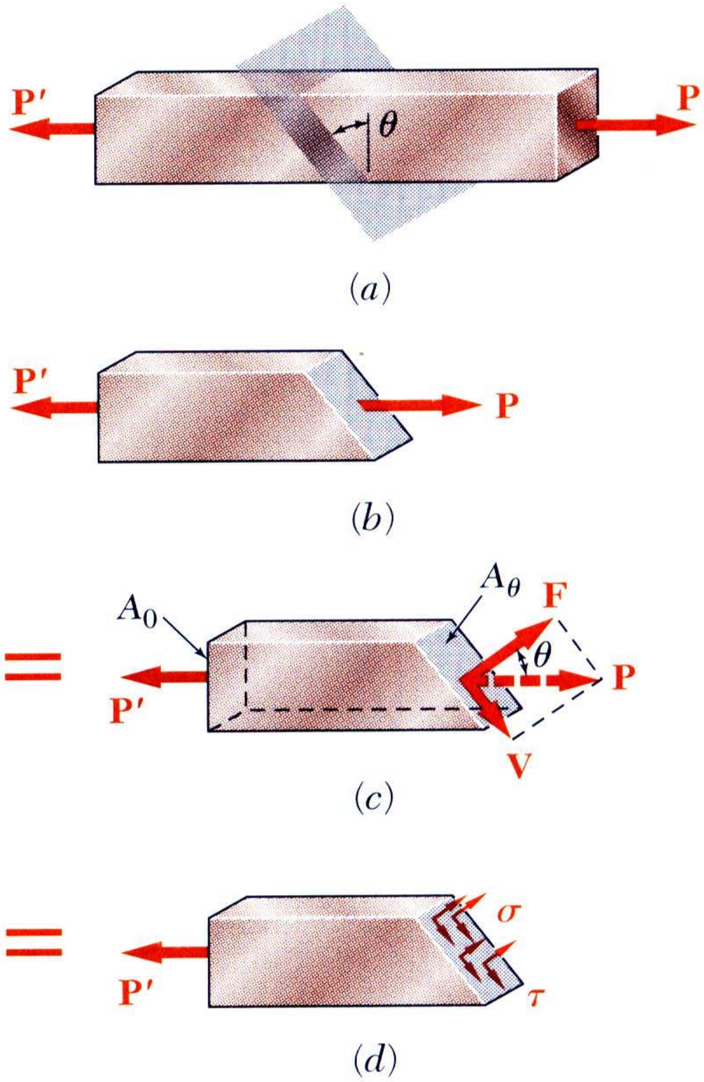
- Bolts, rivets, and pins create stresses on the points of contact or *bearing surfaces* of the members they connect.

- The resultant of the force distribution on the surface is equal and opposite to the force exerted on the pin.

- Corresponding average force intensity is called the bearing stress,

$$\sigma_b = \frac{P}{A} = \frac{P}{t\,d}$$

Stress on an Oblique Plane



- Pass a section through the member forming an angle θ with the normal plane.
- From equilibrium conditions, the distributed forces (stresses) on the plane must be equivalent to the force *P*.
- Resolve *P* into components normal and tangential to the oblique section,

$$F = P\cos\theta \qquad V = P\sin\theta$$

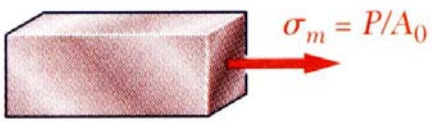
- The average normal and shear stresses on the oblique plane are

$$\sigma = \frac{F}{A_\theta} = \frac{P\cos\theta}{A_0/\cos\theta} = \frac{P}{A_0}\cos^2\theta$$

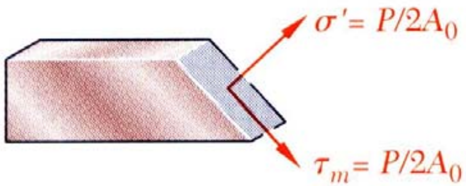
$$\tau = \frac{V}{A_\theta} = \frac{P\sin\theta}{A_0/\cos\theta} = \frac{P}{A_0}\sin\theta\cos\theta$$

Maximum Stresses

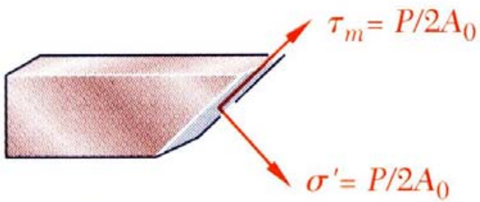
(a) Axial loading



(b) Stresses for $\theta = 0$



(c) Stresses for $\theta = 45°$



(d) Stresses for $\theta = -45°$

- Normal and shearing stresses on an oblique plane

$$\sigma = \frac{P}{A_0}\cos^2\theta \qquad \tau = \frac{P}{A_0}\sin\theta\cos\theta$$

- The maximum normal stress occurs when the reference plane is perpendicular to the member axis,

$$\sigma_{\mathrm{m}} = \frac{P}{A_0} \qquad \tau' = 0$$

- The maximum shear stress occurs for a plane at $\pm$ 45^o with respect to the axis,

$$\tau_m = \frac{P}{A_0}\sin 45\cos 45 = \frac{P}{2A_0} = \sigma'$$

Chapter 4.4

Displacement, Deformation, and Strain

Displacement, deformation, and strain

- Displacement
  - **A vector that represents a movement of a point in a body (due to applied loads) with respect to some reference system of axes**
  - **Translation and/or rotation**
  - **Shape and size of the body do not change**

- Deformation
  - **A vector that represents a movement of a point in a body (due to applied loads) <u>relative to another body point</u>**
  - **The shape and size of the body change (being deformed)**
  - **Volume may be unchanged (special cases)**

- Strain
  - **Intensity of deformation**
  - **Objects of the same materials but different sizes demonstrate different effects when subjected to the same load**
  - **Normal strain (ε): measures the change in <u>size</u> (elongation/contraction)**
  - **Shearing strain (γ): measures the change in <u>shape</u> (angle formed by the sides of a body)**

Normal Strain

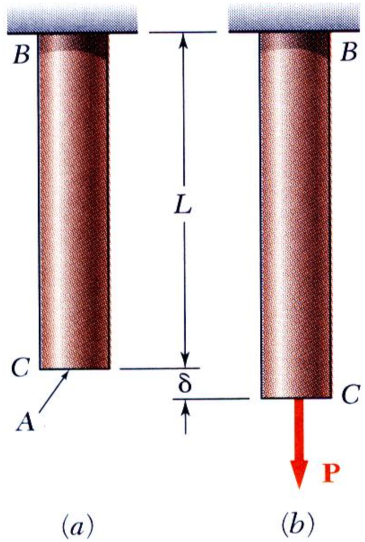


Fig. 2.1

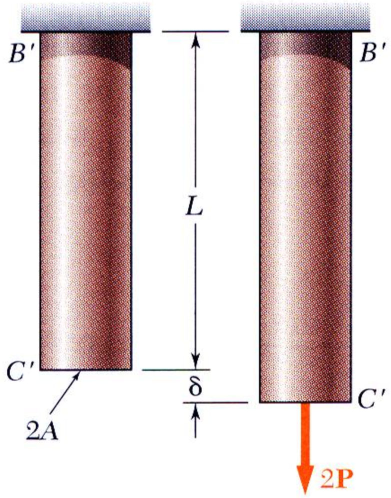


Fig. 2.3

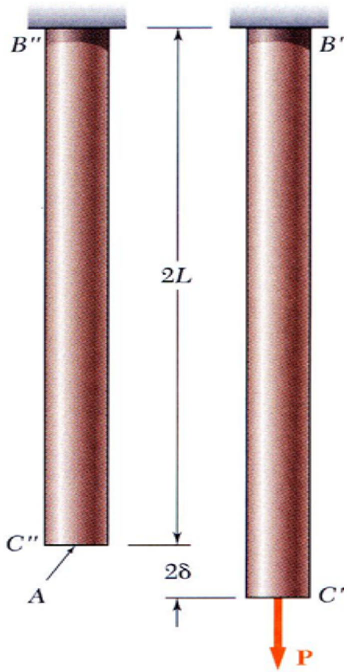


Fig. 2.4

$$\sigma = \frac{P}{A} = \text{stress}$$

$$\varepsilon = \frac{\delta}{L} = \text{normal strain}$$

$$\sigma = \frac{2P}{2A} = \frac{P}{A}$$

$$\varepsilon = \frac{\delta}{L}$$

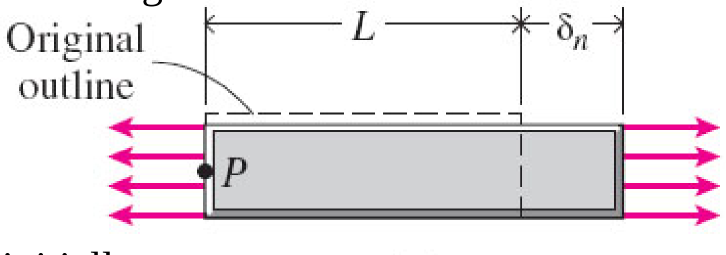
$$\sigma = \frac{P}{A}$$

$$\varepsilon = \frac{2\delta}{2L} = \frac{\delta}{L}$$

Normal and shearing strains

- Normal strain:
  - Average axial strain ←assumed that the deformation is homogeneous
  - Average value along the axial direction

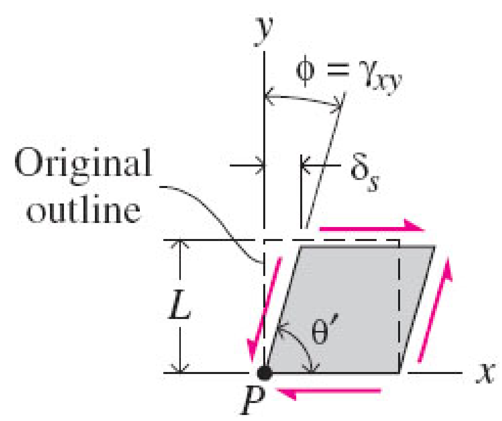
$$\varepsilon_{avg} = \frac{\delta_n}{L}$$



(a)

- Shearing strain
  - θ' = the angle in the deformed state between the two initially orthogonal reference lines

$$\gamma_{avg} = \frac{\delta_s}{L} = \tan\phi \approx \phi = \frac{\pi}{2} - \theta'$$



(b)

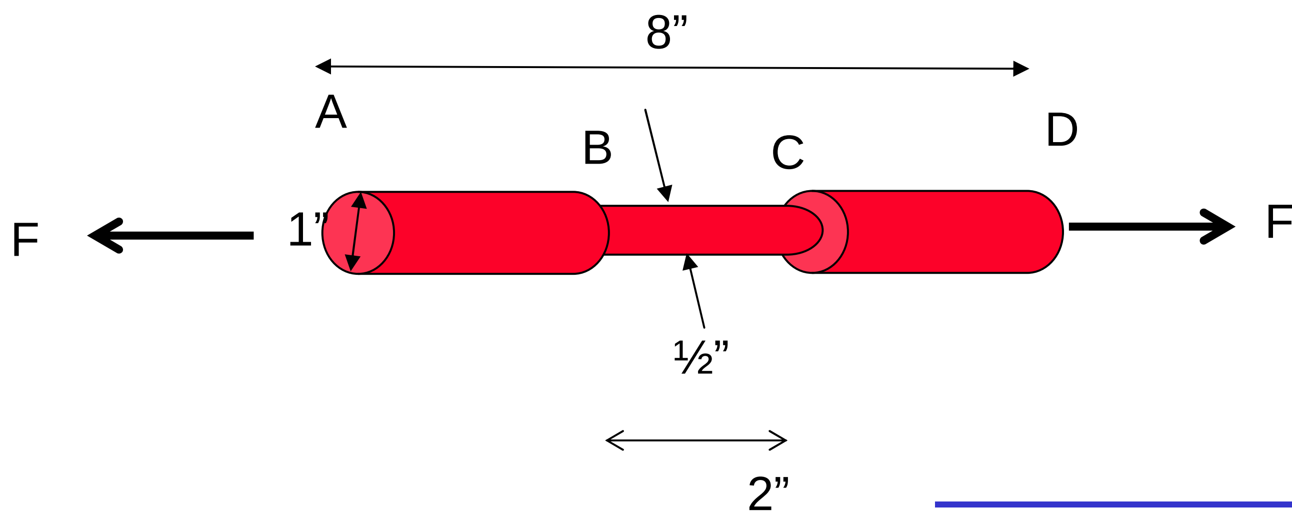
- True axial strain
  - The true local strain at a point in the body

$$\varepsilon(P) = \frac{d\delta_n}{dL}$$

- Units of strain ➔ dimensionless
- Tensile strain == positive, compressive strain == negative

