ENGINEERING MATHEMATICS-I

DIPLOMA COURSE IN ENGINEERING
FIRST SEMESTER

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FOREWORD

We take great pleasure in presenting this book of mathematics to the students of polytechnic colleges. This book is prepared in accordance with the new syllabus under 'M" scheme framed by the Directorate of Technical Education, Chennai.

This book has been prepared keeping in mind, the aptitude and attitude of the students and modern method of education. The lucid manner in which the concepts are explained, make the teaching and learning process more easy and effective. Each chapter in this book is prepared with strenuous efforts to present the principles of the subject in the most easy to understand and the most easy to workout manner.

Each chapter is presented with an introduction, definitions, theorems, explanation, solved examples and exercises given are for better understanding of concepts and in the exercises, problems have been given in view of enough practice for mastering the concept.

We hope that this book serve the purpose keeping in mind the changing needs of the society to make it lively and vibrating. The language used is very clear and simple which is up to the level of comprehension of students.

We extend our deep sense of gratitude to Thiru. R. Sornakumar Coordinator and Principal, Dr. Dharmambal Government Polytechnic College for women, Chennai and to Thiru P.L. Sankar, Convener and Lecturer / SG, Rajagopal Polytechnic College, Gudiyattam who took sincere efforts in preparing and reviewing this book.

Valuable suggestions and constructive criticisms for improvement of this book will be thankfully acknowledged.

AUTHORS

30012 ENGINEERING MATHEMATICS – I
DETAILED SYLLABUS

UNIT—I: ALGEBRA

Chapter - 1.1 DETERMINANTS **7 Hrs.**

Definition and expansion of determinants of order 2 and 3. Properties of determinants (not for examination). Solution of simultaneous equations using Cramer's rule (in 2 and 3 unknowns) - Simple Problems.

Chapter - 1.2 MATRICES **7 Hrs.**

Definition –Singular Matrix, Non-singular Matrix, Adjoint of a matrix and Inverse of a matrix up to 3 x 3 only. Simple Problems. Definition – Rank of a matrix. Finding rank of a matrix by determinant method (matrix of order 3 x 4) Simple Problems.

Chapter - 1.3 BINOMIAL THEOREM **8 Hrs.**

Definition of Factorial notation - Definition of Permutation and Combinations – values of nP_r and nC_r (results only) [not for examination]. Binomial theorem for positive integral index (statement only) - Expansion - Finding of general term, middle term, coefficient of x^n and term independent of x Simple Problems. Binomial Theorem for rational index up to - 3 (statement only), Expansions only for - 1, - 2 and - 3.

UNIT—II: COMPLEX NUMBERS

Chapter - 2.1 ALGEBRA OF COMPLEX NUMBERS **8 Hrs.**

Definition – Real and Imaginary parts, Conjugates, Modulus and amplitude form, Polar form of a complex number, multiplication and division of complex numbers (geometrical proof not needed)– Simple Problems .Argand Diagram – Collinear points, four points forming square, rectangle, rhombus and parallelogram only . Simple Problems.

Chapter - 2.2 DE MOIVER'S THEOREM **7 Hrs.**

Demoivre's Theorem (statement only) – related simple problems.

Chapter - 2.3 ROOTS OF COMPLEX NUMBERS **7 Hrs.**

Finding the n^{th} roots of unity - solving equation of the form $x^n \pm 1 = 0$ where $n \leq 7$. Simple Problems.

UNIT– III: TRIGONOMETRY

Chapter – 3.1 COMPOUND ANGLES **8 Hrs.**

Expansion of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$ [without proof] . Problems using above expansions.

Chapter - 3.2 MULTIPLE ANGLES **7 Hrs.**

Trigonometrical ratios of multiple angles of 2A and 3A and sub multiple angles. Simple Problems.

Chapter - 3.3 SUM AND PRODUCT FORMULAE **7 Hrs.**

Trigonometrical ratios of sum and product formulae. Simple Problems.

UNIT—IV INVERSE TRIGONOMETRIC RATIOS & DIFFERENTIAL CALCULUS – I

Chapter - 4.1 INVERSE TRIGONOMETRIC FUNCTIONS **7 Hrs.**

Definition of inverse trigonometric ratios – Relation between inverse trigonometric ratios. Simple Problems.

Chapter - 4.2 LIMITS **7 Hrs.**

Definition of Limits. Problems using the following results:

(i) $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$ (ii) $\lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ and

(iii) $\lim\limits_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$ (θ - in radians) (results only) . Simple Problems.

Chapter - 4.3 DIFFERENTIATION **8 Hrs.**

Definition – Differentiation of x^n, $\sin x$, $\cos x$, $\tan x$, $\operatorname{cosec} x$, $\sec x$, $\cot x$, $\log x$, e^x, $u \pm v$, uv, uvw, $\frac{u}{v}\ (v \neq 0)$ (results only). Simple problems using the above results.

UNIT—V DIFFERENTIAL CALCULUS – II

Chapter – 5.1 DIFFERENTIATION METHODS **8 Hrs.**

Differentiation of function functions (chain rule), Inverse Trigonometric functions and Implicit functions. Simple Problems.

Chapter - 5.2 SUCCESSIVE DIFFERENTIATION **7 Hrs.**

Successive differentiation up to second order (parametric form not included). Definition of differential equation, order and degree, formation of differential equation. Simple Problems.

Chapter - 5.3 PARTIAL DIFFERENTIATION **7 Hrs.**

Definition – Partial differentiation of two variables up to second order only. Simple Problems.

NOTES

UNIT – I

1.1 DETERMINANTS: Definition and expansion of determinants of order 2 and 3. Properties of determinants (nor for examination). Solution of simultaneous equations using Cramer's rule (in 2 and 3 unknowns)-Simple Problems

1.2 MATRICES: Definition - Singular Matrix, Non-singular Matrix, Ad joint of a matrix and inverse of a matrix up to 3 × 3 only. Simple problems. Definition – Rank of a matrix. Finding rank of a matrix by determinant method (matrix of order 3 × 4)

1.3 BINOMIAL THEOREM: Definition of Factorial notation - Definition of Permutation and Combinations - values of nP_r and nC_r (results only) (not for examination). Binomial theorem for positive integral index (statement only) - Expansion - Finding of general term, coefficient of xn and term independent of x. Simple Problems. Binomial Theorem for rational index up to - 3 (statement only). Expansion only for - 1, - 2 and - 3.

1.1 DETERMINANTS

Definition:

Determinant is a square arrangement of numbers (real or complex) within two vertical lines.

Example: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Order:

$A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, consisting of two rows and two columns is called a determinant of second order. The value of the determinant is $\Delta = a_1b_2 - a_2b_1$.

Example: Let $A = \begin{vmatrix} 2 & -5 \\ 1 & 3 \end{vmatrix}$

$|A| = (2)(3) - (1)(-5)$

$|A| = 6 + 5$

i.e $\Delta = 11$

Determinant of third order.

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of three rows and three columns is called a determinant of third order.

The value of the determinant is obtained by expanding the determinant through any row or column with proper sign attached starting from a_1, the first row first column elements, we will have positive and negative sign alternately.

Example: $\begin{vmatrix} \overset{+}{1} & \overset{-}{-1} & \overset{+}{3} \\ \overset{-}{0} & \overset{+}{4} & \overset{-}{2} \\ \overset{+}{11} & \overset{-}{5} & \overset{+}{-3} \end{vmatrix}$ expand thorugh first row.

$$= +1\begin{vmatrix} 4 & 2 \\ 5 & -3 \end{vmatrix} - (-1)\begin{vmatrix} 0 & 2 \\ 11 & -3 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 11 & 5 \end{vmatrix}$$

$= 1\,(-12 - 10) + 1\,(0 - 22) + 3\,(0 - 44)$

$= -22 - 22 - 132$

$= -176$

Minor of an element:

Minor of an element is a determinant obtained by deleting the row and column in which that element occurs.

Example:

$$\begin{vmatrix} 1 & -1 & 3 \\ 0 & 4 & 2 \\ 11 & 5 & -3 \end{vmatrix}$$

$$\text{Minor of } -1 = \begin{vmatrix} 0 & 2 \\ 11 & -3 \end{vmatrix} = 0 - 22 = -22$$

$$\text{Minor of } 0 = \begin{vmatrix} -1 & 3 \\ 5 & -3 \end{vmatrix} = 3 - 15 = -12$$

Co-factor of an element:

Minor of an element with proper sign attached is called co-factor of that element.

Example:

$$\begin{vmatrix} \overset{+}{3} & \overset{-}{-2} & \overset{+}{1} \\ 2 & 0 & -3 \\ 4 & 5 & 11 \end{vmatrix}$$

$$\text{Co-factor of } -2 = -\begin{vmatrix} 2 & -3 \\ 4 & 11 \end{vmatrix} = -[(22) + 12]$$

$$= -34$$

$$\text{Co-factor of } 0 = +\begin{vmatrix} 3 & 1 \\ 4 & 11 \end{vmatrix}$$

$$= +\ (33 - 4)$$

$$= +\ 29$$

Note: The sign for the element a_{ij} is $(-1)^{i+j}$

Properties of determinant:

Property (1):

The value of the determinant is unaltered by changing rows into columns and vice versa.

$$\text{(i.e)} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Property (2):

If any two rows or columns of a determinant are interchanged then the value of the determinant is changed in its sign only.

$$\text{(i.e)} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

By notation $R_1 \leftrightarrow R_2$

[Generally the rows and columns are denoted by R_1, R_2 ...... and C_1, C_2 ...... respectively]

Property (3):

If any two rows or columns of a determinant are identical or same, then the value of the determinant is zero.

$$\text{(i.e)} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad R_1 \equiv R_2$$

Property (4):

If each element of a row or column of a determinant is multiplied by any constant $k \neq 0$, then the value of the determinant is multiplied by same constant k.

Property (5):

If each element of a row or column is expressed as the sum of two elements then the determinant can be expressed as the sum of two determinant of the same order.

$$\text{i.e.} \begin{vmatrix} a_1 + d_1 & b_1 + d_2 & c_1 + d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property (6):

Any determinant is unaltered when each elements of any row or column is added by the equimultiples of any parallel row or column.

$$\text{i.e.} \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_2 & kb_2 & kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k(0)$$

Property (7):

In a given determinant if two rows or column are identical for a = b, then (a – b) is a factor of the determinant.

$$\text{i.e} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \text{ for } a = b$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ b & b & c \\ b^2 & b^2 & c^2 \end{vmatrix}$$

$$= 0 \qquad c_1 \equiv c_2$$

$\therefore$ (a – b) is a factor.

Solution of simultaneous equations using Cramer's rule:

Consider the linear equations

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

$$\text{let } \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \; ; \; \Delta x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and}$$

$$\Delta y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \text{ then } x = \frac{\Delta x}{\Delta} \; ; \; y = \frac{\Delta y}{\Delta}$$

Provided $\Delta \neq 0$.

x, y are unique solutions of the given equations. This method of solving the line equations is called Cramer's rule.

Similarly for a set of three simultaneous linear equations in x, y and z.

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$, the solution of the system of equations by Cramer's rule is given by

$$x = \frac{\Delta x}{\Delta}; y = \frac{\Delta y}{\Delta}; \text{and } z = \frac{\Delta z}{\Delta}$$

provided $\Delta \neq 0$. Where Δ, Δx, Δy and Δz are the determinents formed in the same way as defined above.

WORKED EXAMPLES

PART – A

1. Solve $\begin{vmatrix} x & 4 \\ 9 & x \end{vmatrix} = 0$.

Solution:

$$\begin{vmatrix} x & 4 \\ 9 & x \end{vmatrix} = 0 \text{ By expanding we have}$$

$$x^2 - 36 = 0$$

$$\text{i.e } x^2 = 36$$

$$\therefore x = \pm 6$$

2. Solve $\begin{vmatrix} x & 2 \\ x & 3x \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} x & 2 \\ x & 3x \end{vmatrix} = 0$$

expand $3x^2 - 2x = 0$

$x\,(3x - 2) = 0$

$x = 0\ ;\qquad 3x - 2 = 0 \qquad 3x = 2$

$x = 0$ or $x = \dfrac{2}{3}$

3. Find the co-factor of 3 in the determinant $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 5 & 6 & 7 \end{vmatrix}$.

Solution:

$$\text{Cofactor of } 3 = A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 5 & 7 \end{vmatrix}$$

$$= (-1)^4\,[7 - 0]$$

$$= +7$$

4. Write down the minor 3 in the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 4 & -4 \\ 5 & 0 & 2 \end{vmatrix} = 0.$

Solution:

$$\text{The minor of } 3 = \begin{vmatrix} -2 & -1 \\ 0 & 2 \end{vmatrix} = -4 - 0$$

$$= -4$$

PART – B

1. Find the value of 'm' when $\begin{vmatrix} m & 2 & 1 \\ 3 & 4 & 2 \\ -7 & 3 & 0 \end{vmatrix} = 0.$

Solution:

Given $\begin{vmatrix} m & 2 & 1 \\ 3 & 4 & 2 \\ -7 & 3 & 0 \end{vmatrix} = 0$

expand along R_1, we have

$m\,(0 - 6) - 2\,(0 + 14) + 1\,(9 + 28) = 0$

$-6m - 28 + 37 = 0$

$-6m + 9 = 0$

$-6m = -9$

$6m = 9$

$m = \dfrac{9}{6} = \dfrac{3}{2}$

2. Using Cramer's rule solve $2x - 3y = 5$; $x - 4y = 8$.

Solution:

$2x - 3y = 5$

$x - 4y = 8.$

where $\Delta = \begin{vmatrix} 2 & -3 \\ 1 & -4 \end{vmatrix}$

$= -8 + 3 = -5$

$\Delta x = \begin{vmatrix} 5 & -3 \\ 8 & -4 \end{vmatrix} = -20 + 24 = 4$

$\Delta y = \begin{vmatrix} 2 & 5 \\ 1 & 8 \end{vmatrix} = 16 - 5 = 11$

By Cramer's rule

$x = \dfrac{\Delta x}{\Delta} = \dfrac{4}{-5}$; $y = \dfrac{\Delta y}{\Delta} = \dfrac{11}{-5} = \dfrac{-11}{5}$

3. Find the values of 'x' when $\begin{vmatrix} x & 6 & 3 \\ 1 & x & 1 \\ -2 & 4 & x \end{vmatrix} = 0.$

Solution:

$\begin{vmatrix} x & 6 & 3 \\ 1 & x & 1 \\ -2 & 4 & x \end{vmatrix} = 0$

$x(x^2 - 4) - 6(x + 2) + 3(4 + 2x) = 0$

$x^3 - 4x - 6x - 12 + 12 + 6x = 0$

$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x = 0, \quad x^2 - 4 = 0 \qquad x^2 = 4$

$x = 0, \quad x = \pm 2$

PART –C

1. Using Cramer's rule. Solve the following simultaneous equations,

$x + y + z = 2$

$2x - y - 2z = -1$

$x - 2y - z = 1$

Solution:

Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix} = 1(1 - 4) - 1(-2 + 2) + 1(-4 + 1)$

$= 1(-3) - 1(0) + 1(-3)$

$= -3 - 3$

$\Delta = -6 \neq 0$

$$\Delta x = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix} = 2\ (1-4) - 1(1+2) + 1(2+1)$$

$$= 2\ (-3) - 1\ (3) + 1(3)$$

$$= -\ 6 - 3 + 3$$

$$\Delta x = -\ 6$$

$$\Delta \text{y} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & -1 \end{vmatrix} = 1\ (1+2) - 2\ (-2+2) + 1\ (2+1)$$

$$= 1\ (3) - 2\ (0) + 1(3)$$

$$= 3 + 3$$

$$\Delta \text{y} = 6$$

$$\Delta \text{z} = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 1 & -2 & 1 \end{vmatrix} = 1\ (-1-2) - 1\ (2+1) + 2\ (-\ 4+1)$$

$$= 1\ (-3) - 1\ (3) + 2(-3)$$

$$= -3 - 3 - 6$$

$$\Delta \text{z} = -12$$

$$x = \frac{\Delta x}{\Delta} = \frac{-6}{-6} = 1$$

$$\text{y} = \frac{\Delta y}{\Delta} = \frac{6}{-6} = -1 \quad ; \quad \text{z} = \frac{\Delta z}{\Delta} = \frac{-12}{-6} = 2$$

1.2 MATRICES

Introduction:

The term matrix was first introduced by a French mathematician CAYLEY in the year 1857. The theory of matrices is one of the powerful tools of mathematics not only in the field of higher mathematics but also in other branches such as applied sciences, nuclear physics, statistics, economics and eletrical circuits.

Definition:

A matrix is a rectangular array of numbers arranged in delete rows and columns enclosed by brackets.

Example:

1) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; 2) $B = \begin{pmatrix} 2 & 1 & 0 \\ -5 & 6 & 7 \\ 1 & 0 & 8 \end{pmatrix}$

Order of a matrix:

If there are 'm' rows and 'n' columns in a matrix, then the order of the matrix is m × n (read as m by n).

Example: $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$

The above matrix A has two rows and three columns. We say that A is a matrix of order 2 × 3.

Types of matrices:

(1) Row matrix:

A matrix having only one row and any number of columns is called a row matrix.

Example: $A = (1 \quad 2 \quad -3)$

(2) Column matrix:

A matrix having only one column and any number of rows is called a column matrix.

Example: $B = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$

(3) Square matrix:

A matrix which has equal number of rows and columns is called a square matrix.

Example: $C = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 4 & -2 \\ 1 & 5 & 6 \end{pmatrix}$ is a square matrix of order 3.

(4) Null matrix (or) Zero matrix:

If all the elements of a matrix are zero, the matrix is called zero or null matrix.

Example: $O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(5) Diagonal matrix:

A square matrix with all the elements equal to zero except those in the leading diagonal is called a diagonal matrix.

$$\text{Example: } D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(6) Unit matrix:

Unit matrix is a square matrix in which the diagonal elements are all ones and all the other elements are zeros.

$$\text{Example: } I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(7) Determinant of a matrix:

Let 'A' be a square matrix. The determinant formed by the elements of A is said to be the determinant of matrix A. This is denoted by $|A|$.

(8) Singular and Non-Singular matrix:

A square matrix 'A' is called a singular matrix if $|A| = 0$. If $|A| \neq 0$, then the matrix A is called a non-singular matrix.

(9) Transpose of a matrix:

If the rows and columns of a matrix are interchanged, then the resultant matrix is called the transpose of the given matrix. It is denoted by A^T.

Algebra of Matrices:

(1) Addition and Subtraction of matrices:

If A and B are any two matrices of the same order, then their sum or difference i.e A + B or A – B is of the same order, and is obtained by adding or subtracting the corresponding elements of A and B.

Example:

$$\text{If } A = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix} \;;\; B = \begin{pmatrix} 2 & 1 \\ -3 & -4 \end{pmatrix} \text{ then}$$

$$A + B = \begin{pmatrix} 4+2 & 3+1 \\ -1-3 & 0-4 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ -4 & -4 \end{pmatrix} \text{ and}$$

$$A - B = A + (-B) \qquad -B = \begin{pmatrix} -2 & -1 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4-2 & 3-1 \\ -1+3 & 0+4 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

(2) Multiplication of matrices:

Two matrices A and B are conformable for multiplication if and only if the number of columns in A is equal to the number of rows in B.

Note : If A is of order $m \times n$ and B is of order $n \times p$ matrices, then AB exists and is of order $m \times p$.

Example:

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ 3 & 4 \\ 4 & -3 \end{pmatrix}$ then

$$AB = \begin{pmatrix} \boxed{1 \quad 2 \quad 3} \\ \boxed{3 \quad 0 \quad 2} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} -1 \\ 3 \\ 4 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 4 \\ -3 \end{matrix}} \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{1 \quad 2 \quad 3}\boxed{\begin{matrix} -1 \\ 3 \\ 4 \end{matrix}} & \boxed{1 \quad 2 \quad 3}\boxed{\begin{matrix} 2 \\ 4 \\ -3 \end{matrix}} \\ \boxed{3 \quad 0 \quad 2}\boxed{\begin{matrix} -1 \\ 3 \\ 4 \end{matrix}} & \boxed{3 \quad 0 \quad 2}\boxed{\begin{matrix} 2 \\ 4 \\ -3 \end{matrix}} \end{pmatrix}$$

$$= \begin{pmatrix} -1+6+12 & 2+8-9 \\ -3+0+8 & 6+0-6 \end{pmatrix}$$

$$AB = \begin{pmatrix} 17 & 1 \\ 5 & 0 \end{pmatrix}$$

Generally $AB \neq BA$

Co-factor matrix:

In a matrix, if all the elements are replaced by the corresponding co-factors of the elements, then the matrix obtained is called the co-factor matrix.

Example: If $A = \begin{pmatrix} 1 & -4 \\ 8 & 3 \end{pmatrix}$

Co-factor of 1 is + 3 Co-factor of – 4 is – 8

Co-factor of 8 is 4 Co-factor of 3 is 1

$\therefore$ Co-factor matrix is $\begin{pmatrix} 3 & -8 \\ 4 & 1 \end{pmatrix}$

Adjoint (or) Adjugate matrix:

The transpose of the co-factor matrix is called the adjoint matrix or adjugate matrix. It is denoted by Adj.A

Exmaple: From the above matrix $A = \begin{pmatrix} 1 & -4 \\ 8 & 3 \end{pmatrix}$ and the co-factor matrix of A is $\begin{pmatrix} 3 & -8 \\ 4 & 1 \end{pmatrix}$, the adjoint of A is $\begin{pmatrix} 3 & 4 \\ -8 & 1 \end{pmatrix}$.

Inverse of a matrix:

Let A be a non-singular square matrix. If there exists a square matrix B, such that AB = BA = I, where I is the unit matrix of the same order as that of A, then B is called the inverse of A and it is denoted by A^{-1}. This can be determined by using the formula.

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

Note:

1. If $|A| = 0$, then there is no inverse for the matrix A.
2. $A^{-1}A = AA^{-1} = I$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$

Rank of a matrix:

Let 'A' be any m × n matrix. It has square sub-matrices of different orders. The order of the largest square submtrix of A whose determinant has a non-zero value is known as the rank of the matrix A.

Generally the rank is denoted by $\rho(A)$, $\rho(A)$ is always minimum of m and n.

Note:

The rank of a matrix is said to be r if

1) It has atleast one non zero minor of order r.
2) Every minor of A of order higher than r is zero.

Example:

1. Find the rank of the matrix $\begin{vmatrix} 1 & 5 & 6 \\ 2 & 3 & 4 \\ -1 & 2 & 2 \end{vmatrix}$.

Solution:

$$\text{Let } A = \begin{vmatrix} 1 & 5 & 6 \\ 2 & 3 & 4 \\ -1 & 2 & 2 \end{vmatrix}$$

$$|A| = 1(6-8) - 5(4+4) + 6(4+3)$$
$$= 1(-2) - 5(8) + 6(7)$$
$$= -2 - 40 + 42$$
$$|A| = 0$$

consider the determinant of the second order

$$\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} = -7 \neq 0$$

Hence the rank of A is 2.

i.e $\rho(A) = 2$

WORKED EXAMPLES

PART – A

1. If $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$; $B = \begin{pmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{pmatrix}$ find A + B.

Solution:

$$A + B = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix} + \begin{pmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0+7 & 2+6 & 3+3 \\ 2+1 & 1+4 & 4+5 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 7 & 8 & 6 \\ 3 & 5 & 9 \end{pmatrix}$$

2. If $A = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$; $B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ find A – B.

Solution:

$$A - B = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1-1 & 3-2 \\ 3-4 & 4-3 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

3. Prove that the matrix $\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ is singular.

Solution:

$$\text{Let } A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$$

$$= -4 + 4$$

$|A| = 0$ $\therefore$ A is singular.

4. Find the rank of $\begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}$.

Solution:

$$\text{Let } A = \begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}$$

Now, $|A| = \begin{vmatrix} 3 & -4 \\ -6 & 8 \end{vmatrix} = 24 - 24 = 0.$ $\therefore \rho(A) \neq 2$

Since each element of A is non zero,

$\rho(A) = 1.$

5. Find the rank of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.

Solution:

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and let the sub matrix of $A = B = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$

$$|B| = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$$

$\therefore \rho(A) = 2$

PART – B

1. If $A=\begin{pmatrix}2 & 3 & 0\\ 5 & 2 & -1\end{pmatrix}$; $B=\begin{pmatrix}3 & -1 & 4\\ 2 & 6 & 7\end{pmatrix}$ find 3A – 2B.

Solution:

$$3A-2B=3\begin{pmatrix}2 & 3 & 0\\ 5 & 2 & -1\end{pmatrix}-2\begin{pmatrix}3 & -1 & 4\\ 2 & 6 & 7\end{pmatrix}$$

$$=\begin{pmatrix}6 & 9 & 0\\ 15 & 6 & -3\end{pmatrix}-\begin{pmatrix}6 & -2 & 8\\ 4 & 12 & 14\end{pmatrix}$$

$$=\begin{pmatrix}6-6 & 9+2 & 0-8\\ 15-4 & 6-12 & -3-14\end{pmatrix}$$

$$3A-2B=\begin{pmatrix}0 & 11 & -8\\ 11 & -6 & -17\end{pmatrix}$$

2. If f(x) = 4x + 2 and A = $\begin{pmatrix}2 & -1\\ 0 & 3\end{pmatrix}$ find f(A).

Solution:

$$f(A) = 4A + 2$$

$$= 4A + 2I$$

$$=4\begin{pmatrix}2 & -1\\ 0 & 3\end{pmatrix}+2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}8 & -4\\ 0 & 12\end{pmatrix}+\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$$

$$f(A)=\begin{pmatrix}10 & -4\\ 0 & 14\end{pmatrix}$$

3. Find the value of ‘a’ so that the matrix $\begin{pmatrix}1 & -2 & 0\\ 2 & a & 4\\ 2 & 1 & 1\end{pmatrix}$ is singular.

