A short training on efficiency and productivity analyses

1. Introduction

- This two day training briefly reviews [upper-undergraduate level] production theory and introduces the concepts and methods of measuring efficiency and productivity (E&P)

- It provides a hands-on training of both production function estimation and E&P measurement

- In our practical applications we will focus on the livestock sector in Ethiopia, which is a least studied area

- Training an attempt to reconcile the difference between the optimizing/rational firm that operates *along* an efficient production possibility function, about which all economics graduates have studied, and firms that actually operate in the real world, most of which operate *around* the production possibility frontier

Outline of training

- The remaining parts of the first section introduce the course
  - Objectives of sections 2 and 3,
  - Assumptions about market structure, which will be used throughout the course
- Second section deals of general production economics
  - Provides a review of the theory of production economics and
  - Applies the theory, particularly we estimate a production function using Stata
- Third section deals with efficiency and productivity
  - Introduces the concepts of efficiency and productivity and relevant others
  - Discusses methods of measuring efficiency and productivity
  - Will use appropriate software to practically apply one, if time permits two, of the efficiency and productivity measuring methods

1.1. Objectives of training

At the end of the second section

- You will be able to recall/understand some of the basic concepts of production economics, including:
  - Production technology, production sets, production functions
  - The short- and long-run and variable and fixed inputs
  - Marginal and average products

- You will be able to estimate a production function using Stata

At the end of the third section

- You will be able to have a good understanding of the concepts of efficiency and productivity
- Identify the main measuring tools of E&P, and
- Be able to estimate [at least] the production possibility frontier

1.2. Assumption on market structure

- A market structure often refers to how the price of a given item is decided, which in turn depends on a number of other factors discussed shortly
- Throughout of this training we assume a perfectly competitive market (PCM) structure
- As we shall see, some of the features of a PCM structure may not hold in livestock production sector in Ethiopia (for that matter, in many other industries).
- However, the structure in a number of industries, including livestock, is closer to PCM than it is monopoly (only one producer), oligopoly (few producers; e.g. two producers (duopoly)), monopolistic competition (may producers selling differentiated products)

Assumptions of a perfectly competitive market

i) *The product is homogeneous:* implies products of all firms are identical/ indistinguishable,

- Homogeneity may hold in products such as eggs, less so in butter and even less so in live animals (e.g. Sheno butter; and Harar cattle, Menz sheep, etc). However, taken alone (as a single product) Harar cattle mostly homogenous.
- For the most part, this assumption is true in farming

ii) *Complete information:* All variables of concern are known to producers and the consumers;

- Producers know prices of inputs and outputs (including prices charged by others) and input-output relations (production functions and variables therein).
- Consumers have perfect information about the price and qualitative aspects of an item

iii) A large number of buyers and sellers in the industry: number of farmers producing/selling and consumers buying each item large.

- Each seller/buyer has an insignificant/small share of the market

iv) Price taking behavior: producers and consumers cannot influence price

- No single firm/consumer is large enough (has large-enough share) to influence price of the commodity. The firm/ consumer can sell/buy as much as it wants at the going market price

v) Free entry and exit of firms: free mobility of resources (inputs or factors of production) exists both in and out of farming.

- Exits likely but entry not easy in many industries including crop and livestock agriculture

However, can view entry as a shift of focus among farmers, e.g. teff farmers starting chicken or egg farming if the latter becomes attractive

➢ The assumptions above guarantee that in a PCM structure the price of a given item is determined by the competition of producers and buyers. If a given firm tries to charge a higher price it will lose all of its customers as consumers know the price charged by others. If [all] existing firms are charging higher prices and earning abnormal profits, which could, for instance, result from growth/spike in demand, then entry/competition of new firms will drive prices down.

2. Production economics
2.1. Introduction
Rough definition of basic concepts

Production: is basically an activity of transforming factor inputs into outputs, whereby the usability of goods and services is increased

Inputs: goods and services used in the production process or factors transformed into outputs

- Can be final outputs of another production process

Outputs: the final good or service that result from a production process

- Can be used as inputs in another firm

Firm: an institution engaged in the production and sale of outputs (goods and services)

- Note, however, that a firm can also be engaged in the purchase/hiring of inputs to be used in the production process

Decision making unit (DMU): firms, workers/laborers, and consumers (economic agents) that are involved in any kind of economic transaction such as the production and sale of outputs, supply/sale of labor, purchase of consumption items, etc

Industry: for our purposes an industry is a collection of firms, workers, and other economic agents engaged in the production and sale (supply) of an item (such as in the automobile or truck

industry, teff value chain) or collection of similar items used for different purposes (such as the chemical industry or agriculture)

- Products could be homogenous or differentiated

Simple mathematical notations

- A linear (convex) combination of variables a and b is given as: $\theta a + (1 - \theta)b$ for $0 \leq \theta \leq 1$. For $a < b$ and $0 < \theta < 1$ it holds that: $a < (\theta a + (1 - \theta)b) < b$.
- Multiplying a vector y by a constant θ such that $0 \leq \theta \leq 1$ radially contracts the vector or $y \leq \theta y$ or makes elements of θy proportionally lower than or equal to elements of y
- Multiplying a vector y by θ such that $1 \leq \theta$ radially expands the vector or $y \geq \theta y$ or makes elements of θy proportionally higher than elements of y

2.2. Representations of a production technology

Simple/informal: a production technology

- Describes the relationship between the feasible bundle of inputs that a farmer can use and the outputs it can produce therefrom
- Is a set containing all combinations of input bundles that are feasible for the farmer to use and corresponding output levels that can be produced from those input bundles

Formal:

➢ A production technology T is given by a set $T \subseteq \mathfrak{R}_+^N \times \mathfrak{R}_+^M$, where T is defined as:

$$T^h = \{(x^h, y^h): \ x^h \ can \ produce \ y^h\}$$

where $\mathfrak{R}_+^N$ and $\mathfrak{R}_+^M$ are an N and M dimensional positive real numbers, respectively, $x^h \in \mathfrak{R}_+^N$ is a vector of N inputs, $y^h \in \mathfrak{R}_+^M$ is a vector of M outputs, and $h \in \{1, 2, ..., H\}$ represents farmer h. From now on we drop the superscript h.

One representation of the production technology is the graph of production technology defined as:

➢ The *graph* of a production technology, GR= $\{(x, y): (x, y) \in T \}$ describes the set of feasible input-output vectors.

The GR assumed to satisfy the following properties:

1. $0^M \in GR$ for all $x^h \in \mathfrak{R}_+^N$ (Inaction is possible: given any input vector it is always possible to produce nothing).

2. $(x, y) \in GR \implies$ then $(x, \theta y) \in GR$ for $0 < \theta \leq 1$ (Weak disposability of outputs: if x can produce y then x can produce any proportional reduction of y).

3. $(x, y) \in GR \implies$ then $(\lambda x, y) \in GR$ for $\lambda \geq 1$ (Weak disposability of inputs: if x can produce y then any proportional increase of x can produce y).

4. For all $x^h \in \mathfrak{R}_+^N$, GR is a bounded set. (Finite inputs can only produce finite outputs).

5. For all $x^h \in \mathfrak{R}_+^N$, GR is a closed set.

A second and third characterization of the production technology is provided by the output and input sets, which are defined as:

➤ The *input sets* of a production technology, $L(y) = \{x : (x, y) \in T, y \in \mathfrak{R}_+^M\}$, describe the sets of input vectors that are feasible for each output vector $y \in \mathfrak{R}_+^M$.

➤ The *output sets* of a production technology, $P(x) = \{y : (x, y) \in T, x \in \mathfrak{R}_+^N\}$, describe the sets of output vectors that are feasible for each input vector $x \in \mathfrak{R}_+^N$.

- Just as the GR is assumed to satisfy certain conditions $L(y)$ and $P(x)$ satisfy certain [mostly equivalent] conditions, which we omit here.