Example Problem 4-8

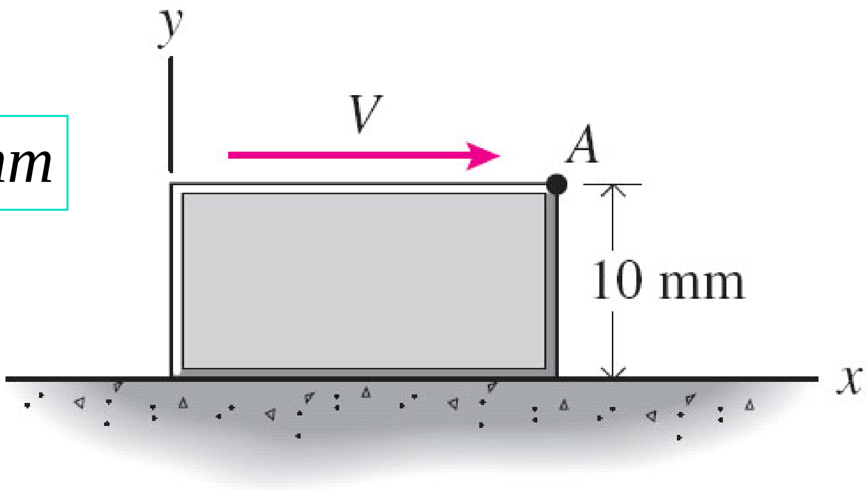
- Given ε_{BC}, compute the elongation of the central portion of the bar
- Given δ_{total} determine the axial strain in the end portions of the bar (basically ε_E)



Example Problem 4-9

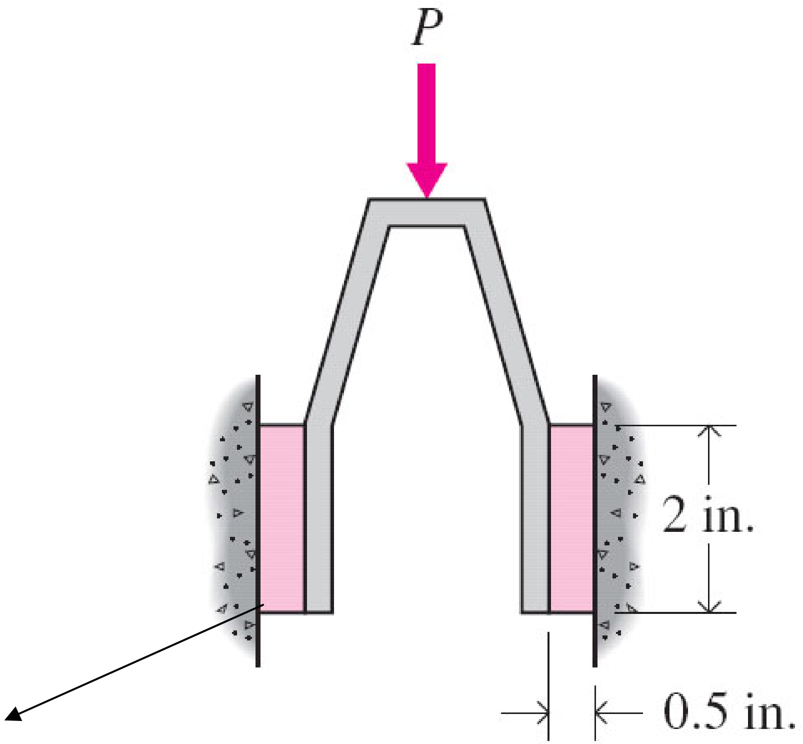
- $\gamma = 1000\ \mu m/m = 10^{-3}$
- Determine the displacement of A (δ_A)

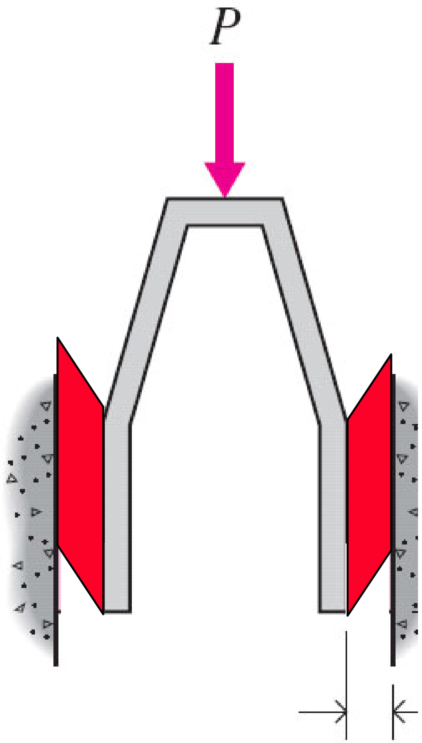
$$\delta_A = \gamma L = 0.001 \times 10 = 0.01 mm$$



Example

- Where is the shearing strain?



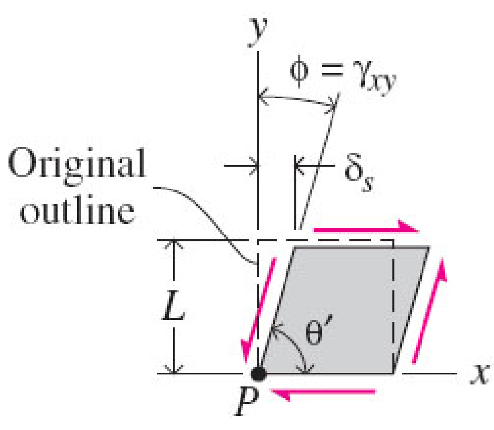


Fixed support
Does not move a bit!!

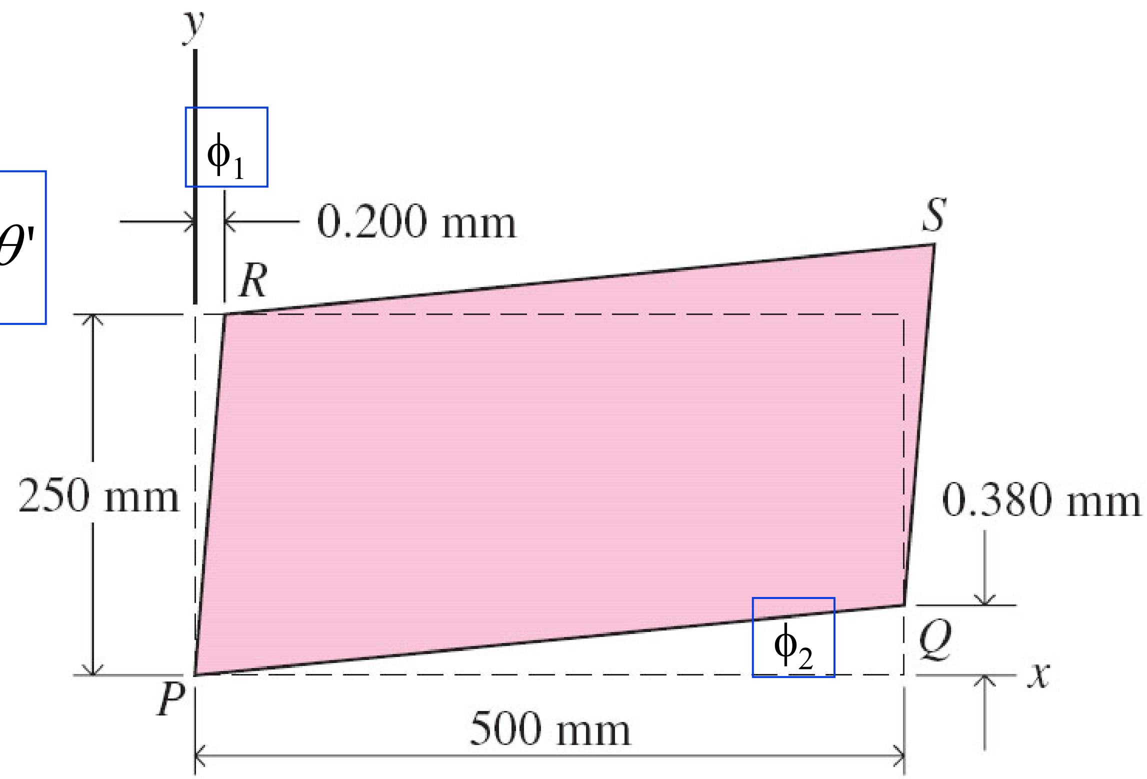
Example

- Determine the shearing strain at P

$$\gamma_{avg} = \frac{\delta_s}{L} = \tan\phi \approx \phi = \frac{\pi}{2} - \theta'$$



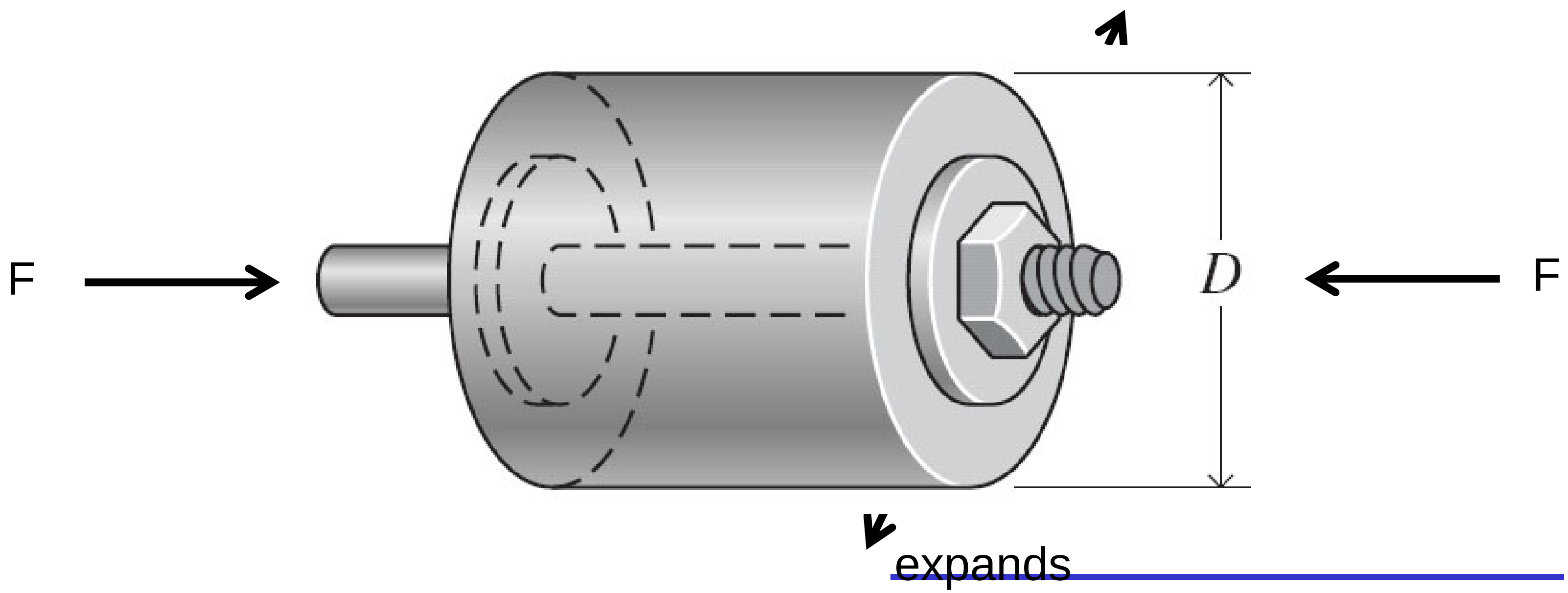
(b)