Solution:

Let $A=\begin{pmatrix}1 & -2 & 0\\ 2 & a & 4\\ 2 & 1 & 1\end{pmatrix}$. The matrix is singular, then | A | = 0.

i.e $\begin{vmatrix}1 & -2 & 0\\ 2 & a & 4\\ 2 & 1 & 1\end{vmatrix} = 0$

expanding through first row

$$1 (a-4) + 2 (2-8) + 0 = 0$$

$$a-4+2(-6) = 0$$

$$a-4-12 = 0$$

$$a-16 = 0$$

$$a = 16$$

4. If $A=\begin{pmatrix}1&0\\2&1\end{pmatrix}$ and $B=\begin{pmatrix}0&3\\2&4\end{pmatrix}$ find AB.

Solution:

$$AB=\begin{pmatrix}1&0\\2&1\end{pmatrix}\begin{pmatrix}0&3\\2&4\end{pmatrix}$$

$$AB=\begin{pmatrix}0+0&3+0\\0+2&6+4\end{pmatrix}=\begin{pmatrix}0&3\\2&10\end{pmatrix}$$

5. Find the inverse of $\begin{pmatrix}2&3\\4&5\end{pmatrix}$.

Solution:

$$\text{Let } A=\begin{pmatrix}2&3\\4&5\end{pmatrix} \therefore\ |A|=\begin{vmatrix}2&3\\4&5\end{vmatrix}=10-12=-2\neq 0$$

$$\text{Adj } A=\begin{pmatrix}5&-3\\-4&2\end{pmatrix}$$

$$A^{-1}=\frac{1}{|A|}\text{Adj } A=\frac{1}{-2}\begin{pmatrix}5&-3\\-4&2\end{pmatrix}$$

6. Find the rank of $\begin{pmatrix}5&-15\\-6&18\\7&-21\end{pmatrix}$.

Solution:

$$\text{Let } A=\begin{pmatrix}5&-15\\-6&18\\7&-21\end{pmatrix}$$

$$\begin{vmatrix}5&-15\\-6&18\end{vmatrix}=90-90=0$$

$$\begin{vmatrix}-6&-15\\7&-21\end{vmatrix}=-105+105=0$$

$$\begin{vmatrix}6&18\\7&21\end{vmatrix}=126-126=0 \qquad \therefore \rho(A)\neq 2$$

$$\therefore \rho(A)=1$$

PART – C

1. If $A=\begin{bmatrix}2&-1&0\\0&-2&1\\1&0&1\end{bmatrix}$ and $B=\begin{bmatrix}-2&1&-1\\1&2&-2\\2&-1&-4\end{bmatrix}$ show that AB = BA.

Solution:

$$AB=\begin{bmatrix}2&-1&0\\0&-2&1\\1&0&1\end{bmatrix}\begin{bmatrix}-2&1&-1\\1&2&-2\\2&-1&-4\end{bmatrix}$$

$$= \begin{bmatrix} -4-1+0 & 2-2+0 & -2+2+0 \\ 0-2+2 & 0-4-1 & 0+4-4 \\ -2+0+2 & 1+0-1 & -1+0-4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 2 & -2 \\ 2 & -1 & -4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4+0-1 & 2-2+0 & 0+1-1 \\ 2+0-2 & -1-4+0 & 0+2-2 \\ 4+0-4 & -2+2+0 & 0-1-4 \end{bmatrix}$$

$$BA = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\therefore AB = BA$$

2. Show that AB ≠ BA if $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3+2 & 2-2 \\ -3+4 & -2-4 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & -6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3-2 & 6+8 \\ 1+1 & 2-4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 14 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 14 \\ 2 & -2 \end{bmatrix}$$

$$\therefore AB \neq BA$$

3. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ and if $f(x) = 3x^2 - 2x + 4$ find f(A).

Solution:

If $f(x) = 3x^2 - 2x + 4$

$f(A) = 3A^2 - 2A + 4I$ (I unit matrix of order 2 × 2)

$$A^2 = A \times A$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6 & 2+2 \\ 3+3 & 6+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix}$$

$\therefore$ f(A) = 3A^2 – 2A + 4I

$$= 3\begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 12 \\ 18 & 21 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ -6 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 21-2+4 & 12-4+0 \\ 18-6+0 & 21-2+4 \end{bmatrix} = \begin{bmatrix} 23 & 8 \\ 12 & 23 \end{bmatrix}$$

4. Find the inverse of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$.

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{vmatrix}$$

= 1 (3 –0) – 1 (6 – 0) – 1 (4 + 1) = 1 (3) – 1 (6) – 1 (5)

= 3 – 6 – 5 = – 8 ≠ 0 A^{-1} exists

Co-factors

$$A_{11} = +\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3 - 0 = 3$$

$$A_{12} = -\begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} = -(6-0) = -6$$

$$A_{13} = +\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = (4+1) = 5$$

$$A_{21} = -\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = -(3+2) = -5$$

$$A_{22} = +\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = (3-1) = 2$$

$$A_{23} = -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -(2+1) = -3$$

$$A_{31} = +\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = (0+1) = 1$$

$$A_{32} = -\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -(0+2) = -2$$

$$A_{32} = +\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (1-2) = -1$$

$$\text{Co-factor matrix } A = \begin{bmatrix} 3 & -6 & 5 \\ -5 & 2 & -3 \\ 1 & -2 & -1 \end{bmatrix}$$

$$\text{Adj . A} = \begin{bmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj A}}{|A|} = \frac{-1}{8}\begin{bmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{bmatrix}$$

5. If $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ show that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

$$AB = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -2 & 7 \\ 1 & -2 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} -2 & 7 \\ 1 & -2 \end{vmatrix} = -3 \neq 0$$

$(AB)^{-1}$ exists

$$\text{Adj (AB)} = \begin{pmatrix} -2 & -7 \\ -1 & -2 \end{pmatrix}$$

$$(AB)^{-1} = \frac{1}{-3}\begin{pmatrix} -2 & -7 \\ -1 & -2 \end{pmatrix}$$

$$(AB)^{-1} = \frac{-1}{3}\begin{pmatrix} -2 & -7 \\ -1 & -2 \end{pmatrix} \qquad \text{.............(i)}$$

$$B = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \quad |B| = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$$

$$\text{Adj B} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$B^{-1} = \frac{1}{-1}\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \quad |A| = \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 3 \neq 0$$

$$\text{Adj A} = \begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1}A^{-1} = -\frac{1}{1}\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1}A^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1}A^{-1} = -\frac{1}{3}\begin{bmatrix} -2 & -7 \\ -1 & -2 \end{bmatrix} \qquad \text{............(ii)}$$

From (i) and (ii) $(AB)^{-1} = B^{-1}A^{-1}$

6. Find the rank of the matrix $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$.

Solution:

Here the rank of the matrix can not exceed 3, the minimum of 3 and 4.

The value of the each of the determinants.

$$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 2 & 0 \\ -4 & 5 & 12 \end{vmatrix} = 2(24-0)-3(60-0)+4(25+8) = 48-180+132 = 0$$

$$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & -1 \\ 5 & 12 & -1 \end{vmatrix} = 0$$

Smilarly we can prove $\begin{vmatrix} 2 & 4 & -1 \\ 5 & 0 & -1 \\ -4 & 12 & -1 \end{vmatrix} = 0$ and $\begin{vmatrix} 2 & 3 & -1 \\ 5 & 2 & -1 \\ -4 & 5 & -1 \end{vmatrix} = 0$

While the second order determinant

$$\begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} = 4-15 = -11 \neq 0$$

Hence the rank of the matrix = 2

1.3 BINOMIAL THEOREM

Definition of Factorial Notation:

The Continued Product of first 'n' natural numbers is called n factorial and is denoted by n! or $\underline{|n}$.

i.e $\underline{|n} = 1 \times 2 \times 3 \times ....(n-1) \times n$

$\underline{|5} = 1 \times 2 \times 3 \times 4 \times 5$

$\underline{|5} = 120$

Zero factorial:

We will require zero factorial for calculating any value which contains zero factorial. It does not make any sense to define it as the product of the integers from 1 to zero. So we define $\underline{|0}$ = 1.

Definition of Permutation and Combination:

First we begin with the following problem. We want to select 2 Carrom players from among 5 players. Let us denote them by the letters A, B, C, D and E.

To select 2 players, we shall first take A and then with him, associate B, C, D and E. That is AB, AC, AD and AE are four types of selection of 2 players. Also starting with B, we have BC, BD and BE, with C, CD and CE, and finally starting with D, we have DE only.

So, totally there are 10 ways. If we denote the number of ways of selection of 2 players out of 5 players, symbolically denoted by $5C_2 = 10$.

Also, let us assume that the selected 2 players want to give pose for a photo. Two chairs are brought as shown below, there are two types of arrangements.

Arrangement –1		Arrangement –2	
A	B	B	A
Chair –1	Chair –2	Chair –1	Chair –2

Thus we see that for one selection, there are two different arrangements, and so for the total of 10 selection, the total number of arrangements is 20.

That is if the number of ways of arrangement of 2 players out of 5 players is denoted by $5P_2$. We have $5P_2 = 20$.

The above "arrangement" and "selection" are usually called "permutation" and "combination".

Where nP_r is the number of ways of arrangement (or permutation) of "r" things out of 'n' things. Also nC_r is the number of ways of selection (or combination) of "r" things out of "n" things.

Note: The value of nP_r and nC_r are given below.

$$nP_r = \frac{\lfloor n}{\lfloor n-r} \quad \text{(or)} \quad \frac{n!}{(n-r)!}$$

$$nC_r = \frac{n!}{(n-r)!\,r!}$$

BINOMIAL THEOREM

Binomial means an expression, consist of two numbers (or) terms connected by plus sign or minus sign.

Example:

$$x + y \;;\; 2x - y \;;\; x^3 - \frac{1}{x} \text{ etc.}$$

In binomial theorem, we deal with the powers of binomial expressions. From School Studies, we know that

$(x + a)^2 = x^2 + 2xa + a^2$ and

$(x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$

we can write the expansion of $(x + a)^3$ as below

$(x + a)^3 = 3C_0\, x^3a^0 + 3C_1\, x^2a^1 + 3C_2\, x^1a^2 + 3C_3\, x^0a^3$.

Where $3C_0 = 3C_3 = 1$ $\quad nC_r = n\,C_{n-r}$

Similarly for $(x + a)^4$, $(x + a)^5$ can be expanded as mentioned above.

Binomial theorem for a positive integral index:

If "n" is any positive integer, then $(x + a)^n = x^n + nC_1\, x^{n-1}a + nC_2\, x^{n-2}\, a^2 + ... + nC_r\, x^{n-r}\, a^r + ... + a^n$.

Note:

1. The total number of terms in the expansion is (n + 1).
2. In each term, sum of the powers of x and a is equal to n.
3. The general term of $(x + a)^n$ is $T_{r+1} = nC_r\, x^{n-r}\, a^r$ $\quad$ i.e $(r + 1)^{th}$ term
4. If the power 'n' is an even integer, there is only one middle term which is at $\left(\frac{n}{2}+1\right)^{th}$ place and if 'n' is odd integer there are two middle terms, which are at $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ places.

Binomial Theorem for rational index:

If x is numerically less than one and n any rational number, then

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1\times 2}x^2+\frac{n(n-1)(n-2)}{1\times 2\times 3}x^3+\ldots\ldots\ldots.$$

Note:

1. The number of terms in the expansion is infinite.
2. Here the notations nC_0, nC_1, nC_2 ...... etc are meaningless. Since n is a rational number.
3. Also here we consider, is expansions for negative integers upto – 3. When the values of 'n' are –1, – 2, –3 the expansions are
   (i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots.$
   (ii) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots.$
   (iii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots.$
   (iv) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots.$
   (v) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots.$
   (vi) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots\ldots\ldots.$

WORKED EXAMPLES

PART –A

1. Write down the number of terms in $\left(x^2+\frac{2}{x}\right)^{11}$.

Solution:

The number of terms is n + 1 = 11 + 1 = 12

2. Which term in $(3x-y)^6$ is the middle term.

Solution:

Here n = 6 (even)

$$\text{Middle term} = \left(\frac{n}{2}+1\right)$$

$$= \left(\frac{6}{2}+1\right) = 3+1$$

$$= 4^{th} \text{ term}$$

∴ 4th term is the middle term.

3. Write down the expansion for $(1-x)^{-1}$.

Solution:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots..$$

4. Expand by using Binomial Theorem $(3x-2y)^3$.

Solution:

$$(3x-2y)^3 = (3x)^3 + 3C_1\,(3x)^2\,(-2y)^1 + 3C_2(3x)^1\,(-2y)^2 + (-2y)^3$$

5. Which are the two middle terms in $\left(2x^2 + \frac{3}{x}\right)^{13}$.

Solution:

Here n = 13 (odd)

Middle terms are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$.

i.e $\frac{13+1}{2} = \frac{14}{2} = 7$ and

$\frac{13+3}{2} = \frac{16}{2} = 8$

$\therefore$ 7^{th} and 8^{th} terms are two middle terms.

PART – B

1. Find the general term in the expansion of $\left(x + \frac{1}{2x}\right)^7$.

Solution:

Let $\left(x + \frac{1}{2x}\right)^7 = (X + A)^n$

Where X = x ; n = 7, $A = \frac{1}{2x}$

general term $T_{r+1} = nC_r X^{n-r} A^r$

i.e $T_{r+1} = 7C_r\, x^{7-r} \left(\frac{1}{2x}\right)^r = 7C_r x^{7-r} \frac{1}{2^r x^r} = 7C_r \frac{x^{7-2r}}{2^r}$

2. Find the 5th term in the expansion of $\left(x + \frac{1}{x}\right)^8$.

Solution:

The general term $T_{r+1} = nC_r\, x^{n-r} a^r$

Put r = 4, we get 5th term T_5. Also $x = x$, a = $\frac{1}{x}$, n = 8.

$T_5 = 8C_4 x^{8-4} \left(\frac{1}{x}\right)^4$

$= 8C_4 x^4 \left(\frac{1}{x}\right)^4$

$= 8C_4 x^4 \frac{1}{x^4}$

$T_5 = 8C_4$

3. Expand in higher powers of x upto three terms $(1 + 3x)^{-2}$.

Solution:

$(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ...$

where $x = 3x$ only

$\therefore (1 + 3x)^{-2} = 1 - 2(3x) + 3\,(3x)^2 - 4(3x)^3$

$= 1 - 6x + 27x^2 - 108x^3 + ...$

PART – C

1. Find the middle terms in the expansion of $\left(2x^2+\frac{1}{x}\right)^{11}$.

Solution:

Since here 'n' is odd number, the total number of terms is even and so there are two middle terms.

$\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms are middle terms.

$\left(\frac{11+1}{2}\right)^{th}$ and $\left(\frac{11+3}{2}\right)^{th}$ i.e 6^{th} and 7^{th} terms.

Put r = 5 and 6 in the general term.

Now, general term is $T_{r+1} = nC_r\, x^{n-r}\, a^r$

$$T_{5+1} = 11C_5 (2x^2)^{11-5}\left(\frac{1}{x}\right)^5$$

$$T_6 = 11C_5 (2x^2)^6\left(\frac{1}{x}\right)^5 = 11C_5 2^6 x^{12}\frac{1}{x^5} = 11C_5 2^6 x^7$$

Put r = 6

$$T_{6+1} = 11C_6 (2x^2)^{11-6}\left(\frac{1}{x}\right)^6$$

$$T_7 = 11C_6 (2x^2)^5\left(\frac{1}{x}\right)^6 = 11C_6 2^5 x^{10}\frac{1}{x^6} = 11C_6 2^5 x^4$$

2. Find the co-efficient of x^{32} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$.

Solution:

Now, general terms is $T_{r+1} = nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = 15C_r (x^4)^{15-r}\left(\frac{-1}{x^3}\right)^r$$

$$= 15C_r x^{60-4r}(-1)^r x^{-3r}$$

$$T_{r+1} = 15C_r (-1)^r x^{60-7r} \qquad \text{.............(i)}$$

To find the co-efficient of x^{32}, put

$$32 = 60 - 7r$$

$$7r = 60 - 32$$

$$7r = 28$$

$$r = \frac{28}{7}$$

$$r = 4$$

Applying r = 4 in the equation (i), we get

$$T_{4+1} = 15C_4 (-1)^4 x^{60-7(4)}$$

$$T_5 = 15C_4 x^{32}$$

$\therefore$ co-efficient of x^{32} is $15C_4$.

3. Find the term independent of x in the expansion of $\left(x-\frac{1}{x}\right)^{10}$.

Solution:

Now, general term $T_{r+1} = nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = 10C_r x^{10-r}\left(\frac{-1}{x}\right)^r$$

$$= 10C_r x^{10-r}(-1)^r x^{-r}$$

$$T_{r+1} = 10C_r(-1)^r x^{10-2r} \quad \text{...............(i)}$$

To find independent term, put $10 - 2r = 0$

i.e $10 - 2r = 0$

$10 = 2r$

$5 = r, \quad r = 5$

Put $r = 5$ in equation (i) we get

$$T_{5+1} = 10C_5(-1)^5 x^{10-2(5)}$$

$$= 10C_5 (-1)\, x^0$$

$T_6 = -10C_5$ i.e The term independent of x is $-10C_5$.

4. Write the first three terms in the expansion of $(3 - 4x)^{-3}$.

Solution:

$$(3-4x)^{-3} = 3^{-3}\left(1-\frac{4}{3}x\right)^{-3}$$

$$= \frac{1}{3^3}\left[1+(-3)\left(-\frac{4}{3}x\right)+\frac{(-3)(-3-1)}{1.2}\left(\frac{-4}{3}x\right)^2+\text{..........}\right]$$

$$= \frac{1}{27}\left[1+4x+\frac{(-3)(-4)}{2}\left(\frac{16x^2}{9}\right)+\text{.........}\right]$$

$$= \frac{1}{27}\left[1+4x+\frac{32x^2}{3}+\text{.........}\right]$$

EXERCISES

PART – A

1. Solve $\begin{vmatrix} x & 2 \\ x & 3x \end{vmatrix} = 0$.

2. Solve $\begin{vmatrix} x & 8 \\ 2 & x \end{vmatrix} = 0$.

3. Find the value of the determinant $\begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix}$.

4. Find the value of the minor of 3 in the determinant $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 1 \\ 5 & -1 & 0 \end{vmatrix}$.

5. Find the co-factor value of 5 in the determinant $\begin{vmatrix} 1 & 1 & 3 \\ -1 & 0 & 5 \\ 4 & 2 & 0 \end{vmatrix}$.

6. If $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 & 4 \\ 10 & 0 & 4 \end{bmatrix}$ find A – B.

7. Show that the matrix $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 2 & -4 & 6 \end{bmatrix}$ is singular.

8. Find the adjoint of $\begin{bmatrix} 3 & 2 \\ -3 & 4 \end{bmatrix}$.

9. How many terms are there in the expansion of $(2x - y)^{15}$.

10. Which are the middle terms in the expansion of $\left(3x^2 - \frac{1}{x}\right)^{21}$.

11. Expand upto three terms of $(1 + x)^{-3}$.

12. Find the rank of $\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$.

13. Find the rank of $\begin{pmatrix} -5 & -6 \\ 20 & 24 \end{pmatrix}$.

14. Find the rank of $\begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & -8 \end{pmatrix}$.

15. Find the rank of $\begin{pmatrix} 1 & 2 \\ 3 & -8 \\ -4 & 5 \end{pmatrix}$.

16. Find the rank of $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$.

PART – B

1. Find the value of 'x' if $\begin{vmatrix} 1 & -1 & 2 \\ 5 & 3 & x \\ 2 & 1 & 4 \end{vmatrix} = 0$.

2. Solve by using Cramer's rule: $x - y = 1$; $2x - 3y = -1$.

3. If $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ find AB.

4. Find the inverse of $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$.

5. Find x if the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 5 & -1 & -2 \\ 2 & x & 6 \end{bmatrix}$ is singular.

6. Find the general term in the expansion of $\left(x + \frac{2}{x^2}\right)^{10}$.

7. Find the 5th term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{15}$.

8. Write the first 3 terms of $(1 - 2x)^{-2}$.

9. Find the rank of $\begin{pmatrix} 3 & -4 & 1 \\ -6 & 8 & -2 \end{pmatrix}$.

10. Find the rank of $\begin{pmatrix} 1 & -2 \\ -3 & 6 \\ 5 & -10 \end{pmatrix}$.

11. Find the rank of $\begin{pmatrix} 2 & 5 & 7 \\ 7 & 1 & 6 \\ 5 & -4 & -1 \end{pmatrix}$.

12. Find the rank of $\begin{pmatrix} -3 & 2 & 0 \\ 7 & -1 & 11 \\ 5 & 4 & 22 \end{pmatrix}$.

PART –C

1. Solve: $3x - y + 2x = 8$; $x - y + z = 2$ and $2x + y - z = 1$ using Cramer's rule.
2. Solve: $x + 2y - z = -3$; $3x + y + z = 4$ and $x - y + 2z = 6$, using Cramer's rule.
3. Find the inverse of $\begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & -3 \\ 6 & -2 & -1 \end{pmatrix}$.
4. If $A = \begin{pmatrix} 3 & 6 \\ 1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ 2 & 3 \end{pmatrix}$ verify $(AB)^{-1} = B^{-1}A^{-1}$.

5. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ find AB and BA.

6. If $A = \begin{pmatrix} 3 & -5 \\ -4 & 2 \end{pmatrix}$ show that $A^2 - 5A - 14I = 0$.

7. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$.

8. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$.

9. Find the middle terms in the expansion of $\left(x^3 + \frac{2}{x^3}\right)^{11}$.

10. Find the co-efficient of $\left(x^{-5}\right)$ in the expansion of $\left(x - \frac{3}{5x^2}\right)^7$.

11. Find the term independent of x in the expansion of $\left(x + \frac{1}{x}\right)^{10}$.

12. Find the term independent of x in the expansion of $\left(3\sqrt{x} - \frac{2}{x^2}\right)^{10}$.

ANSWERS

PART – A

1) $x = 0, \frac{2}{3}$ 2) x = + 4, – 4 3) one 4) – 1 5) + 2 6) $\begin{bmatrix} 0 & -1 & -5 \\ -10 & 2 & -3 \end{bmatrix}$

8) $\begin{bmatrix} 4 & -2 \\ 3 & 3 \end{bmatrix}$ 9) 16 10) 11th and 12th 11) $1 - 3x + 6x^2$ 12) 1

13) 1 14) 2 15) 2 16) 3

PART – B

1) $x = 10$ 2) $x = 4$; y = 3 3) $\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ 4) $\frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$ 5) $x = \frac{68}{17}$

6) $10C_r\, x^{10-3r}\, 2^r$ 7) $15C_4\, x^{30-3r}$ 8) $1 + 4x + 12x^2$

9) 1 10) 1 11) 2 12) 2

PART – C

1) (1, 3, 4)

2) (1, – 1, 2)

3) $-\frac{1}{15}\begin{bmatrix} -9 & -3 & 0 \\ -16 & -7 & 5 \\ -22 & -4 & 5 \end{bmatrix}$

5) $AB = \begin{bmatrix} 5 & 6 & 13 \\ 1 & 6 & 8 \\ -1 & 0 & -2 \end{bmatrix}$; $BA = \begin{bmatrix} 2 & -1 & -1 \\ 5 & -1 & 2 \\ 8 & 2 & 8 \end{bmatrix}$

7) rank = 2

8) rank = 3

9) $11C_5\ 32x^3$, $11C_6\ \frac{64}{x^3}$

10) $7C_5\left(\frac{3}{5}\right)^4$

11) $10C_5$

12) $10C_2\ 2^2\ 3^8$

UNIT – II COMPLEX NUMBERS

2.1 ALGEBRA OF COMPLEX NUMBERS:

Definition – Real and Imaginary parts, Conjugates, Modulus and amplitude form, Polar form of a complex number, multiplication and division of complex numbers (geometrical proof not needed) – Simple Problems. Argan Diagram – Collinear points, four points forming square, rectangle, rhombus and parallelogram only. Simple problems.

2.2 DE MOIVRE'S THEOREM

Demoivre's Theorem (Statement only) – related simple problems.

2.3 ROOTS OF COMPLEX NUMBERS

Finding the nth roots of unity - solving equation of the form $x^n \pm 1 = 0$ where $n \leq 7$. Simple problems.

2.1 ALGEBRA OF COMPLEX NUMBERS

Introduction:

Let us consider the quadratic equation $ax^2 + bx + c = 0$. The solution of this equation is given by the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which is meaningful only when $b^2 - 4ac > 0$. Because the square of a real number is always positive and it cannot be negative. If it is negative, then the solution for the equation extends the real number system to a new kind of number system that allows the square root of negative numbers. The square root of – 1 is denoted by the symbol i, called the imaginary unit, which was first introduced in mathematics by the famous Swiss mathematician, Leonhard Euler in 1748. Thus for any two real numbers a and b, we can form a new number a + ib is called a **complex number**. The set of all complex numbers denoted by C and the nomenclature of a complex number was introduced by a German mathematician C.F. Gauss.

Definition: Complex Number

A number which is of the form a + ib where $a, b \in R$ and $i^2 = -1$ is called a complex number and it is denoted by z. If z = a + ib then *a* is called the real part of z and *b* is called the imaginary part of z and are denoted by Re(z) and Im(z).

For example, if z = 3 + 4i then Re (z) = 3 and Im (z) = 4.

Note:

In the complex number z = a + ib we have,

(i) If a = 0 then z is purely imaginary

(ii) If b = 0 then z is purely real.

(iii) z = a + ib = (a, b) any complex number can be expressed as an ordered pair.

Conjugate of a complex number:

If z = a + ib then the conjugate of z is defined by a – ib and it is denoted by $\bar{z}$. Thus, if z = a + ib then $\bar{z} = a - ib$.

Results:

(i) $\overline{\overline{z}} = z$

(ii) $a = \text{Re}(z) = \dfrac{z+\overline{z}}{2}$ & $b = I_m(z) = \dfrac{z-\overline{z}}{2}$

(iii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(iv) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(v) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

(vi) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ where $z_2 \neq 0$

(vii) $\overline{z^n} = (\overline{z})^n$

Algebra of complex numbers:

(i) Addition of two complex numbers:

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers then their sum is defined as

$z_1 + z_2 = a + ib + c + id = (a + c) + i\,(b + d) \in C$

$z + \overline{z} = 2a$ Real number.

(ii) Difference of two complex numbers:

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers then their difference is defined as

$z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i\,(b - d) \in C$

$z - \overline{z} = 2ib$ Imaginary number.

(iii) Multiplication of two compelx numbers:

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers then their product is defined as,

$$\begin{aligned} z_1\,z_2 &= (a + ib)\,(c + id) \\ &= a\,c + i\,ad + i\,bc + i^2\,bd \\ &= (ac - bd) + i\,(ad + bc) \in C \end{aligned}$$

$z\,\overline{z} = (a + ib)\,(a - ib) = a^2 + b^2$

(iv) Division of two complex numbers:

Let $z_1 = a + ib$ and $z_2 = c + id \neq 0$ be any two complex numbers then their quotient is defined as

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \left[\frac{ac+bd}{c^2+d^2}\right] + i\left[\frac{bc-ad}{c^2+d^2}\right]$$

Modulus of a complex number:

If $z = a + ib$ is a complex number then the modulus (or) absolute value of z is defined as $\sqrt{a^2 + b^2}$ and is denoted by $|z|$. Thus, if $z = a + ib$ then $|z| = \sqrt{a^2 + b^2}$.

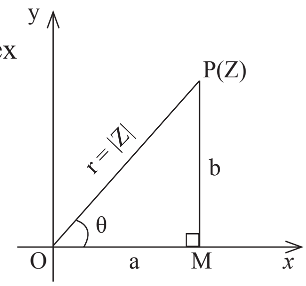
Note:

(i) $|\overline{z}| = |z| = \sqrt{a^2 + b^2}$

(ii) $|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$

(iii) $\text{Re}(z) \leq |z|$ and $\text{Im}(z) \leq |z|$

Polar form of a Complex Number:



Let (r, θ) be the Polar co-ordinates of the point P representing the complex number $z = a + ib$. Then from the fig. we get,

$$\cos\theta = \frac{OM}{OP} = \frac{a}{r} \text{ and } \sin\theta = \frac{PM}{OP} = \frac{b}{r}$$

$\Rightarrow a = r\cos\theta$ and $b = r\sin\theta$

where $r = \sqrt{a^2 + b^2} = |a + ib|$ is called the **modulus** of $z = a + ib$.