- The input-output correspondences can be recovered from one another by means of the relationships (Fare, Grosskopf, and Lovell, 1994)
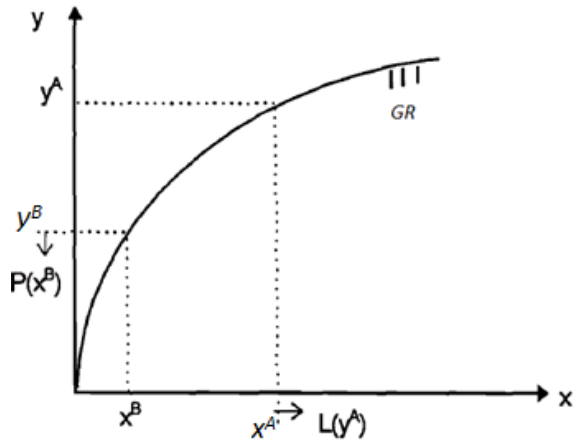
$L(y) = \{x : y \in P(x)\}$ and

$P(x) = \{y : x \in L(y)\},$

- Moreover, the input-output correspondences can be recovered from the graph of the technology

$L(y) = \{x : (x, y) \in GR\}$ and

$P(x) = \{y : (x, y) \in GR\},$

- Consider the case of a single-input single-output firm. Then we can represent the three definitions graphically as:

Figure 1. The input and output sets of production technology *(M = 1, N= 1)*

- Conditions 1-5 listed above imply that the input requirement set of y^A or $L(y^A)$ lies to the right of x^A in Figure 1 above
- Similarly, the output set of x^B or $P(x^B)$ lies below y^B

2.3. Production functions

- A number of functions/concepts most of us are familiar with are derived or are instances of the input and output set representation of a production technology
- For instance,
  - ➢ **Input isoquants**, defined as: Isoq $L(y) = \{x: x \in L(y), \lambda x \notin L(y), \lambda < 1\}$, describe the sets of input vectors capable of producing each output vector y but which, when radially contracted [proportionally reduced], become incapable of producing output vector y
  - ➢ **Input efficient subsets**, defined as: Eff $L(y) = \{x: x \in L(y), x' \leq x \Rightarrow x' \notin L(y)$, describe the sets of input vectors capable of producing each output vector *y* but which, when contracted in any dimension, become incapable of producing output vector *y*.
- Output isoquants and output efficient subsets can be defined in an analogues fashion
- Moreover, together with prices of inputs and outputs sets of a production technology provide other representation of a production technology

- For instance,
  - ➢ The input set of a production technology ($L(y)$), together with prices of inputs (w), provides the cost function as: $C(y, w) = \min_x wx$ subject to $x \in L(y)$
- Similarly, the output sets can be used to define output isoquants and revenue functions
- More importantly, the input and output sets can be used to define the production function/frontier as:
  - ➢ A *production frontier* is a function $f(x) = \max\{y: y \in P(x)\} = \max\{y: x \in L(y)\}$

A production frontier is assumed to satisfy certain conditions:

1. $f(0) = 0$; No free lunch or essentiality
2. $f(x) > 0 \Rightarrow f(\lambda x) \to +\infty$ as $\lambda \to +\infty$; outputs are non-decreasing in inputs
3. $f(\lambda x) > f(x), \lambda > 1$ for $x \in \mathfrak{R}_+^N$; weak monotonicity of inputs
4. f is concave on $x \in \mathfrak{R}_+^N$ or $f(\theta x^0 + (1 - \theta)x^1) \geq \theta f(x^0) + (1 - \theta)f((x^1)$ for all $0 \leq \theta \leq 1$ or a linear combination of input vectors x^0 and x^1 is capable of producing no less output than the linear combination of the outputs produced from the inputs

- The production frontier provides the upper boundary of production possibilities, and the input-output combination of each producer is located on or beneath the production frontier.
- Suppose we have a single-input single-output firm. Coelli et al. (2005) (Figure 2.1) represent the properties/conditions above as:

Figure 2. Production function of a single-input single-output firm
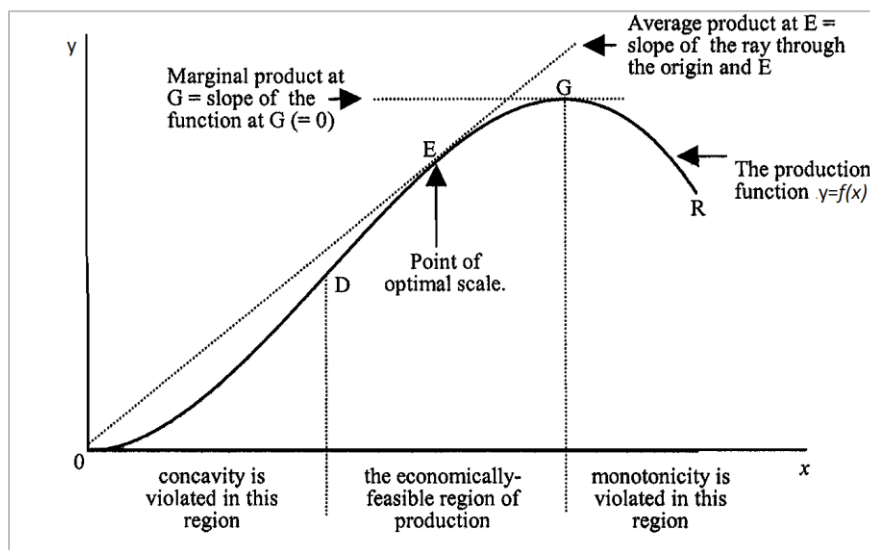
Source: Coelli et al. (2005)

Average and marginal products

Average product

- Output produced per unit of input type n used (yield, labor productivity,…)
- Given a production frontier $y = f(x_1, x_2, \ldots, x_N)$, the average product of x_n (AP_{x_n}) is given as: $AP_{x_n} = f(x_1, x_2, \ldots, x_N)/x_n$
- For a single-input single-output firm the AP ($f(x)/x$) also serves as a measure of productivity of input x/productivity of the firm (more on this in the next section)

Marginal product

- At a given level of production the marginal product (MP) of a given input is the change in output that results from a unit change in the use of the input while holding the use of other inputs constant or it is given as: $MP_{x_2} = \Delta f(x_1, x_2)/\Delta x_2$
- Given a differentiable production function the MP can be computed for infinitesimally small changes in input use (or as $\Delta x_2 \to 0$). Using calculus the MP is given as: $MP_{x_2} = \partial f(x_1, x_2)/\partial x_2$
- A concept related to MP that we will encounter is the elasticity of output WRT a given input,
- It measures the percentage change in output for a given percentage change in input and is given as: $E_1 = \dfrac{\partial f(x_1, x_2)}{\partial x_1} \dfrac{x_1}{f(x_1, x_2)}$

Example (2.2.3 in Coelli et al. (2005)): suppose we have a two-input single output Cobb-Douglas production function given by $y = 2x_1^{0.5} x_2^{0.4}$. Then

$MP_1 = \dfrac{\partial y}{\partial x_1} = x_1^{-0.5} x_2^{0.4}$ and $MP_2 = \dfrac{\partial y}{\partial x_2} = 0.8 x_1^{0.5} x_2^{-0.6}$

$E_1 = x_1^{-0.5} x_2^{0.4} \left(\dfrac{x_1}{2x_1^{0.5} x_2^{0.4}} \right) = 0.5$

$E_2 = 0.8 x_1^{0.5} x_2^{-0.6} \left(\dfrac{x_2}{2x_1^{0.5} x_2^{0.4}} \right) = 0.4$

2.4. The short- and long-run and variable and fixed inputs

- Marginal and average product computations above *assumed* at least one input is held fixed,
- In reality there are cases in which a producer cannot change use levels of at least one input (such as land in crop production in the middle of meher, buildings in manufacturing, etc.)