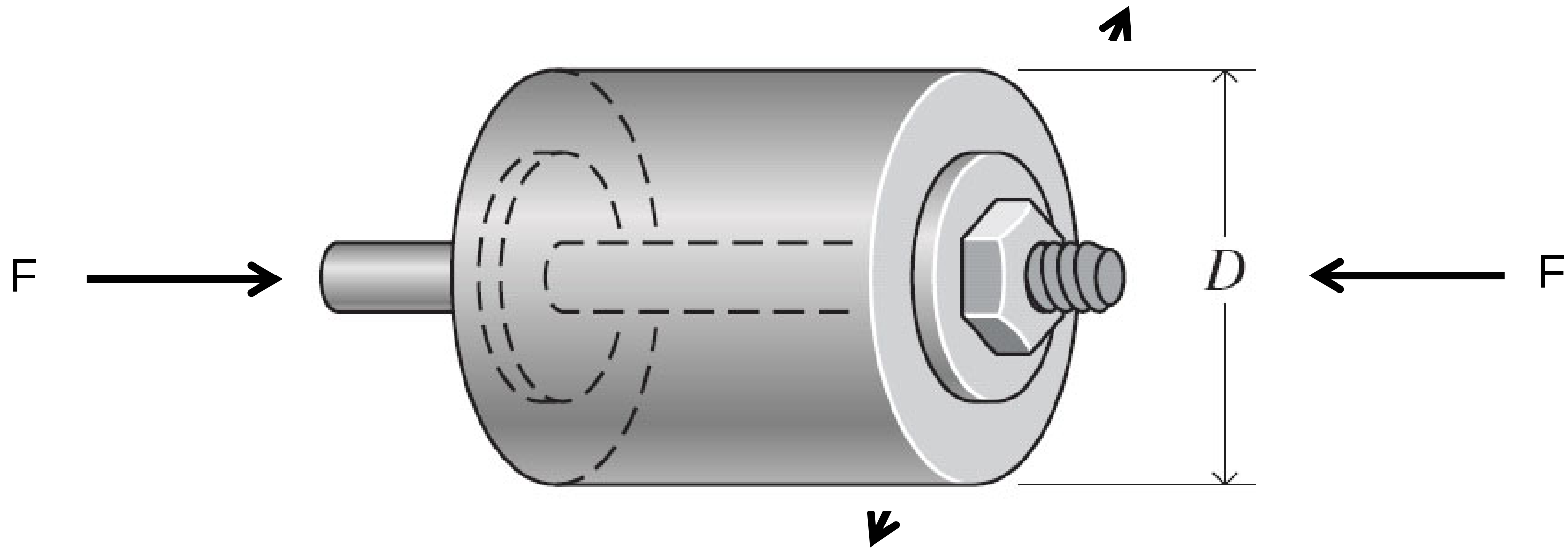


$$\phi = \phi_1 + \phi_2$$

Example

- Normal strain along a diameter = the ratio of the net diameter change to the original diameter
- Circumferential strain = the ratio of the net circumference change to that of the original circumferential





SOLUTION

(a) $$\varepsilon_D = \frac{2.15 - 2.00}{2.00} = 75.0 \times 10^{-3} \text{ in./in.}$$

(b) $$\varepsilon_C = \frac{\pi(2.15) - \pi(2.00)}{\pi(2.00)} = 75.0 \times 10^{-3} \text{ in./in.}$$

4-5 Stress-strain-temperature relationships (constitutive relationship)

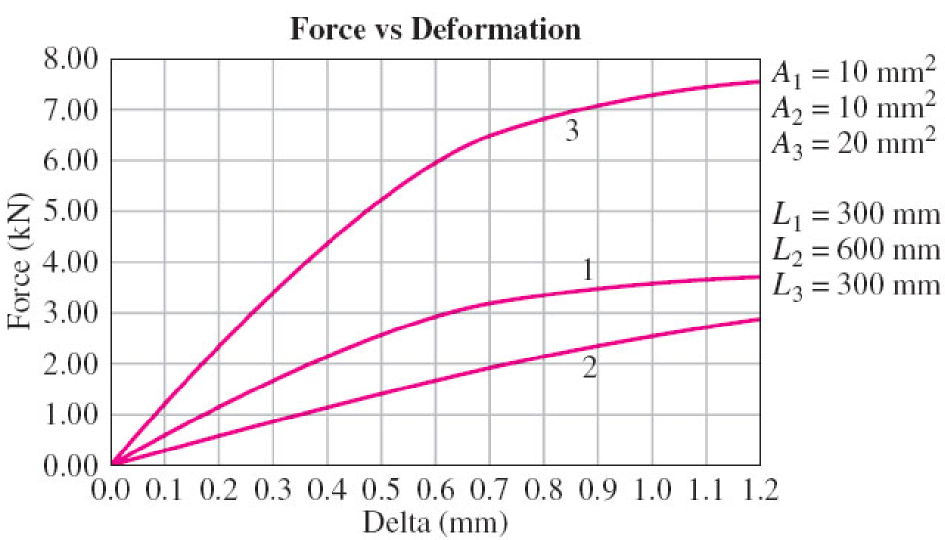
- Stress vs. strain
- Thermal strain
- Deformation of axially loaded members

Stress vs. strain relationship

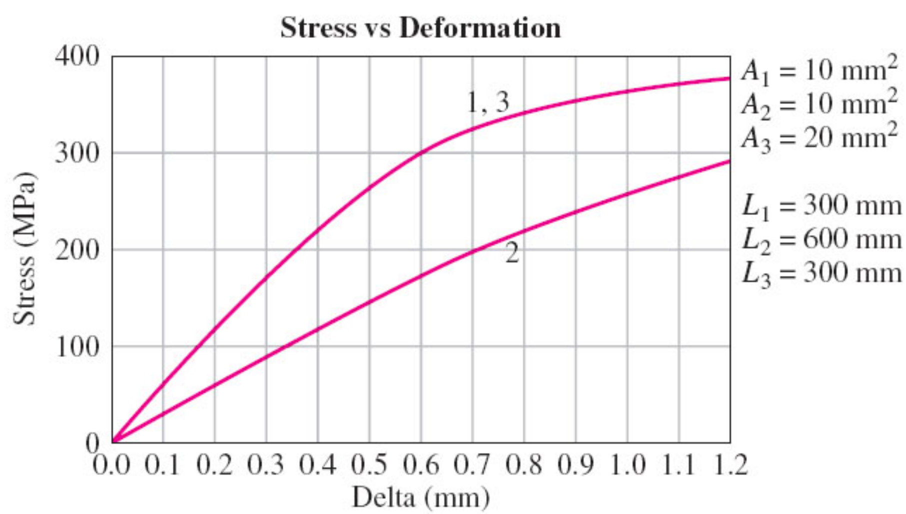
- Structural analysis and design requires understanding of the system of the applied forces and the material behavior
- The behavior of a material can be studied by means of mechanical testing
- Stress vs. strain diagrams are often used to describe the material behavior
- Stress vs. strain diagrams are supposedly/theoretically identical for the same material, but technically there is always some differences

Why stress vs. strain?

- Force vs. deformation and stress vs. deformation diagrams cannot uniquely describe the material behavior
  - Force depends on the application area
  - Displacement depends on the length of the specimens



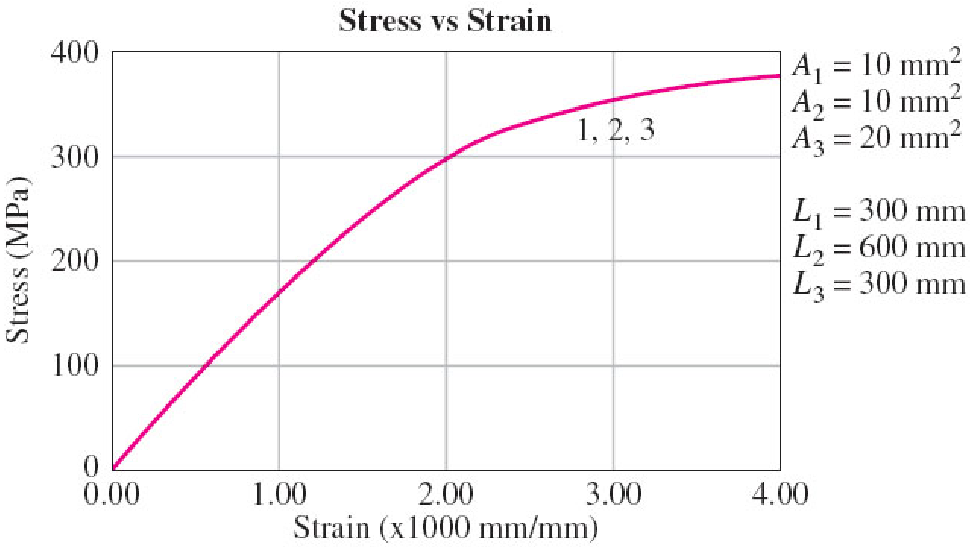
(*a*)



(*b*)

Why stress vs. strain?

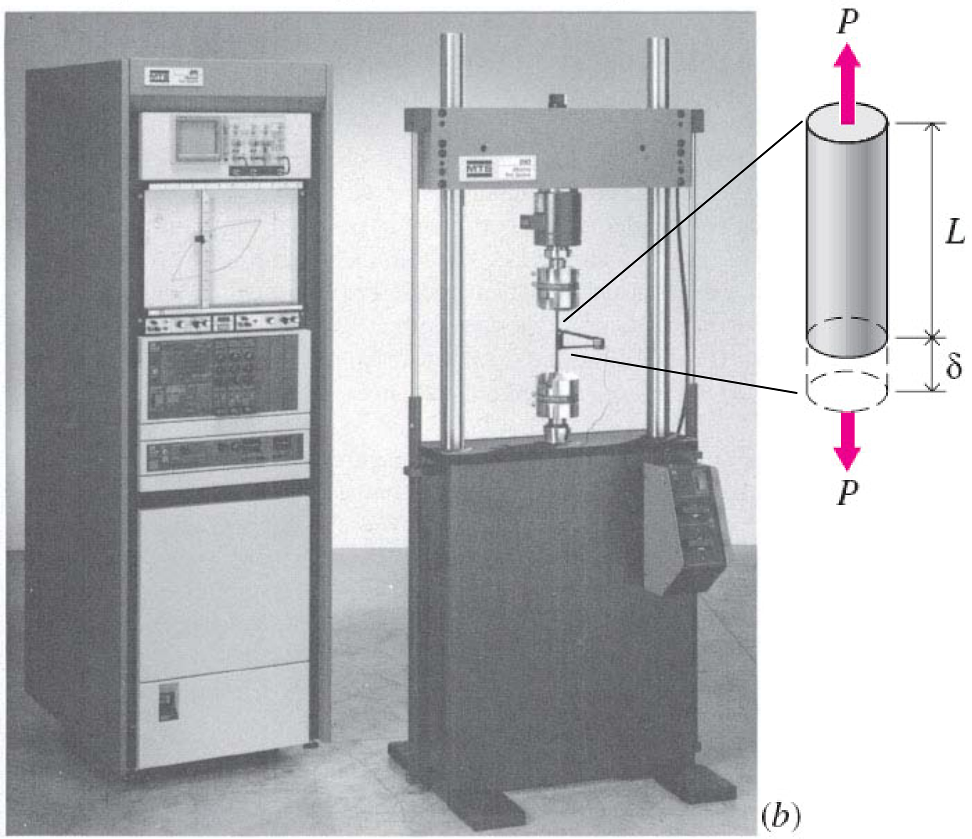
- When the stress vs. strain diagrams are used, the curves are merging ➔ diminishing the effects of size of the samples



(c)

The tensile test

- Uniaxial loading tester allows us to study the behavior of materials under tension
- The applied force is measured by means of load cells
- The stress is calculated utilizing the cross section area of the sample
- The deformation can be measured from the motion of the grips where the sample is attached to
- Utilizing the original length, the strain can be calculated
- Alternatively, a strain gauge may be used
- The stress vs. strain diagram can be obtained



The Tensile Test
(Normal Stress-Normal Strain)

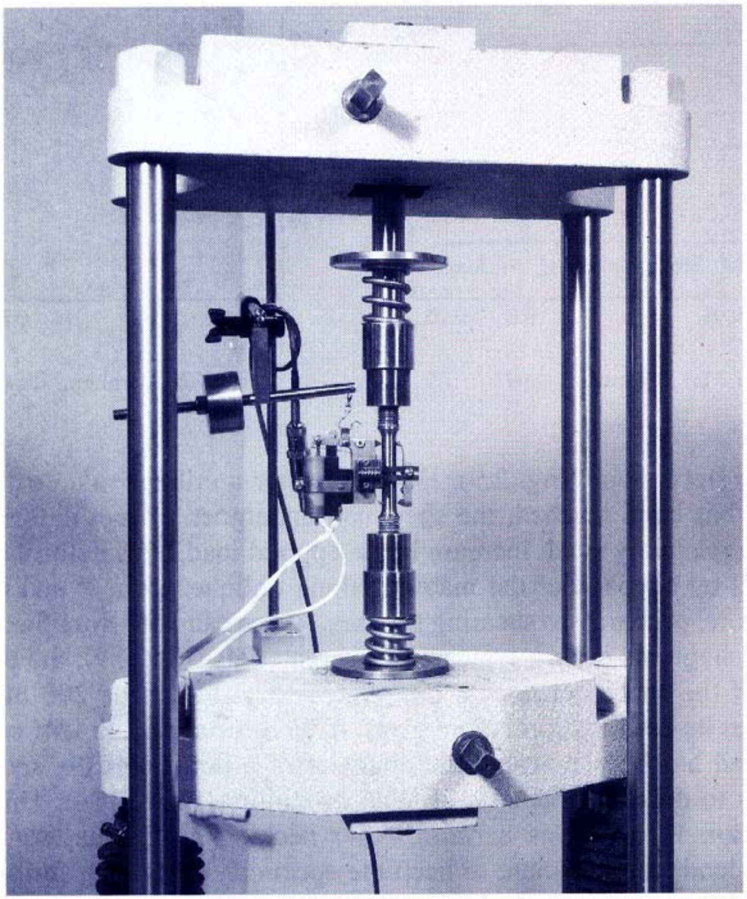


Fig. 2.7 This machine is used to test tensile test specimens, such as those shown in this chapter.