Also, $\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$ is called the **amplitude** or **argument** of $z = a + ib$ and denoted by amp(z) or arg(z) and is measured as the angle in positive sense. Thus, $\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$.

Hence $z = a + ib = r(\cos\theta + i\sin\theta)$ is called the Polar form or the modulus amplitude form of the complex number.

Theorems of Complex numbers:

1) The product of two complex numbers is a complex number whose modulus is the product of their modulii and whose amplitude is the sum of their amplitudes

   i.e, $|z_1 z_2| = |z_1||z_2|$

   and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

2) The quotient of two complex numbers is a complex number whose modulus is the quotient of their modulii and whose amplitude is the difference of their amplitudes.

   i.e $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ wehre $Z_2 \neq 0$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

Euler's formula:

The symbol $e^{i\theta}$ is defined by $e^{i\theta} = \cos\theta + i\sin\theta$ is known as Euler's formula.

If $z \neq 0$ then $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. This is called the exponential form of the complex number z.

Note: If $z = re^{i\theta}$ then $\bar{z} = re^{-i\theta}$.

Multiplication and Division of complex numbers (Geometrical proof not needed)

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

be any two complex numbers in Polar form then their product is given by

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Also the division of the above two complex numbers is given by

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \text{ where } z_2 \neq 0.$$

WORKED EXAMPLES

PART – A

1. If $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$ find $z_1 + z_2$.

Solution:

Given: $z_1 = 2 + 3i$ & $z_2 = 4 - 5i$

$$\begin{aligned} z_1 + z_2 &= (2 + 3i) + (4 - 5i) \\ &= 2 + 3i + 4 - 5i \\ &= (2 + 4) + (3i - 5i) \end{aligned}$$

$$\Rightarrow \boxed{z_1 + z_2 = 6 - 2i}$$

2. If $z_1 = 3 - 4i$ and $z_2 = -2 + 3i$ find the value of $2z_1 - 3z_2$.

Solution:

Given: $z_1 = 3 - 4i$ & $z_2 = -2 + 3i$

$$\begin{aligned} 2z_1 - 3z_2 &= 2(3 - 4i) - 3(-2 + 3i) \\ &= 6 - 8i + 6 - 9i \end{aligned}$$

$$\Rightarrow \boxed{2z_1 - 3z_2 = 12 - 17i}$$

3. Express: $(3 + 2i)(4 + 2i)$ in $a + ib$ form.

Solution:

$$\begin{aligned} (3 + 2i)(4 + 2i) &= 12 + 6i + 8i + 4i^2 \\ &= 12 + 14i - 4 \\ &= 8 + 14i = a + ib \text{ form.} \end{aligned}$$

4. Find the real and imaginary parts of $\dfrac{1}{3+2i}$.

Solution:

$$\text{Let } z = \frac{1}{3+2i} = \frac{1}{3+2i} \times \frac{3-2i}{3-2i}$$

$$= \frac{3-2i}{(3)^2 - (2i)^2}$$

$$= \frac{3-2i}{9+4}$$

$$= \frac{3-2i}{13}$$

$$\Rightarrow \boxed{z = \frac{3}{13} - \frac{2i}{13}}$$

$$\therefore \text{Re}(z) = \frac{3}{13} \ \& \ \text{Im}(z) = \frac{-2}{13}$$

5. Find the conjugate of $\frac{1}{1+i}$.

Solution:

$$\text{Let } z = \frac{1}{1+i} = \frac{1}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{1-i}{(1)^2 - (i)^2}$$

$$= \frac{1-i}{1+1}$$

$$= \frac{1-i}{2}$$

$$\Rightarrow \boxed{z = \frac{1}{2} - \frac{i}{2}}$$

$$\therefore \text{Conjugate}: \bar{z} = \frac{1}{2} + \frac{i}{2}$$

6. Find the modulus and amplitude of 1 + i.

Solution:

Let z = 1 + i

Here a = 1 & b = 1

$$\text{Modulus}: |z| = \sqrt{a^2 + b^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\text{and amp}(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1)$$

$$\Rightarrow \theta = 45^\circ$$

PART –B

1. Find the real and imaginary parts of $\frac{4+5i}{3-2i}$.

Solution:

$$\text{Let } z = \frac{4+5i}{3-2i} = \frac{4+5i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{12+8i+15i+10i^2}{(3)^2 - (2i)^2}$$

$$= \frac{12+23i-10}{9+4}$$

$$= \frac{2+23i}{13}$$

$$z = \frac{2}{13} + \frac{23i}{13}$$

$$\therefore \text{Re}(z) = \frac{2}{13} \quad \& \quad \text{Im}(z) = \frac{23}{13}$$

2. Express the complex number $\frac{1}{3-2i}+\frac{1}{2-3i}$ in a + ib form.

Solution:

$$\text{Let } z = \frac{1}{3-2i}+\frac{1}{2-3i}$$

$$= \frac{1}{3-2i}\times\frac{3+2i}{3+2i}+\frac{1}{2-3i}+\frac{2+3i}{2+3i}$$

$$= \frac{3+2i}{3^2+2^2}+\frac{2+3i}{2^2+3^2}$$

$$= \frac{3+2i+2+3i}{13}$$

$$= \frac{5+5i}{13}$$

$$z = \frac{5}{13}+\frac{5}{13}i = a+ib \text{ form}$$

3. Find the modulus and argument of the complex number $\frac{1-i}{1+i}$.

Solution:

$$\text{Let } z = \frac{1-i}{1+i} = \frac{1-i}{1+i}\times\frac{1-i}{1-i}$$

$$= \frac{1-i-i+i^2}{(1)^2-(i)^2}$$

$$= \frac{1-2i-1}{1+1}$$

$$= \frac{-2i}{2}$$

$$z = -i \text{ where a = 0 \& b = } -1$$

$$\text{Modulus}: |z| = \sqrt{a^2+b^2} = \sqrt{(0)^2+(-1)^2} = \sqrt{1} = 1$$

$$\text{Argument}: \tan\theta = \frac{b}{a} = \frac{-1}{0} = \infty$$

$$\theta = \tan^{-1}(\infty) = 90^o$$

The complex number – i = (0, – 1) lies IIIrd Quadrant.

Hence amplitude = $180^o + 90^o = 270^o$.

PART – C

1. Find the real and imaginary parts of the complex number $\frac{(1+i)\,(2-i)}{1+3i}$.

Solution:

$$\text{Let } z = \frac{(1+i)\,(2-i)}{1+3i}$$

$$= \frac{2-i+2i-i^2}{1+3i}$$

$$= \frac{2+i+1}{1+3i}$$

$$= \frac{3+i}{1+3i} \times \frac{1-3i}{1-3i}$$

$$= \frac{3-9i+i-3i^2}{(1)^2-(3i)^2}$$

$$= \frac{3-8i+3}{1+9}$$

$$= \frac{6-8i}{10}$$

$$= \frac{3-4i}{5}$$

$$z = \frac{3}{5} - \frac{4i}{5} = a + ib \text{ form.}$$

$$\therefore \text{Re}(z) = \frac{3}{5} \text{ \& Im}(z) = -\frac{4}{5}$$

2. Express the complex number $\frac{i-4}{3-2i} + \frac{4i+1}{2-3i}$ in a + ib form.

Solution:

$$\text{Let } z = \frac{i-4}{3-2i} + \frac{4i+1}{2-3i}$$

$$= \frac{i-4}{3-2i} \times \frac{3+2i}{3+2i} + \frac{4i+1}{2-3i} \times \frac{2+3i}{2+3i}$$

$$= \frac{3i-12+2i^2-8i}{3^2+2^2} + \frac{8i+2+12i^2+3i}{2^2+3^2}$$

$$= \frac{-5i-14}{13} + \frac{11i-10}{13}$$

$$= \frac{6i-24}{13}$$

$$= \frac{-24+6i}{13} = \frac{-24}{13} + \frac{6i}{13} = a + ib \text{ form}$$

3. Find the modulus and amplitude of $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$.

Solution:

$$\text{Let } z = \frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$$

$$= \frac{1+3\sqrt{3}i}{\sqrt{3}+2i} \times \frac{\sqrt{3}-2i}{\sqrt{3}-2i}$$

$$= \frac{\sqrt{3}-2i+9i-6\sqrt{3}i^2}{(\sqrt{3})^2-(2i)^2}$$

$$= \frac{\sqrt{3}+7i+6\sqrt{3}}{3+4}$$

$$= \frac{7\sqrt{3}+7i}{7}$$

$$= \frac{7(\sqrt{3}+i)}{7}$$

$$z = \sqrt{3} + i = a + ib \text{ form}$$

Here $a = \sqrt{3}$ & $b = 1$

$\therefore$ Modulus : $|z| = \sqrt{a^2 + b^2} = \sqrt{(3)^2 + (1)^2} = \sqrt{3} + \sqrt{1} = \quad 4 = 2$

Amplitude : $\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^o$

4. Find the modulus and argument of the complex number $\frac{5-i}{2-3i}$.

Solution:

Let $z = \frac{5-i}{2-3i}$

$= \frac{5-i}{2-3i} \times \frac{2+3i}{2+3i}$

$= \frac{10+15i-2i-3i^2}{(2)^2-(3i)^2}$

$= \frac{10+13i+3}{4+9}$

$= \frac{13+13i}{13}$

$= \frac{13(1+i)}{13}$

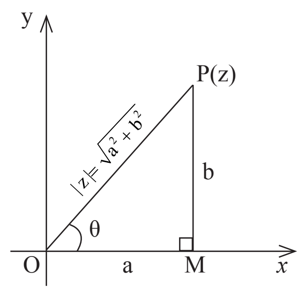
$z = 1 + i = a + ib$ form

Here $a = 1$ & $b = 1$

Modulus : $|z| = \sqrt{a^2 + b^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$

Amplitude : $\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1)$

$\Rightarrow \theta = 45^o$



Argand Diagram

Every complex number a + ib can be considered as an ordered pair (a, b) of real numbers, we can represent such number by a point in xy-plane called the complex plane and such a representation is also known as the argand diagram. The complex number $z = a + ib$ represented by P(z) then the distance between z and the origin is the modulus. i.e $|z| = \sqrt{a^2 + b^2}$

Here the set of real numbers $(x, 0)$ corresponds to the x-axis called real axis and the set of Imaginary numbers (0, y) correponds to the y-axis called the imaginary axis.

Result:

The distance between the two complex numbers z_1 and z_2 is $|z_1 - z_2|$. Thus, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

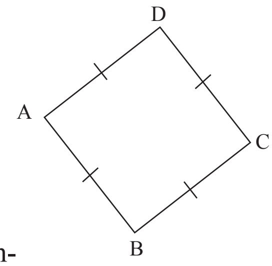
Collinear Points:

If A, B and C are any three points representing the complex numbers $x_1 + iy_1$, $x_2 + iy_2$ and $x_3 + iy_3$ respectively, are collinear then the required condition is, the area of ΔABC is zero.

i.e. $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$

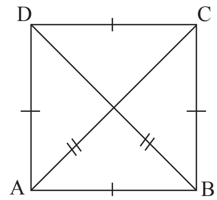
i.e $x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) = 0$

$$\text{i.e.} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$



Condition for square:

If A, B, C and D are any four complex numbers representing the vertices of a square then the required condition is



(i) AB = BC = CD = DA

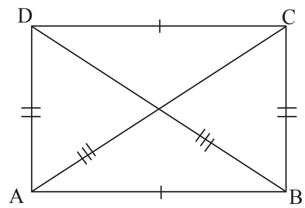
i.e four sides are equal.

(ii) AC = BD

i.e the diagonals are equal

Condition for rectangle

If A, B, C and D are any four complex numbers represents the vertices of a rectangle then the required condition is,



(i) AB = CD and BC = DA.

i.e Opposite sides are equal.

(ii) AC = BD

i.e the diagonals are equal.

Condition for rhombus

If A, B, C and D are any four complex numbers represents the vertices of a rhombus then the required condition is

AB = BC = CD = DA

i.e four sides are equal.

Condition for Parallelogram

If A, B, C and D are any four complex numbers represents the vertices of a parallelogram then the required condition is either,

mid-point of the diagonal AC = mid.point of the diagonal BD.

(or) The length of the opposite sides are equal. i.e AB = CD & BC = DA.

Note:

(i) To find length of the sides and diagonals of square, rectangle, rhombus and parallelogram apply the distance formula.

i.e distance: $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ etc.

(ii) To find the mid.point of the diagonals of the parallelogram apply middle point formula,

$$(x, y) = \left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right]$$

WORKED EXAMPLES

PART – A

1. Find the distance between the complex numbers 2 + i and 1 –2i.

Solution:

Let $z_1 = 2 + i$ & $z_2 = 1 - 2i$

$$\therefore \text{Distance}: z_1z_2 = \sqrt{(2-1)^2 + (1+2)^2}$$

$$= \sqrt{(1)^2 + (3)^2}$$

$$= \sqrt{1+9}$$

$$\Rightarrow \boxed{z_1z_2 = \sqrt{10}}$$

PART – B

1. Show that the complex numbers 1 + 2i, 3 – 2i and 6 – 8i are collinear.

Solution:

Given complex numbers are

A : 1 + 2i = (1, 2)

B : 3 – 2i = (3, – 2)

& C : 6 – 8i = (6, – 8)

Condition for collinear is $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

$$\text{LHS}: \begin{vmatrix} 1 & 2 & 1 \\ 3 & -2 & 1 \\ 6 & -8 & 1 \end{vmatrix} = 1\begin{vmatrix} -2 & 1 \\ -8 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 6 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 6 & -8 \end{vmatrix}$$

$$= 1(-2+8) - 2(3-6) + 1(-24+12)$$

$$= 1(6) - 2(-3) + 1(-12)$$

$$= 6 + 6 - 12$$

$$= 12 - 12$$

$$= 0$$

∴ The given complex numebrs are collinear.

2. Show that the complex numbers – 1 + i, 3 + 2i, 2 + 2i and – 2 + i form a parallelogram.

Solution:

Let the given complex numbers are

A : – 1 + i = (– 1, 1)

B : 3 + 2i = (3, 2)

C : 2 + 2i = (2, 2)

D : – 2 + i = (– 2, 1)

$$\text{Mid-point of AC} = \left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right] = \left[\frac{-1+2}{2}, \frac{1+2}{2}\right] = \left[\frac{1}{2}, \frac{3}{2}\right] \text{............(1)}$$

$$\text{Mid-point of BD} = \left[\frac{3-2}{2}, \frac{2+1}{2}\right] = \left[\frac{1}{2}, \frac{3}{2}\right] \text{............(2)}$$

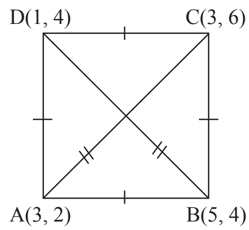
From (1) & (2), Mid-point of AC = Mid-point of BD.

⇒ Given complex numbers form a parallelogram.

PART – C

1. Prove that the points representing the complex numbers 3 + 2i, 5 + 4i, 3 + 6i and 1 + 4i form a square.

Solution:



Let the given complex number be

A : 3 + 2i = (3, 2)

B : 5 + 4i = (5, 4)

C : 3 + 6i = (3, 6)

& D : 1 + 4i = (1, 4)

Now,

$$AB = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(3-5)^2+(2-4)^2} = \sqrt{(-2)^2+(-2)^2} = \sqrt{4+4} = \sqrt{8}$$

$$BC = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(5-3)^2+(4-6)^2} = \sqrt{(2)^2+(-2)^2} = \sqrt{4+4} = \sqrt{8}$$

$$CD = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(3-1)^2+(6-4)^2} = \sqrt{(2)^2+(2)^2} = \sqrt{4+4} = \sqrt{8}$$

and $$DA = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(1-3)^2+(4-2)^2} = \sqrt{(-2)^2+(2)^2} = \sqrt{4+4} = \sqrt{8}$$

$$\text{Also, } AC = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(3-3)^2+(2-6)^2} = \sqrt{(0)^2+(-4)^2} = \sqrt{16} = 4$$

$$BD = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(5-1)^2+(4-4)^2} = \sqrt{(4)^2+(0)^2} = \sqrt{16} = 4$$

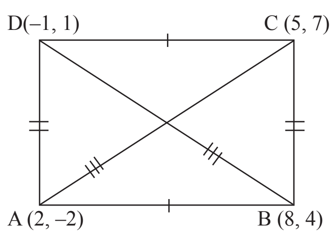
Here the sides AB = BC = CD = DA = $\sqrt{8}$

& diagonals AC = BD = 4

∴ The given complex numbers form a square.

2. Prove that the points representing the complex numbers 2 –2i, 8 + 4i, 5 + 7i and – 1 + i form a rectangle.

Solution:



Let the given complex numbers be,

A : 2 –2i = (2, – 2)

B : 8 + 4i = (8, 4)

C : 5 + 7i = (5, 7)

& D : – 1 + i = (– 1, 1)

Now, $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(2-8)^2 + (-2-4)^2} = \sqrt{(-6)^2 + (-6)^2} = \sqrt{36+36} = \sqrt{72}$

Now, $BC = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(8-5)^2 + (4-7)^2} = \sqrt{(3)^2 + (-3)^2} = \sqrt{9+9} = \sqrt{18}$

$CD = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(5+1)^2 + (7-1)^2} = \sqrt{(6)^2 + (6)^2} = \sqrt{36+36} = \sqrt{72}$

& $DA = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(-1-2)^2 + (1+2)^2} = \sqrt{(-3)^2 + (3)^2} = \sqrt{9+9} = \sqrt{18}$

Also, $AC = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(2-5)^2 + (-2-7)^2} = \sqrt{(-3)^2 + (-9)^2} = \sqrt{9+81} = \sqrt{90}$

$BD = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(8+1)^2 + (4-1)^2} = \sqrt{(9)^2 + (3)^2} = \sqrt{81+9} = \sqrt{90}$

Here the sides $AB = CD = \sqrt{72}$

$BC = DA = \sqrt{18}$

and diagonals $AC = BD = \sqrt{90}$

∴ The given complex numbers form a rectangle.

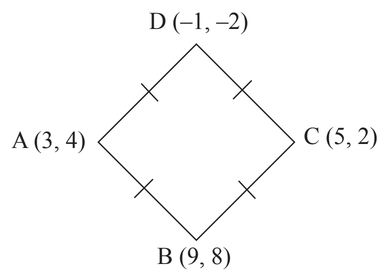
3. Prove that the points 3 + 4i, 9 + 8i, 5 + 2i and – 1 – 2i form a rhombus in the argand diagram.

Solution:

Let the given complex numbers be,

A : 3 + 4i = (3, 4) B : 9 + 8i = (9, 8)

C : 5 + 2i = (5, 2) & D : – 1 – 2i = (– 1, – 2)



$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(3-9)^2 + (4-8)^2} = \sqrt{(-6)^2 + (-4)^2} = \sqrt{36+16} = \sqrt{52}$

$BC = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(9-5)^2 + (8-2)^2} = \sqrt{(4)^2 + (6)^2} = \sqrt{16+36} = \sqrt{52}$

$CD = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(5+1)^2 + (2+2)^2} = \sqrt{(6)^2 + (4)^2} = \sqrt{36+16} = \sqrt{52}$

& $DA = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(-1-3)^2 + (-2-4)^2} = \sqrt{(-4)^2 + (-6)^2} = \sqrt{16+36} = \sqrt{52}$

Here the sides, $AB = BC = CD = DA = \sqrt{52}$.

∴ The given complex number form a rhombus.

2.2 DE-MOIVRE'S THEOREM

DeMoivre's Thoerem: (Statement only)

(i) If 'n' is an integer positive or negative then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

(ii) If 'n' is a fraction, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Results:

1) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

2) $(\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - i\sin n\theta)$

3) $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

4) $\dfrac{1}{\cos\theta - i\sin\theta} = (\cos\theta - i\sin\theta)^{-1} = \cos\theta + i\sin\theta$

5) $\sin\theta + i\cos\theta = \cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right)$

Note:

1) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

2) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3) = \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$

3) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots (\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots\ldots + \theta_n)$

WORKED EXAMPLES

PART – A

1. If $z = \cos 30^o + i\sin 30^o$ what is the value of z^3.

Solution:

$z^3 = [\cos 30^o + i\sin 30^o]^3$

$= \cos 90^o + i\sin 90^o$

$= 0 + i(1) = i$

2. If $z = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$ what is the value of z^8.

Solution:

$$z^8 = \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^8$$

$$= \cos 8\left(\frac{\pi}{2}\right) + i\sin 8\left(\frac{\pi}{2}\right)$$

$$= \cos 4\pi + i\sin 4\pi$$

$$= 1 + i(0) = 1$$

3. If $z = \cos 45^o - i\sin 45^o$ what is the value of $\dfrac{1}{z}$.

Solution:

$$\frac{1}{z} = z^{-1}$$

$= [\cos 45 - i \sin 45^o]^{-1}$

$= \cos 45^o + i \sin 45^o$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

4. If $\frac{1}{z}$ = cos 60° + i sin 60° what is the value of z.

Solution:

$$z = \frac{1}{\frac{1}{z}}$$

$$= \frac{1}{\cos 60^o + i \sin 60^o} = (\cos 60^o + i \sin 60^o)^{-1}$$

$$= \cos 60^o - i \sin 60^o$$

$$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

5. Find the value of $\frac{\cos 3\theta + i \sin 3\theta}{\cos\theta - i \sin\theta}$.

Solution:

$$\frac{\cos 3\theta + i \sin 3\theta}{\cos\theta - i \sin\theta} = (\cos 3\theta + i \sin 3\theta)(\cos\theta - i \sin\theta)^{-1}$$

$$= (\cos 3\theta + i \sin 3\theta)(\cos\theta + i \sin\theta)$$

$$= \cos(3\theta + \theta) + i \sin(3\theta + \theta)$$

$$= \cos 4\theta + i \sin 4\theta$$

6. Simplify: (cos 20° + i sin 20°) (cos 30° + i sin 30°) (cos 40° + i sin 40°)

Solution:

$(\cos 20^o + i \sin 20^o)(\cos 30^o + i \sin 30^o)(\cos 40^o + i \sin 40^o)$

$= \cos(20^o + 30^o + 40^o) + i \sin(20^o + 30^o + 40^o)$

$= \cos 90^o + i \sin 90^o$

$= 0 + i(1) = i$

7. If x = cos θ + i sin θ find $x + \frac{1}{x}$.

Solution:

Given: x = cos θ + i sin θ

$$\Rightarrow \quad \frac{1}{x} = \cos\theta - i \sin\theta$$

$$\therefore x + \frac{1}{x} = \cos\theta + i \sin\theta + \cos\theta - i \sin\theta$$

$$= 2\cos\theta$$

8. If x = cos α + i sin α and y = cos β + i sin β find xy.

Solution:

$$xy = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$
$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

9. If $a = \cos\alpha + i\sin\alpha$ and $b = \cos\beta + i\sin\beta$ find $\frac{a}{b}$.

Solution:

$$\frac{a}{b} = a(b)^{-1}$$
$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)^{-1}$$
$$= (\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)$$
$$= \cos(\alpha-\beta) + i\sin(\alpha-\beta)$$

10. Find the product of $3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

Solution:

$$3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \times 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$
$$= 12\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\right]$$
$$= 12\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$
$$= 12\,[1 + i(0)] = 12$$

PART – B

1. If $x = \cos\theta + i\sin\theta$ find the value of $x^m + \frac{1}{x^m}$.

Solution:

Given: $x = \cos\theta + i\sin\theta$

$\Rightarrow x^m = (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

also, $\frac{1}{x^m} = (x^m)^{-1}$

$$= [\cos m\theta + i\sin m\theta]^{-1}$$

$$\Rightarrow \frac{1}{x^m} = \cos m\theta - i\sin m\theta$$

$$\therefore x^m + \frac{1}{x^m} = \cos m\theta + i\sin m\theta + \cos m\theta - i\sin m\theta = 2\cos m\theta$$

2. If $a = \cos\alpha + i\sin\alpha$ and $b = \cos\beta + i\sin\beta$ find $ab + \frac{1}{ab}$.

Solution:

Given: $a = \cos\alpha + i\sin\alpha$

& $b = \cos\beta + i\sin\beta$

$ab = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$\Rightarrow ab = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

and $\frac{1}{ab} = \cos(\alpha+\beta) - i\sin(\alpha+\beta)$

$$\therefore ab + \frac{1}{ab} = \cos(\alpha+\beta) + i\sin(\alpha+\beta) + \cos(\alpha+\beta) - i\sin(\alpha+\beta)$$
$$= 2\cos(\alpha+\beta)$$

3. Prove that $(\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)$.

Solution:

We have $\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$

$\& \cos\theta = \sin\left(\frac{\pi}{2}-\theta\right)$

LHS: $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n$

$= \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$

4. If x = cos 3α + i sin 3α, y = cos 3β + i sin 3β find the value of $\sqrt[3]{xy}$.

Solution:

Given: x = cos 3α + i sin 3α

& y = cos 3β + i sin 3β

xy = (cos 3α + i sin 3α) (cos 3β + i sin 3β)

= cos (3α + 3β) + i sin (3α + 3β)

⇒ xy = cos 3 (α + β) + i sin 3 (α + β)

$\sqrt[3]{xy} = (xy)^{1/3}$

$= [\cos 3(\alpha+\beta) + i\sin 3(\alpha+\beta)]^{1/3}$

$= \cos\frac{1}{3}.3(\alpha+\beta) + i\sin\frac{1}{3}.3(\alpha+\beta)$

$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

5. If a = cos α + i sin α, b = cos β + i sin β and c = cos γ + i sin γ find the value of $\frac{ab}{c}$.

Solution:

Given: a = cos α + i sin α

b = cos β + i sin β

& c = cos γ + i sin γ

$\therefore \frac{ab}{c} = ab(c)^{-1}$

$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)^{-1}$

$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma - i\sin\gamma)$

$\Rightarrow \frac{ab}{c} = \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)$

PART – C

1. Simplify: $\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2(\cos 4\theta + i\sin 4\theta)^{-3}}$

Solution:

$$\frac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 3\theta - i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2 (\cos 4\theta + i\sin 4\theta)^{-3}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{3\times 2} (\cos\theta + i\sin\theta)^{4\times -3}}{(\cos\theta + i\sin\theta)^{2\times 3} (\cos\theta + i\sin\theta)^{-3\times 4}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{6} (\cos\theta + i\sin\theta)^{-12}}{(\cos\theta + i\sin\theta)^{6} (\cos\theta + i\sin\theta)^{-12}}$$

$$= (\cos\theta + i\sin\theta)^{6-12-6+12}$$

$$= (\cos\theta + i\sin\theta)^{0}$$

$$= \cos 0 + i\sin 0$$

$$= 1 + i(0) = 1$$

2. Simplify : $\dfrac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 4\theta - i\sin 4\theta)^3}{\cos 3\theta + i\sin 3\theta}$ when $\theta = \dfrac{\pi}{9}$.