- While such a period may vary in length from one firm to another it generally is called a *short-run*
- Periods in which a firm can change all inputs are called the *long-run*
- Inputs that cannot be increased or decreased in the short-run (or can be altered only in the long-run) are called fixed inputs. Those that can be increased in the short-run (and in the long-run) are called variable inputs
- If a firm increases all inputs by a factor of k and outputs increase by a factor greater/equal/less than k then the firm is said to have increasing/constant/decreasing returns to scale. That is: if

$f(\lambda x_1, \lambda x_2) > \lambda f(x_1, x_2) \Longleftrightarrow$ IRS

$f(\lambda x_1, \lambda x_2) = \lambda f(x_1, x_2) \Longleftrightarrow$ CRS

$f(\lambda x_1, \lambda x_2) < \lambda f(x_1, x_2) \Longleftrightarrow$ DRS

- Continuing the example: given $y = 2x_1^{0.5} x_2^{0.4}$ and $\lambda = 3$ we find that $f(3x_1, 3x_2) = 3^{0.9}(f(x_1, x_2))$, which is less than $3(2x_1^{0.5} x_2^{0.4})$, indicating this is a DRS
- In particular, a Cobb-Douglas production function has a fixed returns to scale given by the sum of the elasticities of outputs (in our example: 0.9=0.5+0.4).

2.5. Practical application: estimating a production function

- In this section we use Stata to estimate a milk production function,
- We will use the Cobb-Douglas (CD) model for our production function estimation exercise
- Given the CD production function, a farmer that produces y using N types of inputs is specified as: $y = f(x_1, x_2, \dots, x_N) = A x_1^{\alpha_1} x_2^{\alpha_2} \dots x_N^{\alpha_N}$.
- Log-linearizing the function gives: $\log y = \alpha_0 + \alpha_1 \log x_1 + \alpha_2 \log x_2 + \dots + +\alpha_N \log x_N$.
- So we will actually estimate the log-linearized version of the CD production function
- For that purpose we first need to convert the variables into their natural logarithms,
  - Note: Most often we encounter zero values for one or more variables. Since the logarithm of a zero value is undefined Stata returns a missing value when computing logs of zeros. So to protect observations with zero values for some variables from being dropped from the analysis we often replace logs of the zeros by logs of a small positive number such as ln(0.00001).
- To estimate both the simple CD production function and the stochastic production frontier

(discussed in the next section) we will be using a dataset named "MilkProd&InputUseData.dta", which is provided in the "Data" folder of the course materials,

- The dataset has the relevant variables converted to logs

- The data is taken from different parts of CSA's annual Agricultural Sample Survey pertaining to livestock production.

- In accordance with IFPRI's agreement with CSA to not give the data for a third party we have removed all identifiers associated with the households. The remaining two identifiers in the dataset (*year* and *zone_id*) are dummy variables meant to be used in the analyses. The year variable indicates that the data pertains to two years; and the zone_id variable indicates that the HHs in the dataset come from 12 zones. However, the years and zones cannot be identified from the aforementioned variables,

- There are two ways of opening the aforementioned data file
  - Double clicking the file or
  - Calling the file from a do file, which is advisable as this enables you to record and save all of your work for your future references,

- We will be using the do file named "Practical Examples.do" to conduct the analyses. The do file is provided in the course materials folder

- When you select and run the line: "use "D:\Users\fbachewe\Desktop\Prodvty&Effic Training\Course materials\Data\MilkProd&InputUseData", clear" Stata calls the data file,

- However, this command calls only from the directory where I saved the file. You need to first change this directory to the folder where you saved the milk production file so as Stata calls it from where you saved the milk production file.

- After opening the data file we follow the few steps in the do file to estimate the production function, store the results, and save them into an xml file

3.   Productivity and Efficiency

- This section first provides simple definitions of efficiency, productivity, and related concepts and then provides the technical definitions
- The second subsection discusses input and output distance functions, which are key in efficiency and productivity analyses
- The section that follows deal with issues related with efficiency and efficiency measurement while also providing practical exercises
- Finally, we provide a formal definition of productivity and discuss issues related with measuring levels and changes in productivity

3.1.   Introduction

- This part provides an intuitive definition of relevant concepts mainly from Coelli et al. (2005)
- Concepts defined here are formalized in the remaining sections; so a firm grasp here helpful

Definitions

- ***Productivity***: is defined as the ratio of the output(s) that a firm produces to the input(s) it uses:

    Productivity= outputs/inputs   (1)

- Productivity of a single-input single-output firm is equal to the average product: $f(x)/x$
- However, firms almost always use more type of inputs than one and often produce more than one type of output (e.g. milk, eggs, crops).
- In such cases the average product of one of the inputs (e.g. land productivity/yield) defined in a similar fashion $(f(x_1, x_2)/x_2)$ provides only a partial [an incomplete] measure of productivity
- A measure of productivity that considers all outputs and all inputs, such as that at equation (1), is often referred to as ***total factor productivity*** (TFP)
- Measuring TFP requires using methods that aggregate multiple inputs into a single input index and multiple outputs into a single output index (more on this latter)
- Therefore, TFP can be defined as:

    TFP=Output index/input index   (2)

- The terms productivity and efficiency, erroneously used interchangeably
- Let us use a one-input one-output technology and the associated production frontier (Figure

3.1) to fix ideas,

- In Figure 3.1, points B, C, and D represent efficient points whereas point A represents an inefficient point.

- A firm operating at point A is inefficient because technically it could increase output to the level associated with point B without requiring more input or could use less input and produce the same quantity, such as the level at point C.

- More generally, firms that operate on the production frontier are technically efficient, as alluded in the definition of the production frontier ($f(x) = \max\{y: y \in P(x)\}$) and technically inefficient firms operate beneath the frontier.

Figure 3.1. Productivity, technical efficiency, and scale economies
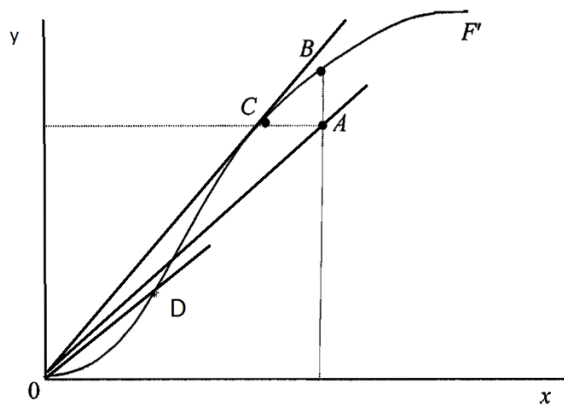
Source: Modified from Figures 1 and 2 of Coelli et al. (2005)

- A *technically efficient* firm can produce more output only by using more inputs or its outputs must decline if it uses less inputs than it is currently using

- In a multi-input multi-output setting a technically efficient firm can increase the production of one item only by reducing the production of another or increasing its use of one or more inputs

- In Figure 3.1, the productivity of each firm is also given by the slope of a line ($y/x = f(x)/x$) that passes through the origin and the point the firm operates

- Productivity of the firm at A improves if it moves to B because the slope of the line through B (not shown) is higher and productivity of A improves even better if it moves to C

- The latter implies that the firm at B, which is technically efficient, has a lower productivity than that at C

- Similarly, the firm at D, which is technically efficient, has a lower productivity than the

technically inefficient firm A, a clear evidence of the difference between the two concepts

- Given the production frontier in Figure 3.1 point C is the (technically) optimal scale since operation at any other point results in a lower productivity.

- Since changing the scale of firms' operations can often be difficult to achieve quickly, technical efficiency and productivity are in some cases given a short- and long-run interpretations.

- When one considers productivity comparisons through time, an additional source of productivity change, called technical change, is possible.