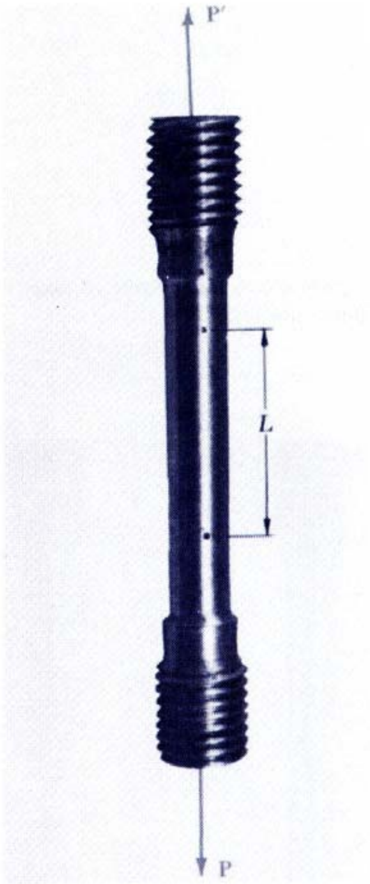
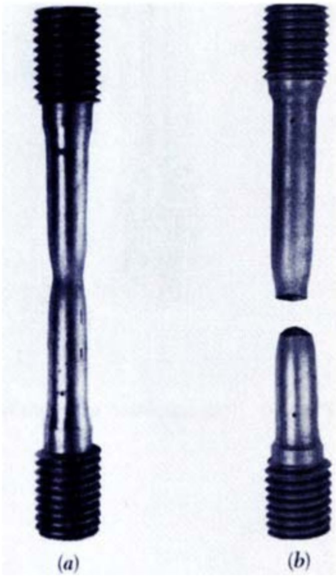
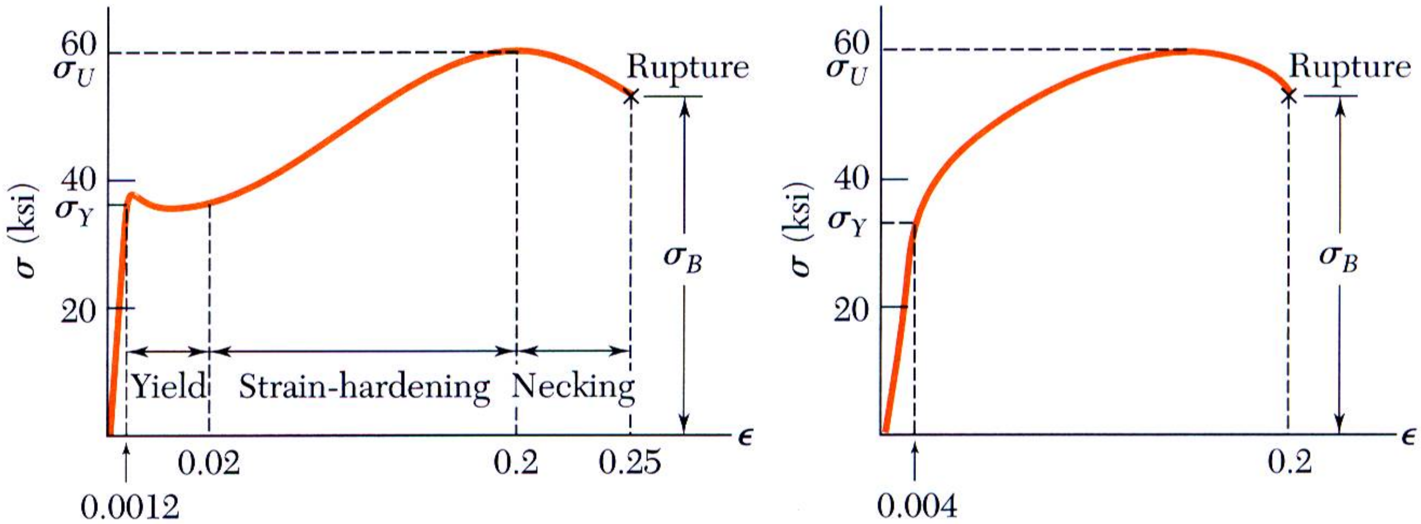


Fig. 2.8 Test specimen with tensile load.

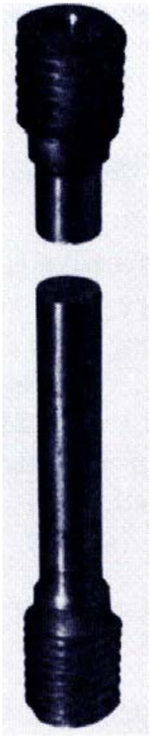
Stress-Strain Diagram: Ductile Materials





(a) Low-carbon steel (b) Aluminum alloy

Stress-Strain Diagram: Brittle Materials



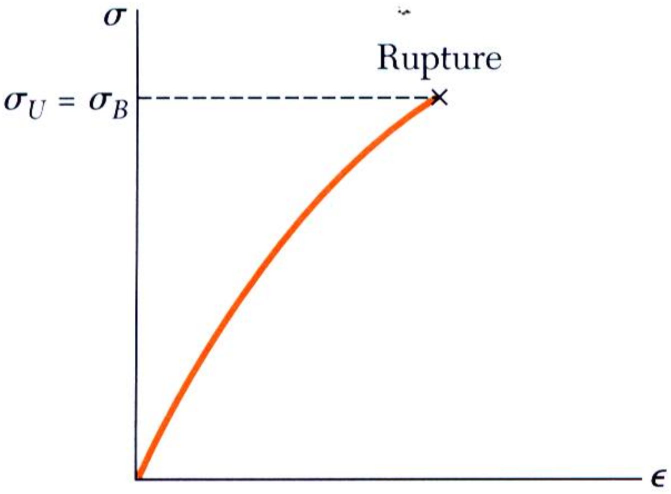
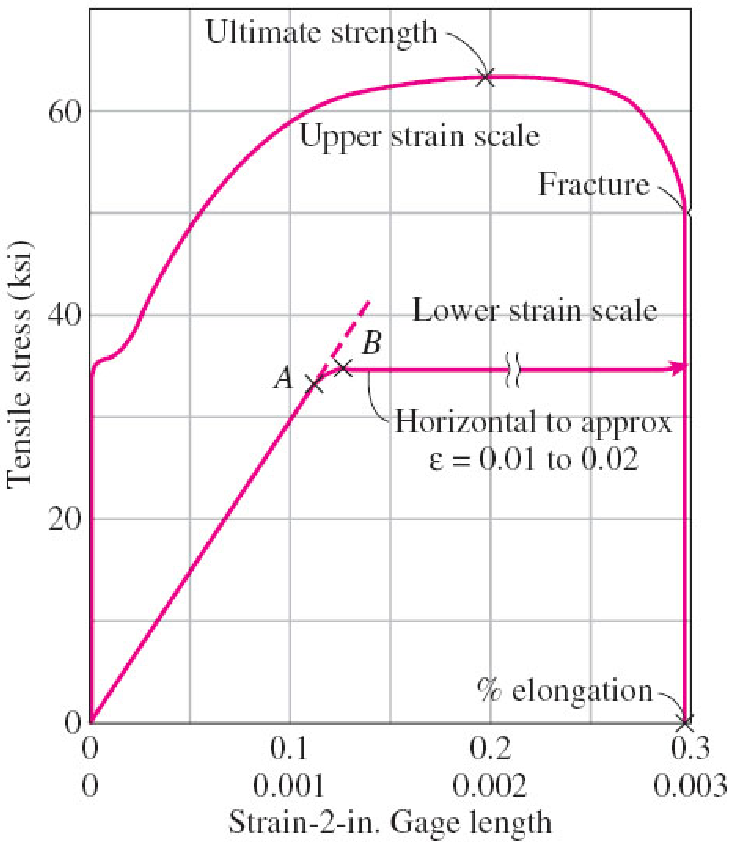
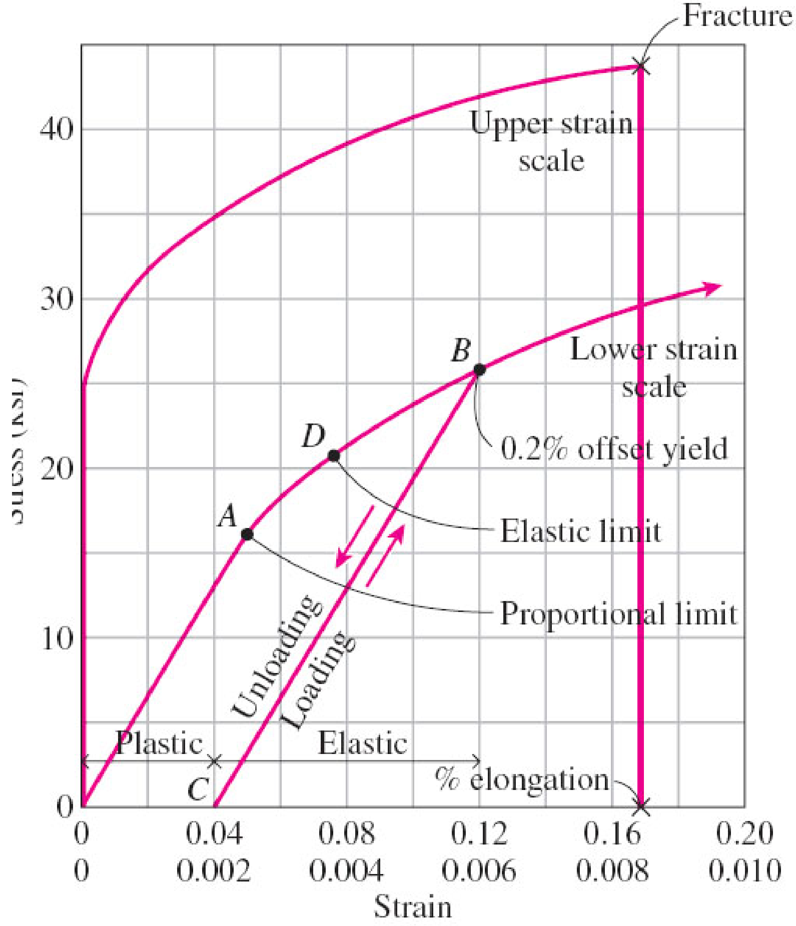


Fig. 2.11 Stress-strain diagram for a typical brittle material.