Solution:

$$\frac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 4\theta - i\sin 4\theta)^3}{\cos 3\theta + i\sin 3\theta}$$

$$= \frac{(\cos\theta + i\sin\theta)^{3\times 2} (\cos\theta + i\sin\theta)^{3\times -4}}{(\cos\theta + i\sin\theta)^{3}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{6} (\cos\theta + i\sin\theta)^{-12}}{(\cos\theta + i\sin\theta)^{3}}$$

$$= (\cos\theta + i\sin\theta)^{6-12-3}$$

$$= (\cos\theta + i\sin\theta)^{-9}$$

$$= \cos 9\theta - i\sin 9\theta \qquad \text{when } \theta = \frac{\pi}{9}$$

$$= \cos 9\left(\frac{\pi}{9}\right) - i\sin 9\left(\frac{\pi}{9}\right)$$

$$= \cos\pi - i\sin\pi$$

$$= -1 - i(0) = -1$$

3. Prove that $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta \cdot$

Solution:

$$\text{LHS}: \left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \left[\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta} \times \frac{i}{i}\right]^4$$

$$= (i)^4 \left[\frac{\cos\theta + i\sin\theta}{i\sin\theta + i^2\cos\theta}\right]^4$$

$$= 1\left[\frac{\cos\theta + i\sin\theta}{-\cos\theta + i\sin\theta}\right]^4$$

$$= \left[\frac{\cos\theta + i\sin\theta}{-(\cos\theta - i\sin\theta)}\right]^4$$

$$= \left[\frac{\cos\theta + i\sin\theta}{(\cos\theta + i\sin\theta)^{-1}}\right]^4$$

$$= \left[\left(\cos\theta + i\sin\theta\right)^{1+1}\right]^4$$

$$= \left[(\cos\theta + i\sin\theta)^2\right]^4$$

$$= (\cos\theta + i\sin\theta)^8$$

$$= \cos 8\theta + i\sin 8\theta = \text{RHS}$$

4. Prove that $\left[\dfrac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right]^n = \cos n\theta + i\sin n\theta$.

Solution:

Let $z = \cos\theta + i\sin\theta$

$$\Rightarrow \frac{1}{z} = \cos\theta - i\sin\theta$$

$$\text{LHS}: \left[\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right]^n$$

$$= \left[\frac{1+z}{1+\frac{1}{z}}\right]^n$$

$$= \left[\frac{1+z}{\frac{z+1}{z}}\right]^n$$

$$= \left[\frac{z(1+z)}{(1+z)}\right]^n$$

$$= z^n$$

$$= (\cos\theta + i\sin\theta)^n$$

$$= \cos n\theta + i\sin n\theta = \text{RHS}$$

5. Show that $\left[\dfrac{1+\sin A + i\cos A}{1+\sin A - i\cos A}\right]^n = \cos n\left(\dfrac{\pi}{2} - A\right) + i\sin n\left(\dfrac{\pi}{2} - A\right)$.

Solution:

Let $z = \sin A + i\cos A$

$$\Rightarrow z = \cos\left(\frac{\pi}{2} - A\right) + i\sin\left(\frac{\pi}{2} - A\right)$$

$$\therefore \frac{1}{z} = \cos\left(\frac{\pi}{2} - A\right) - i\sin\left(\frac{\pi}{2} - A\right) = \sin A - i\cos A$$

$$\text{LHS}\left[\frac{1+\sin A+i\cos A}{1+\sin A-i\cos A}\right]^n$$

$$=\left[\frac{1+z}{1+\frac{1}{z}}\right]^n$$

$$=\left[\frac{1+z}{\frac{z+1}{z}}\right]^n$$

$$=\left[\frac{z(1+z)}{(1+z)}\right]^n$$

$$=(z)^n$$

$$=\left[\cos\left(\frac{\pi}{2}-A\right)+i\sin\left(\frac{\pi}{2}-A\right)\right]^n$$

$$=\cos n\left(\frac{\pi}{2}-A\right)+i\sin n\left(\frac{\pi}{2}-A\right)$$

6. If $a = \cos\theta + i\sin\theta$, $b = \cos\phi + i\sin\phi$ prove that

(i) $\cos(\theta+\phi)=\frac{1}{2}\left[ab+\frac{1}{ab}\right]$

(ii) $\sin(\theta-\phi)=\frac{1}{2i}\left[\frac{a}{b}-\frac{b}{a}\right]$

Solution:

Given $a = \cos\theta + i\sin\theta$

& $b = \cos\phi + i\sin\phi$

Now, $ab = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

$\Rightarrow ab = \cos(\theta+\phi) + i\sin(\theta+\phi)$ .....(1)

also, $\frac{1}{ab}=(ab)^{-1}=\left[\cos(\theta+\phi)+i\sin(\theta+\phi)\right]$

$\Rightarrow \frac{1}{ab}=\cos(\theta+\phi)-i\sin(\theta+\phi)$.........(2)

$\therefore$ (1) + (2) $\Rightarrow$

$$ab+\frac{1}{ab}=\cos(\theta+\phi)+i\sin(\theta+\phi)+\cos(\theta+\phi)-i\sin(\theta+\phi)$$

$\Rightarrow ab+\frac{1}{ab}=2\cos(\theta+\phi)$

$\Rightarrow \cos(\theta+\phi)=\frac{1}{2}\left[ab+\frac{1}{ab}\right]$

(ii) $\frac{a}{b}=a(b)^{-1}$

$=(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)^{-1}$

$=(\cos\theta+i\sin\theta)(\cos\phi-i\sin\phi)$

$$\Rightarrow \frac{a}{b} = \cos(\theta - \phi) + i\sin(\theta - \phi) ....(3)$$

$$\text{also, } \frac{b}{a} = \left(\frac{a}{b}\right)^{-1} = \left[\cos(\theta - \phi) + i\sin(\theta - \phi)\right]^{-1}$$

$$\Rightarrow \frac{b}{a} = \cos(\theta - \phi) - i\sin(\theta - \phi) \quad ....(4)$$

$$(3) - (4) \Rightarrow$$

$$\frac{a}{b} - \frac{b}{a} = \cos(\theta - \phi) + i\sin(\theta - \phi) - \cos(\theta - \phi) + i\sin(\theta - \phi)$$

$$\Rightarrow \frac{a}{b} - \frac{b}{a} = 2i\sin(\theta - \phi)$$

$$\Rightarrow \sin(\theta - \phi) = \frac{1}{2i}\left[\frac{a}{b} - \frac{b}{a}\right]$$

7. If a = cos x + i sin x, b = cos y + i sin y prove that

(i) $\sqrt{ab} + \frac{1}{\sqrt{ab}} = 2\cos\left(\frac{x+y}{2}\right)$

(ii) $\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = 2\cos\left(\frac{x-y}{2}\right)$

Solution:

Given: a = cos x + i sin x

& b = cos y + i sin y

(i) Now, ab = (cos x + i sin x) (cos y + i sin y)

$\Rightarrow$ ab = cos (x + y) + i sin (x + y)

$$\therefore \sqrt{ab} = (ab)^{1/2} = \left[\cos(x+y) + i\sin(x+y)\right]^{1/2}$$

$$\Rightarrow \sqrt{ab} = \cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right) \quad ....(1)$$

$$\text{also, } \frac{1}{\sqrt{ab}} = \left(\sqrt{ab}\right)^{-1} = \left[\cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right)\right]^{-1}$$

$$\Rightarrow \frac{1}{\sqrt{ab}} = \cos\left(\frac{x+y}{2}\right) - i\sin\left(\frac{x+y}{2}\right) \quad ......(2)$$

$$\therefore (1) + (2) \Rightarrow$$

$$\sqrt{ab} + \frac{1}{\sqrt{ab}} = \cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right) + \cos\left(\frac{x+y}{2}\right) - i\sin\left(\frac{x+y}{2}\right)$$

$$\Rightarrow \sqrt{ab} + \frac{1}{\sqrt{ab}} = 2\cos\left(\frac{x+y}{2}\right)$$

$$\text{(ii) } \frac{a}{b} = a(b)^{-1} = (\cos x + i\sin x)(\cos y + i\sin y)^{-1}$$

$$= (\cos x + i\sin x)(\cos y - i\sin y)$$

$$\Rightarrow \quad \frac{a}{b} = \cos(x-y) + i\sin(x-y)$$

$$\therefore \sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/2} = [\cos(x-y) + i\sin(x-y)]^{1/2}$$

$$\Rightarrow \quad \sqrt{\frac{a}{b}} = \cos\left(\frac{x-y}{2}\right) + i\sin\left(\frac{x-y}{2}\right) \quad ....(3)$$

$$\text{also,} \quad \sqrt{\frac{b}{a}} = \left(\sqrt{\frac{a}{b}}\right)^{-1} = \left[\cos\left(\frac{x-y}{2}\right) + i\sin\left(\frac{x-y}{2}\right)\right]^{-1}$$

$$\Rightarrow \quad \sqrt{\frac{b}{a}} = \cos\left(\frac{x-y}{2}\right) - i\sin\left(\frac{x-y}{2}\right) \quad ......(4)$$

$\therefore (3) + (4) \Rightarrow$

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \cos\left(\frac{x-y}{2}\right) + i\sin\left(\frac{x-y}{2}\right) + \cos\left(\frac{x-y}{2}\right) - i\sin\left(\frac{x-y}{2}\right)$$

$$\Rightarrow \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = 2\cos\left(\frac{x-y}{2}\right)$$

8. If a = cos α + i sin α, b = cos β + i sin β and c = cos γ + i sin γ find the value of $\frac{ab}{c} - \frac{c}{ab}$.

Solution:

Given:

a = cos α + i sin α

b = cos β + i sin β

& c = cos γ + i sin γ

Now,

$$\frac{ab}{c} = ab(c)^{-1}$$

$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)^{-1}$$

$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma - i\sin\gamma)$$

$$\Rightarrow \quad \frac{ab}{c} = \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma) \quad ......(1)$$

$$\text{also,} \ \frac{c}{ab} = \left[\frac{ab}{c}\right]^{-1}$$

$$= [\cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)]^{-1}$$

$$\Rightarrow \quad \frac{c}{ab} = \cos(\alpha+\beta-\gamma) - i\sin(\alpha+\beta-\gamma) \quad ....(2)$$

$\therefore (1) - (2) \Rightarrow$

$$\frac{ab}{c} - \frac{c}{ab} = \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma) - \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)$$

$$\Rightarrow \frac{ab}{c} - \frac{c}{ab} = 2i\sin(\alpha+\beta-\gamma)$$

9. If $x + \frac{1}{x} = 2\cos\theta$ prove that (i) $x^n + \frac{1}{x^n} = 2\cos n\theta$ (ii) $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

Solution:

Given: $x+\frac{1}{x}=2\cos\theta$

$\frac{x^2+1}{x}=2\cos\theta$

$x^2+1=2x\cos\theta$

$x^2-2x\cos\theta+1=0$

Here a = 1, b = –2 cos θ & c = 1

$$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{2\cos\theta\pm\sqrt{(-2\cos\theta)^2-4\times1\times1}}{2\times1}$$

$$=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}$$

$$=\frac{2\cos\theta\pm\sqrt{4(\cos^2\theta-1)}}{2}$$

$$=\frac{2\cos\theta\pm\sqrt{4(-\sin^2\theta)}}{2}$$

$$=\frac{2\cos\theta\pm i2\sin\theta}{2}$$

$$=\frac{2[\cos\theta\pm i\sin\theta]}{2}$$

$\Rightarrow x=\cos\theta\pm i\sin\theta$

Consider $x=\cos\theta+i\sin\theta$

$\therefore x^n=(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

& $\frac{1}{x^n}=\cos n\theta-i\sin n\theta$

(i) $x^n+\frac{1}{x^n}=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$

$\Rightarrow x^n+\frac{1}{x^n}=2\cos n\theta$

(ii) $x^n-\frac{1}{x^n}=\cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta$

$\Rightarrow x^n-\frac{1}{x^n}=2i\sin n\theta$

10. Show that $(1+i)^n+(1-i)^n=2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$.

Solution:

Let 1 + i = r (cos θ + i sin θ) = r cos θ + i sin θ

Equating real & imaginary parts on both sides

$r\cos\theta = 1 \quad \& \quad r\sin\theta = 1$

Now, $(r\cos\theta)^2 + (r\sin\theta)^2 = (1)^2 + (1)^2$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 1 + 1$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 2$

$\Rightarrow r^2(1) = 2$

$\Rightarrow r^2 = 2 \quad \Rightarrow \boxed{r = \sqrt{2}}$

Also, $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{1}{1}$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \tan^{-1}(1)$

$\Rightarrow \boxed{\theta = \dfrac{\pi}{4}}$

$$\therefore 1 + i = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] \quad ....(1)$$

Similarly we can prove that

$$1 - i = \sqrt{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right] ....(2)$$

LHS: $(1 + i)^n + (1 - i)^n$

$$= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n + \left[\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^n$$

$$= \left(\sqrt{2}\right)^n\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^n + \left(\sqrt{2}\right)^n\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]^n$$

$$= \left(\sqrt{2}\right)^n\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right]$$

$$= 2^{\frac{n}{2}}2\cos\frac{n\pi}{4}$$

$$= 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$$

$$= 2^{\frac{n+2}{2}}\cos\frac{n\pi}{4} = \text{RHS}$$

2.3 ROOTS OF COMPLEX NUMBERS

Definition:

A number ω is called the n^{th} root of a complex number z, if $\omega^n = z$ and we write $\omega = z^{\frac{1}{n}}$.

Working rule to find the n^{th} roots of a complex numbers:

Step (I) : Write the given complex number in Polar form.

Step (II) : Add "2kπ" to the argument.

Step (III) : Apply Demoivre's theorem

Step (IV) : Put k = 0, 1, ... upto (n – 1).

Illustration:

Let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z = r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]$ where $k \in I$

$$\therefore z^{\frac{1}{n}} = \{r[\cos(2k\pi+\theta)+i\sin(2k\pi+\theta)]\}^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}[\cos(2k\pi+\theta)+i\sin(2k\pi+\theta)]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi+\theta}{n}\right)+i\sin\left(\frac{2k\pi+\theta}{n}\right)\right] \text{ where } K = 0, 1, 2, ..., n-1.$$

Only these values of k will give 'n' different values of $z^{\frac{1}{n}}$ provided $z \neq 0$.

To find the nth roots of unity

$1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$

$$\therefore n^{th} \text{ roots of unity} = 1^{1/n} = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$$

$$= \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \text{ where } k = 0, 1, 2, ..., n-1$$

The roots are,

for k = 0; $R_1 = \cos 0 + i\sin 0 = 1 + i0 = 1 = e^{i0}$

$$k = 1;\ R_2 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{i\frac{2\pi}{n}} = \omega \text{ (say)}$$

$$k = 2;\ R_3 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n} = e^{i\frac{4\pi}{n}} = \left[e^{i\frac{2\pi}{n}}\right]^2 = \omega^2$$

$$k = n-1;\ R_n = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n} = e^{i\frac{2(n-1)\pi}{n}} = \omega^{n-1}$$

$\therefore$ The nth roots of unity are

$$e^{i0}, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, ..., e^{i\frac{2(n-1)\pi}{n}}$$

i.e., $1, \omega, \omega^2, ..., \omega^{n-1}$.

Result:

If ω is nth roots of unity then

(i) $\omega^n = 1$

(ii) Sum of the roots is zero.

i.e $1 + \omega + \omega^2 + ... + \omega^{n-1} = 0$

(iii) The roots are in G.P with common ratio ω.

(iv) The arguments are in A.P with common difference $\frac{2\pi}{n}$.

(v) The product of the roots is $(-1)^{n+1}$.

To find cube roots of unity

$$\text{Let } x = (1)^{1/3}$$
$$= (\cos 0 + i\sin 0)^{1/3}$$
$$= (\cos 2k\pi + i\sin 2k\pi)^{1/3}$$
$$= \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right) \text{ where } k = 0, 1, 2$$

∴ The roots are

for k = 0; $R_1 = \cos 0 + i \sin 0 = 1 + i0 = 1$

$$k = 1;\quad R_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$k = 2;\quad R_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

The cube roots of unity are $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Result:

If we denote the second root $R_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ by ω then the other root ,

$R_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ becomes ω^2

$$\text{Thus, } R_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \omega$$
$$R_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \omega^2$$

∴ The cube roots of unit are 1, ω, ω^2

Note:

If ω is cube roots of unity then (i) $\omega^3 = 1$, (ii) $1 + \omega + \omega^2 = 0$

Fourth roots of unity

$$\text{Let } x = (1)^{1/4}$$
$$= (\cos 0 + i\sin 0)^{1/4}$$
$$= (\cos 2k\pi + i\sin 2k\pi)^{1/4}$$
$$= \cos\left(\frac{2k\pi}{4}\right) + i\sin\left(\frac{2k\pi}{4}\right) \text{ where } k = 0, 1, 2, 3$$

∴ The roots are,

for k = 0; $R_1 = \cos 0 + i \sin 0 = 1 + i0 = 1$

$$k=1;\quad R_2=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=0+i(1)=i=\omega\ \text{(say)}$$

$$k=2;\quad R_3=\cos\pi+i\sin\pi=-1+i(0)=-1=\omega^2$$

$$k=3;\ R_4=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}=0+i(-1)=-i=\omega$$

The fourth roots of unity are 1, i, – 1, – i (i.e.) 1, ω, ω^2, ω^3.

Note:

(i) The sum of the fourth roots of unity is zero. i.e $1+\omega+\omega^2+\omega^3=0$ and $\omega^4=1$.

(ii) The value of ω used in cube roots of untiy and in fourth roots of unity are different.

Sixth roots of unity

$$\text{Let } x=(1)^{1/6}$$

$$=(\cos 0+i\sin 0)^{1/6}$$

$$=(\cos 2k\pi+i\sin 2k\pi)^{1/6}$$

$$=\cos\left(\frac{2k\pi}{6}\right)+i\sin\left(\frac{2k\pi}{6}\right)\ \text{where } k=0,1,2,3,4,5$$

$\therefore$ The six roots are

for k = 0; $R_1=\cos 0+i\sin 0=e^{i0}=1$

$$k=1;\quad R_2=\cos\frac{2\pi}{6}+i\sin\frac{2\pi}{6}=e^{i\frac{2\pi}{6}}=\omega$$

$$k=2;\quad R_3=\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6}=e^{i\frac{4\pi}{6}}=\omega^2$$

$$k=3;\quad R_4=\cos\frac{6\pi}{6}+i\sin\frac{6\pi}{6}=e^{i\frac{6\pi}{6}}=\omega^3$$

$$k=4;\quad R_5=\cos\frac{8\pi}{6}+i\sin\frac{8\pi}{6}=e^{i\frac{8\pi}{6}}=\omega^4$$

$$k=5;\quad R_6=\cos\frac{10\pi}{6}+i\sin\frac{10\pi}{6}=e^{i\frac{10\pi}{6}}=\omega^5$$

$\therefore$ The sixth roots of unity are $e^{i0}, e^{i\frac{2\pi}{6}}, e^{i\frac{4\pi}{6}}, e^{i\frac{6\pi}{6}}, e^{i\frac{8\pi}{6}}, e^{i\frac{10\pi}{6}}$

i.e, 1, ω, ω^2, ω^3, ω^4, ω^5.

Note:

The sum of the sixth roots of unity is zero. i.e $1+\omega+\omega^2+\omega^3+\omega^4+\omega^5=0$ and $\omega^6=1$

Note: $1=\cos 0+i\sin 0$

$-1=\cos\pi+i\sin\pi$

$$i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$$

$$-i=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$$

WORKED EXAMPLES

PART – A

1. If ω is a cube roots of unity, find the value of $\omega^4+\omega^5+\omega^6$.

Solution:

If ω is cube roots of unity then $\omega^3 = 1$.

$$\therefore \omega^4 + \omega^5 + \omega^6 = \omega^3 . \omega + \omega^3 . \omega^2 + \omega^3 . \omega^3$$
$$= (1)\,\omega + (1)\,\omega^2 + (1)\,(1)$$
$$= \omega + \omega^2 + 1$$
$$= 1 + \omega + \omega^2 = 0$$

2. Simplify: $(1 + \omega)(1 + \omega^2)$ where ω is cube roots of unity.

Solution:

$$(1 + \omega)(1 + \omega^2) = 1 + \omega^2 + \omega + \omega^3$$
$$= [1 + \omega + \omega^2] + \omega^3$$
$$= 0 + 1$$
$$= 1$$

3. Solve: $x^2 - 1 = 0$

Solution:

Given: $x^2 - 1 = 0$

$$x^2 = 1$$
$$x = (1)^{1/2} = [\cos 0 + i \sin 0]^{1/2}$$
$$= [\cos 2k\pi + i \sin 2k\pi]^{1/2}$$
$$= \cos\left(\frac{2k\pi}{2}\right) + i \sin\left(\frac{2k\pi}{2}\right) \text{ where } k = 0,1$$

4. Find the value of $(i)^{1/3}$.

Solution:

Let $x = (i)^{1/3}$

$$= \left[\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right]^{1/3}$$
$$= \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i \sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{1/3}$$
$$= \cos\frac{1}{3}\left(2k\pi + \frac{\pi}{2}\right) + i \sin\frac{1}{3}\left(2k\pi + \frac{\pi}{2}\right) \quad \text{where } k = 0, 1, 2.$$

5. Find the value of $(-1)^{1/3}$.

Solution:

Let $x = (-1)^{1/3}$

$$= (\cos\pi + i \sin\pi)^{1/3}$$
$$= [\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)]^{1/3}$$
$$= \cos\frac{1}{3}(2k\pi + \pi) + i \sin\frac{1}{3}(2k\pi + \pi) \text{ where } k = 0, 1, 2$$

6. Find the value of $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$.

Solution:

We have $\frac{-1+i\sqrt{3}}{2} = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ = cos 120° + i sin 120°

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^3 = (\cos 120^o + i\sin 120^o)^3$$

$$= \cos 360^o + i\sin 360^o$$

$$= 1 + i(0)$$

$$= 1$$

PART – B

1. Find the cube roots of unity.

Solution:

Let 'x' be the cube roots of unity.

i.e. $x^3 = 1$

$$x = (1)^{1/3}$$

$$= (\cos 0 + i\sin 0)^{1/3}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{1/3}$$

$$= \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3} \text{ where } k = 0,1,2$$

For k = 0; x = cos 0 + i sin 0 = 1 + i0 = 1

$$k = 1;\ x = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

$$k = 2;\ x = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

2. Find all the values of $(i)^{2/3}$.

Solution:

Let $x = (i)^{2/3}$

$$= \left[\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]^{2/3}$$

$$= \left[\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^2\right]^{1/3}$$

$$= \left[\cos 2\left(\frac{\pi}{2}\right) + i\sin 2\left(\frac{\pi}{2}\right)\right]^{1/3}$$

$$= [\cos\pi + i\sin\pi]^{1/3}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/3}$$

$$= \cos\left(\frac{2k\pi + \pi}{3}\right) + i\sin\left(\frac{2k\pi + \pi}{3}\right) \quad \text{where } k = 0, 1, 2$$

when $k = 0$; $x = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

$k = 1$; $x = \cos\dfrac{3\pi}{3} + i\sin\dfrac{3\pi}{3}$

$k = 2$; $x = \cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}$

3. Solve: $x^2 + 16 = 0$

Solution:

Given: $x^2 + 16 = 0$

$$x^2 = -16 = 16 \times -1$$

$$x = (16)^{1/2}(-1)^{1/2}$$

$$= 4[\cos\pi + i\sin\pi]^{1/2}$$

$$= 4[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/2}$$

$$= 4\left[\cos\left(\frac{2k\pi + \pi}{2}\right) + i\sin\left(\frac{2k\pi + \pi}{2}\right)\right] \quad \text{where } k = 0, 1$$

when $k = 0$; $x = 4\left[\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right]$

$k = 1$; $x = 4\left[\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}\right]$

4. If ω is the cube roots of unity then prove that $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$.

Solution:

If ω is the cube roots of unity then $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

LHS : $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5$

$$= (1 + \omega^2 - \omega)^5 + (1 + \omega - \omega^2)^5$$

$$= (-\omega - \omega)^5 + (-\omega^2 - \omega^2)^5$$

$$= (-2\omega)^5 + (-2\omega^2)^5$$

$$= (-2)^5\,\omega^5 + (-2)^5\,(\omega^2)^5$$

$$= -32\omega^2 - 32\omega$$

$$= -32\,(\omega^2 + \omega)$$

$$= -32\,(-1) = 32 = \text{RHS}$$

PART – C

1. Solve: $x^7 + 1 = 0$

Solution:

Given: $x^7 + 1 = 0$

$$x^7 = -1$$

$$x = (-1)^{1/7}$$

$$= (\cos\pi + i\sin\pi)^{1/7}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/7}$$

$$= \cos\left(\frac{2k\pi + \pi}{7}\right) + i\sin\left(\frac{2k\pi + \pi}{7}\right) \quad \text{where } k = 0, 1, 2, 3, 4, 5, 6,$$

∴ The values are

$$\text{when } k = 0;\ x = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$$

$$k = 1;\ x = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}$$

$$k = 2;\ x = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$$

$$k = 3;\ x = \cos\frac{7\pi}{7} + i\sin\frac{7\pi}{7}$$

$$k = 4;\ x = \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}$$

$$k = 5;\ x = \cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7}$$

$$k = 6;\ x = \cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}$$

2. Solve: $x^6 - 1 = 0$

Solution:

Given: $x^6 - 1 = 0$

$$x^6 = 1$$

$$x = (1)^{1/6}$$

$$= (\cos 0 + i\sin 0)^{1/6}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{1/6}$$

$$= \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6} \quad \text{where } k = 0, 1, 2, 3, 4, 5$$

∴ The values are,

when $k = 0$; $x = \cos 0 + i\sin 0$

$k = 1$; $x = \cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}$

$k = 2$; $x = \cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}$

$k = 3$; $x = \cos\frac{6\pi}{6} + i\sin\frac{6\pi}{6}$

$k = 4$; $x = \cos\frac{8\pi}{6} + i\sin\frac{8\pi}{6}$

$k = 5$; $x = \cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6}$

3. Solve: $x^8 + x^5 + x^3 + 1 = 0$

Solution:

Given: $x^8 + x^5 + x^3 + 1 = 0$

$x^5 (x^3 + 1) + 1 (x^3 + 1) = 0$

$(x^5 + 1) (x^3 + 1) = 0$

$x^5 + 1 = 0$; $x^3 + 1 = 0$

Case (i)

$x^5 + 1 = 0$

$x = (-1)^{1/5}$

$= (\cos\pi + i\sin\pi)^{1/5}$

$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/5}$

$= \cos\left(\frac{2k\pi + \pi}{5}\right) + i\sin\left(\frac{2k\pi + \pi}{5}\right)$ where $k = 0, 1, 2, 3, 4$

The roots are,

when $k = 0$; $x = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$k = 1$; $x = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

$k = 2$; $x = \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}$

$k = 3$; $x = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$

$k = 4$; $x = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$

Case (ii) :

$$x^3 + 1 = 0$$

$$x = (-1)^{1/3}$$

$$= (\cos\pi + i\sin\pi)^{1/3}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/3}$$

$$= \cos\left(\frac{2k\pi + \pi}{3}\right) + i\sin\left(\frac{2k\pi + \pi}{3}\right) \quad \text{where } k = 0, 1, 2$$

$$\text{when } k = 0;\ x = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$k = 1;\ x = \cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}$$

$$k = 2;\ x = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

4. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and also prove that the product of the four values is 1.