- *Technical change* refers to advances in technology that may cause an upward shift in the production frontier

- This means all firms can technically produce more output now for each level of input use, relative to what was possible in previous periods

- Therefore, increases in productivity across periods (across years or two long-runs) may result from efficiency improvements, exploitation of scale economies, technical change, or from some combination of these factors.

- *Allocative efficiency* involves selecting the mix of inputs (e.g., labour and capital) that produce a given quantity of output at minimum cost (given the input prices which prevail)

- Allocative and technical efficiency combine to provide an overall *economic efficiency* measure.

- This training deals only with technical efficiency as the time does not allow to cover allocative efficiency, which adds a price dimension to the issue

3.2. Output and input distance functions

This part is mainly from Kumbakhar and Lovell (2000)

- Output and input distance functions generally measure the distance between the point a firm is operating and the relevant frontier

- Distance functions are simple, intuitive, and central in the study of efficiency measurement


Input distance functions

  ➢ An *input distance function* is a function: $D_I(y, x) = \max\{\lambda : x/\lambda \in L(y)\}$.

- An input distance function adopts an input-conserving approach to the measurement of the

distance from the point a producer is operating to the boundary of production possibilities or input requirement set.

- $D_I(y, x)$ gives the maximum amount by which a producer's input vector can be radially contracted and still remain feasible for the output vector it produces.

Figure 3.2. Input distance functions of a one- and two-input firm

N=1   N=2
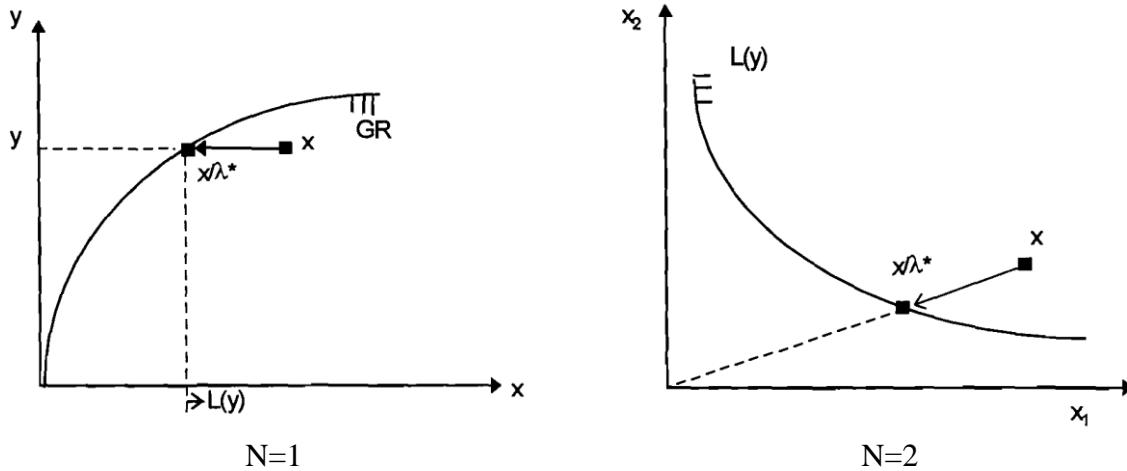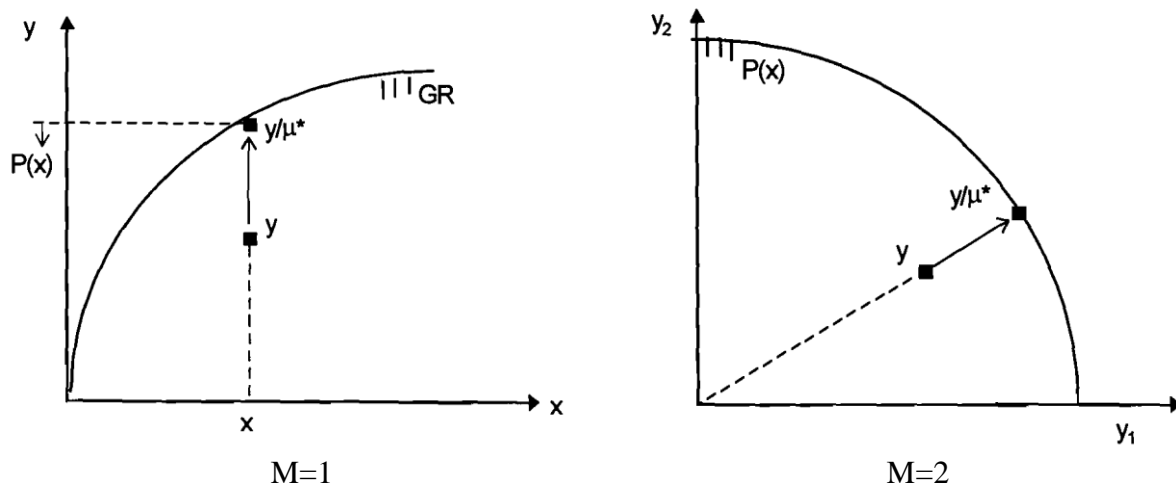
Source: Figures 2.8 and 2.9 of Kumbakhar and Lovell (2000)

- In Figure 3.2 (left panel) the scalar input x is feasible for output y, but y can be produced with smaller input x/λ^*, and so $D_I(y, x) = \lambda^* > 1$.
- Similarly, in the right panel the input vector x is feasible for output y, but y can be produced with the radially contracted input vector x/λ^*, and so $D_I(y, x) = \lambda^* > 1$.
- $D_I(y, x)$ moves input use levels in the interior of L(y), such as x, to the minimum possible at the boundary of $L(Y_j)$ along a ray passing through the origin and x
- Since the input distance function $D_I(y, x)$ is defined in terms of L(y), which satisfy certain properties, the input distance function satisfies a corresponding set of properties
- The definition of $D_I(y, x)$ and Figure 3.2 imply that $L(y) = \{x: D_I(y, x) \geq 1\}$ and $Isoq\ L(y) = \{x: D_I(y, x) = 1\}$
- The input isoquant is one possible standard against which to measure technical efficiency of input use and corresponds to the set of input vectors with input distance function value of unity.
- All other feasible input vectors have input distance function values greater than unity.

Output distance functions

➤ An *output distance function* is a function: $D_O(x, y) = \min\{\mu : y/\mu \in P(x)\}$.

- An output distance function adopts an output-expanding approach to the measurement of the distance from the point a producer is operating to the boundary of production possibilities.

- $D_O(x, y)$ gives the minimum amount by which a producer's output vector can be radially expanded and still remain producible with a given input vector.

Figure 3.3. Output distance functions of a one- and two-output firm

M=1   M=2

Source: Figures 2.10 and 2.11 of Kumbakhar and Lovell (2000)

- In Figure 3.3 (left panel) the scalar output y can be produced using input x, but so can a larger output y/μ^* and so $D_O(x, y) \leq 1$.

- Similarly, in the right panel the output vector *y* is producible with input *x*, but so is the radially expanded output vector y/μ^* and so $D_O(x, y) \leq 1$.

- $D_O(x, y)$ provides the ratio of y and the optimal output that can be produced using x. Specifically, $(1/\mu^*)$ represents the largest factor by which both output types can proportionally increase holding input use fixed.

- Since the output distance function $D_O(y, x)$ is defined in terms of P(x), which satisfy certain properties, the output distance function satisfies a corresponding set of properties

- $D_O(x, y)$ and Figure 3.3 imply $P(x) = \{y : D_O(x, y) \leq 1\}$ and $Isoq\ P(x) = \{y : D_O(x, y) = 1\}$

- The output isoquant (PPF) is one possible standard against which to measure technical efficiency of output production and corresponds to output distance function value of unity.

- All other feasible output vectors have output distance function values less than unity.