Typical stress vs. strain diagrams

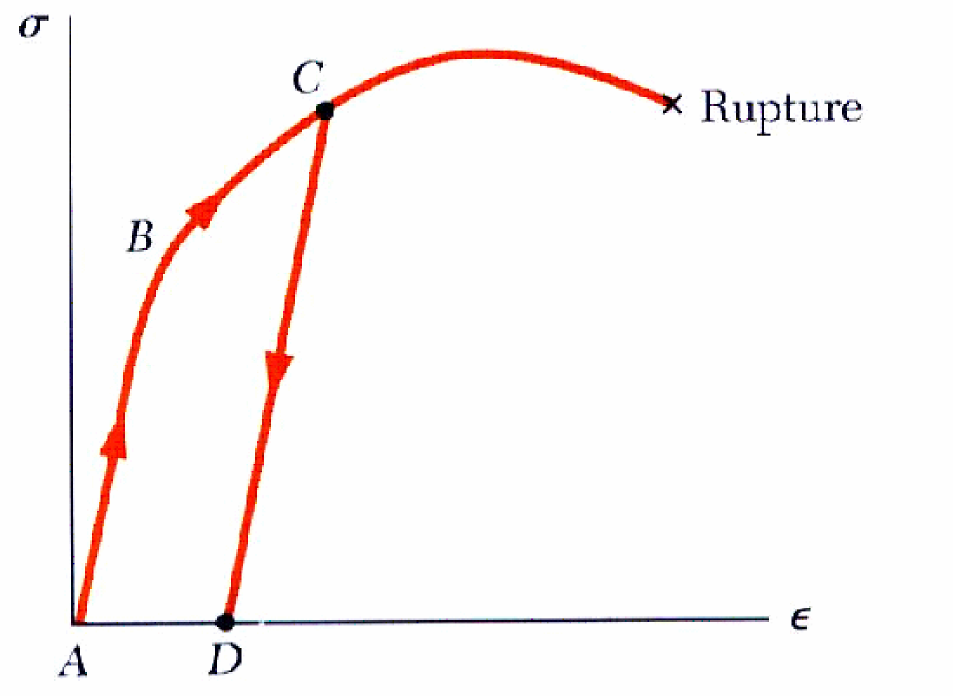


(*a*) Structural steel



(*b*) Magnesium alloy

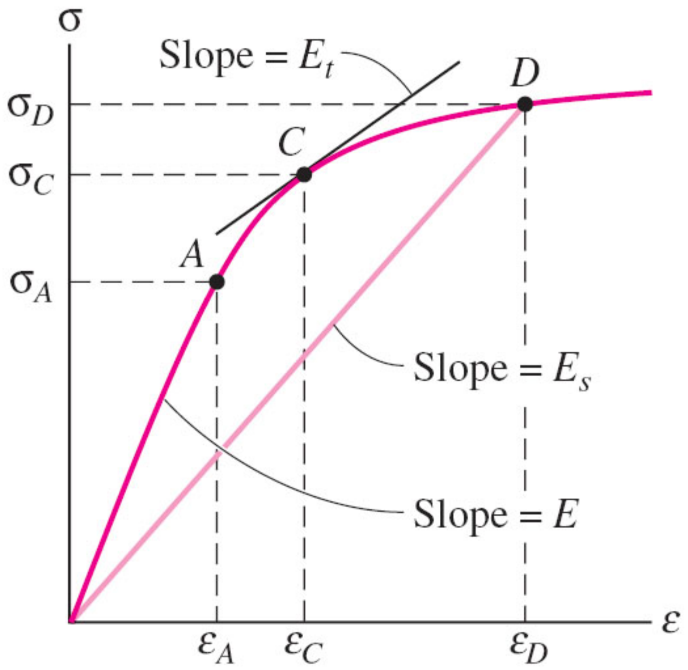
Elastic vs. Plastic Behavior



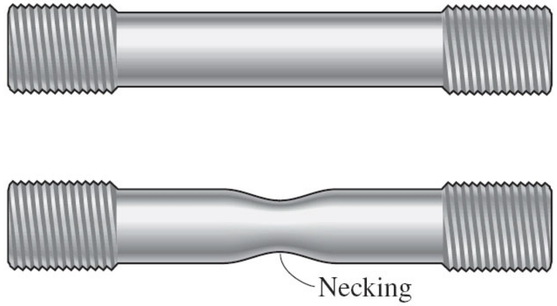
- If the strain disappears when the stress is removed, the material is said to behave *elastically*.
- The largest stress for which this occurs is called the *elastic limit*.
- When the strain does not return to zero after the stress is removed, the material is said to behave *plastically*.

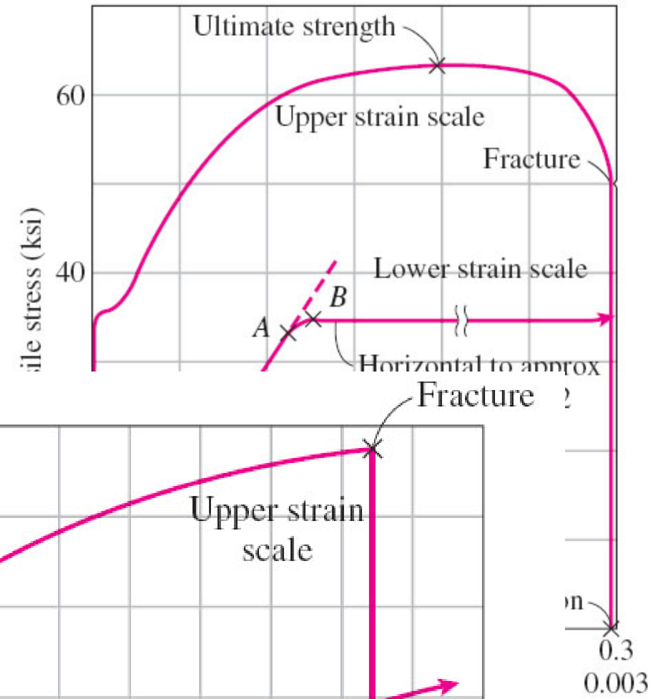
Modulus

- Modulus are used to "quantify" the "strength" of a material
- Young's modulus = elastic modulus (E)
  - The slope of the linear portion of the curve
  - A = proportional limit
- Tangential modulus (E_t)
  - The slope of the stress vs. strain curve at any selected strain
- Secant modulus (E_s)
  - The slope of the line connecting the origin to any point on the stress vs. strain curve (practically, beyond the proportional limit)

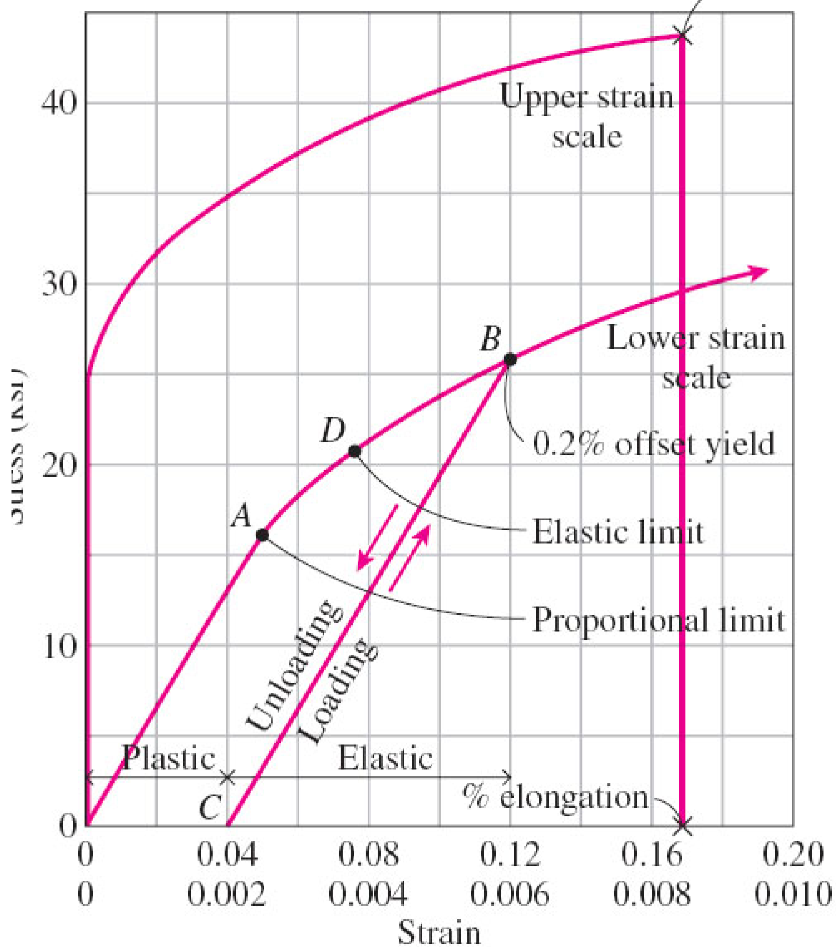


Special points on the curve





- A = proportional limit
- D = elastic limit
  - Beyond this point, the material is no longer elastic
- B= Yield point (in fig. a)
  - A stress level beyond which the material would demonstrate high strain for a small stress (perform like a plastic)
- B= Yield strength (point B in fig. b)
  - Stress that will induce permanent set (an offset to the original length)
  - In fig. b, line OC = the offset, line BC is parallel to OA
- Ultimate strength (see in fig. a)
  - The maximum engineering stress before rupture
  - Different from the true stress due to 'necking'



(b) Magnesium alloy

Linearly Elastic region

- Elastic: Strain is gone when the load is gone
- Stress vs. strain is linear
  - E = Young's modulus (elastic modulus, modulus of elasticity) can be used
- The Hooke's law
  - The most primitive stress vs. strain relationship
  - The Hooke's law is valid only in the elastic region
- For shearing,
  - use G = modulus of rigidity or shear modulus

$$\sigma = E\varepsilon$$

$$\tau = G\gamma$$

Poisson's ratio

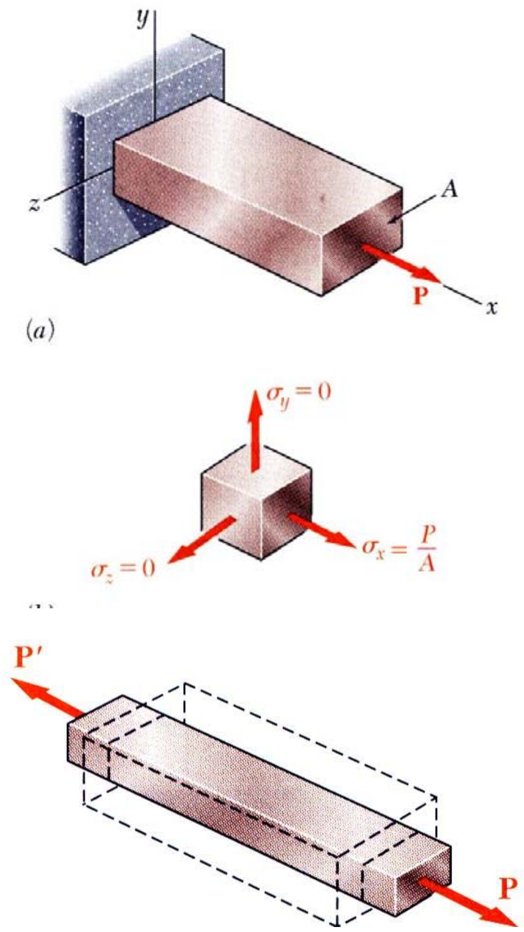
- A constant stated in 1811 by Siméon D. Poisson
- A material loaded in one direction will undergo strains perpendicular to the direction of the load in addition to those parallel to the load
- The ratio between the two strains = Poisson's ratio (ν)
- ε_{lat} = lateral strain = ε_t = tranverse strain
- ε_{long} = longitudinal strain = ε_a = axial strain
- The sign of strain is positive when the strain is outward

$$\nu = -\frac{\varepsilon_{lat}}{\varepsilon_{long}} = -\frac{\varepsilon_t}{\varepsilon_a}$$

- Relates the G to E

$$E = 2(1+\nu)G$$

Poisson's Ratio



- For a slender bar subjected to axial loading:

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \sigma_y = \sigma_z = 0$$

- The elongation in the x-direction is accompanied by a contraction in the other directions. Assuming that the material is isotropic (no directional dependence),

$$\varepsilon_y = \varepsilon_z \neq 0$$

- Poisson's ratio is defined as

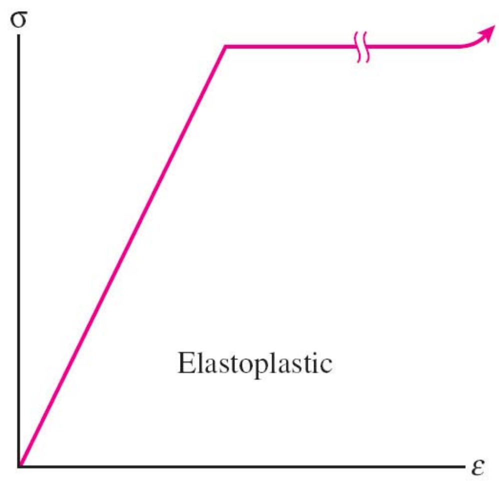
$$\nu = \left|\frac{\text{lateral strain}}{\text{axial strain}}\right| = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