Solution:

$$\text{Let } a + ib = \frac{1}{2} + i\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta) \quad ....(1)$$

$$\text{Here } a = \frac{1}{2} \quad \& \quad b = \frac{\sqrt{3}}{2}$$

Modulus:

$$r = \sqrt{a^2 + b^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

Arugument:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left[\frac{\sqrt{3}/2}{1/2}\right] = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$\therefore$(1) becomes,

$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\Rightarrow \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{3/4}$$

$$= \left[\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3\right]^{\frac{1}{4}}$$

$$= \left[\cos 3\left(\frac{\pi}{3}\right) + i\sin 3\left(\frac{\pi}{3}\right)\right]^{1/4}$$

$$= [\cos\pi + i\sin\pi]^{1/4}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/4}$$

$$= \cos\left(\frac{2k\pi + \pi}{4}\right) + i\sin\left(\frac{2k\pi + \pi}{4}\right) \quad \text{where } k = 0, 1, 2, 3$$

$\therefore$ The values are,

when $k = 0$; $R_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$k = 1$; $R_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$k = 2$; $R_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

$k = 3$; $R_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

Product of the four values

$R_1 \times R_2 \times R_3 \times R$

$$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$$

$$= \cos\frac{16\pi}{4} + i\sin\frac{16\pi}{4}$$

$$= \cos 4\pi + i\sin 4\pi$$

$$= 1 + i(0) = 1$$

EXERCISE

PART – A

1. If $z_1 = -1 + 2i$ and $z_2 = -3 + 4i$ find $3z, - 4z_2$.
2. If $z_1 = (2, 3)$ and $z = (5, 7)$ find $4z_1 + 3z_2$.
3. If $z_1 = (-3, 5)$ and $z_2 = (1, -2)$ find $z_1 z_2$.
4. If $z_1 = 1 + i$ and $z_2 = 1 - i$ find z_1 / z_2.
5. If $z_1 = 2 + i$ and $z_2 = 1 + i$ find z_2 / z_1.
6. Express the following complex numbers in a + ib form.

   (i) $\frac{1}{4+3i}$ (ii) $\frac{2}{3-i}$ (iii) $(4+5i)(5+7i)$
7. Find the real and imaginary parts of the following complex numbers

   (i) $\frac{1}{2-i}$ (ii) $\frac{1}{2+3i}$ (iii) $\frac{1}{i-3}$ (iv) $\frac{1+i}{1-i}$
8. Find the complex conjugate of the following:

   (i) $(2-3i)(7+11i)$ (ii) $\frac{4}{1-i}$ (iii) $\frac{1-i}{1+i}$ (iv) $\frac{2}{i-5}$
9. Find the modulus and argument (or) amplitude of the following:

   (i) $\sqrt{3}+i$ (ii) $-1+i$ (iii) $\sqrt{3}-i$ (iv) $1-\sqrt{-3}$ (v) $\frac{1}{2}+i\frac{\sqrt{3}}{2}$

   (vi) $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$ (vii) $1-i\sqrt{3}$
10. Find the distance between the following two complex numbers

    (i) 2 + 3i and 3 – 2i (ii) 4 + 3i and 5 – 6i (iii) 2 – 3i and 5 + 7i (iv) 1 + i and 3 – 2i
11. State DeMoivre's theorem.
12. Simplify the following:

    (i) $(\cos\theta + i\sin\theta)^3 (\cos\theta + i\sin\theta)^{-4}$

    (ii) $(\cos\phi + i\sin\phi)^5 (\cos\phi + i\sin\phi)^{-6}$

    (iii) $(\cos\theta - i\sin\theta)^4 (\cos\theta + i\sin\theta)^7$
13. Find the value of the following:

    (i) $\frac{\cos 5\theta + i\sin 5\theta}{\cos 3\theta - i\sin 3\theta}$ (ii) $\frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta}$

    (iii) $\frac{\cos 10\theta + i\sin 10\theta}{\cos 7\theta + i\sin 7\theta}$ (iv) $\frac{\cos 4\theta + i\sin 4\theta}{\cos\theta + i\sin\theta}$

    (v) $\frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^4}$ (vi) $\frac{\cos 6\theta + i\sin 6\theta}{(\cos\theta - i\sin\theta)^4}$
14. Simplify: $\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)$

15. Simplify: $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

16. Find the product of $5\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ and $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$.

17. Find the product of $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ and $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

17.(a) If $z_1 = 5\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ and $z_2 = 3\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ find Z_1Z_2.

18. If $x = \cos\theta + i\sin\theta$ find $x-\frac{1}{x}$.

19. If $a = \cos\theta + i\sin\theta$, $b = \cos\phi + i\sin\phi$ find ab.

20. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ find $\frac{b}{a}$.

21. If $a = \cos x + i\sin x$, $b = \cos y + i\sin y$ find $\sqrt{ab}$.

22. If ω is the cube root of unity find the value of $1 + \omega^2 + \omega^4$.

23. If ω is the fourth root of unity find the value $\omega^4 + \omega^5 + \omega^6 + \omega^7$.

24. If ω is the six root of unity find the value of $\omega^2 + \omega^4 + \omega^6$.

25. Solve: $x^2 + 1 = 0$

26. Find the value of (i) $(i)^{1/2}$ (ii) $(1)^{1/3}$ (iii) $(-1)^{1/2}$

27. Find the value of $\left(\frac{-1-i\sqrt{3}}{2}\right)^3$.

PART – B

1. Express the following complex numbers in a + ib form.

   (i) $\frac{1}{1+i}+\frac{1}{1-i}$ (ii) $\frac{2+3i}{4-i}$ (iii) $\frac{1+i}{(1-i)^2}$ (iv) $\frac{4+3i}{1-i}$

2. Find the real and imaginary parts of the following complex numbers

   (i) $\frac{1+2i}{1-i}$ (ii) $\frac{(2-i)^2}{1+i}$ (iii) $\frac{2+i}{1+4i}$

3. Find the conjugate the following complex numbers

   (i) $\frac{13}{11+12i}$ (ii) $\frac{1+i}{1-i}$ (iii) $\frac{1}{2-i}+\frac{1}{2+i}$

4. Find the modulus and amplitude of the following complex numbers

   (i) $\frac{1+i}{1-i}$ (ii) $\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$ (iii) $2+2\sqrt{3}i$ (iv) $-\sqrt{2}+\sqrt{2}i$

5. Show that the following complex numbers are collinear.

   (i) $1 + 3i$, $5 + i$, $3 + 2i$
   (ii) $4 + 2i$, $7 + 5i$, $9 + 7i$
   (iii) $1 + 3i$, $2 + 7i$, $-2 - 9i$

6. Show that the following complex numbers form a parallelogram.

(i) 2 – 2i, 8 + 4i, 5 + 7i, – 1 + i

(ii) 3 + i, 2 + 2i, – 2 + i, – 1

(iii) 1 – 2i, –1 + 4i, 5 + 8i, 7 + 2i

7. If x = cos θ + i sin θ find the value of

(i) $x^n + \frac{1}{x^n}$ (ii) $x^m - \frac{1}{x^m}$ (iii) $x^3 + \frac{1}{x^3}$ (iv) $x^5 + \frac{1}{x^5}$ (v) $x^2 - \frac{1}{x^2}$ (vi) $x^8 - \frac{1}{x^8}$

8. If a = cos x + i sin x, b = cos y + i sin y find $ab - \frac{1}{ab}$.

9. If x = cos 2α + i sin 2α, y = cos 2β + i sin 2β find $\sqrt{xy}$.

10. If x = cos α + i sin α, y = cos β + i sin β find $x^m y^n$.

11. Find the cube roots of 8.

12. Find the all the values of $(-1)^{2/3}$.

PART – C

1. Find the real and imaginary parts of the following complex numbers

(i) $\frac{(1+i)(1+2i)}{1+3i}$ (ii) $\frac{(1+2i)^3}{(1+i)(2-i)}$ (iii) $\left(\frac{1-i}{1+i}\right)^3$ (iv) $\frac{3}{4+3i}+\frac{i}{3-4i}$

(vii) $\frac{3}{3+4i}+\frac{i}{5-2i}$ (viii) $\frac{4}{3+2i}+\frac{2}{5-4i}$

(ix) $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$ (x) $\frac{1}{1+\cos\theta+i\sin\theta}$

2. Express the following complex numbers in a + ib form.

(i) $\frac{(1+i)(1-2i)}{(1+3i)}$ (ii) $\frac{(1+i)(1+2i)}{(1+4i)}$ (iii) $\frac{(1+i)(3+i)^2}{(2-i)^2}$

(iv) $\frac{3}{4+3i}+\frac{i}{3-4i}$ (v) $\frac{2+3i}{1-i}$ (vi) $\frac{7-5i}{(2+3i)^2}$

3. Find the conjugate of the following complex numbers

(i) $\frac{(1+i)(2-i)}{(2+i)^2}$ (ii) $\frac{(1+i)(2+i)}{(3+i)}$ (iii) $\frac{1-i}{3+2i}$ (iv) $\frac{3+i}{2+5i}$ (v) $\frac{5-i}{2-3i}$

4. Find the modulus and argument of the following complex numbers

(i) $\frac{1+\sqrt{3}i}{1+i}$ (ii) $\frac{2-i}{3+7i}$ (iii) $\frac{1+i\sqrt{3}}{1-i}$

(iv) $\frac{(1+i)(1+2i)}{1+3i}$ (v) $\frac{i-3}{i-1}$ (vi) $\frac{1-i}{1+i}$

5. Prove that the following complex numbers form a square.

(i) 9 + i, 4 + 13i, – 8 + 8i, – 3 – 4i (ii) 2 + i, 4 + 3i, 2 + 5i, 3i

(iii) – 1, 3i, 3 + 2i, 2 – i (iv) 4 + 5i, 1 + 2i, 4 – i, 7 + 2i

6. Show that the following complex numbers form a rectangle.

(i) 1 + 2i, – 2 + 5i, 7i, 3 + 4i (ii) 4 + 3i, 12 + 9i, 15 + 5i, 7 – i

(iii) 1 + i, 3 + 5i, 4 + 4i, 2i (iv) 8 + 4i, 5 + 7i, – 1 + i, 2 – 2i

7. Show that the following complex number form a rhombus.

(i) 8 + 5i, 16 + 11i, 10 + 3i, 2 – 3i (ii) 6 + 4i, 4 + 5i, 6 + 3i, 8 + i

(iii) 1 + i, 2 + i, 2 + 2i, 1 + 2i

8. Simplify the following using De Moivre's theorem

(i) $\dfrac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^2(\cos 5\theta - i\sin 5\theta)^{-3}}$

(ii) $\dfrac{(\cos 2\theta - i\sin 2\theta)^3(\cos 3\theta + i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2(\cos 5\theta - i\sin 5\theta)^{-3}}$

(iii) $\dfrac{(\cos 2\theta - i\sin 2\theta)^7(\cos 3\theta + i\sin 3\theta)^{-5}}{(\cos 4\theta + i\sin 4\theta)^2(\cos 5\theta - i\sin 5\theta)^{-6}}$

(iv) $\dfrac{(\cos 3\theta + i\sin 3\theta)^2(\cos 4\theta - i\sin 4\theta)^3}{(\cos\theta + i\sin\theta)^3}$ when $\theta = \dfrac{\pi}{9}$

(v) $\dfrac{(\cos x - i\sin x)^3(\cos 3x + i\sin 3x)^5}{(\cos 2x - i\sin 2x)^5(\cos 5x + i\sin 5x)^7}$ when $x = \dfrac{2\pi}{13}$

(vi) $\dfrac{(\cos 5\theta - i\sin 5\theta)\,(\cos 2\theta - i\sin 2\theta)^{-3}}{(\cos\theta + i\sin\theta)^5(\cos 3\theta + i\sin 3\theta)^{-5}}$ when $\theta = \dfrac{2\pi}{11}$

9. Show that $\left[\dfrac{\cos\theta + i\sin\theta}{\sin\theta - i\cos\theta}\right]^4 = 1$

10. Show that $\left[\dfrac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right]^3 = \cos 3\theta + i\sin 3\theta$

11. Prove that $\left[\dfrac{1+\sin\dfrac{\pi}{8} + i\cos\dfrac{\pi}{8}}{1+\sin\dfrac{\pi}{8} - i\cos\dfrac{\pi}{8}}\right]^8 = 1\cdot$

12. If a = cos α + i sin α, b = cos β + i sin β and c = cos γ + i sin γ find the value of $\dfrac{ab}{c} + \dfrac{c}{ab}$.

13. If x = cos 3α + i sin 3α, y = cos 3β + i sin 3β prove that

(i) $\sqrt[3]{xy} + \dfrac{1}{\sqrt[3]{xy}} = 2\cos(\alpha+\beta)$ (ii) $\sqrt[3]{xy} - \dfrac{1}{\sqrt[3]{xy}} = 2i\sin(\alpha+\beta)$

14. If x = cos α + i sin α, y = cos β + i sin β, z = cos γ + i sin γ and if x + y + z = 0 show that

(i) cos 3α + cos 3β + cos 3γ = 3 cos (α + β + γ)

(ii) sin 3α + sin 3β + sin 3γ = 3 sin (α + β + γ)

15. If x = cos α + i sin α, y = cos β + i sin β and z = cos γ + i sin γ and if x + y + z = 0 prove that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 0$

16. If $x + \dfrac{1}{x} = 2\cos\theta$ and $y + \dfrac{1}{y} = 2\cos\phi$ show that $\dfrac{x^m}{y^m} + \dfrac{y^n}{x^m} = 2\cos(m\theta - n\phi)$.

17. If x = cos α + i sin α, y = cos β + i sin β prove that $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$.

18. If 'n' is a positive integer, prove that $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\frac{n\pi}{3}$.

19. If 'n' is a positive integer prove that $(\sqrt{3}+i)^n - (\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6}$.

20. Prove that $(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$.

21. Solve: $x^4 + 1 = 0$

22. Solve: $x^5 + 1 = 0$

23. Solve: $x^6 + 1 = 0$

24. Solve: $x^4 - 1 = 0$

25. Solve: $x^5 - 1 = 0$

26. Solve: $x^7 - 1 = 0$

27. Solve: $x^5 + x^3 + x^2 + 1 = 0$

28. Solve: $x^8 - x^5 + x^3 - 1 = 0$

29. Solve: $x^7 + x^4 + x^3 + 1 = 0$

30. Solve: $x^7 - x^4 + x^3 - 1 = 0$

ANSWERS

PART – A

(1) 9 – 10i (2) (23, 33) (3) 7 – i (4) i (5) $\frac{3+i}{5}$ (6) (i) $\frac{4}{25} - \frac{3i}{25}$ (ii) $\frac{3}{5} + \frac{i}{5}$ (iii) $-15 + 53i$

(7) (i) $Re(z) = \frac{1}{5}, Im(z) = \frac{1}{5}$ (ii) $Re(z) = \frac{2}{13}, Im(z) = \frac{-3}{13}$

(iii) $Re(z) = \frac{-3}{10}, Im(z) = \frac{-1}{10}$ (iv) $Re(Z) = 0, Im(z) = 1$

(8) (i) $\overline{Z} = 47 - i$ (ii) $\overline{Z} = 2(1-i)$ (iii) $\overline{Z} = i$ (iv) $\overline{Z} = \frac{-5+i}{13}$

(9)

(i) $r = 2; \theta = \frac{\pi}{6}$ (ii) $r = \sqrt{2}; \theta = -\frac{\pi}{4}$ (iii) $r = 2; \theta = -\frac{\pi}{6}$

(iv) $r = 2; \theta = -\frac{\pi}{3}$ (v) $r = 1; \theta = \frac{\pi}{3}$ (vi) $r = 1; \theta = \frac{\pi}{3}$ (vii) $r = 2; \theta = -\frac{\pi}{3}$

10) (i) $\sqrt{26}$ (ii) $\sqrt{82}$ (iii) $\sqrt{109}$ (iv) $\sqrt{13}$

12) (i) cos θ – i sin θ (ii) cos 11ϕ + i sin 11ϕ (iii) cos 3θ + i sin 3θ

13) (i) cos 8θ + i sin 8θ (ii) cos 5θ + i sin 5θ (iii) cos 3θ + i sin 3θ

(iv) cos 3θ + i sin 3θ (v) cos 4θ + i sin 4θ (vi) cos 10θ + i sin 10θ

14) – 1 15) –1 16) $5(-1-i\sqrt{3})$ 17) i 17(a) – 15 18) 2i sin θ

19) $\cos(\theta+\phi) + i\sin(\theta+\phi)$ 20) $\cos(\beta-\alpha) + i\sin(\beta-\alpha)$

21) $\cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right)$ 22) 0 23) 0 24) $-\omega-\omega^3-\omega^5$

25) $x = \cos\left(\frac{2k\pi+\pi}{2}\right) + i\sin\left(\frac{2k\pi+\pi}{2}\right)$ where k = 0, 1

26) (i) $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4},\ \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

(ii) $\cos 0 + i\sin 0,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

(iii) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2},\ \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

27) 1

PART – B

1) (i) 1 (ii) $\frac{11+10i}{17}$ (iii) $\frac{-1+i}{2}$ (iv) $\frac{1+7i}{2}$

2) (i) $Re(Z) = -\frac{1}{2},\quad I_m(Z) = \frac{3}{2}$

(ii) $Re(Z) = \frac{1}{2},\quad I_m(Z) = \frac{-9}{2}$

(iii) $Re(Z) = \frac{6}{17},\quad I_m(Z) = -\frac{7}{17}$

3) (i) $\overline{Z} = \frac{13}{23}(11+12i)$ (ii) $\overline{Z} = -i$ (iii) $\overline{Z} = \frac{4}{5}$

4) (i) $r = 1, \theta = \infty$ (ii) $r = \frac{3}{2}, \theta = 45^o$

(iii) $r = 4;\ \theta = \frac{\pi}{3}$ (iv) $r = 2,\ \theta = \frac{-\pi}{4}$

7) (i) 2 cos nθ (ii) 2i sin mθ (iii) 2 cos3θ (iv) 2 cos5θ (v) 2i sin 2θ (vi) 2i sin 8θ

8) 2i sin (x –y) 9) cos (x + y) + i sin (x – y) 10) cos (mα + nβ) + i sin (mα + nβ)

11) $2[\cos 0 + i\sin 0],\ 2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right],\ 2\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right]$

12) $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3},\ \cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3}$

PART – C

1) (i) $\text{Re}(z)=\frac{4}{5}, \text{Im}(z)=\frac{3}{5}$ (ii) $\text{Re}(z)=\frac{-7}{2}, \text{Im}(z)=\frac{1}{2}$

(iii) $\text{Re}(z)=0, \ \text{Im}(z)=1$ (iv) $\text{Re}(z)=\frac{8}{25}, \ \text{Im}(z)=-\frac{6}{25}$

(v) $\text{Re}(z)=1, \ \text{Im}(z)=\frac{1}{2}$ (vi) $\text{Re}(z)=\frac{26}{41}, \ \text{Im}(z)=-\frac{29}{41}$

(vii) $\text{Re}(z)=\frac{366}{725}, \text{Im}(z)=\frac{-298}{725}$ (viii) $\text{Re}(z)=\frac{622}{533}, \ \text{Im}(z)=\frac{-224}{533}$

(ix) $\text{Re}(z)=\sqrt{3}, \ \text{Im}(z)=1$ (x) $\text{Re}(z)=\frac{1}{2}, \ \text{Im}(z)=-\frac{1}{2}\tan\frac{\theta}{2}$

2) (i) $\frac{8-9i}{29}$ (ii) $\frac{193+149i}{41}$ (iii) $\frac{1}{2}-i$

(iii) $\frac{23-24i}{65}$ (v) $\frac{17+7i}{26}$ (vi) -1

3) (i) $\overline{Z}=\frac{13}{25}+\frac{9i}{25}$ (ii) $\overline{Z}=\frac{3}{5}-\frac{4i}{5}$ (iii) $\overline{Z}=\frac{1}{13}+\frac{5i}{13}$

(iv) $\overline{Z}=\frac{11}{29}-\frac{13}{29}i$ (v) $\overline{Z}=1-i$

4) (i) $|Z|=2, \ \theta=\tan^{-1}(1-\sqrt{3})$ (ii) $|Z|=\frac{\sqrt{290}}{58}, \ \theta=\tan^{-1}(17)$

(iii) $|Z|=\sqrt{2}, \ \theta=\tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$ (iv) $|Z|=\sqrt{\frac{5}{2}}, \ \theta=\tan^{-1}=(9)$

(v) $|Z|=\sqrt{5}, \ \theta=\tan^{-1}\left(\frac{1}{2}\right)$ (vi) $|Z|=1, \theta=-\frac{\pi}{2}$

8) (i) $\cos 19\theta - i \sin 19\theta$ (ii) $\cos 15\theta - i \sin 15\theta$ (iii) $\cos 107\theta - i \sin 107\theta$

(iv) -1 (v) 1 (vi) 1

12) $2\cos(\alpha+\beta-\gamma)$

21) $\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}, \ \cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}, \ \cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}, \ \cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}$

22) $\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}, \ \cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}, \ \cos\frac{5\pi}{5}+i\sin\frac{5\pi}{5}, \ \cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}, \cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$

23) $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}, \ \cos\frac{3\pi}{6}+i\sin\frac{3\pi}{6}, \ \cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}, \ \cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6},$

$\cos\frac{9\pi}{6}+i\sin\frac{9\pi}{6}, \ \cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}$

24) $\cos 0+i\sin 0, \cos\frac{2\pi}{4}+i\sin\frac{2\pi}{4}, \cos\frac{4\pi}{4}, i\sin\frac{4\pi}{4}, \ \cos\frac{6\pi}{4}+i\sin\frac{6\pi}{4}$

25) $\cos 0+i\sin 0, \cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}, \cos\frac{4\pi}{5}, i\sin\frac{4\pi}{5}, \ \cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}, \cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}$

26) $\cos 0+i\sin 0, \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}, \cos\frac{4\pi}{7}, i\sin\frac{4\pi}{7}, \ \cos\frac{6\pi}{7}+i\sin\frac{6\pi}{7}, \cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}, \cos\frac{9\pi}{7}+i\sin\frac{9\pi}{7}$

27) Case (i) $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $\cos\frac{3\pi}{3}+i\sin\frac{3\pi}{3}$, $\cos\frac{9\pi}{7}+i\sin\frac{9\pi}{7}$

Case (ii) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$, $\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$

28) Case (i) $\cos 0+i\sin 0$, $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$

Case (ii) $\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$, $\cos\frac{5\pi}{5}+i\sin\frac{5\pi}{5}$,

$\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}$, $\cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$

29) $\cos\left(\frac{2k\pi+\pi}{4}\right)+i\sin\left(\frac{2k\pi+\pi}{4}\right)$, $k=0,1,2,3$.

$\cos\left(\frac{2k\pi+\pi}{3}\right)+i\sin\left(\frac{2k\pi+\pi}{3}\right)$, $k=0,1,2$

30) $\cos\left(\frac{2k\pi+\pi}{4}\right)+i\sin\left(\frac{2k\pi+\pi}{4}\right)$, $k=0,1,2,3$

$\cos\left(\frac{2k\pi+\pi}{3}\right)+i\sin\left(\frac{2k\pi+\pi}{3}\right)$, $k=0,1,2$

UNIT – III TRIGONOMETRY

3.1 COMPOUND ANGLES

3.1 Expansion of sin (A ± B), cos (A ± B) and tan (A ± B) (without proof). Problems using above expansions.

3.2 Trigonometrical ratios of multiple angles of 2A and 3A and sub-multiple angles. Simple problems.

3.3 Trigonometrical ratios of sum and product formulae. Simple problems.

Introduction:

Trigonometry is one of the oldest branches of Mathematics. The word Trigonometry is derived from the Greek words 'Trigonon' and 'metron' means measurement of angles. In olden days Trigonometry was mainly used as a tool for studying astronomy. In earlier stages Trigonometry was mainly concerned with angles of a triangle. But now it has its applications in various branches of science such as surveying, engineering, navigations etc. For the study of higher mathematics, knowledge of Trigonometry is essential.

Trigonometrical ratios:

There are six Trigonometrical ratios sine, cosine, tangent, cotangent, secant and cosecant shortly written as sin θ, cos θ, tan θ, cot θ, sec θ and cosec θ.

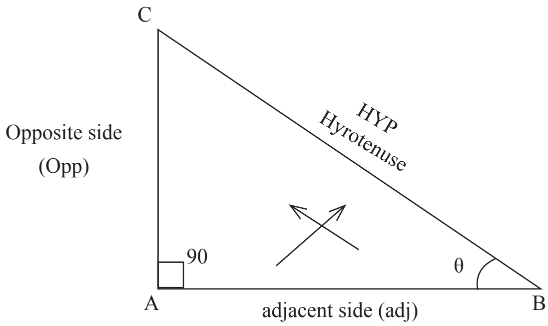


Fig.3.11

The side opposite to angle θ (Theta) is called opposite side. The side opposite to 90^o is called Hypotenuse. The other side is adjacent side.

$$\sin\theta = \frac{\text{Opp}}{\text{Hyp}} \qquad \text{cosec}\theta = \frac{1}{\sin\theta} = \frac{\text{Hyp}}{\text{Opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{Hyp}} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{\text{Hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{Opp}}{\text{adj}} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\text{adj}}{\text{opp}}$$

Note:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Fundamental trigonometrical identities

1) $\sin^2\theta + \cos^2\theta = 1$ 2) $1 + \tan^2\theta = \sec^2\theta$ 3) $1 + \cot^2\theta = \text{cosec}^2\theta$

Trigonometrical ratios of known angles

θ	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Signs of Trigonometrical ratios

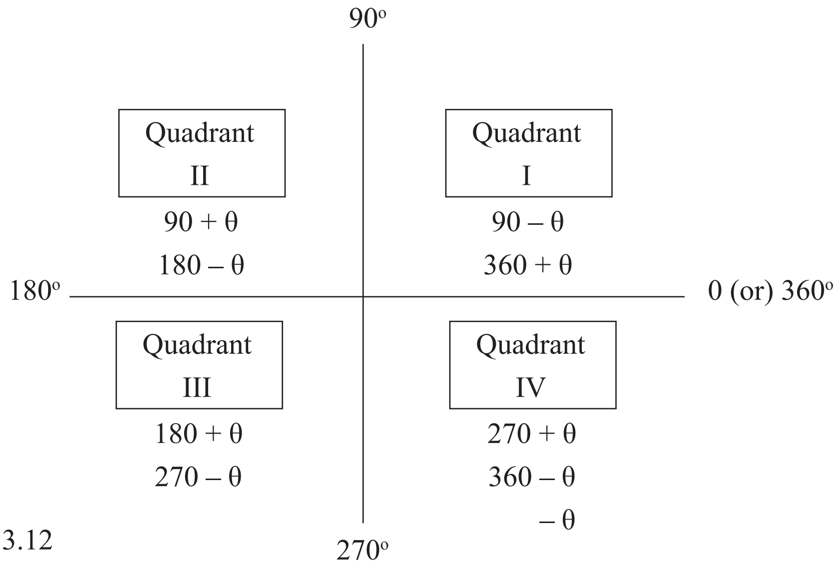


Fig.3.12

Quadrant	Signs of ratios	Remember
I	All are +ve	All
II	Sin θ and cosec $\theta = \frac{1}{\sin\theta}$ are +ve; Other ratios are – ve.	Silver
III	tan θ and cot θ are + ve; other ratios are – ve.	Tea
IV	cos θ and sec $\theta = \frac{1}{\cos\theta}$ are + ve; other ratios are –ve.	Cups

Table 3.11

Trigonometrical ratios of related or allied Angles:

The basic angle is θ and angles associated with θ by a right angle (or) its multiples are called related angle or allied angles. Thus $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ are known as related or allied angles.