Notes:

- The value of input and output distance functions are obtained by solving certain linear programming problems that search for the optimal [scalar] values of the distance functions

- However, these scalar values are not interesting in themselves, as much as they are when used to measure efficiency.

- In any case, as they stand (before solved), the I&O distance functions are some complicated functions of the input requirement and output possibilities sets, respectively, and the latter could be recovered from the former, just as the former are constructed using the former

- If both inputs and outputs are weakly disposable it can be shown that $D_I(y, x) \geq 1$ if, and only if, $D_O(x, y) \leq 1$

- Furthermore, given a constant returns to scale technology: $D_I(y, x) = 1/D_O(x, y)$

3.3. Measurement of efficiency

Efficiency measures

- Efficiency measures have their roots in the works of Debreu (1951), Koopmans (1951), Farrell (1957), Charnes and Cooper (1957), and Shephard (1970)

First, formal definitions of technical efficiency:

- ➢ E1: An *output-input* vector $(y, x) \in$ GR is *technically efficient* if and only if $(y', x') \notin$ GR for $(y', -x') \geq (y, x)$.

- ➢ E2: An *input* vector $x \in L(y)$ is *technically efficient* if and only if $x' \notin L(y)$ for $x' \leq x$.

- ➢ E3: An *output* vector $y \in P(x)$ is *technically efficient* if and only if $y' \notin P(x)$ for $y' \leq y$.

- E1 calls a feasible output-input vector technically efficient if, and only if, no increase in any output and/or decrease in any input is feasible.

- For a fixed output vector E2 calls a feasible input vector technically efficient if, and only if, no reduction in any input is feasible.

- Given a fixed input vector, E3 calls a feasible output vector technically efficient if, and only if, no increase in any output is feasible

- Thus technical efficiency is defined in terms of membership in an efficient subset.

Given the definitions of input-output, input, and output technical efficiency (TE) the logical

question is: how do we measure TE? We have input and output oriented measures:

> An input-oriented measure of TE is a function $TE_I(y, x) = \min\{\theta: \theta x \in L(y)\}$.

> An output-oriented measure of TE is a function $TE_O(x, y) = [\max(\phi: \phi y \in P(x)\}]^{-1}$.

$TE_I(y, x)$ satisfies the following properties:

1. $TE_I(y, x) \leq 1$
2. $TE_I(y, x) = 1 \Longleftrightarrow x \in Isoq\ L(y)$
3. $TE_I(y, x)$ is non-increasing in x
4. $TE_I(y, x)$ is homogenous of degree -1 in x
5. $TE_I(y, x)$ is invariant with respect to the units x and y are measured

$TE_O(x, y)$ satisfies the following properties:

1. $TE_O(x, y) \leq 1$
2. $TE_O(x, y) = 1 \Longleftrightarrow y \in Isoq\ P(x)$
3. $TE_O(x, y)$ is non-decreasing in y
4. $TE_O(x, y)$ is homogenous of degree +1 in x
5. $TE_O(x, y)$ is invariant with respect to the units x and y are measured

- The input- and output-oriented measures of efficiency defined above, together with previous discussions, imply the following relationships:

1. $TE_I(x, y)$ and $TE_O(x, y)$ assign the same technical efficiency score to a producer if, and only if, the technology is homogeneous of degree +1 (CRS)
2. An input-oriented measure of TE is given by $TE_I(y, x) = \min\{\theta: D_I(y, \theta x) \geq 1\}$.
3. An output-oriented measure of TE is given by $TE_O(x, y) = [\max\{\phi: D_O(x, \phi y) \leq 1\}]^{-1}$.
4. $TE_I(y, x) = [D_I(y, x)]^{-1}$ and $TE_O(x, y) = D_O(x, y)$

- The definitions of the distance functions and I & O efficiency measures provided above presuppose that we know the production frontier, input requirement set, and so forth so that compare farmers operating at sub optimal levels against those operating at the frontier
- But in reality this frontier is unobservable; we only have a set of data points from which to construct a representative frontier.

- Two methods are used to construct such a frontier: stochastic frontier analyses (SFA) and data envelopment analyses (DEA)

- The methods differ to what they attribute differences in outputs per unit input produced.

- SFA attributes such differences to either or both of random shocks and efficiency differences

  o SFA not only allows to test if differences in output result from either or both factors and measure the contribution of each factor but also it enables indicate factors associated with such differences.

- DEA considers differences in outputs per unit input produced as reflections only of efficiency differences.

  o DEA allows decompose such differences in efficiency into input and output oriented technical and scale efficiencies.

- In the following we first describe the SFA and practically apply the method, which is followed by a similar description and application of the DEA method.

3.3.1. Stochastic production frontiers

- Stochastic frontier analyses is a method of estimating stochastic production, cost, revenue, and so forth functions. In this training we study stochastic production functions;

- Consider I (i=1, 2,, …,I) farmers that produce M (m=1, 2, …, M) types of livestock outputs using N (n=1, 2, …, N) types of inputs.

- We assume that the production technology is uniform across farmers.

- Let the optimal output vector $y^* \in \Re_+^M$ corresponding to each input vector $x \in \Re_+^N$ be given by the maximum level of output produced by one or more farmers using x.

- Suppose also the relationship between the optimal output vector, y^*, that can be produced using a given input vector, x, is given by:

$y^* = F(x, \beta)$

where $y \in [y_1^*, y_2^*, …, y_M^*]$ is a 1xM output vector, $x \in [x_1, x_2, …, x_N]$ is a 1xN input vector, and β is an NX1 vector of production function parameters.

- The actual output, y_i, that farmer i produces using the same input vector may differ from the optimal level, y^*. Suppose the input-output relationship of farmer i is represented by:

$y_i = F(X, \beta)$ for all $i \in [1,2, …, I]$

- SFA and DEA differ in how they explain some measure of the difference between y_i and y^*, such as $y^* - y_i$ or y_i/y^*.
- The underlying assumption of SFA that output differences result from efficiency differences or random shocks, or both is incorporated in the production frontier using an error term that accounts for both components.
- The stochastic production frontier (SPF) is often specified as:

$$y_i = F(X, \beta) = f(X, \beta) \cdot \exp(V_i - U_i) \text{ for all } i \in [1, 2, \dots, I] \tag{3.1}$$

- In (3.1), $f(X, \beta)$, often called the deterministic component, is uniform across farmers.
- $\exp(V_i - U_i)$ part is called the composed error term
- The $exp(V_i)$ part of the composed error term represents the deviation of y_i from y^* due to producer specific random shocks.
- Such shocks could lead to output levels higher or lower than the optimum. As a result V_i can be positive or negative and mostly assumed to be symmetrically distributed with zero mean.
- In equation (3.1) the term $exp(-U_i)$ accounts for the deviation of y_i from y^* due to farmer i's level of efficiency relative to optimal farmer/s.
- Hence U_i is zero for technically efficient producers at the production frontier and takes increasingly positive values with decline in efficiency.
- Under certain assumptions on V_i and U_i, parameters of the SPF given at (3.1) can be estimated using maximum likelihood methods (Kumbhakar and Lovell. 2000).
- Let us index the optimal farmer/s associated with x by k, such that $y^* = y_k = F(X, \beta)$. Then, given (3.1), the technical efficiency of farmer i relative to k is given by:

$TE_j = \frac{y_i}{y_k} = \frac{f(X,\beta) \cdot \exp(V_i - U_j)}{f(X,\beta) \cdot \exp(V_k - U_k)}$ for all $i \in [1, 2, \dots, I]$

- Given $U_k = 0$ and allowing for the random component that varies across producers, the technical efficiency of farmer i is given by:
- $TE_i = \exp(-U_i)$ for all $i \in [1, 2, \dots, I]$
- Given $U_i \geq 0$, $TE_i = 1$ if $U_i = 0$ and TE_i declines as U_i increases with inefficiency.
- The $U_i's$ are assumed to be a function of farmer and location specific variables that affect technical efficiency while δ is an mX1 vector of unknown parameters or $U_i \sim N(Z_i\delta, \sigma_U^2)$.
- The inefficiency equation associated with the SPF given at (3.1) is then specified as:

$$U_i = Z_i \delta + W_i \tag{3.2}$$

where W_i is assumed to be a normally distributed random variable with mean of zero and variance of σ_W^2 or $W_i \sim N(0, \sigma_W^2)$.