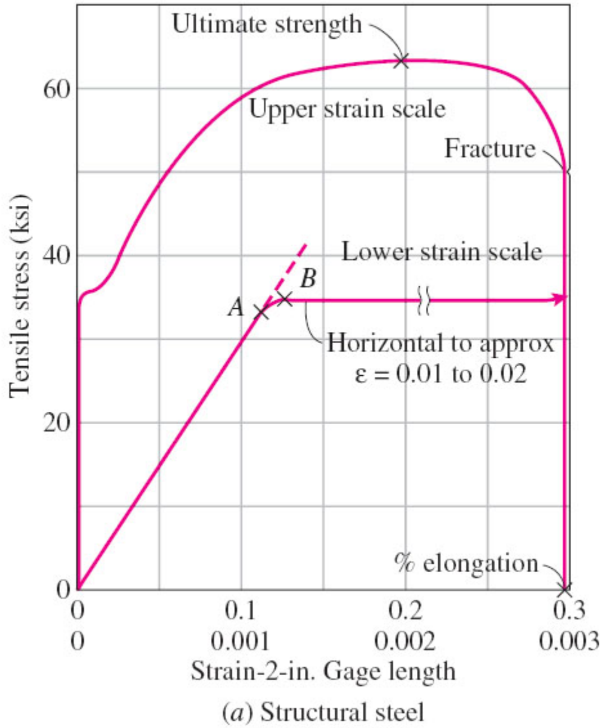
- *E*, *G*, and ν related by

$$E = 2(1+\nu)G$$

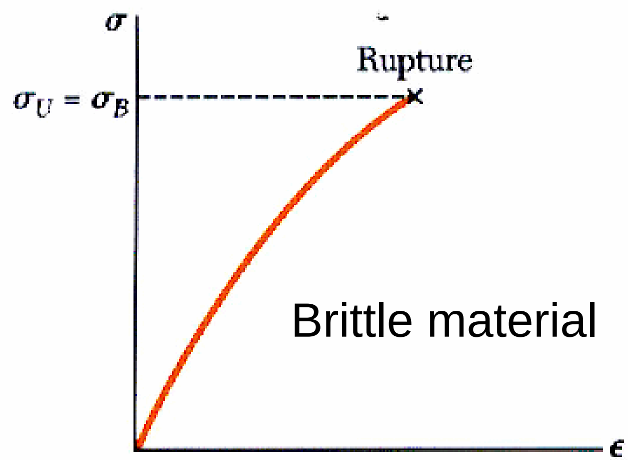
Nonlinear behaviors

- Nonlinear elastic
  - Elastic but not linear
  - Rubbers
  - Bio soft tissues
- Elastoplastic:
  - Nonlinear but bi-linear
  - Easy to formulate
- Ductile
  - Able to sustain plastic deformation (high strain but low stress)
  - Gradual stages prior to rupture ➔ *safe*
  - Steels, Plastic
- Brittle
  - The opposite of ductile
  - No gradual stages before rupture
  - Concrete, Alloys, Bones, cold steel
- Strain hardening or strain stiffening
- Strain softening



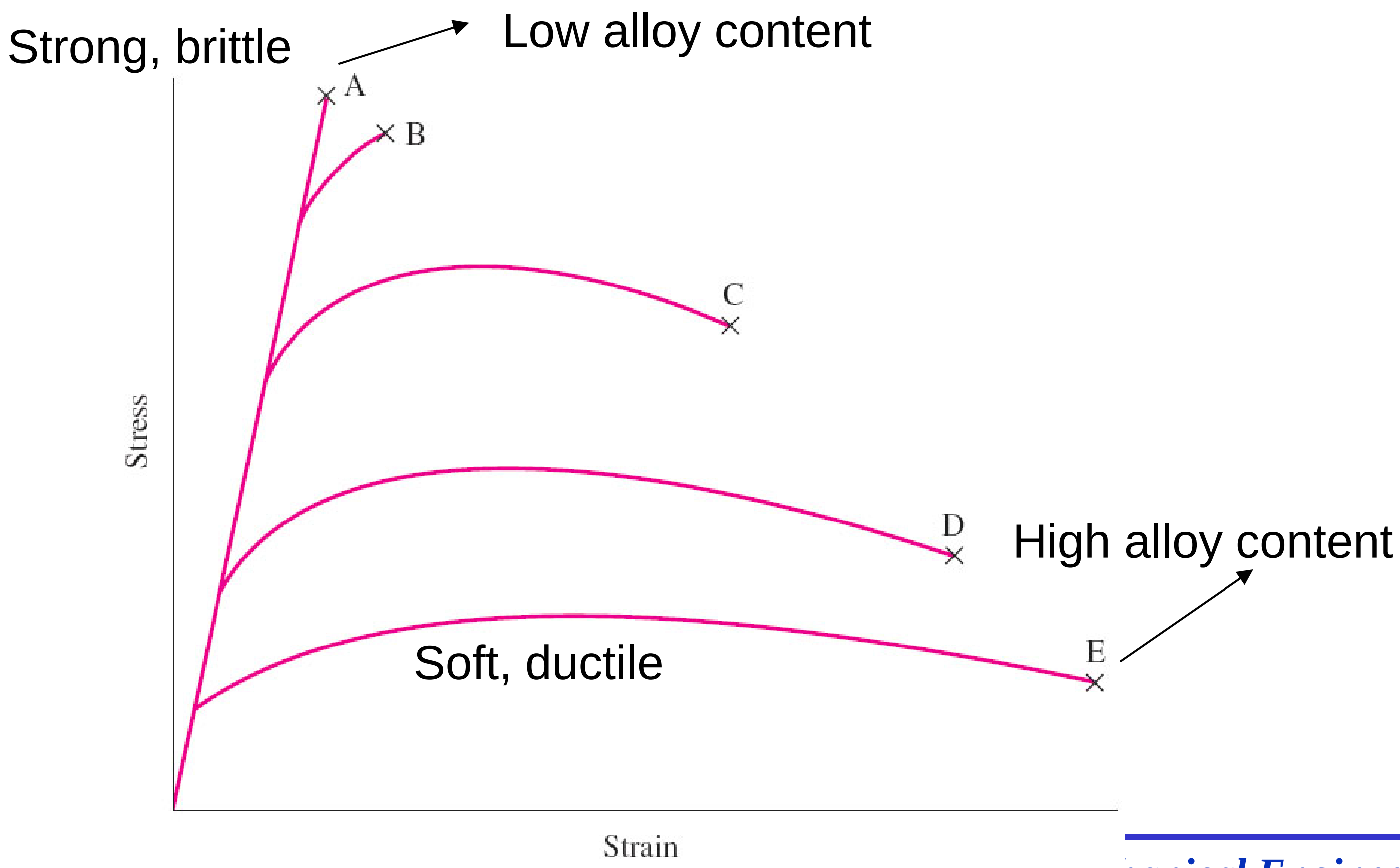


Ductile material



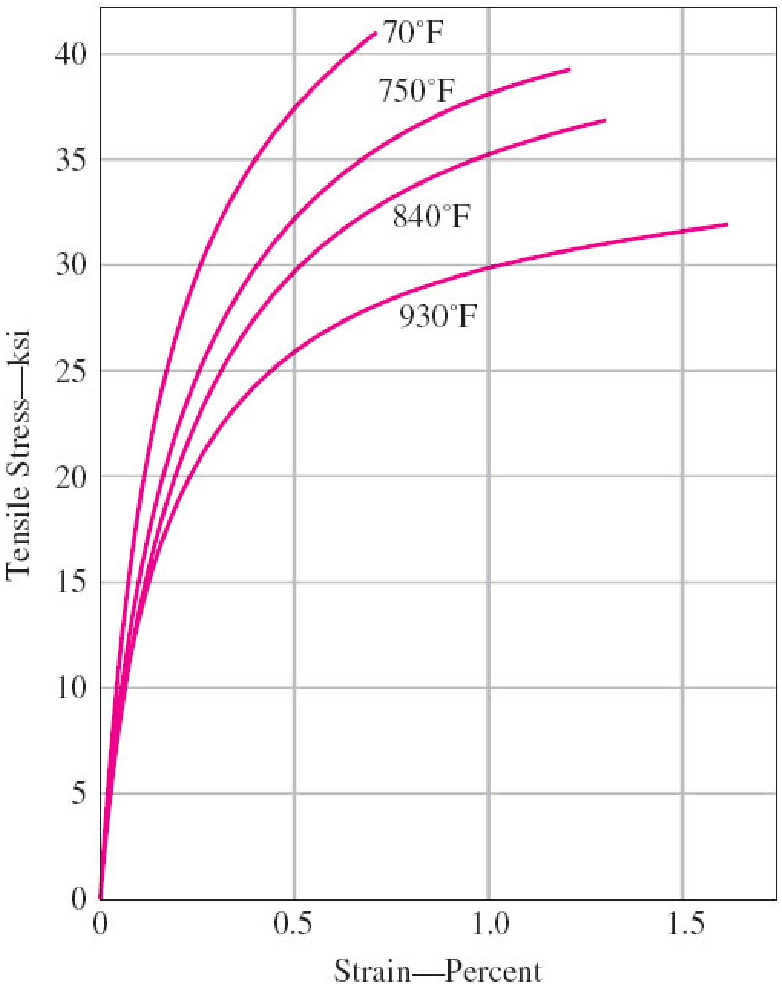
Effect of composition

- Example: high alloy content causes the steel to become ductile



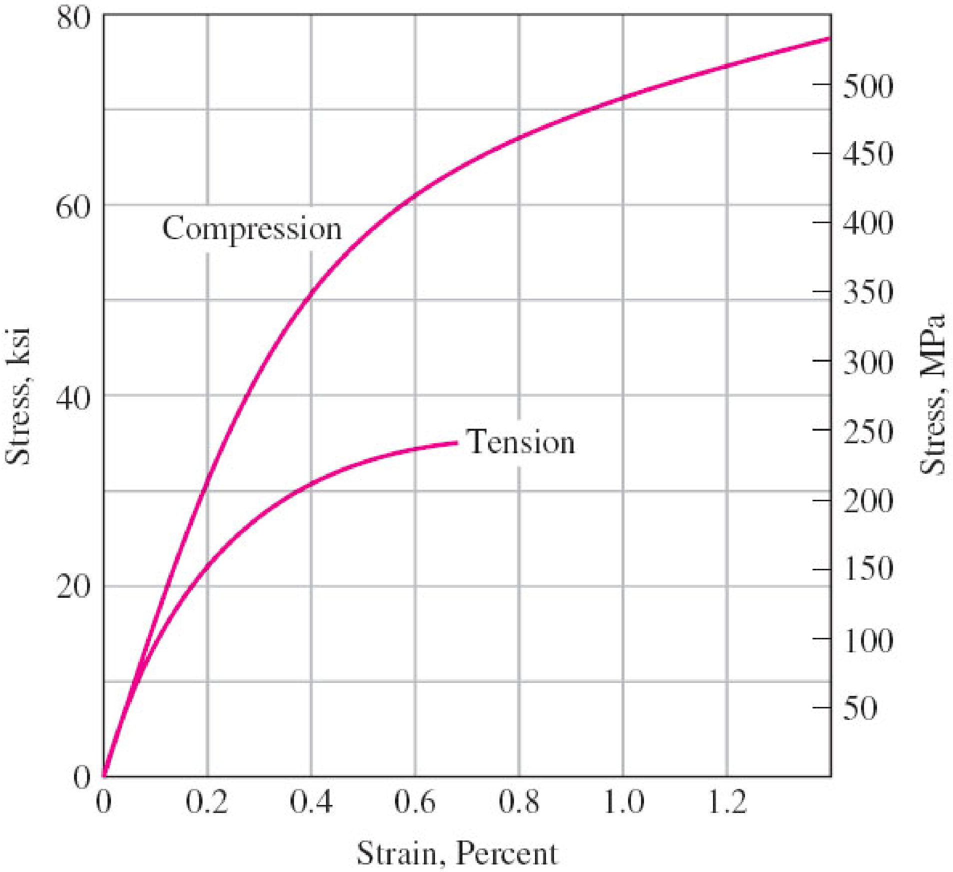
Effects of temperature

- High temperature causes the material to become ductile



Effects of loading direction

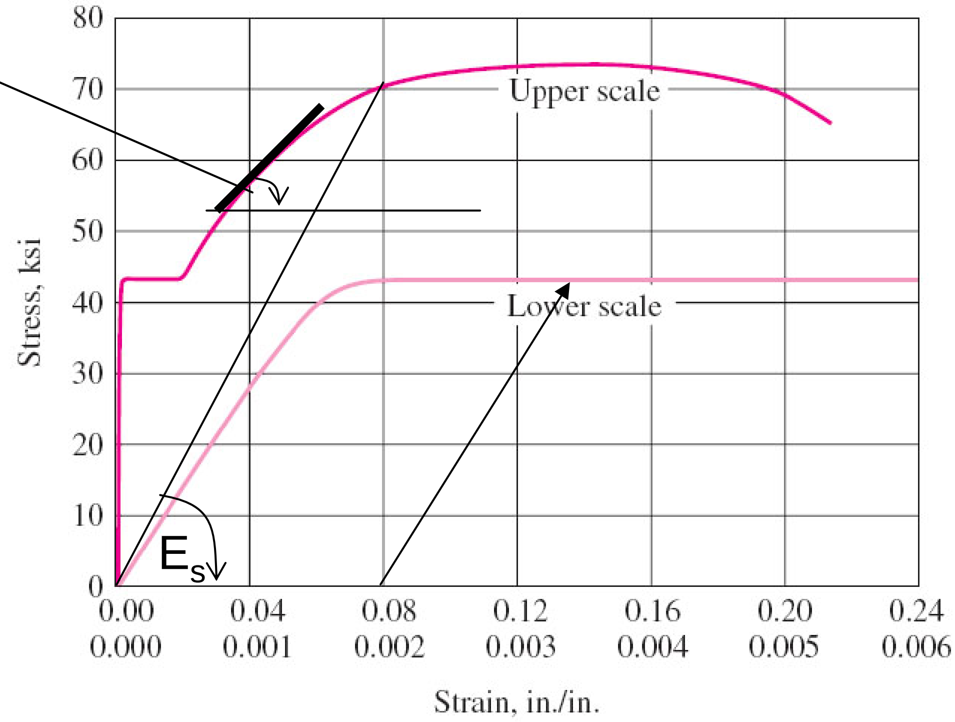
- For ductile materials, tension and compression behaviors are assumed to be the same
- For brittle materials, they are different