Trigonometrical ratios of any Angle

Working Rule for $180 \pm \theta$ and $360 \pm \theta$

1. Determine the sign.
2. Change of ratio-**Do not change**

Refer Fig.3.12

Ratio 180º (or) 360º	Falls in Quadrant Refer Fig.3.12	Sign Refer Table 3.11	Change of ratio (Do not change)
1) sin (180 – θ)	II (Silver)	sin θ is + ve	+ sin θ
2) tan (180 + θ)	III (Tea)	tan θ is + ve	+ tan θ
3) sec (360 – θ)	IV (Cups)	sec θ is +ve	+ sec θ
4) tan (180 – θ)	II (Silver)	tan θ is – ve	– tan θ
5) cos (180 + θ)	III (Tea)	cos θ is – ve	– cos θ
6) cosec (360 – θ)	IV (Cups)	cosec θ is – ve	– cosec θ

Working Rule for 90 ± θ and 270 ± θ

1. Determine the sign
2. **Change of ratio** as follows:

sin θ will change as cos θ and cos θ will change as sin θ

tan θ will change as cot θ and cot θ will change as tan θ

sec θ will change as cosec θ and cosec θ will change as sec θ

Ratio 90º (or) 270º	Refer Fig.3.12 Quadrant	Referable 3.11 Sign	Change of ratio
1) sin (90 – θ)	I (All)	sin θ is + ve	+ cos θ
2) cosec (90 + θ)	II (Silver)	cosec θ is + ve	+ sec θ
3) cot (270 – θ)	III (Tea)	cot θ is +ve	+ tan θ
4) sin (270 + θ)	IV (Cups)	sinθ is – ve	– cos θ
5) tan (270 + θ)	IV (Cups)	tan θ is – ve	– cot θ
6) cos (270 – θ)	III (Tea)	cos θ is – ve	– sin θ

Examples:

1) $\sin 180^{\circ} = \sin(\overset{\text{III}}{180} + 0) = -\sin 0^{\circ} = 0$

2) $\cos 180^{\circ} = \cos(\overset{\text{III}}{180} + 0) = -\cos 0^{\circ} = -1$

3) $\tan 225^{\circ} = \tan(\overset{\text{III}}{180} + 45) = +\tan 45^{\circ} = 1$

4) $\sin 120^{\circ} = \sin(\overset{\text{II}}{180} - 60) = +\sin 60^{\circ} = \frac{\sqrt{3}}{2}$

5) $\cos 120^{\circ} = \cos(\overset{\text{II}}{180} - 60) = -\cos 60^{\circ} = -\frac{1}{2}$

6) $\cot 210 = \cot(\overset{\text{III}}{180} + 30) = +\cot 30^{\circ} = \frac{1}{\left(\frac{1}{\sqrt{3}}\right)} = \sqrt{3}$

7) $\left.\begin{aligned} \sin 270^{\circ} &= \sin(\overset{\text{III}}{180} + 90) = -\sin 90^{\circ} = -1 \\ &= \sin(\overset{\text{IV}}{270} + 0) = -\cos 0 = -1 \end{aligned}\right\}$

8) $\left.\begin{aligned} \cos 240 &= \cos(\overset{\text{III}}{180} + 60) = -\cos 60 = -\frac{1}{2} \\ &= \cos(\overset{\text{III}}{270} - 30) = -\sin 30 = -\frac{1}{2} \end{aligned}\right\}$

9) $\underset{(\text{Remove multiangles of } 360^{\circ})}{\sin 600} = \sin 240 = \sin(\overset{\text{III}}{180} + 60) = -\sin 60 = -\frac{\sqrt{3}}{2}$

10) $\underset{(\text{Remove multiangles of } 360^{\circ})}{\cos 720} = \cos 0 = 1$

Note:

– θ falls in Quadrant IV where cos θ and sec θ are + ve. Therefore cos (– θ) = cos θ and sec (– θ) = sec θ ; sin (– θ) = – sin θ ; tan (– θ) = – tan θ ; cot (– θ) = – cot θ ; cosec (– θ) = – cosec θ.

Compound Angles:

If an angle is expressed as the algebraic sum or difference of two or more angles, then it is called compound angle.

Formulae:

1) $\sin(A + B) = \sin A \cos B + \cos A \sin B$
2) $\sin(A - B) = \sin A \cos B - \cos A \sin B$
3) $\cos(A + B) = \cos A \cos B - \sin A \sin B$
4) $\cos(A - B) = \cos A \cos B + \sin A \sin B$
5) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
6) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Results:

1. Prove that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$

$$\sin(A + B) = \underbrace{\sin A \cos B}_{x} + \underbrace{\cos A \sin B}_{y} = x + y \text{ (say)}$$

$$\sin(A - B) = \underbrace{\sin A \cos B}_{x} - \underbrace{\cos A \sin B}_{y} = x - y$$

LHS $= \sin(A + B)\sin(A - B) = (x + y)(x - y) = x^2 - y^2$

$= (\sin A \cos B)^2 - (\cos A \sin B)^2$

$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$

$= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B$

$= \sin^2 A - \sin^2 B =$ RHS

$\sin^2 B + \cos^2 B = 1$ $\cos^2 B = 1 - \sin^2 B$
$\sin^2 A = \cos^2 A = 1$ $\cos^2 A = 1 - \sin^2 A$

2. Prove that $\cos(A + B)\cos(A - B) = \cos^2 B - \sin^2 A = \cos^2 A - \sin^2 B$.

$$\cos(A + B) = \underbrace{\cos A \cos B}_{x} - \underbrace{\sin A \sin B}_{y} = x - y$$

$$\cos(A - B) = \underbrace{\cos A \cos B}_{x} + \underbrace{\sin A \sin B}_{y} = x + y$$

LHS $= \cos(A + B)\cos(A - B)$

$= (x - y)(x + y) = x^2 - y^2$

$= (\cos A \cos B)^2 - (\sin A \sin B)^2$

$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$

$= (1 - \sin^2 A)\cos^2 B - \sin^2 A (1 - \cos^2 B)$

$= \cos^2 B - \sin^2 A \cos^2 B - \sin^2 A + \sin^2 A \cos^2 B$

$= \cos^2 B - \sin^2 A =$ RHS

$= (1 - \sin^2 B) - (1 - \cos^2 A)$

$= 1 - \sin^2 B - 1 + \cos^2 A$

$= \cos^2 A - \sin^2 B$

$\sin^2 A + \cos^2 A = 1$ $\cos^2 A = 1 - \sin^2 A$
$\sin^2 B = \cos^2 B = 1$ $\sin^2 B = 1 - \cos^2 B$

$\sin^2 B + \cos^2 B = 1$ $\cos^2 B = 1 - \sin^2 B$
$\sin^2 A = \cos^2 A = 1$ $\sin^2 A = 1 - \cos^2 A$

3.1 WORKED EXAMPLES

PART – A

1. Find the value of $\sin 65^o \cos 25^o + \cos 65^o \sin 25^o$.

Solution:

$\sin 65^o \cos 25^o + \cos 65^o \sin 25^o$

$= \sin (65^o + 25^o) = \sin 90^o = 1$

sin (A + B) formula

2. Find the value of $\sin 40^o \cos 10^o – \cos 40^o \sin 10^o$.

Solution:

$\sin 40^o \cos 10^o – \cos 40^o \sin 10^o$

$= \sin (40^o –10^o) = \sin 30^o = \frac{1}{2}$.

sin (A – B) formula

3. Find the value of $\cos 50^o \cos 40^o – \sin 50^o \sin 40^o$.

Solution:

$\cos 50^o \cos 40^o – \sin 50^o \sin 40^o$

$= \cos (50^o + 40^o) = \cos 90^o = 0$

cos (A – B) formula

4. What is the value of $\cos 70^o \cos 10^o + \sin 70^o \sin 10^o$?

Solution:

$\cos 70^o \cos 10^o + \sin 70^o \sin 10^o$

$= \cos (70^o – 10^o) = \cos 60^o = \frac{1}{2}$

cos (A + B) formula

5. Find the value of $\frac{\tan 20^o + \tan 25^o}{1 - \tan 20^o \tan 25^o}$.

Solution:

$\frac{\tan 20^o + \tan 25^o}{1 - \tan 20^o \tan 25^o}$

$= \tan (20^o + 25^o) = \tan 45^o = 1$

tan (A + B) formula

6. Find the value of $\frac{\tan 135^o - \tan 75^o}{1 + \tan 135^o \tan 75^o}$.

Solution:

$\frac{\tan 135^o - \tan 75^o}{1 + \tan 135^o \tan 75^o}$

tan (A – B) formula

$= \tan (135^o – 75^o) = \tan 60^o = \sqrt{3}$

PART – B

1. Find the value of $\cos (60^o – A) \cos (30^o + A) – \sin (60^o – A) \sin (30^o + A)$.

Solution:

$\cos (60^o – A) \cos (30^o + A) – \sin (60^o – A) \sin (30^o + A)$

$= \cos x \cos y – \sin x \sin y$

$= \cos (x + y) = \cos 90^o = 0$

$x = 60^o – A$ $y = 30^o + A$
$x + y = 90^o$

2. Find the value of sin 15°.

Solution:

$$\sin 15^\circ = \sin (45^\circ - 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

3. Find hte value of cos 75°.

Solution:

$$\cos 75^\circ = \cos (45^\circ + 30^\circ)$$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

4. Prove that $\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)} = \tan A$.

Solution:

$$\text{LHS} = \frac{\sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B}{\cos A \cos B - \sin A \sin B + \cos A \cos B + \sin A \sin B}$$

$$= \frac{2 \sin A \cos B}{2 \cos A \cos B}$$

$$= \frac{\sin A}{\cos A} = \tan A = \text{RHS}$$

5. Prove that sin (A + B) sin (A – B) + sin (B + C) sin (B – C) + sin (C + A) sin (C – A) = 0.

Solution:

$$\text{LHS} = \sin (A + B) \sin (A - B) + \sin (B + C) \sin (B - C) + \sin (C + A) \sin (C - A)$$

$$= \sin^2 A - \sin^2 B + \sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A = 0 = \text{RHS}$$

6. If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$ find the value of tan (A + B).

Solution:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}$$

$$= \frac{\frac{3+2}{6}}{1-\frac{1}{6}} = \frac{\frac{5}{6}}{\frac{6-1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

7. Find the value of tan 105°, with out using tables.

Solution:

tan 105° = tan (60° + 45°)

$$= \frac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ \tan 45^\circ}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}\times 1} = \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

8. Prove that tan 20° + tan 25° + tan 20° tan 25° = 1.

Solution:

20° + 25° = 45°

tan (20° + 25°) = tan 45°

$$\frac{\tan 20^\circ + \tan 25^\circ}{1 - \tan 20^\circ \tan 25^\circ} = 1$$

Cross multiplying, tan 20° + tan 25° = 1 – tan 20° tan 25°

tan 20° + tan 25° + tan 20° tan 25° = 1

PART – C

1. If A and B are acute and if $\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{5}}$ prove that A + B = $\frac{\pi}{4}$.

Solution:

Given: $\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{5}}$

$\cos A = \sqrt{1-\sin^2 A}$	$\cos B = \sqrt{1-\sin^2 B}$
$= \sqrt{1-\left(\frac{1}{\sqrt{10}}\right)^2}$	$= \sqrt{1-\left(\frac{1}{\sqrt{5}}\right)^2}$
$= \sqrt{1-\frac{1}{10}}$	$= \sqrt{1-\frac{1}{5}}$
$= \sqrt{\frac{10-1}{10}}$	$= \sqrt{\frac{5-1}{5}}$
$= \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$	$= \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$

$\sin A = \frac{1}{\sqrt{10}}$	$\sin B = \frac{1}{\sqrt{5}}$
$\cos A = \frac{3}{\sqrt{10}}$	$\cos B = \frac{2}{\sqrt{5}}$

sin (A + B) = sin A cos B + cos A sin B

$$= \frac{1}{\sqrt{10}} \times \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} \times \frac{1}{\sqrt{5}}$$

$$= \frac{2}{\sqrt{50}} + \frac{3}{\sqrt{50}}$$

$$= \frac{5}{\sqrt{50}} = \frac{5}{\sqrt{25 \times 2}} = \frac{5}{5\sqrt{2}}$$

$$\sin(A+B) = \frac{1}{\sqrt{2}} = \sin 45^\circ$$

$$A + B = 45^\circ = \frac{180^\circ}{4} = \frac{\pi}{4}$$

2. If A and B are acute and if $\cos A = \frac{1}{7}$ and $\cos B = \frac{13}{14}$, prove that $A - B = 60^o = \frac{\pi}{3}$.

Solution:

Given:

$$\cos A = \frac{1}{7} \quad \text{and} \quad \cos B = \frac{13}{14}$$

$$\sin A = \sqrt{1-\cos^2 A}$$
$$= \sqrt{1-\left(\frac{1}{7}\right)^2}$$
$$= \sqrt{1-\frac{1}{49}}$$
$$= \sqrt{\frac{49-1}{49}}$$
$$= \sqrt{\frac{48}{49}} = \sqrt{\frac{16\times 3}{49}}$$
$$= \frac{4\sqrt{3}}{7}$$

$$\sin B = \sqrt{1-\cos^2 B}$$
$$= \sqrt{1-\left(\frac{13}{14}\right)^2}$$
$$= \sqrt{1-\frac{169}{196}}$$
$$= \sqrt{\frac{196-169}{196}}$$
$$= \sqrt{\frac{27}{196}} = \sqrt{\frac{9\times 3}{14\times 14}}$$
$$= \frac{3\sqrt{3}}{14}$$

$\sin A = \frac{4\sqrt{3}}{7}$	$\sin B = \frac{3\sqrt{3}}{14}$
$\cos A = \frac{1}{7}$	$\cos B = \frac{13}{14}$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$
$$= \frac{1}{7}\times\frac{13}{14}+\frac{4\sqrt{3}}{7}\times\frac{3\sqrt{3}}{14}$$
$$= \frac{13}{98}+\frac{12\times 3}{98}$$
$$= \frac{13+36}{98} = \frac{49}{98} = \frac{1}{2}$$
$$\cos(A-B) = \frac{1}{2} = \cos 60^o$$
$$A - B = 60^o = \frac{180^o}{3} = \frac{\pi}{3}$$

3. If $A + B = 45^o$, prove that $(1 + \tan A)(1 + \tan B) = 2$ and hence deduce the value of $\tan 22\frac{1}{2}^o$.

Solution:

Given: $A + B = 45^o$

Taking tan on both sides

$\tan(A + B) = \tan 45^o$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$\tan A + \tan B = 1 - \tan A \tan B$

$\tan A + \tan B + \tan A \tan B = 1$ ....(1)

LHS = $(1 + \tan A)(1 + \tan B)$

$= 1 + \tan B + \tan A + \tan A \tan B$

$= 1 + 1$ using (1)

$= 2$ = RHS

Put A = $22\frac{1}{2}^{o}$, B = $22\frac{1}{2}^{o}$. Then A + B = $22\frac{1}{2}^{o} + 22\frac{1}{2}^{o} = 45^{o}$.

$(1 + \tan 22\frac{1}{2}^{o})(1 + \tan 22\frac{1}{2}^{o}) = 2$

(say) x x

$x^2 = 2$

$x = \sqrt{2}$

$1 + \tan 22\frac{1}{2} = \sqrt{2}$

$\tan 22\frac{1}{2} = \sqrt{2} - 1$

4. If A and B are acute angle and if tan A = $\frac{n}{n+1}$ and tan B = $\frac{1}{2n+1}$, show that A + B = $\frac{\pi}{4}$.

Solution:

Given: tan A = $\frac{n}{n+1}$ and tan B = $\frac{1}{2n+1}$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{n}{(n+1)} + \frac{1}{(2n+1)}}{1 - \frac{n}{(n+1)} \times \frac{1}{(2n+1)}}$$

$$= \frac{\frac{n(2n+1) + 1(n+1)}{(n+1)(2n+1)}}{\frac{(n+1)(2n+1) - n \times 1}{(n+1)(2n+1)}}$$

$$= \frac{2n^2 + n + n + 1}{2n^2 + n + 2n + 1 - n}$$

$$= \frac{2n^2 + 2n + 1}{2n^2 + 2n + 1} = 1$$

$$\tan(A+B) = 1 = \tan 45^{o}$$

$$A + B = 45^{o} = \frac{180^{0}}{4} = \frac{\pi}{4}$$

5. If tan A – tan B = p and cot B – cot A = q, show that cot (A – B) = $\frac{1}{p} + \frac{1}{q}$.

Solution:

RHS = $\frac{1}{p} + \frac{1}{q}$

$$= \frac{1}{\tan A - \tan B} + \frac{1}{\cot B - \cot A}$$

$$= \frac{1}{\tan A - \tan B} + \frac{1}{\frac{1}{\tan B} - \frac{1}{\tan A}}$$

$$= \frac{1}{\tan A - \tan B} + \frac{1}{\frac{\tan A - \tan B}{\tan A \tan B}}$$

$$= \frac{1 + \tan A \tan B}{\tan A - \tan B}$$

$= \cot(A - B)$

= RHS

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\cot(A-B) = \frac{1 + \tan A \tan B}{\tan A - \tan B}$$

3.2 MULTIPLE AND SUB-MULTIPLE ANGLES

Trigonometric ratios of multiple angles of 2A in terms to that of A.

1. Prove that sin 2A = 2 sin A cos A

We know that

sin (A + B) = sin A cos B + cos A sin B

Put B = A

sin (A + A) =sin A cos A + cos A sin A

sin 2A = sin A cos A + sin A cos A

$\boxed{\sin 2A = 2\sin A \cos A}$ Formula for sin 2A

2. Prove that cos 2A = $\cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$.

We know that

cos (A + B) = cos A cos B – sin A sin B

Put B = A

cos (A + A) = cos A cos (A) – sin A sin (A)

$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$ Formula for cos 2A .... (i)

$= \cos^2 A - (1 - \cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A$

$\boxed{\cos 2A = 2\cos^2 A - 1}$ Formula for cos2A .... (ii)

$= 2(1 - \sin^2 A) - 1$

$= 2 - 2\sin^2 A - 1$

$\boxed{\cos 2A = 1 - 2\sin^2 A}$ Formula for cos 2A .... (iii)

3. Prove that $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$.

Solution:

We know that

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Put B = A

$$\tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\boxed{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

4. Prove that

a) $\sin^2 A = \frac{1-\cos 2A}{2}$

b) $\cos^2 A = \frac{1+\cos 2A}{2}$

c) $\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$

Solution:

(a) $\text{RHS} = \frac{1-\cos 2A}{2} = \frac{1-(1-2\sin^2 A)}{2}$ using formula for cos 2A – (iii)

$= \frac{\not{1} - \not{1} + 2\sin^2 A}{2} = \frac{\not{2}\sin^2 A}{\not{2}}$

$= \sin^2 A = \text{LHS}$

(b) $\text{RHS} = \frac{1+\cos 2A}{2} = \frac{1+(2\cos^2 A - 1)}{2}$ using formula for cos 2A – (ii)

$= \frac{2\cos^2 A}{2} = \cos^2 A = \text{LHS}$

(c) Formula for cos 2A – (iii)

$\text{RHS} = \frac{1-\cos 2A}{1+\cos 2A} = \frac{1-(1-2\sin^2 A)}{1+2\cos^2 A - 1}$

$= \frac{\not{1} - \not{1} + 2\sin^2 A}{2\cos^2 A} = \frac{\not{2}\sin^2 A}{\not{2}\cos^2 A}$

$= \frac{\sin^2 A}{\cos^2 A} = \tan^2 A = \text{LHS}$

<u>To express sin 2A and cos 2A in terms of tan A</u>

Prove that

(a) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

(b) $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

Solution:

(a) $\text{RHS} = \frac{2\tan A}{1+\tan^2 A} = \frac{\frac{2\sin A}{\cos A}}{1+\frac{\sin^2 A}{\cos^2 A}}$

$= \frac{\frac{2\sin A}{\cos A}}{\frac{\cos^2 A + \sin^2 A}{\cos^2 A}} = \frac{\frac{2\sin A}{\cos A}}{\frac{1}{\cos^2 A}}$

$= \dfrac{2\sin A}{\cos A} \times \dfrac{\cos^2 A}{1}$ $\boxed{\text{cancelling} \cos A}$

$= 2\sin A \cos A$ Formula

$= \sin 2A = \text{LHS}$ $\boxed{\sin 2A = 2\sin A \cos A}$

SUB-MULTIPLE ANGLES

If **A** is any angle, then **A/2** is called sub-multiple angle.

Prove that

a) $\sin A = 2\sin\left(A/2\right)\cos\left(A/2\right)$

b) $\cos A = \cos^2\left(A/2\right) - \sin^2\left(A/2\right)$

$= 2\cos^2\left(A/2\right) - 1$

$= 1 - 2\sin^2\left(A/2\right)$

a) We know that

sin 2A = 2 sin A cos A

Divide the angles on both sides by 2

$\sin\left(\dfrac{2A}{2}\right) = 2\sin\left(A/2\right)\cos\left(A/2\right)$

$\sin A = 2\sin\left(A/2\right)\cos\left(A/2\right)$

(b) We know that

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

Divide the angles on both sides by 2

$\cos A = \cos^2\left(\dfrac{A}{2}\right) - \sin^2\left(\dfrac{A}{2}\right)$ ....(i)

$= 2\cos^2\left(\dfrac{A}{2}\right) - 1$ ....(ii)

$= 1 - 2\sin^2\left(\dfrac{A}{2}\right)$ ....(iii)

Prove that

(a) $\sin^2\left(\dfrac{A}{2}\right) = \dfrac{1-\cos A}{2}$

(b) $\cos^2\left(\dfrac{A}{2}\right) = \dfrac{1+\cos A}{2}$

(c) $\tan^2\left(\dfrac{A}{2}\right) = \dfrac{1-\cos A}{1+\cos A}$

(a) $$\text{RHS} = \frac{1-\cos A}{2} = \frac{1-\left[1-2\sin^2\left(\frac{A}{2}\right)\right]}{2}$$

$$= \frac{1-1+2\sin^2\left(\frac{A}{2}\right)}{2} = \frac{2\sin^2\left(\frac{A}{2}\right)}{2}$$ using Formula (iii)

$$= \sin^2\left(\frac{A}{2}\right) = \text{LHS}$$

(b) $$\text{RHS} = \frac{1+\cos 2A}{2} = \frac{1+2\cos^2\left(\frac{A}{2}\right)-1}{2}$$ using Formula (ii)

$$= \frac{2\cos^2\left(\frac{A}{2}\right)}{2} = \cos^2\left(\frac{A}{2}\right) = \text{LHS}$$

$$\text{RHS} = \frac{1-\cos A}{1+\cos A} = \frac{1-\left[1-2\sin^2\left(\frac{A}{2}\right)\right]}{1+2\cos^2\left(\frac{A}{2}\right)-1}$$ using Formula (iii)

$$= \frac{1-1+2\sin^2\left(\frac{A}{2}\right)}{2\cos^2\left(\frac{A}{2}\right)} = \frac{2\sin^2\left(\frac{A}{2}\right)}{2\cos^2\left(\frac{A}{2}\right)}$$ using Formula (ii)

$$= \tan^2\left(\frac{A}{2}\right) = \text{LHS}$$

To express sin A, cos A, tan A in terms of $\tan\left(\frac{A}{2}\right)$.

Prove that

a) $$\sin A = \frac{2\tan\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)}$$

b) $$\cos A = \frac{1-\tan^2\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)}$$

c) $$\tan A = \frac{2\tan\left(\frac{A}{2}\right)}{1-\tan^2\left(\frac{A}{2}\right)}$$

We have already proved that

$$\sin(2A) = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos(2A) = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

Divide the angles on both sides by 2 on the above three results.