3.3.2. Practical application: SPF

- In this section we use Stata to estimate a stochastic production frontier (SPF)

- While there is a Stata inbuilt command (frontier) to estimate a SPF, we will be using a more convenient user written code (sfcross) that provides more options and returns with more results

- For those that do not have the ado and accompanying files they are included in the sfcross folder of the course materials we provided

- The PDF file in the sfcross folder reviews the theory behind SFA and discuss how to use the different features of sfcross

- We will estimate a stochastic milk production frontier and estimate technical efficiency in milk production using the same dataset used in the CD production function estimation: "MilkProd&InputUseData.dta",

- In the stochastic milk production frontier we will also use the deterministic component used in the CD production function ($f(X, \beta)$)

- To estimate the SPF let us open the do file "Practical Examples.do".

- Do not forget changing the folder directory to open the data file

- Then we follow the steps in the SFA practice part of the do file, which starts at about line 34

-

3.3.3. Data envelopment analyses

- DEA involves the use of linear programming methods to construct a non-parametric piece-wise linear convex surface (or frontier) over the data.

- Efficiency measures are then calculated relative to this surface.

- Suppose we have i=1, 2, …,I farmers each using N (n=1, 2, …, N) types of inputs and producing M (m=1, 2, …, M) types of outputs.

- Let x_i be the N×1 input vector of farmer i and q_i be an M×1 output vector of i,

- Let X represent the N×I matrix of all farmers and Q, an M×I matrix of outputs

- Then, assuming a CRS technology and using the duality in linear programming, the input-oriented efficiency of firm i is obtained by solving the linear programming problem (LPP):

$$\min_{\theta,\lambda} \theta \tag{3.3}$$

Subject to
$$-q_i + Q\lambda \geq 0$$
$$\theta x_i - X\lambda \geq 0$$
$$\lambda \geq 0$$

where θ is a scalar and λ, is a I×1 vector of constants,

- The value of θ obtained from the LPP above is the efficiency score for the ith firm.

- The solution of satisfies: $\theta^* < 1$ ($\theta^* = 1$) for a technically inefficient (efficient) farmer

- Note that the LPP must be solved I times, once for each firm in the sample.

- Intuitively, the LPP seeks a value of θ that radially contracts input use levels of farmer i as much as possible while keeping i within the feasible input set (first constraint) and that output level of i is less than or equal to that produced by all farmers in the data (first constraint).

- The inner boundary of this set is a piece-wise linear isoquant, determined by the observed data points (i.e., all the firms in the sample).

Example 1: Suppose there are 5 farmers that produce 1 type of output using 2 input types. Then the LPP that solves the input-oriented efficiency score of farmer 3 is stated as:

$$\min_{\theta,\lambda} \theta$$

Subject to
$$-q_3 + (q_1\lambda_1 + q_2\lambda_2 + q_3\lambda_3 + q_4\lambda_4 + q_5\lambda_5) \geq 0$$
$$\theta x_{13} - (x_{11}\lambda_1 + x_{12}\lambda_2 + x_{13}\lambda_3 + x_{14}\lambda_4 + x_{15}\lambda_5) \geq 0$$
$$\theta x_{23} - (x_{21}\lambda_1 + x_{22}\lambda_2 + x_{23}\lambda_3 + x_{24}\lambda_4 + x_{25}\lambda_5) \geq 0$$

$$[\lambda_1 \; \lambda_2 \; \lambda_3 \; \lambda_4 \; \lambda_5] \geq \mathbf{0}$$

- The CRS assumption is appropriate when all firms are operating at an optimal scale.

- However, firms may not operate at optimal scale for one or another reason (imperfect competition, government regulations, constraints on finance, etc.)

- Various authors suggested adjusting the CRS DEA model to account for variable returns to scale (VRS) situations.

- Using CRS DEA when in fact some firms have VRS makes the estimated efficiency scores misleading as they include a scale inefficiency effect

- The CRS LPP can be easily modified to account for VRS by adding the convexity constraint: $\sum_{i=1}^{I} \lambda_i = l_1' \lambda = 1$ where l_1' is a column vector of ones with dimension of I or l_1' an I×1 vector of ones. Then the LPP of input oriented efficiency measure, assuming VRS, is given as:

$$\min_{\theta,\lambda} \theta \qquad\qquad (3.4)$$

Subject to
$$-q_i + Q\lambda \geq 0$$
$$\theta x_i - X\lambda \geq 0$$
$$l_1'\lambda = 1$$
$$\lambda \geq 0$$

- The presence of $l_1'\lambda = 1$ ensures that an inefficient firm is compared only against firms of a similar scale

- By recognizing the presence of scale differences the LPP at (3.4) constructs a frontier that envelopes the data points more tightly than the one constructed by the LPP that assumes CRS

- Thus the LPP at (3.4) provides technical efficiency scores that are greater than or equal to those obtained using the CRS model.

- Output oriented measure of TE is computed by setting up a similar LP problems. You can refer about these in Coelli et al. (2015).

3.3.4. Practical application of DEA

Example 1.1: suppose the input-output combination of 5 farmers that produce 1 type of output using 2 input types is given as follows:
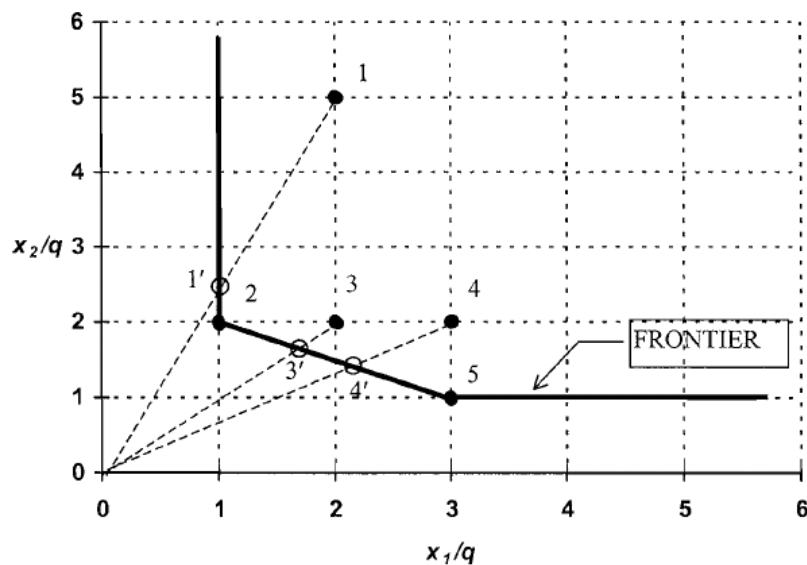
Table 3.1. Example data for CRS DEA example

firm	q	x_1	x_2	x_1/q	x_2/q
1	1	2	5	2	5
2	2	2	4	1	2
3	3	6	6	2	2
4	1	3	2	3	2
5	2	6	2	3	1

Source: Table 1 of Coelli et al. (2005)

Then the frontier constructed by the data points is given in Figure 3.1

Figure 3.1. CRS DEA input-oriented example

Source: Figure 6.2 of Coelli et al. (2005)

- To measure the performance of each firm in technical and allocative efficiency we use the software called DEAP, which is developed by Tim J. Coelli at Center for Efficiency and Productivity Analysis, Department of Economics, University of New England.
- DEAP is a freely downloadable software (Coelli, 1996). We provide all elements of the software as well as the dataset that will be used in our practical example in the DEAP folder of

the course materials. The folder also contains a PDF file that goes through the theory of DEA that we discussed and also serves as a manual for the software.