Example

- Given: stress vs. strain diagram (beware of the lower and upper scales)
- Initially the dia = 0.25”
- Determine:
  - Yield strength at 0.2% offset (0.002 strain)
  - Tangent modulus at 60 ksi (E_t)
  - Secant modulus at 70 ksi (E_s)
  - The <u>true stress</u> at 70 ksi if the diame of the specimen was 0.22”

Must find the loading P



Thermal strain

- When unrestrained, most engineering materials expand when heated and contract when cooled
- Coefficient of thermal expansion (CTE)
  - α = thermal strain due to a one degree (1^o) change in temperature
  - α is a material property (and it may depend on T)
- Thermal strain

$$\varepsilon_T = \alpha \Delta T$$

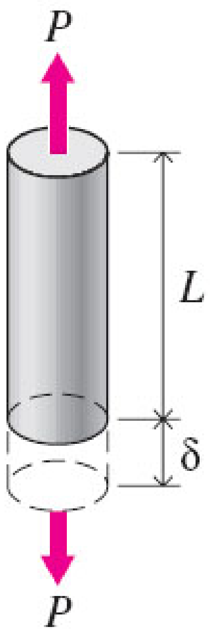
- Total strain

$$\varepsilon = \varepsilon_\sigma + \varepsilon_T = \frac{\sigma}{E} + \alpha \Delta T$$

- Please follow example problems 4-11 and 4-12

Deformation of axially loaded members (linear materials)

- Uniform member – single load



Definition of stress $$\sigma = \frac{P}{A}$$

Definition of strain $$\delta = \varepsilon L$$

Constitutive relation $$\varepsilon = \frac{\sigma}{E}$$

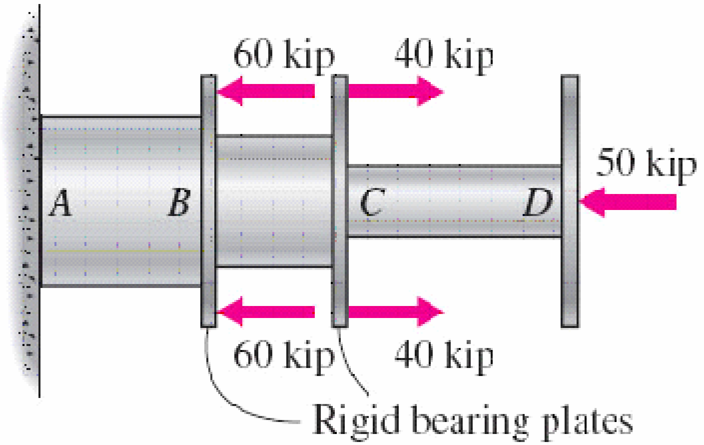
$$\delta = \varepsilon L = \frac{\sigma L}{E} = \frac{PL}{EA}$$

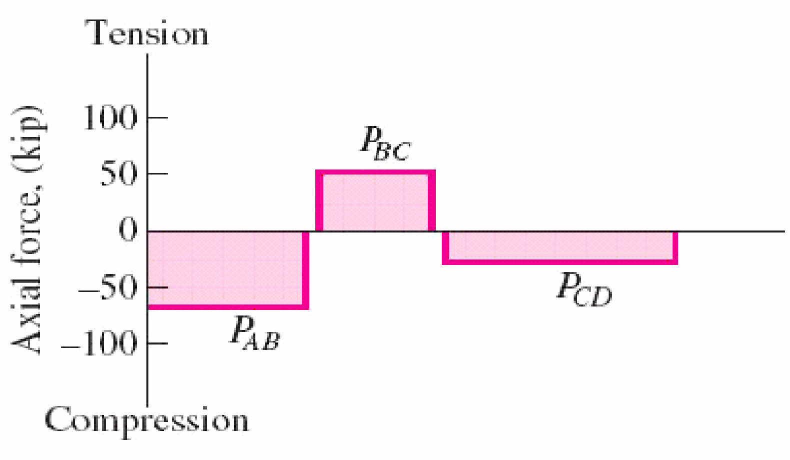
E = Modulus of elasticity
A = cross sectional area
P = tension/compression force
L = undeformed length
ε = normal strain
σ = normal stress

Deformation of axially loaded members

- Multiple loads/sizes

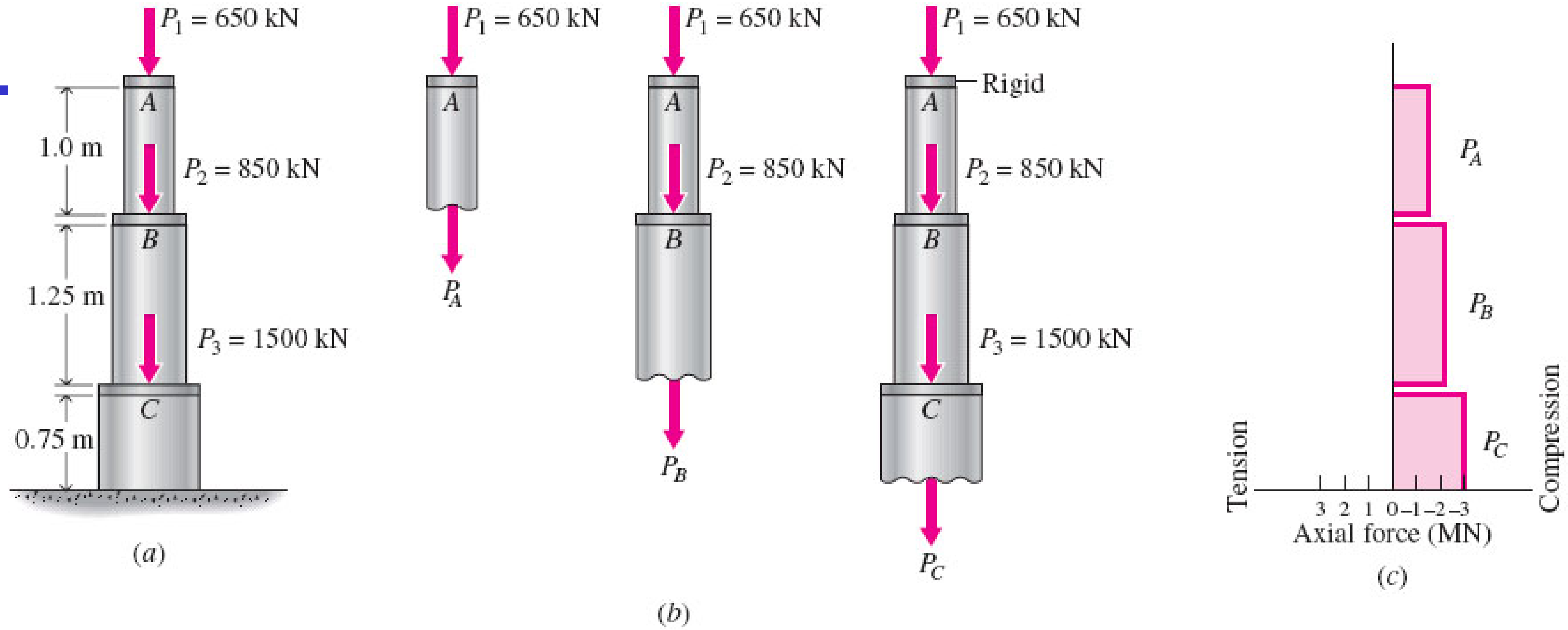
$$\delta = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} \frac{P_i L_i}{E_i A_i}$$





(b)

Example

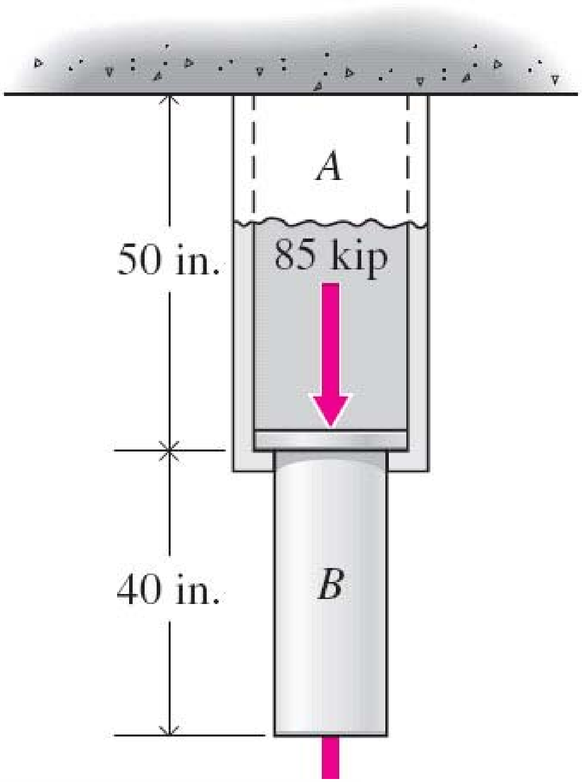


- Always assume that the internal force is in "tension"
- 1st find the internal force; P_A, P_B, and P_C from FBDs
- Determine the displacement of each member or use the total displacement formulation

Example:

- Given:
  - A is a tube
  - B is a solid cylinder
- Determine the total deflection

$$\delta = \delta_A + \delta_B$$



SOLUTION

$E_{st} = 29,000$ ksi $\qquad E_{al} = 10,600$ ksi

$P_A = 205$ kip (T) $\qquad P_B = 120$ kip (T)

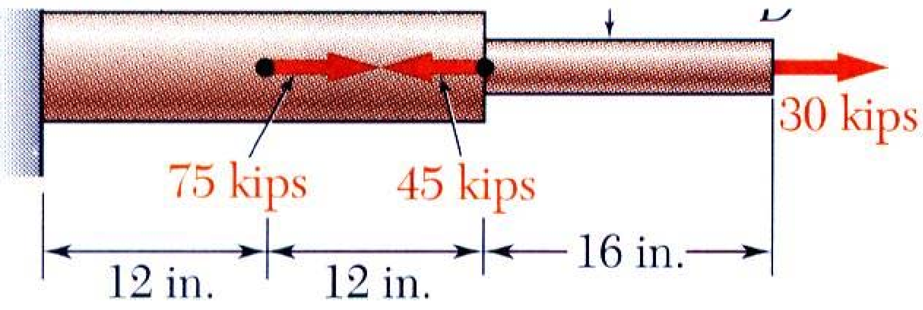
ip

(a) $$\delta_A = \frac{PL}{EA} = \frac{(205\times10^3)(50)}{(29\times10^6)\left[\pi(6^2-4.5^2)/4\right]} = 0.0286 \text{ in.}$$

(b) $$\delta_B = \frac{(120\times10^3)(40)}{(10.6\times10^6)\left[\pi(4^2)/4\right]} = 0.0360 \text{ in.}$$

$$\delta_{total} = \delta_A + \delta_B = 0.0646 \text{ in.}$$

Example Problem



$$E = 29\times10^{-6}\,\text{psi}$$

$$D = 1.07\text{ in.}\quad d = 0.618\text{ in.}$$

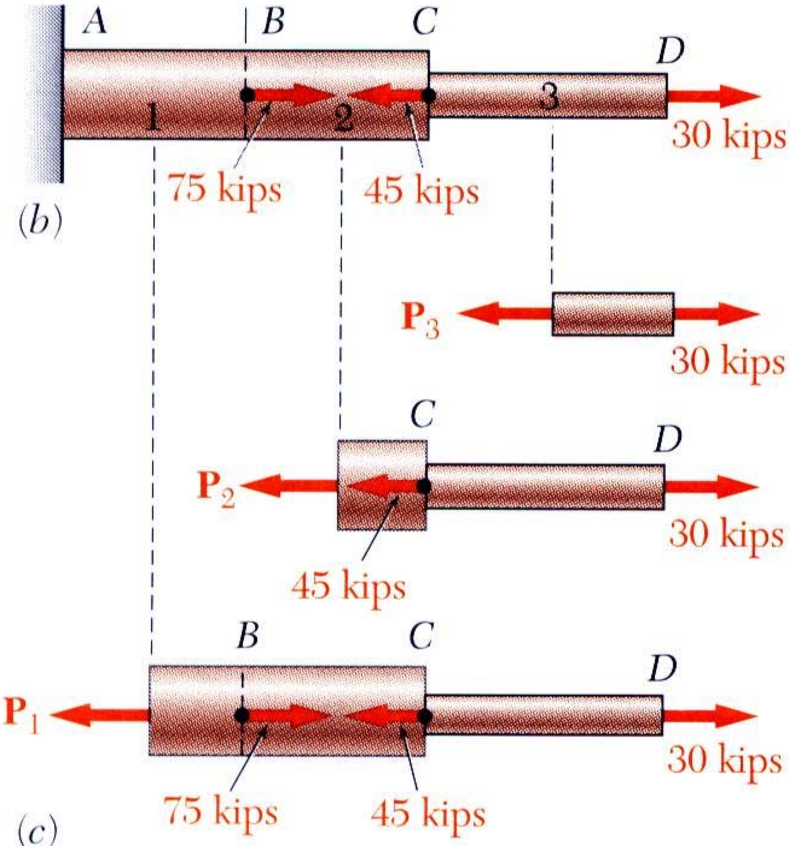
Determine the deformation of the steel rod shown under the given loads.