Then we get,

a) $\sin A = \dfrac{2\tan\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)}$

b) $\cos A = \dfrac{1-\tan^2\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)}$

c) $\tan A = \dfrac{2\tan\left(\frac{A}{2}\right)}{1-\tan^2\left(\frac{A}{2}\right)}$

Multiple angles of 3A

Prove that

$\boxed{\sin 3A = 3\sin A - 4\sin^3 A}$ Formula

We know that

sin (A + B) = sin A cos B + cos A sin B

Put B = 2A

sin (A + 2A) = sin A cos (2A) + cos A sin (2A)

$$\begin{aligned}\sin 3A &= \sin A\,(1 - 2\sin^2 A) + \cos A\,(2\sin A\cos A)\\ &= \sin A - 2\sin^3 A + 2\sin A\cos^2 A\\ &= \sin A - 2\sin^3 A + 2\sin A\,(1 - \sin^2 A)\\ &= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A\\ \sin 3A &= 3\sin A - 4\sin^3 A\end{aligned}$$

Prove that

$\boxed{\cos 3A = 4\cos^3 A - 3\cos A}$ Formula

We know that

cos (A + B) = cos A cos B – sin A sin B

Put B = 2A

cos (A + 2A) = cos A cos 2A – sin A sin 2A

$$\begin{aligned}\cos 3A &= \cos A\,(2\cos^2 A - 1) - \sin A\,(2\sin A\cos A)\\ &= 2\cos^3 A - \cos A - 2\cos A\sin^2 A\\ &= 2\cos^3 A - \cos A - 2\cos A\,(1 - \cos^2 A)\\ &= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A\\ &= 4\cos^3 A - 3\cos A\end{aligned}$$

Prove that $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

Solution:

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

We know that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Put $B = 2A$

$$\tan(A+2A) = \frac{\tan A + \tan 2A}{1 - \tan A \,.\, \tan 2A}$$

$$= \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A \times \dfrac{2\tan A}{1-\tan^2 A}}$$

$$= \frac{\dfrac{\tan A(1-\tan^2 A) + 2\tan A}{1-\tan^2 A}}{\dfrac{1-\tan^2 A - 2\tan^2 A}{1-\tan^2 A}}$$

$$= \frac{\tan A - \tan^3 A + 2\tan A}{1 - 3\tan^2 A}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

3.2 WORKED EXAMPLES

PART – A

1. Find the value of 2 sin15° cos 15°.

   Solution:

   We know that sin 2A = 2 sin A cos A

   Put A = 15°

   sin (2 × 15°) = 2in 15° cos 15°

   sin 30° = 2 sin 15° cos 15°

   2 sin 15° cos 15° = sin 30° = $\dfrac{1}{2}$

2. Find the value of 2 sin 75° cos 75°.

   Solution:

   2 sin 75° cos 75° = sin (2 × A)

   = sin (2 × 75°)

   = sin 150° = sin (180 – 30)

   = + sin 30° = $\dfrac{1}{2}$

3. Find the value of $\cos^2 15^o - \sin^2 15^o$.

Solution:

We know that

$$\cos^2 A - \sin^2 A = \cos 2A$$

Put $A = 15^o$

$$\cos^2 15^o - \sin^2 15^o = \cos(2 \times 15^o)$$

$$= \cos 30^o = \frac{\sqrt{3}}{2}$$

4. Find the value of $1 - 2\sin^2 22\tfrac{1}{2}^o$.

Solution:

$$1 - 2\sin^2 A = \cos 2A$$

$$1 - 2\sin^2 22\tfrac{1}{2}^o = \cos(2 \times 22\tfrac{1}{2}^o) = \cos 45^o = \frac{1}{\sqrt{2}}$$

5. Prove that $\dfrac{2\tan 15^o}{1+\tan^2 15^o} = \dfrac{1}{2}$

Solution:

We know that

$$\frac{2\tan A}{1+\tan^2 A} = \sin 2A$$

Put $A = 15^o$

$$\frac{2\tan 15^o}{1+\tan^2 15^o} = \sin 2(15^o)$$

$$= \sin 30^o = \frac{1}{2}$$

6. Find the value of $\dfrac{1-\tan^2 15^o}{1+\tan^2 15^o}$.

Solution:

We know that

$$\frac{1-\tan^2 A}{1+\tan^2 A} = \cos 2A$$

$$\frac{1-\tan^2 15^o}{1+\tan^2 15^o} = \cos(2\times 15) = \cos 30 = \frac{\sqrt{3}}{2}$$

7. Prove that $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$.

Solution:

$$\text{LHS} = \frac{\sin 2A}{1+\cos 2A}$$

$$= \frac{2\sin A\cos A}{1+2\cos^2 A - 1}$$

Formula
$\sin 2A = 2\sin A\cos A$
$\cos 2A = 2\cos^2 A - 1$

$$= \frac{\not{2} \sin A \cos A}{\not{2} \cos^2 A}$$

$$= \frac{\sin A}{\cos A} \quad \boxed{\text{cancelling } \cos A}$$

$$= \tan A = \text{LHS}$$

8. Prove that $\dfrac{\sin A}{1 - \cos A} = \cot\left(\dfrac{A}{2}\right)$.

Solution:

$$\text{LHS} = \frac{\sin A}{1 - \cos A}$$

Formulae
$\sin 2A = 2 \sin A \cos A$
$\cos 2A = 1 - 2\sin^2 A$

Dividing angles by 2

$$= \frac{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{1 - \left(1 - 2\sin^2 \frac{A}{2}\right)}$$

$\sin A = 2\sin A/2 \cos A/2$
$\cos A = 1 - 2\sin^2 A/2$

$$= \frac{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{1 - 1 + 2\sin^2\left(\frac{A}{2}\right)}$$

$$= \frac{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{2\sin^2\left(\frac{A}{2}\right)}$$

$$= \frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)} = \cot\left(\frac{A}{2}\right)$$

9. Prove that $\left(\sin\dfrac{A}{2} - \cos\dfrac{A}{2}\right)^2 = 1 - \sin A \cdot$

Solution:

$$\text{LHS} = \left(\sin\frac{A}{2} - \cos\frac{A}{2}\right)^2 \quad \boxed{(a-b)^2 = a^2 + b^2 - 2ab}$$

$$= \sin^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{A}{2}\right) - 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$

$$= 1 - \sin A$$

$$= \text{RHS}$$

$\sin^2\theta + \cos^2\theta = 1$
$2\sin A/2 \cos A/2 = \sin 2(A/2)$
$= \sin A$

10. Find the value of $3 \sin 20^o - 4 \sin^3 20^o$.

We know that

$\sin 3A = 3 \sin A - 4 \sin^3 A$

Put A = 20

$\sin (3 \times 20^o) = 3 \sin 20^o - 4 \sin^3 20^o$

$3 \sin 20^o - 4\sin^3 20^o = \sin (3 \times 20^o)$

$= \sin 60^o = \dfrac{\sqrt{3}}{2}$

11. Find that value of $4 \cos^3 10^o - 3 \cos 10^o$.

We know that

$4 \cos 3A - 3 \cos A = \cos 3A$

$4 \cos^3 10^o - 3 \cos 10^o = \cos (3 \times 10^o)$

$= \cos 30^o = \dfrac{\sqrt{3}}{2}$

12. Find the value of $3 \sin 40^o - 4 \sin^3 40^o$.

Solution:

We know that

$3 \sin A - 4 \sin 3A = \sin 3A$

$3 \sin 40^o - 4 \sin^3 40^o = \sin (3 \times 40^o)$

$= \sin 120^o$

$= \sin (180^o - 60^o)$

$= \sin 60^o = \dfrac{\sqrt{3}}{2}$

PART – B

1. Prove that tan A + cot A = 2 cosec 2A.

Solution:

LHS = tan A + cot A

$$\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{1}{\sin A \cos A} = \frac{2}{2 \sin A \cos A} \quad \boxed{\text{Multiply and Divide by 2}}$$

$$= \frac{2}{\sin 2A} \quad \boxed{\because \sin 2A = 2 \sin A \cos A}$$

$$= 2\text{cosec } 2A \quad \boxed{\because \frac{1}{\sin A} = \text{cosec } A}$$

= RHS

2. Prove that $\dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A$.

Solution:

$$\text{LHS} = \frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A}$$

Formula

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = 2 \cos^2 A - 1$

$= \frac{\sin A + 2\sin A \cos A}{\cancel{1} + \cos A + 2\cos^2 A - \cancel{1}}$

sin A is common in Nr.
cos A is common in Dr.

$= \frac{\sin A[1 + 2\cos A]}{\cos A[1 + 2\cos A]}$

$= \frac{\sin A}{\cos A}$ [cancelling 1 + 2 cosA]

$= \tan A = \text{RHS}$

3. Prove that $\cos^4 A - \sin^4 A = \cos 2A$.

Solution:

LHS $= (\cos^2 A)^2 - (\sin^2 A)^2$ $a^2 - b^2 = (a + b)(a - b)$

$= (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$

$= 1(\cos 2A) = \cos 2A = \text{RHS}$

PART – C

1. Prove that $\frac{1 - \cos 2A + \sin 2A}{1 + \cos 2A + \sin 2A} = \tan A$.

Solution:

LHS $= \frac{1 - \cos 2A + \sin 2A}{1 + \cos 2A + \sin 2A}$

Formulae
$\sin 2A = 2\sin A \cos A$
$\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

$= \frac{1 - (1 - 2\sin^2 A) + 2\sin A \cos A}{\cancel{1} + 2\cos^2 A - \cancel{1} + 2\sin A \cos A}$

$= \frac{\cancel{1} - \cancel{1} + 2\sin^2 A + 2\sin A \cos A}{2\cos^2 A + 2\sin A \cos A}$

$= \frac{2\sin A(\sin A + \cos A)}{2\cos A(\cos A + \sin A)}$

2sin A is common in Nr
2cos A is common in Dr

$= \frac{\sin A}{\cos A} = \tan A = \text{RHS}$

2. Prove that $\frac{1 - \cos A + \sin A}{1 + \cos A + \sin A} = \tan\left(\frac{A}{2}\right)$.

Solution:

LHS $= \frac{1 - \cos A + \sin A}{1 + \cos A + \sin A}$

$\sin A = 2\sin\left(A/2\right)\cos\left(A/2\right)$
$\cos A = 1 - 2\sin^2\left(A/2\right)$
$= 2\cos^2\left(A/2\right) - 1$

$= \frac{1 - \left[1 - 2\sin^2\left(A/2\right)\right] + 2\sin\left(A/2\right)\cos\left(A/2\right)}{1 + \left[2\cos^2\left(A/2\right) - 1\right] + 2\sin\left(A/2\right)\cos\left(A/2\right)}$

$= \frac{1 - 1 + 2\sin^2\left(\frac{A}{2}\right) + 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{1 + 2\cos^2\left(\frac{A}{2}\right) + 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$

$2\sin\left(A/2\right)$ is common in Nr
$2\cos\left(A/2\right)$ is common in Dr

$$= \frac{2\sin\left(A/2\right)\left[\sin\left(A/2\right) + \cos\left(A/2\right)\right]}{2\cos\left(A/2\right)\left[\cos\left(A/2\right) + \sin\left(A/2\right)\right]}$$

$$= \frac{\sin\left(A/2\right)}{\cos\left(A/2\right)} = \tan\left(A/2\right) = \text{LHS}$$

3. If $\tan A = \frac{1}{3}$ and $\tan B = \frac{1}{7}$ show that $2A + B = \frac{\pi}{4}$.

Solution:

Given $\tan A = \frac{1}{3} \quad \tan B = \frac{1}{7}$

Formula for $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$

$$= \frac{2\left(\frac{1}{3}\right)}{1 - \left(\frac{1}{3}\right)^2}$$

$$= \frac{\frac{2}{3}}{1 - \frac{1}{9}} = \frac{\frac{2}{3}}{\frac{9-1}{9}}$$

$$= \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{\not{2}}{\not{3}} \times \frac{\not{9}\,3}{\not{8}\,4}$$

$$\tan 2A = \frac{3}{4} \qquad \text{...........(1)}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Put $A = 2A$ on both sides

$$\tan(2A+B) = \frac{\tan 2A + \tan B}{1 - \tan 2A \tan B}$$

$$= \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \times \frac{1}{7}} \qquad \text{using (1)}$$

$$= \frac{\frac{21+4}{28}}{\frac{28-3}{28}} = \frac{\frac{25}{28}}{\frac{25}{28}}$$

$$= \frac{25}{28} \times \frac{28}{25} = 1$$

$$\tan(2A + B) = 1 = \tan 45^\circ$$

$$2A + B = 45^\circ = \frac{180^\circ}{4} = \frac{\pi}{4}$$

4. If $\sin\theta = \frac{3}{5}$ find the value of sin 3θ.

Solution:

Given: $\sin\theta = \frac{3}{5}$

We know that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$= 3\left(\frac{3}{5}\right) - 4\left(\frac{3}{5}\right)^3$$

$$= \frac{9}{5} - 4\left(\frac{27}{125}\right) = \frac{9}{5} - \frac{108}{125}$$

$$= \frac{225-108}{125} = \frac{117}{125}$$

5. If $\cos A = \frac{1}{3}$ find the value of cos 3A.

Solution:

Given : $\cos A = \frac{1}{3}$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$= 4\left(\frac{1}{3}\right)^3 - 3\left(\frac{1}{3}\right)$$

$$= 4\left(\frac{1}{27}\right) - 1 = \frac{4}{27} - 1$$

$$= \frac{4-27}{27} = \frac{-23}{27}$$

6. Prove that $\frac{\sin 3A}{1+2\cos 2A} = \sin A$.

Solution:

$$\text{LHS} = \frac{\sin 3A}{1+2\cos 2A}$$

Formulae
$\sin 3A = 3\sin A - 4\sin^3 A$
$\cos 2A = 1 - 2\sin^2 A$

$$= \frac{3\sin A - 4\sin^3 A}{1+2(1-2\sin^2 A)}$$

sin A is common in Nr

$$= \frac{\sin A(3-4\sin^2 A)}{1+2-4\sin^2 A}$$

$$= \frac{\sin A(3-4\sin^2 A)}{(3-4\sin^2 A)}$$

$$= \sin A = \text{RHS}$$

7. Prove that 4 sin A sin (60 + A) sin (60 – A) = sin3A.

Solution:

LHS = 4 sin A sin (60 + A) sin (60 – A)

= 4 sin A [$\sin^2 60 - \sin^2 A$]

Formula:

$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

$$= 4\sin A\left[\frac{3}{4} - \sin^2 A\right]$$

$$\boxed{\begin{aligned}&\sin 60 = \frac{\sqrt{3}}{2}\\ &\sin^2 60 = \frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2} = \frac{3}{4}\end{aligned}}$$

$$= 4\sin A\left[\frac{3-4\sin^2 A}{4}\right]$$

$$= \sin A\left[3-4\sin^2 A\right]$$

$$= 3\sin A - 4\sin^3 A = \text{RHS}$$

8. Prove that $\dfrac{\sin 3A}{\sin A} - \dfrac{\cos 3A}{\cos A} = 2$

Solution:

$$\text{LHS} = \frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A}$$

$$= \frac{(3\sin A - 4\sin^3 A)}{\sin A} - \frac{(4\cos^3 A - 3\cos A)}{\cos A}$$

$$= \frac{\sin A[3-4\sin^2 A]}{\sin A} - \frac{\cos A[4\cos^2 A - 3]}{\cos A}$$

$$= [3-4\sin^2 A] - [4\cos^2 A - 3]$$

$$= 3 - 4\sin^2 A - 4\cos^2 A + 3$$

$$= 6 - 4(\sin^2 A + \cos^2 A)$$

$$= 6 - 4(1) \qquad \boxed{\because \sin^2 A + \cos^2 A = 1}$$

$$= 6 - 4 = 2 = \text{RHS}$$

9. Prove that $\dfrac{\cos^3 A - \cos 3A}{\cos A} + \dfrac{\sin^3 A + \sin 3A}{\sin A} = 3$.

Solution:

$$\text{LHS} = \frac{\cos^3 A - \cos 3A}{\cos A} + \frac{\sin^3 A + \sin 3A}{\sin A} = 3$$

Formulae

$$\boxed{\begin{aligned}&\sin 3A = 3\sin A - 4\sin^3 A\\ &\cos 3A = 4\cos^3 A - 3\cos A\end{aligned}}$$

$$= \frac{\cos^3 A - (4\cos A + 3\cos A)}{\cos A} + \frac{\sin^3 A + 3\sin A - 4\sin^3 A}{\sin A}$$

$$= \frac{\cos A[\cos^2 A - 4\cos^2 A + 3]}{\cos A} + \frac{\sin A[\sin^2 A + 3 - 4\sin^2 A]}{\sin A}$$

$$= 3 - 3\cos^2 A + 3 - 3\sin^2 A$$

$$= 6 - 3(\cos^2 A + \sin^2 A) \qquad \boxed{\because \cos^2 A + \sin^2 A = 1}$$

$$= 6 - 3(1) = 6 - 3 = 3 = \text{RHS}$$

10. Prove that $\cos 20^{o} \cos 40^{o} \cos 80^{o} = \frac{1}{8}$.

Solution:

LHS $= \cos 20^{o} \cos 40^{o} \cos 80^{o}$

$= \cos 80^{o} \cos 40^{o} \cos 20^{o}$

$= \cos(60^{o} + 20^{o})\cos(60^{o} + 20^{o})\cos 20^{o}$

$= [\cos^2 20 - \sin^2 60]\cos 20^{0}$

Formula :
$\cos(A+B)\cos(A-B)$
$= \cos^2 B - \sin^2 A$

$= \left[\cos^2 20 - \frac{3}{4}\right]\cos 20^{0}$

$\sin 60^{o} = \frac{\sqrt{3}}{2} \sin^2 60 = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{3}{4}$

$= \frac{(4\cos^2 20 - 3)}{4}\cos 20^{o}$

$= \frac{1}{4}(4\cos^3 20^{o} - 3\cos 20^{o})$

$= \frac{1}{4}\cos 3A [\because \cos 3A = 4\cos^3 A - 3\cos A]$

$= \frac{1}{4}\cos(3 \times 20) = \frac{1}{4}\cos 60 = \frac{1}{4} \times \frac{1}{2}$

$= \frac{1}{8} =$ RHS

3.3 SUM AND PRODUCT FORMULAE

We know that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \quad \text{------ (1)}$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \quad \text{------ (2)}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad \text{------ (3)}$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \quad \text{------ (4)}$$

Adding (1) and (2), we get

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B \quad \text{------ (5)}$$

Subtracting (2) from (1), we get

$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B \quad \text{------ (6)}$$

Adding (3) and (4), we get

$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B \quad \text{------ (7)}$$

Subtracting (4) from (3), we get

$$\cos(A+B) - \cos(A-B) = -2 \sin A \sin B \quad \text{------ (8)}$$

If we put,

$$A + B = C \quad \text{------ (i)}$$

$$A - B = D \quad \text{------ (ii)}$$

then

$$(i) + (ii) \Rightarrow 2A = C + D \Rightarrow A = \frac{C+D}{2}$$

$$(i) - (ii) \Rightarrow 2B = C - D \Rightarrow B = \frac{C-D}{2}$$

Then the results (5), (6), (7) and (8) will become

$$\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right) \quad \text{\_\_\_\_\_(9)}$$

$$\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right) \quad \text{\_\_\_\_\_(10)}$$

$$\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right) \quad \text{\_\_\_\_\_(11)}$$

$$\cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right) \quad \text{\_\_\_\_\_(12)}$$

The above results are known as sum and product formulae (because their LHS is sum & difference and RHS is product). These formulae can be remembered by the short way

$$\boxed{\begin{array}{l} S + S = 2SC \\ S - S = 2CS \\ C + C = 2CC \\ C - C = -2SS \end{array}}$$

where

$S = \sin$

$C = \cos$

3.3 WORKED EXAMPLES

PART – A

1. Express the following as sum or difference:

   (i) 2 sin 2A cos A (ii) 2 cos 5A sin 3A (iii) 2 cos 3A cos 2A (iv) 2 sin 3A sin A

   Solution:

   (i) 2 sin 2A cos A $\boxed{2SC = S + S}$

   $= \sin(\overset{\text{Add}}{2A + A}) + \sin(\overset{\text{Subtract}}{2A - A})$

   $= \sin 3A + \sin A$

   (ii) $2 \cos 5A \sin 3A$ $\boxed{2CS = S - S}$

   $= \sin(\overset{\text{Add}}{5A + 3A}) - \sin(\overset{\text{Subtract}}{5A - 3A})$

   $= \sin 7A + \sin 2A$

   (iii) 2 cos 3A cos 2A $\boxed{2CC = C + C}$

   = cos (3A + 2A) + cos (3A – 2A)

   = cos 5A + cos A

   (iv) 2 sin 3A sin A $\boxed{C - C = -2SS}$

   = [2 sin 3A sin A]

   = – [cos (3A + A) – cos (3A – A)]

   = – [cos4A – cos 2A] = cos 2A – cos 4A

2. Express the following as product

   i) sin 4A + sin 2A (ii) sin 5A – sin 3A iii) cos 7A + cos 3A iv) cos 4A – cos 2A

   Solution: Formula-(9)

   i) sin 4A + sin 2A $\boxed{S + S = 2SC}$

   $$= 2\sin\left(\frac{4A + 2A}{2}\right)\cos\left(\frac{4A - 2A}{2}\right)$$

   $$= 2\sin\left(\frac{6A}{2}\right)\cos\left(\frac{2A}{2}\right)$$

   $$= 2\sin 3A \cos A$$

   ii) sin 5A – sin 3A $\overset{\text{Formula}-(10)}{\boxed{S - S = 2CS}}$

   $$= 2\cos\left(\frac{5A + 3A}{2}\right)\sin\left(\frac{5A - 3A}{2}\right)$$

   $$= 2\sin\left(\frac{8A}{2}\right)\cos\left(\frac{2A}{2}\right) = 2\cos 4A \sin A,$$

   iii) cos 7A + cos 3A $\overset{\text{Formula}-(11)}{\boxed{C + C = 2CC}}$

   $$= 2\cos\left(\frac{7A + 3A}{2}\right)\cos\left(\frac{7A - 3A}{2}\right)$$

   $$= 2\cos\left(\frac{10A}{2}\right)\cos\left(\frac{4A}{2}\right) = 2\cos 5A \cos 2A$$

iv) cos 4A – cos 2A   Formula–(12) $\boxed{C - C = -2SS}$

$$= -2\sin\left(\frac{4A + 2A}{2}\right)\sin\left(\frac{4A - 2A}{2}\right)$$

$$= -2\sin\left(\frac{6A}{2}\right)\sin\left(\frac{2A}{2}\right)$$

$$= -2\sin(3A)\sin(A)$$

3. Write sin 5A – sin 7A in product form.

Solution:

$$\sin 5A - \sin 7A \qquad \boxed{S - S = 2CS}$$

$$= 2\cos\left(\frac{5A + 7A}{2}\right)\sin\left(\frac{5A - 7A}{2}\right)$$

$$= 2\cos\left(\frac{12A}{2}\right)\sin\left(\frac{-2A}{2}\right)$$

$$= 2\cos(6A)\sin(-A) \qquad \boxed{\begin{array}{c}\because \sin(-\theta) = -\sin\theta \\ \text{as } (-\theta) \text{ is Quadrant-IV}\end{array}}$$

$$= 2\cos 6A(-\sin A)$$

$$= -2\cos 6A \sin A$$

4. Write the product 2 cos 4A cos 6A as sum or difference of trigonometric function.

Solution:

$$2\cos 4A \cos 6A \qquad \boxed{C + C = 2CC}$$

$$= \cos(4A + 6A) + \cos(4A - 6A)$$

$$= \cos 10A + \cos(-2A) \qquad \boxed{\begin{array}{c}\cos(-\theta) = +\cos\theta \\ \text{as } (-\theta) \text{ is Quadrant-IV}\end{array}}$$

$$= \cos 10A + \cos 2A$$

PART –B

1. Prove that sin 50° – sin 70° + sin 10° = 0

Solution:

$$\text{LHS} = \boxed{\sin 50^o + \sin 10^o} - \sin 70^o \qquad \boxed{S + S = 2SC}$$

$$= 2\sin\left(\frac{50^o + 10^o}{2}\right)\cos\left(\frac{50^o - 10^o}{2}\right) - \sin 70^o$$

$$= 2\sin\left(\frac{60^o}{2}\right)\cos\left(\frac{40^o}{2}\right) - \sin 70^o$$

$$= 2\sin(30^o)\cos(20^o) - \sin(\overset{\text{I}}{90^o} - 20^o)$$

$$= 2 \times \frac{1}{2}\cos 20^o - \cos 20^o \qquad \boxed{\text{change of ratio for } 90^o}$$

$$= \cos 20^o - \cos 20^o$$

$$= 0 = \text{RHS}$$

2. Prove that $\cos 20^o + \cos 100^o + \cos 140^o = 0$

Solution:

$$\text{LHS} = \boxed{\cos 140^o + \cos 100^o} + \cos 20^o \qquad \boxed{C + C = 2CC}$$

$$= 2\cos\left(\frac{140^o + 100^o}{2}\right)\cos\left(\frac{140^o - 100^o}{2}\right) + \cos 20^o$$

$$= 2\cos\left(\frac{240^o}{2}\right)\cos\left(\frac{40^o}{2}\right) + \cos 20^o$$

$$= 2\cos 120^o \cos 20^o + \cos 20^o$$

$$= 2 \times -\frac{1}{2}\cos 20^o + \cos 20^o \qquad \boxed{\begin{aligned}&\cos 120^o = \cos(\overset{\text{II}}{180^o} - 60^o)\\ &= -\cos 60 = -\frac{1}{2}\end{aligned}}$$

$$= -\cos 20^o + \cos 20^o$$

$$= 0$$

3. Prove that $\cos A + \cos(120^o + A) + \cos(120^o - A) = 0$.

Solution:

$$\text{LHS} = \cos(120^o + A) + \cos(120^o - A) + \cos A$$

$$= 2\cos\left[\frac{120^o + A + 120^o - A}{2}\right]\cos\left[\frac{120^o + A - (120^o - A)}{2}\right] + \cos A$$

$$= 2\cos\left[\frac{240^o}{2}\right]\cos\left[\frac{120^0 + A - 120 + A}{2}\right] + \cos A$$

$$= 2\cos(120^o)\cos\left(\frac{2A}{2}\right) + \cos A$$

$$= 2 \times -\frac{1}{2}\cos A + \cos A$$

$$= -\cos A + \cos A$$

$$= 0 = \text{RHS}$$

4. Prove that $\dfrac{\sin 2A - \sin 2B}{\cos 2A + \cos 2B} = \tan(A + B)\cdot$

Solution:

$$\text{LHS} = \frac{\sin 2A - \sin 2B}{\cos 2A + \cos 2B} \qquad \boxed{\begin{aligned}S - S &= 2CS\\ C + C &= 2CC\end{aligned}}$$

$$= \frac{2\cos\left(\frac{2A + 2B}{2}\right)\sin\left(\frac{2A - 2B}{2}\right)}{2\cos\left(\frac{2A + 2B}{2}\right)\cos\left(\frac{2A - 2B}{2}\right)}$$

$$= \frac{2\cos(A + B)\sin(A - B)}{2\cos(A + B)\cos(A - B)}$$

$$= \tan(A - B) = \text{RHS}$$

PART – C

1. Prove that $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$.

Solution:

$$\text{LHS} = (\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$$

$\boxed{C + C = 2CC}$ $\boxed{S + S = 2SC}$

Formulae

$$= \left[2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2 + \left[2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2$$

$$= 4\cos^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right) + 4\sin^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right)$$

$\boxed{4\cos^2\left(\frac{\alpha-\beta}{2}\right) \text{ is common for both the terms}}$

$$=4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left[\cos^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\left(\frac{\alpha+\beta}{2}\right)\right]$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right) \times 1 \quad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right) = \text{RHS}$$

2. If sin x + sin y = a and cos x + cos y = b. Prove that $\tan^2\left(\frac{x-y}{2}\right) = \frac{4-a^2-b^2}{a^2+b^2}$.