- This software, which is a DOS based systems, works as follows:

1. For each analyses to be conducted we need to prepare two files: a data file and an instruction file. All of the files are to be prepared as a text file.

- The instruction file is prepared according to the format prepared by the authors of the software. It contains spaces for number of inputs, outputs, periods, firms, RTS, etc. to be filled in. You need to first save a copy of the ins file from the authors and replace the values by those appropriate for your problem. You may need to put the extension "ins" at the end of the filename to remember that it is an instruction file.

- The data file also is prepared according to the authors' formatting requirement. All outputs need to be listed first, followed by inputs. You do not include variable names.

- If you are conducting panel data analyses all panel members should be observed at all periods. In the case of panel data the first entries all members of the panel in the first period; then all members in the second period, etcetera.

- ***All file names need to be 12 characters or less***.

2. Show how to prepare the two files

3. To estimate the TE of the firms in the example above double click the DOS executable file "DEAP.EXE". Then insert "Eg1-ins.txt" at the cursor where it reads "Enter instruction file name:" and enter

- Then the output file of the exercise (example 1), "eg1-out.txt" appears almost instantaneously,

4. Double click the output file (eg1-out.txt).

- As you see the output file contains all of the results that you expected (TE estimates under the heading "EFFICIENCY SUMMARY") and estimates of λ's, which are discerned by looking the two results under "SUMMARY OF PEERS:" and "SUMMARY OF PEER WEIGHTS:".

- Additional results included are: input and output slacks, input and output targets, and radial and slack movements of O/I by firm. We summarize some of the results below.

- Recall that the estimates of λ's indicate the weights of firms (e.g. firms 2 and 5) that each firm (firm 4) is compared with. So across firm 4 we have weights for firms 2 and 5.

- This is a LP problem and in such a problem, as you recall, slacks could arise if there are inputs that are wasted at optimum. In this case a radial contraction of input use of firm 1 leaves it at

$(x_1, x_2) = (1, 2.5)$ while firm 2 produces the same quantity using $(x_1, x_2) = (1, 2)$, indicating firm 1 has a slack 0.5 units of input 2 (see Figure 3.1).

Table 3.2 Results of DEA analyses using DEAP

firm	θ	λ_1	λ_2	λ_3	λ_4	λ_5	IS_1	IS_2	OS
1	0.500	-	0.500	-	-	-	0.00	0.50	0.00
2	1.000	-	1.000	-	-	-	0.00	0.00	0.00
3	0.833	-	1.000	-	-	0.500	0.00	0.00	0.00
4	0.714	-	0.214	-	-	0.286	0.00	0.00	0.00
5	1.000	-	-	-	-	1.000	0.00	0.00	0.00
Average	0.810						0.00	0.10	0.00

Example 1.2

1. Now, let us re-conduct the analyses by only changing the distance function into output oriented distance function:

- To do that first open the instruction file "Eg1-ins.txt". Then change the output file name (line 2) to "eg12-out.txt", change the type of distance function (line 7) from 0 to 1, and save the instruction file as "Eg12-ins.txt".

- Then open the DEAP.EXE file and insert the name of the new instruction file (Eg12-ins.txt) and enter

2. Open the new output file (eg12-out.txt).

- Looking at the new results you might be surprised to find that the TE estimates are identical. This is because the technology assumed is CRS and as you recall output and input oriented measure assign the same measure of efficiency if the technology is CRS.

- However, you begin seeing differences when you look at the peer weights and thereafter.

- For instance, the target outputs and inputs are different from those obtained in the previous analysis. In particular, since the output oriented measure indicates the maximum output to which inefficient firms' output can expand to, we see that output targets of firms 1, 3, and 4 have changed.

- As discussed above, firm 1 has a slack in input 2. This shows in also this analysis. That is, at its current input use level firm 1 can expand its output to 2 (from 1) and still have a slack of 1 in input 2.

Table 3.3 Results of DEA analyses using DEAP (Output-oriented)

firm	Actual			Target		
	q	x_1	x_2	q	x_1	x_2
1	1	2	5	2	2	4
2	2	2	4	2	2	4
3	3	6	6	3.6	6	6
4	1	3	2	1.4	3	2
5	2	6	2	2	6	2

Example 2: Now let us use DEA to estimate productivity in egg production.

• The data for this analysis is provided in the file "EggProd_2014_15.xlsx". This is a zonal level aggregated data. Although we derive this data from farmer level datasets it can also be extracted from CSA annual reports (Livestock and Livestock Characteristics (Vol-II)). So we did not remove the identifiers.

• Open this file to "Eggs" sheet. This sheet has all the data (2014 and 2015) including variable names. The 2014, 2015, and 2014_15 sheets contain the data for the years implied by the sheets.

• The names of the variables in the "Eggs" sheet indicate there is one output (eggs000-egg production in thousands; EggLying_Tot (total number of egg lying hens); PoultExt (number using poultry extension); and Pct_Imp (percent of hybrid and foreign egg lying chicken. You may have observed a repetition of the number of 0.05 (particularly, in PoultExt and Pct_Imp), which replace zero entries of the inputs.

• Now save the data in the 2014 sheet as a text file (give it such a name as "EgY1dta.txt")

• Open the "Eg1-ins.txt" file and save it under a new name "EgY1ins.txt". Change the data file name (line 1) to "EgY1dta.txt"; change the output file name (line 2) to "EgY1out.txt"; change the number of firms to 68; and the number of inputs to 3. Re-save the file after these changes

• Open the DOS executable (DEAP.EXE) file and insert the name of the instruction file you just created and enter

• Double click the new/instantaneously created output file (EgY1ins.txt)

• Now I want some volunteer/s that will discuss the results for us

3.4. Measurement of productivity and productivity change

- At the introduction to this section we indicated that a measure of productivity of a firm that produces multiple outputs using multiple inputs is given as: TFP= Output index/input index

- The definition above indicates that productivity is essentially a level concept, which can be used to compare the performance of two firms at a given point in time.

- One can also compare the productivity performance or measure the change in productivity performance of a firm at two points in time/periods

- Productivity change refers to movements in productivity of a firm over time

- Such a measure of productivity of a firm with multiple outputs and multiple inputs (MO-MI) requires using methods that aggregate the multiple inputs into a single input index and multiple outputs into a single output index

- Aggregation of outputs and inputs gives rise to index number problems.

- As discussed below there also is a problem of price indices and these issues are even more complicated when considering multiple periods (changes in productivity)

- This area of production economics is not a well settled area and open to different interpretations by different authors

- Furthermore, I do not think the material is appropriate for such a training

- So, we will discuss the methods of measuring productivity levels and changes in productivity only briefly.

3.4.1. Measuring and comparing productivity levels

- How do we compare the TFP performance of two firms that produce multiple outputs using multiple inputs?

- A simple TFP measure for firms with MO-MI is to look at the profitability of the firms, defined as the firms' revenue divided by its input cost

- Suppose we have two firms producing output vectors q_1 and q_2 using inputs x_1 and x_2, respectively. Suppose the corresponding output and input price vectors are given by (p_1, p_2) and (w_1, w_2). Then the *profitability ratio* of firm i, where i=1, 2 is given by:

$$\pi_i = \frac{p_i' q_i}{w_i' x_i} = \frac{\sum_{m=1}^{M} p_{mi} q_{mi}}{\sum_{n=1}^{N} w_{ni} x_{ni}} \tag{3.5}$$

where m=1, 2, …,M and n=1, 2, …,N represent the number of outputs and inputs, respectively.

- Then a measure of relative performance is given by the ratio π_2/π_1.