SOLUTION:

- Divide the rod into components at the load application points.
- Apply a free-body analysis on each component to determine the internal force
- Evaluate the total of the component deflections.

SOLUTION:

- Divide the rod into three components:



$L_1 = L_2 = 12\text{ in.}$ $L_3 = 16\text{ in.}$

$A_1 = A_2 = 0.9\text{ in}^2$ $A_3 = 0.3\text{ in}^2$

- Apply free-body analysis to each component to determine internal forces,

$$P_1 = 60\times10^3\text{lb}$$

$$P_2 = -15\times10^3\text{lb}$$

$$P_3 = 30\times10^3\text{lb}$$

- Evaluate total deflection,

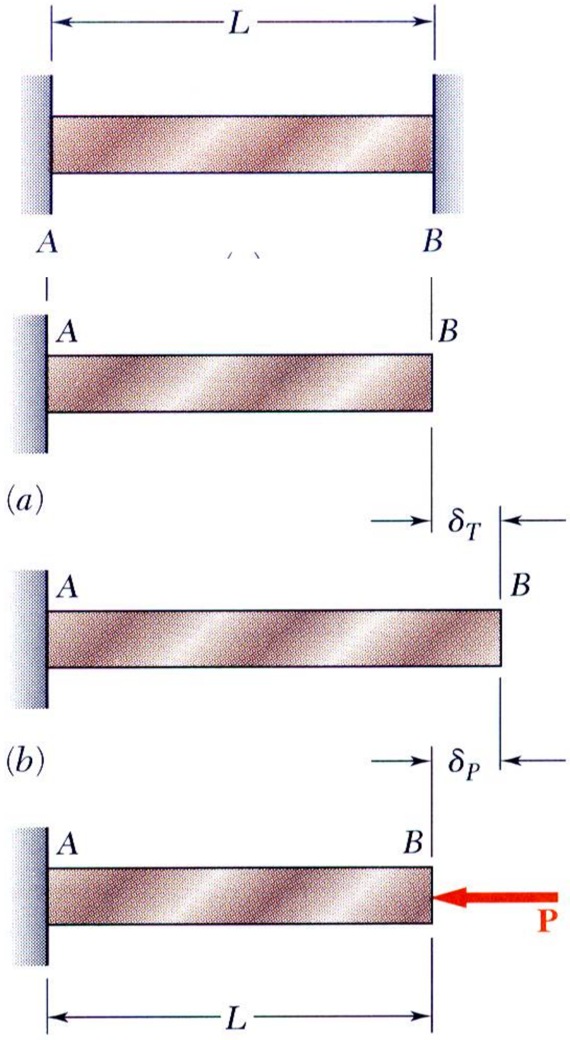
$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1L_1}{A_1} + \frac{P_2L_2}{A_2} + \frac{P_3L_3}{A_3}\right)$$

$$= \frac{1}{29\times10^6}\left[\frac{\left(60\times10^3\right)12}{0.9} + \frac{\left(-15\times10^3\right)12}{0.9} + \frac{\left(30\times10^3\right)16}{0.3}\right]$$

$$= 75.9\times10^{-3}\text{in.}$$

$$\boxed{\delta = 75.9\times10^{-3}\text{ in.}}$$

Thermal Strains and Stresses



- A temperature change results in a change in length or *thermal strain*. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.
- Treat the additional support as redundant and apply the principle of superposition.

$$\delta_T = \alpha(\Delta T)L \qquad \delta_P = \frac{PL}{AE}$$

$$\alpha = \text{thermal expansion coef.}$$

- The thermal deformation and the deformation from the redundant support must be compatible.

$$\delta = \delta_T + \delta_P = 0$$

$$\alpha(\Delta T)L + \frac{PL}{AE} = 0$$

$$\delta = \delta_T + \delta_P = 0$$

$$P = -AE\alpha(\Delta T)$$

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$

Example Problem

A 10-m section of steel [$E = 200$ GPa and $\alpha = 11.9(10^{-6})/°C$] rail has a cross-sectional area of 7500 mm^2. Both ends of the rail are tight against adjacent rails that, for this problem, can be assumed to be rigid. The rail is supported against lateral movement. For an increase in temperature of 50°C, determine

(a) The normal stress in the rail.

(b) The internal force on a cross section of the rail.

Solution:

- The change in length resulting from the temperature change is

$$\delta = \epsilon_T L = \alpha L \Delta T = 11.9(10^{-6})(10)(50) = 5.95(10^{-3}) \text{ m} = 5.95 \text{ mm}$$

- The stress needed to resist a change in length 0f 5.95 mm is

$$\sigma = \frac{E\delta}{L} = \frac{200(10^9)(5.95)(10^{-3})}{10} = 119.0(10^6)\ \mathrm{N/m^2} = 119.0\ \mathrm{MPa}$$

- The Internal force on the cross section of the rail will be

$$F = \sigma A = 119.0(10^6)(7500)(10^{-6}) = 892.5(10^3)\ \mathrm{N} \cong 893\ \mathrm{kN} \blacksquare$$

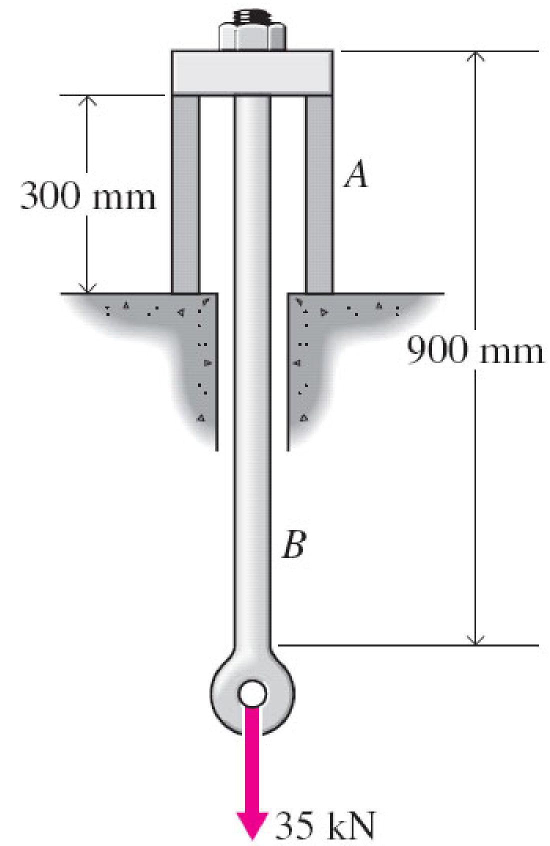
Example: design problem

- Given:
  - A is a tube (E=73 GPa, OD = 75 mm)
  - B is a solid cylinder (E=200 GPa, d=25 mm)
  - Load P is 35 kN
  - Maximum deflection at the end of bar B is 0.4 mm

  (Anyway, whose deflection is this?)

- <u>What is the thickness of A?</u>

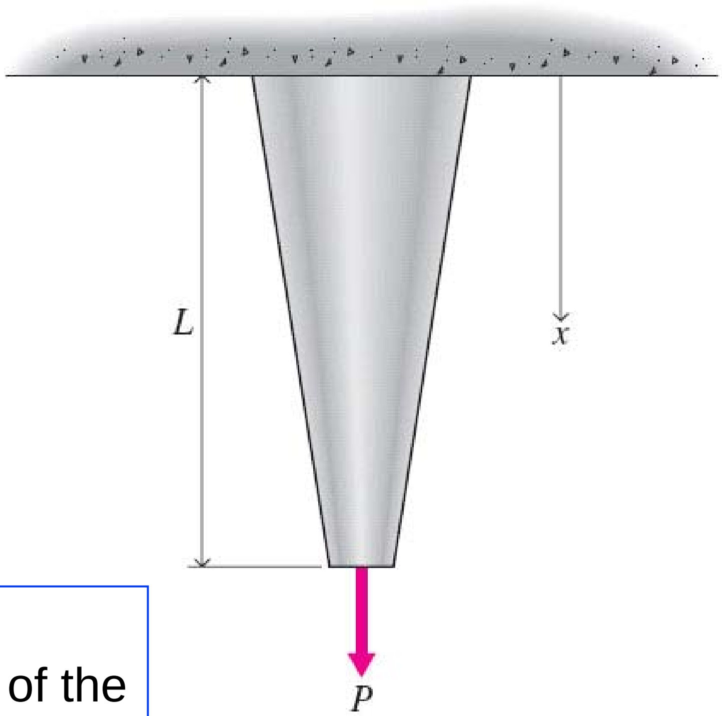
- Notes:
- Tube A experiences shortening
- Cylinder B experiences extension
- Total Displacement = $\delta_A + \delta_B$



Deformation of axially loaded members

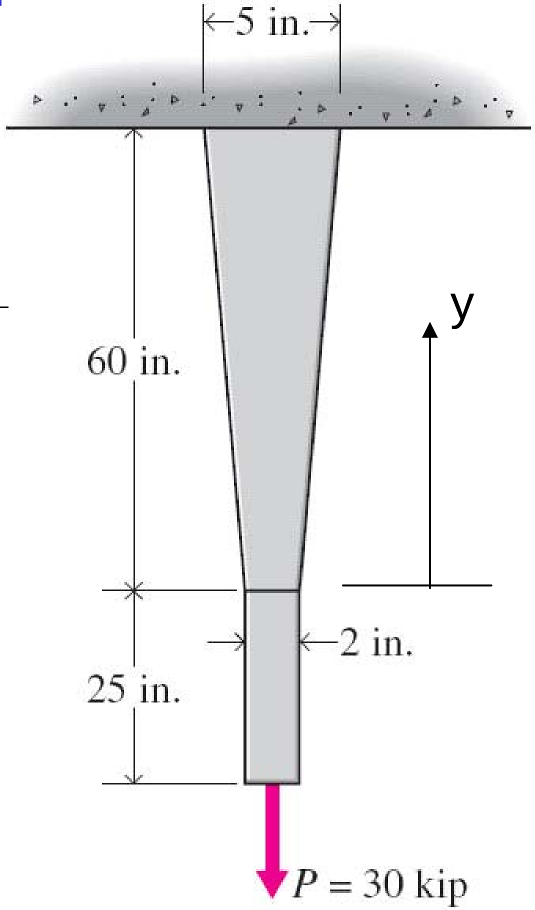
- Nonuniform deformation

$$\delta = \int_0^L \frac{P_x}{E_x A_x}\,dx$$



The subscript "x" indicates that the loading P, cross section A, and modulus E are functions of the distance from the base of the member.

Example



4-69 A structural steel (see Appendix A for properties) bar of rectangular cross section consists of uniform and tapered sections as shown in Fig. P4-69. The width of the tapered section varies linearly from 2 in. at the bottom to 5 in. at the top. The bar has a constant thickness of ½ in. Determine the elongation of the bar resulting from application of the 30-kip load P. Neglect the weight of the bar.

SOLUTION

$E = 29{,}000 \text{ ksi}$

$b = 2 + \frac{3y}{60} \text{ in.} \qquad 0 < y < 60 \text{ in.}$

b = width of the tapered section (varies with y)

$$\delta = \frac{(30\times10^3)(25)}{(29\times10^6)(2\times0.5)} + \frac{30\times10^3}{29\times10^6}\int_0^{60}\frac{dy}{\left(2+\frac{3y}{60}\right)(0.5)}$$

$$= 0.02586 + 1.03448\times10^{-3}\left[40\ln(40+y)\right]_0^{60}$$

$$= 0.02586 + 41.37931\times10^{-3}\left[4.60517 - 3.68888\right]$$

$\delta = 0.0638 \text{ in.}$ .... **Ans.**

$$\delta = \frac{PL}{EA}$$

$$\delta = \frac{P}{E}\int_0^L \frac{1}{A_x}\,dx$$