- Though π_1 and π_2 are scalar measures of total productivity, a strict comparison of π_1 and π_2 is difficult since the output and input prices faced by these firms could be different

- The only option here is to adjust the value aggregates in equation (3.5) for differences in price levels. Such an adjustment requires that the value aggregates in the numerator and the denominator of equation (3.5) are deflated by suitable price deflators or price index numbers, which poses yet another problem.

- Suppose we have the simple case of two firms with a single input and single output. Suppose the data for the two firms is given by (p₁, q₁, w₁, x₁) and (p₂, q₂, w₂, x₂).

- In this case, a comparison of the profitability ratios is given by

$$\frac{\pi_2}{\pi_1} = \frac{(p_2 q_2 / w_2 x_2)}{(p_1 q_1 / w_1 x_1)} = \frac{(p_2 q_2 / p_1 q_1)}{(w_2 x_2 / w_1 x_1)} \qquad (3.6)$$

- Equation (3.6) provides a comparison of profitability between firms 2 and 1.

- To make an adjustment for differences in prices faced by the two firms suppose we take p_2/p_1 as the output price index and divide the numerator by that index. Take also w_2/w_1 as the input price index and divide the denominator by that index. Then equation (3.6) becomes:

$$\frac{\pi_2^*}{\pi_1^*} = \frac{(q_2 / q_1)}{(x_2 / x_1)} = \frac{(q_2 / x_2)}{(q_1 / x_1)} \qquad (3.7)$$

- Equation (3.7) indicates that a comparison of productivity of firms 1 and 2 using the profitability ratio simply reduces to a ratio of output level differences measured by q_2/q_1 to the input ratio x_2/x_1. (That is, if the firms are a single-output single-input firms)

- If firm 2 produces 50 per cent more output than firm 1 and uses only 25 per cent more input, then a measure of the productivity level of firm 2 relative to 1 is given by 1.50/1.25 = 1.2. This is consistent with what is understood by productivity differences between firms.

- The last part of equation (3.7) also shows that the relative profitability is indeed a ratio of productivity of firms 2 and 1 as measured by output per unit of input.

- However, the example above is a lot simpler than the simplest of problems we face in reality.

- In particular, adjusting price level differences is not straightforward when there are multiple inputs and outputs.

- In such a case, we need to make use of appropriate price deflators constructed using an appropriate index number methodology

  - An important point recall is that a tool/method/formula that is used to assess productivity

performance needs to be devoid of price effects

- In particular, if two firms are using the same technology, then their performance in productivity (and profitability) at a given point in time should depend only on the *technical, allocative* and *scale efficiency* levels of the two firms and not on the prices of inputs and outputs.

3.4.2. Measuring productivity change

- This subsection considers the issue of measuring changes in productivity.
- Let us consider the problem of measuring productivity change for a firm from period s to period t. We assume that the firm makes use of the state of knowledge, as represented by production technologies S^s and S^t in periods s and t. Suppose the firm produces outputs q_s and q_t using inputs x_s and x_t, respectively. In some cases, we may have information on output and input prices, which are represented by output price vectors, p_s and p_t and input price vectors, w_s and w_t, in periods s and t, respectively.
- Given these data on this firm, how do we measure productivity change? Among several approaches to measure productivity change Coelli et al. (2005) consider four alternatives:

1. Hicks-Moorsteen TFP index (HM-TFP): This approach simply uses a measure of output growth, net of growth in inputs. If output has doubled over the period s to t, and if this output growth was achieved using only a 60 percent growth in input use, we conclude that the firm has achieved productivity growth rate of 25% or 2/1.6=1.25.

- The HM-TFP index represents a fairly simple TFP index that measures the growth in output, net of growth in inputs. If output growth and input growth are measured using output and input quantity index numbers, then the index is given by= HM TFP index=output index/input index.

- The HM index can be made operational once appropriate measures of output and input growth are selected. This index is also closely related to the profitability based index and the Malmquist TFP index (discussed below).

- This index is easy to measure and interpret. However, the index does not allow identify the main sources of productivity growth or does not allow identify whether changes in TFP resulted from technical change or to improvements in efficiency.

2. Profitability ratio based TFP index: This approach extends the profitability approach

discussed above and measure productivity change using growth in profitability after making appropriate adjustments for movements in input and output prices over the period *s* to *t*.

- Note that this method, as well as the others, can be used to compare the productivity performance of a given firm across periods and to compare the performance of two firms at a given period in time.

3. Malmquist TFP Index (Caves, Christensen and Diewert, 1982a; 1982b): measures productivity by comparing the observed outputs in period s and t with the maximum level of outputs (keeping the output mix constant) that can be produced using x_s and x_t operating under the reference technology.

- With respect to the reference technology, suppose the firm produced 70 percent of the maximum feasible output for the given input vector, x_s, in period s. Suppose also that in period t, the firm produced 30 per cent above the maximum feasible output for the given input vector, x_t, then a measure of productivity change from period s to t is given by the ratio 1.30/0.70=1.857.

- This method essentially uses the output and input distance functions defined earlier to measure the productivity performance of a firm across two periods. As a result, it enables identify the source of changes in TFP (technical change or improvements in efficiency)

- For instance, change in productivity, based on period s technology, can be defined using the output oriented distance function as:

$$m_O^s(q_s, q_t, x_s, x_t) = \frac{D_O^s(q_t, x_t)}{D_O^s(q_s, x_s)}$$

- Suppose the firm was technical and scale efficient in period s or $D_O^s(q_s, x_s) = 1$. Then $m_O^s(\cdot) = D_O^s(q_t, x_t)$. Suppose also in period t the firm has a higher level of productivity than is implied by the period-s technology or $D_O^s(q_t, x_t) > 1$ then $m_O^s(\cdot) > 1$

- If instead the firm was 20% of technically inefficient and had 10% scale inefficiency (or $D_O^s(q_s, x_s) = 0.7$). Suppose also in period t the firm is both technically and scale efficient as implied by the period-s technology or $D_O^s(q_t, x_t) = 1$ then $m_O^s(\cdot) = 1.429$. Out of the (43%) gain in productivity 29% and 14% is due to improvements in technical and scale efficiency.

4. Measurement by sources of productivity change: The first three methods define what a productivity change constitutes and then try to decompose the sources (top-down approach).

- The fourth approach to productivity measurement (Balk, 2001) tries to identify all the sources of productivity change and then constructs a measure of the growth in total factor productivity.

- This approach, which can be considered as a bottoms-up approach, starts with a list of all possible sources of productivity growth and then examines the best possible way of measuring each of these sources and combines them to derive a measure of productivity change.

- The resulting measure of productivity change should satisfy the basic property, stated in equation (3.8) below, that the productivity change measure must be homogeneous of degree +1 in outputs and homogeneous of degree -1 in inputs

An important requirement for the TFP Index

- Irrespective of which approach is employed to measure performance in TFP it is important that the index used satisfies the following property. If a firm produces the same quantity of output in two periods, *s* and *t*, but the input use is *decreased* by a proportion then the TFP index should increase accordingly.

- For instance, if the firm uses 25 percent lower inputs in period t than those used in period s to produce the same quantity (period t inputs are only 75 percent of the inputs in period s) then the TFP index should be equal to 1/0.75.

- Similarly, if the outputs increase by a given percent while input levels are fixed, then the TFP index should increase by the same percent as the increase in the outputs. If outputs in period t are 30% higher than those in period *s* with input use fixed then the TFP index should be equal to 1.3.

- Suppose we represent the TFP index by a function $F(x_t, q_t, x_s, q_s)$ with period s and period *t* input and output vectors as arguments. Then, for $x_t = \lambda x_s$ and $q_t = \mu q_s$ the TFP index should satisfy the property:

$$F(\lambda x_s, \mu q_s, x_s, q_s) = \mu/\lambda \text{ for all } \mu, \lambda > 0 \tag{3.8}$$

- That is, the TFP index is homogeneous of degree +1 in q and -1 in x.