Solution:

Given a = sin x + sin y b = cos x + cos y

$\therefore \boxed{a^2 + b^2}$

$$= [\sin x + \sin y]^2 + [\cos x + \cos y]^2$$

Formula $\boxed{S + S = 2SC}$ $\boxed{C + C = 2CC}$

$$= \left[2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right]^2 + \left[2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right]^2$$

$$= 4\sin^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right) + 4\cos^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right)$$

$\boxed{4\cos^2\left(\frac{x-y}{2}\right) \text{ is common to both are terms}}$

$$= 4\cos^2\left(\frac{x-y}{2}\right)\left[\sin^2\left(\frac{x+y}{2}\right) + \cos^2\left(\frac{x+y}{2}\right)\right]$$

$$= 4\cos^2\left(\frac{x-y}{2}\right) \times 1 \qquad \boxed{\because \sin^2\theta + \cos^2\theta = 1}$$

$$a^2 + b^2 = 4\cos^2\left(\frac{x-y}{2}\right)$$

$$\text{RHS} = \frac{4 - a^2 - b^2}{(a^2 + b^2)} = \frac{4 - (a^2 + b^2)}{(a^2 + b^2)}$$

$$= \frac{4 - 4\cos^2\left(\frac{x-y}{2}\right)}{4\cos^2\left(\frac{x-y}{2}\right)} = \frac{\cancel{4}\left[1 - \cos^2\left(\frac{x-y}{2}\right)\right]}{\cancel{4}\cos^2\left(\frac{x-y}{2}\right)}$$

$$= \frac{\sin^2\left(\frac{x-y}{2}\right)}{\cos^2\left(\frac{x-y}{2}\right)} \qquad \boxed{\because 1 - \cos^2\theta = \sin^2\theta}$$

$$= \tan^2\left(\frac{x-y}{2}\right) = \text{RHS}$$

3. If sin x + sin y = a and cos x + cos y = b prove that sin (x + y) = $\frac{2ab}{a^2 + b^2}$

Solution:

$$a^2 + b^2 = 4\cos^2\left(\frac{x-y}{2}\right) \quad \text{[Refer Example 2]}$$

2ab = 2 (sin x + sin y) (cos x + cos y)

$$\boxed{S + S = 2SC} \qquad \boxed{C + C = 2CC}$$

$$= 2\left[2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right].\,2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$= 8\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right)$$

$$\text{RHS} = \frac{2ab}{a^2 + b^2} = \frac{2\cancel{8}\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right)\cancel{\cos^2}\left(\frac{x-y}{2}\right)}{\cancel{4}\,\cancel{\cos^2}\left(\frac{x-y}{2}\right)}$$

$$= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$= 2\sin\theta\cos\theta = \sin 2\theta \text{ where } \theta = \frac{x+y}{2}$$

$$= \sin\left[\cancel{2} \times \frac{(x+y)}{\cancel{2}}\right]$$

$$= \sin(x+y) = \text{LHS}$$

4. Prove that $\frac{\sin A + \sin 2A + \sin 3A}{\cos A + \cos 2A + \cos 3A} = \tan 2A$.

Solution:

$$\text{LHS} = \frac{\overset{S+S=2SC}{\boxed{\sin 3A + \sin A}} + \sin 2A}{\underset{}{\overset{C+C=2CC}{\boxed{\cos 3A + \cos A}}} + \cos 2A}$$

$$= \frac{2\sin\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)+\sin 2A}{2\cos\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)+\cos 2A}$$

$$= \frac{2\sin\left(\frac{4A}{2}\right)\cos\left(\frac{2A}{2}\right)+\sin 2A}{2\cos\left(\frac{4A}{2}\right)\cos\left(\frac{2A}{2}\right)+\cos 2A}$$

$$= \frac{2\sin(2A)\cos A+\sin 2A}{2\cos 2A\cos A+\cos 2A}$$

sin 2A is common
cos 2A is common

$$= \frac{\sin 2A[\cancel{2\cos A}+1]}{\cos 2A[\cancel{2\cos A}+1]}$$

$$= \tan 2A = \text{RHS}$$

EXERCISE

PART – A

1. Show that cos (– 330°) cos 420° = $\frac{\sqrt{3}}{4}$.
2. Show that cos 780° sin 750° = $\frac{1}{4}$.
3. Find the value of sin 40° cos 20° + cos 40° sin 20°.
4. Find the value of sin 70° cos 10° – cos 70° sin 10°.
5. Find the value of cos 75° cos 15° + sin 75° sin 15°.
6. Find the value of cos 75° cos 15° – sin 75° sin 15°.
7. Find the value of $\frac{\tan 22^o + \tan 23^o}{1-\tan 22^o \tan 23^o}$.
8. Find the value of $\frac{\tan 65^o - \tan 20^o}{1+\tan 65^o \tan 20^o}$.
9. Find the value of 2 sin 15° cos 15°.
10. Find the value of 2 $\cos^2 15^o$ –1.
11. Find the value of 1 – 2 $\sin^2 15^o$.
12. Find the value of $\frac{2\tan 22\,{}^1\!/_2{}^o}{1-\tan^2 22\,{}^1\!/_2{}^o}$.
13. Prove that $\frac{\sin 2A}{1-\cos 2A} = \cot A$.
14. Prove that $(\sin A + \cos A)^2 = 1 + \sin 2A$.
15. Prove that $\frac{\sin A}{1-\cos\ A} = \cot\left(\frac{A}{2}\right)$.
16. Find the value of 3 sin10° – 4 $\sin^3 10$.
17. Find th value of 4 $\cos^3 20^o$ – 3 cos 20°.

18. Find the value of $\dfrac{3\tan 20^{o} - \tan^{3} 20^{o}}{1 - 3\tan^{2} 20^{o}}$.

19. Express the products as a sum (or) difference

    i) 2 sin 4A cos 2A   ii) 2 cos8A cos 6A   iii) 2 sin 3A cos 6A   iv) 2 sin 5A sin 3A

20. Express the following sum (or) difference as product

    i) sin 10A + sin 4A   ii) sin 5A – sin 3A   iii) cos 10A + cos 4A   iv) cos 10A – cos 4A

PART – B

1. Prove that cos (30^{o} + A) cos (30^{o} – A) – sin (30^{o} + A) sin (30^{o} – A) = $\frac{1}{2}$.
2. Prove that cos (45^{o} + A) cos (45^{o} – A) + sin (45^{o} + A) sin (45^{o} – A) = cos 2A.
3. Prove that sin A + sin (120^{o} + A) + sin (120^{o} – A) = 0.
4. If tan A = $\frac{5}{6}$ adn tan B = $\frac{1}{11}$ find the value of tan (A + B).
5. Prove that $\dfrac{\sin(A-B)}{\sin A \sin B} + \dfrac{\sin(B-C)}{\sin B \sin C} + \dfrac{\sin(C-A)}{\sin C \sin A} = 0$.
6. If A + B + C = 180^{o}, prove tan A + tan B + tan C = tan A tan B tan C.
7. If tan A = $\frac{1}{2}$, find the value of tan 2A.
8. If tan A = $\frac{1}{3}$, find the value of sin 2A and cos 2A.
9. Prove that cot A – cot 2A = cosec 2A.
10. If $\tan\left(\frac{A}{2}\right) = \frac{1}{5}$ find the value of tan A.
11. If $\tan\left(\frac{A}{2}\right) = \frac{3}{5}$ find the value of sin A and cos A.
12. If sin A = $\frac{4}{5}$, find sin 3A.
13. If cos θ = $\frac{3}{5}$, find cos 3θ.
14. If tan θ = 3, find tan 3θ.

Prove the following:

15. sin 10^{o} + sin 50^{o} – sin 70^{o} = 0.
16. sin 78^{o} – sin 18^{o} + cos 132^{o} = 0.
17. cos 80^{o} + cos 40^{o} – cos 20^{o} = 0.
18. cos 52^{o} + cos 68^{o} + cos 172^{o} = 0.

Prove the following:

19. $\dfrac{\sin 3A - \sin A}{\cos 3A + \cos 5A} = \cot 2A$.
20. $\dfrac{\sin 7A - \sin 5A}{\cos 7A + \cos 5A} = \tan A$.
21. $\dfrac{\sin 3A + \sin A}{\cos 3A + \cos A} = \tan 2A$
22. $\dfrac{\cos B - \cos A}{\sin A - \sin B} = \tan\left(\dfrac{A+B}{2}\right)$

PART – C

1. If $\sin A = \frac{3}{5}$, $\cos B = \frac{12}{13}$ find i) $\sin(A + B)$ ii) $\cos(A + B)$.
2. If $\sin A = \frac{8}{17}$, $\sin B = \frac{5}{13}$ prove that $\sin(A + B) = \frac{171}{221}$.
3. If $\sin A = \frac{4}{5}$, $\cos B = \frac{15}{17}$ find $\cos(A - B)$.
4. If $A + B = 45^o$, prove that $(\cot A - 1)(\cot B - 1) = 2$. Deduce $\cot 22\frac{1}{2}^o$.
5. If $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$, show that $\tan(2\alpha + \beta) = 3$.
6. Prove that $\frac{1+\cos 2A+\sin 2A}{1-\cos 2A+\sin 2A} = \cot A$.
7. Prove that $\frac{1+\cos A+\sin A}{1-\cos A+\sin A} = \cot\left(\frac{A}{2}\right)$.
8. Prove that $\frac{\sin 3\theta}{\sin\theta} + \frac{\cos 3\theta}{\cos\theta} = 4\cos 2\theta$.
9. Prove that $\frac{\sin 3A+\sin^3 A}{\cos^3 A-\cos 3A} = \cot A$.
10. Prove that $\sin 20^o \sin 40^o \sin 80^o = \frac{\sqrt{3}}{8}$.
11. Prove that $\cos 10^o \cos 50^o \sin 70^o = \frac{\sqrt{3}}{8}$.
12. Prove that $\sin 10^o \sin 50^o \sin 70^o = \frac{1}{8}$.
13. Prove that $\frac{\sin 2A+\sin 5A-\sin A}{\cos 2A+\cos 5A+\cos A} = \tan 2A$.
14. Prove that $\frac{\cos 2A+\cos 5A+\cos A}{\sin 2A+\sin 5A-\sin A} = \cot 2A$.
15. Prove that $\frac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A} = \tan 4A$

Prove the following:

16. $(\cos A + \cos B)^2 + (\sin A - \sin B)^2 = 4\cos^2\left(\frac{A+B}{2}\right)$
17. $(\cos A - \cos B)^2 + (\sin A + \sin B)^2 = 4\sin^2\left(\frac{A+B}{2}\right)$
18. $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$
19. If $\sin x + \sin y = a$ and $\cos x + \cos y = b$, prove that $\sec^2\left(\frac{x-y}{2}\right) = \frac{4}{a^2+b^2}$
20. If $\sin x - \sin y = a$ and $\cos x - \cos y = b$, prove that $\sec^2\left(\frac{x-y}{2}\right) = \frac{4}{4-(a^2+b^2)}$.

ANSWERS

PART – A

3) $\frac{\sqrt{3}}{2}$ 4) $\frac{\sqrt{3}}{2}$ 5) $\frac{1}{2}$ 6) 0 7) 1 8) 1 9) $\frac{1}{2}$ 10) $\frac{\sqrt{3}}{2}$ 11) $\frac{\sqrt{3}}{2}$ 12) 1

16) $\frac{1}{2}$ 17) $\frac{1}{2}$ 18) $\sqrt{3}$ 19) i) $\sin 6\theta + \sin 2\theta$ (ii) $\cos 14\theta + \cos 2\theta$

(iii) $\sin 9A - \sin 3A$ iv) $\cos 2A - \cos 8A$ 20) i) $2 \sin 7A \cos 3A$ (ii) $2 \cos 4A \sin A$

(iii) $2 \cos 7A \cos 3A$ iv) $-2 \sin 7A \sin 3A$.

PART – B

4) 1 7) $\frac{4}{3}$ 8) $\frac{3}{5}$ and $\frac{4}{5}$ 10) $\frac{5}{12}$ 11) $\frac{15}{17}$ and $\frac{8}{17}$ 12) $\frac{44}{125}$ 13) $\frac{-117}{125}$

14) $\frac{9}{13}$

PART – C

1) (i) $\frac{56}{65}$ (ii) $\frac{33}{65}$ 3) $\frac{77}{85}$

UNIT – IV

INVERSE TRIGONOMETRIC RATIOS AND DIFFERENTIAL CALCULUS-I

4.1 INVERSE TRIGONOMETRIC FUNCTIONS:

Definition of inverse trigonometric ratios – Relation between inverse trigonometric ratios. Simple problems.

4.2 LIMITS

Definition of Limits. Problems using the following results:

(i) $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x-a} = na^{n-1}$ (ii) $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$ and (iii) $\lim\limits_{\theta\to 0}\dfrac{\tan\theta}{\theta} = 1$ (θ - in radians) (results only).

Simple Problems.

4.3 DIFFERENTIATION

Definition – Differentiation of x^n, sin x, cos x, tan x, cosec x, sec x, cot x, log x, ex, u ± v, uv, uvw, $\frac{u}{v}$ (v ≠ 0) (results only) Simple problems using the above results.

4.1 INVERSE TRIGONOMETRIC FUNCTIONS

Definition:

A function which is the reverse process of a trigonometric function is called the inverse trigonometric function.

The domain of a trigonometric function is the set angles and the range is the set of real numbers. In case of inverse trigonometric function, the domain is the set of real numbers and the range is the set of angles.

Inverse trigonometric function of sin x is denoted as $\sin^{-1}x$. Similarly $\cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ and $\text{cosec}^{-1}x$ are the inverse trigonometric functions of cos x, tan x, sec x, cot x and cosec x respectively.

Examples:

We know that

$\sin 30^o = \frac{1}{2}$ $\quad \therefore \sin^{-1}\left(\frac{1}{2}\right) = 30^o$

$\cos 0^o = 1$ $\quad \therefore \cos^{-1}(1) = 0$

$\tan 30^o = \frac{1}{\sqrt{3}}$ $\quad \therefore \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^o$

$\cot 45^o = 1$ $\quad \therefore \cot^{-1}(1) = 45^o$

$\sec 30^o = \frac{2}{\sqrt{3}} \quad \therefore \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = 30^o$

$\text{cosec} 90^o = 1 \quad \therefore \text{cosec}^{-1}(1) = 90^o$

If x = sin θ. θ = $\sin^{-1}x$.

$\sin^{-1}x$ is an angle.

There is a difference between $\sin^{-1}x$ and $(\sin x)^{-1}$. $\sin^{-1}x$ is the inverse trigonometric function of sin θ where as $(\sin x)^{-1}$ is the reciprocal of sin x. i.e, $(\sin x)^{-1} = \frac{1}{\sin x} = \text{cosec } x$.

Principal Value

Among all the values, the numerically least value of the inverse trigonometric function is called principal value.

Examples:

1) $\sin 30^o = \frac{1}{2}$, $\sin 150^o = \frac{1}{2}$, $\sin 390^o = \frac{1}{2}$

$\sin^{-1}\left(\frac{1}{2}\right)$ = 30^o, 150^o, 390^o, ...

The least positive value is 30; which is called the principal value of $\sin^{-1}\left(\frac{1}{2}\right)$.

2) $\cos 60^o = \frac{1}{2}$, $\cos 300^o = \frac{1}{2}$, $\cos(-60^o) = \frac{1}{2}$, $\cos 420^o = \frac{1}{2}$

$\cos^{-1}\left(\frac{1}{2}\right)$ = 60^o, 300^o, -60^o, 4 = 0

The principal value of $\cos^{-1}\left(\frac{1}{2}\right)$ is 60^o.

Function	Domain	Range of Principal value of θ
$\sin^{-1} x$	$-1 \le x \le 1$	$-\pi/2 \le \theta \le \pi/2$
$\cos^{-1} x$	$-1 \le x \le 1$	$0 \le \theta \le \pi$
$\tan^{-1} x$	$(-\infty, \infty)$	$-\pi/2 \le \theta \le \pi/2$
$\cot^{-1} x$	$(-\infty, \infty)$	$-\pi/2 < \theta < \pi/2$
$\sec^{-1} x$	(–1, 1) except 0	$0 < \theta < \pi, \theta \ne \pi/2$
$\text{cosec}^{-1} x$	(–1, 1) except 0	$-\pi/2 < \theta < \pi/2, \theta \ne 0$

Properties

Property (1) :

(a) $\sin^{-1}(\sin x) = x$

(b) $\cos^{-1}(\cos x) = x$

(c) $\tan^{-1}(\tan x) = x$

(d) $\cot^{-1}(\cot x) = x$

(e) $\sec^{-1}(\sec x) = x$

(f) $\text{cosec}^{-1}(\text{cosec } x) = x$

Proof:

(a) Let $\sin x = y$ ——— (1)

$\therefore \quad x = \sin^{-1} y$

$= \sin^{-1} (\sin x)$ [from (1)

$\therefore \quad \sin^{-1} (\sin x) = x$

Similarly other results can be proved.

Property (2):

(a) $\sin (\sin^{-1} x) = x$
(b) $\cos (\cos^{-1} x) = x$
(c) $\tan (\tan^{-1} x) = x$
(d) $\cot (\cot^{-1} x) = x$
(e) $\sec (\sec^{-1} x) = x$
(f) $\text{cosec} (\text{cosec}^{-1} x) = x$

Proof:

(b) Let $\cos^{-1}x = y$ ——— (1)

Then $x = \cos y$

$= \cos (\cos^{-1}x)$ [From (1)

$\therefore \cos (\cos^{-1}x) = x$

Similarly other results can be proved.

Property (3):

(a) $\sin^{-1} \left(\frac{1}{x}\right) = \text{cosec}^{-1} (x)$

(b) $\cos^{-1} \left(\frac{1}{x}\right) = \sec^{-1} (x)$

(c) $\tan^{-1} \left(\frac{1}{x}\right) = \cot^{-1} (x)$

(d) $\cot^{-1} \left(\frac{1}{x}\right) = \tan^{-1} (x)$

(e) $\sec^{-1} \left(\frac{1}{x}\right) = \cos^{-1} (x)$

(f) $\text{cosec}^{-1} \left(\frac{1}{x}\right) = \sin^{-1} (x)$

Proof:

(c) Let $\tan^{-1} \left(\frac{1}{x}\right) = y$ ——— (1)

$\therefore \quad \frac{1}{x} = \tan y$

$\therefore \quad x = \frac{1}{\tan y} = \cot y$

$\therefore \quad y = \cot^{-1}(x)$ ——— (2)

From (1) & (2), $\tan^{-1} \left(\frac{1}{x}\right) = \cot^{-1} (x)$

Similarly other results can be proved.

Property (4):

(a) $\sin^{-1}(-x) = -\sin^{-1} x$
(b) $\cos^{-1}(-x) = \pi - \cos^{-1} x$
(c) $\tan^{-1}(-x) = -\tan^{-1} x$
(d) $\cot^{-1}(-x) = -\cot^{-1} x$
(e) $\sec^{-1}(-x) = \pi - \sec^{-1} x$
(f) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$

Proof:

(f) Let $y = \text{cosec}^{-1}(-x)$ ———— (1)
$\therefore \text{cosec } y = -x$
i.e., $x = -\text{cosec } y$
$= \text{cosec}(-y)$ [$\because$ $\text{cosec}(-y) = \text{cosec}(2\pi - y) = -\text{cosec } y$]
i.e., $\text{cosec}^{-1}(x) = -y$
$= -\text{cosec}^{-1}(-x)$ [From (1)
$\therefore \text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$
(e) Let $y = \sec^{-1}(-x)$ ———— (1)
$\therefore \sec y = -x$
i.e., $x = -\sec y$
$= \sec(\pi - y)$ [$\sec(\pi - y) = -\sec y$
i.e., $\sec^{-1}x = \pi - y$
$\therefore y = \pi - \sec^{-1}x$ ———— (2)
From (1) & (2), $\sec^{-1}(-x) = \pi - \sec^{-1}x$
Similarly other results can be proved.

Property (5):

$\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$

$\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$

$\sec^{-1}x + \text{cosec}^{-1}x = \dfrac{\pi}{2}$

Proof:

Let $\theta = \tan^{-1}x$ ———— (1)
$\therefore x = \tan\theta$

$= \cot\left(\dfrac{\pi}{2} - \theta\right)$ [$\because \cot(90^\circ - \theta) = \tan\theta$]

$= \cot^{-1} x = \dfrac{\pi}{2} - \theta$

$= \dfrac{\pi}{2} - \tan^{-1} x$ [From (1)

$\therefore \tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$

Similarly other results can be proved.

Property (6):

If $xy < 1$, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Let $A = \tan^{-1}x$ $\quad\therefore\ x = \tan A$ ———— (1)

Let $B = \tan^{-1}y$ $\quad\therefore\ y = \tan B$ ———— (2)

Now,

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{x+y}{1-xy}$$

$$\therefore (A+B) = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

i.e, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ [From (1) & (2)

Property (7):

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

Proof:

Let $A = \sin^{-1}x$ $\quad\therefore\ x = \sin A$ ——— (1)

Let $B = \sin^{-1}y$ $\quad\therefore\ y = \sin B$ ——— (2)

$$\sin(A+B) = \sin A\cos B + \cos A \sin B$$

$$= \sin A\sqrt{1-\sin^2 B} + \sqrt{1-\sin^2 A}\ \sin B$$

$$= x\sqrt{1-y^2} + \sqrt{1-x^2}\,y$$

$$A + B = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

i.e, $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$ [From (1) & (2)

Property (8):

$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$$

Proof:

Let $A = \cos^{-1}x$ $\quad\therefore\ x = \cos A$ ——— (1)

Let $B = \cos^{-1}y$ $\quad\therefore\ y = \cos B$ ——— (2)

Now, $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \cos A\cos B - \sqrt{1-\cos^2 A}\sqrt{1-\cos^2 B}$$

$$= xy - \sqrt{1-x^2}\ \sqrt{1-y^2}$$

i.e., $A + B = \cos^{-1}\left[xy - \sqrt{1-x^2}\ \sqrt{1-y^2}\right]$

From (1) & (2)

$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\ \sqrt{1-y^2}\right]$$

WORKED EXAMPLES

PART – A

1. Find the principal value of

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

(iv) $\cot^{-1}(-1)$ (v) $\sec^{-1}(-2)$ (vi) $\text{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution:

(i) Let $x = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\therefore \sin x = \frac{1}{\sqrt{2}}$

i.e, $\sin x = \sin 45^{o}$ $\left[\because \sin 45^{o} = \frac{1}{\sqrt{2}}\right]$

$\therefore x = 45^{o}$

$\therefore \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^{o}$

(ii) Let $x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\therefore \cos x = \frac{\sqrt{3}}{2}$

$\cos x = \cos 30^{o}$ $\left[\because \cos 30 = \frac{\sqrt{3}}{2}\right]$

$\therefore x = 30^{o}$

$\therefore \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^{o}$

(iii) Let $x = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$\therefore \tan x = -\frac{1}{\sqrt{3}}$

$= -\tan 30^{o}$ $\left[\because \tan 30 = \frac{1}{\sqrt{3}}\right]$

i.e., $\tan x = \tan^{o}(-30^{o})$ $[\because \tan(-\theta) = -\tan\theta]$

$\therefore x = -30^{o}$

$\therefore \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30^{o}$

(iv) Let $x = \cot^{-1}(-1)$

$\therefore \cot x = -1$

$= -\cot 45^o$

i.e., $\cot x = \cot(-45^o)$

$\therefore x = -45^o$

$\therefore \cot^{-1}(-1) = -45^o$

(v) Let $x = \sec^{-1}(-2)$

$\therefore \sec x = -2$

$= -\sec 60^o$

i.e., $\sec x = \sec(120^o)$ $\quad [\because \sec(120) = \sec(180-60) = -\sec 60$

$\therefore x = -120^o$

$\therefore \sec^{-1}(-2) = 120^o$

(v) Let $x = \text{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$\therefore \text{cosec } x = \frac{2}{\sqrt{3}}$

$= \text{cosec } 60^o$ $\quad [\because \text{cosec } 60 = \frac{1}{\sin 60} = \frac{1}{(\sqrt{3}/2)} = \frac{2}{\sqrt{3}}$

$\therefore x = 60^o$

$\therefore \text{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = 60^o$

2. Prove that $\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$

Solution:

LHS $= \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$

$= \tan^{-1} x + \cot^{-1}(x)$ $\quad$ [By property 3

$= \frac{\pi}{2}$ $\quad$ [By property 5

PART - B

1. Prove that $\sin^{-1}(\sqrt{1-x^2}) = \cos^{-1} x$.

Solution:

Put $x = \cos\theta$ $\quad \therefore \theta = \cos^{-1}x$ ———— (1)

Now, LHS $= \sin^{-1}\left(\sqrt{1-x^2}\right)$

$= \sin^{-1}\left(\sqrt{1-\cos^2\theta}\right)$

$= \sin^{-1}\left(\sqrt{1-\sin^2\theta}\right)$

$= \sin^{-1}(\sin\theta)$

$= \theta$ $\quad$ [By Property 1

$= \cos^{-1}x$ $\quad$ [By (1)

2. Prove that $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{2}{5}\right)=\tan^{-1}\left(\frac{11}{13}\right)$.

Solution:

We know, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\therefore \tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{2}{5}\right)=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{2}{5}}{1-\frac{1}{3}\cdot\frac{2}{5}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{5+6}{15}}{\frac{15-2}{15}}\right)$$

$$=\tan^{-1}\left(\frac{11}{13}\right)$$

PART – C

1. Show that $2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution:

Let $x=\tan\theta$ $\quad\therefore\ \theta=\tan^{-1}x$

$$\text{R.H.S}=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$=\cos^{-1}(\cos 2\theta)$$

$$=2\theta$$

$$=2\tan^{-1}x=\text{L.H.S}$$

2. Show that $\tan^{-1}\left(\frac{3x-x^2}{1-3x^2}\right)=3\tan^{-1}x$.

Solution:

Let $x=\tan\theta$ $\quad\therefore\ \theta=\tan^{-1}x$ ———— (1)

$$\text{LHS}=\tan^{-1}\left(\frac{3x-x^2}{1-3x^2}\right)$$

$$=\tan^{-1}\left[\frac{3\tan\theta-\tan^2\theta}{1-3\tan^2\theta}\right]$$

$$=\tan^{-1}(\tan 3\theta)$$

$$=3\theta$$

$$=3\tan^{-1}x \qquad \text{[From (1)}$$

=RHS

3. Show that $2\tan^{-1}\left(\frac{2}{3}\right) = \tan^{-1}\left(\frac{12}{15}\right)$

Solution:

$$\text{LHS} = 2\tan^{-1}\left(\frac{2}{3}\right)$$

$$= \tan^{-1}\left(\frac{2}{3}\right) + \tan^{-1}\left(\frac{2}{3}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}+\frac{2}{3}}{1-\frac{2}{3}\cdot\frac{2}{3}}\right)$$

$$= \tan^{-1}\left[\frac{\frac{4}{3}}{\frac{9-4}{9}}\right]$$

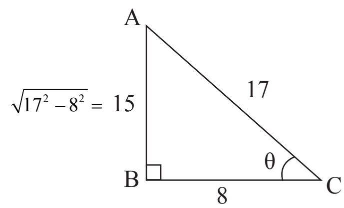
$$= \tan^{-1}\left(\frac{4}{3}\times\frac{9}{5}\right)$$

$$= \tan^{-1}\left(\frac{12}{5}\right) = \text{RHS}$$

4. Evaluate $\tan\left[\cos^{-1}\left(\frac{8}{17}\right)\right]$.

Solution:

$$\text{Let } \cos^{-1}\left(\frac{8}{17}\right) = \theta \qquad ....(1)$$



$$\therefore \cos\theta = \frac{8}{17}$$

$$AB^2 = AC^2 - BC^2 = 17^2 - 8^2$$

$$= 289 - 64 = 225$$

$$\therefore AB = \sqrt{225} = 15$$

$$\therefore \tan\theta = \frac{AB}{BC} = \frac{15}{8}$$

$$\therefore \theta = \tan^{-1}\left(\frac{15}{8}\right)$$

$$\text{From (1) \& (2), } \cos^{-1}\left(\frac{8}{17}\right) = \tan^{-1}\left(\frac{15}{8}\right)$$

$$\therefore \tan\left[\cos^{-1}\left(\frac{8}{17}\right)\right] = \tan\left[\tan^{-1}\left(\frac{15}{8}\right)\right]$$

$$= \frac{15}{8}$$

4.2 LIMITS

Mlathematical quantities can be divided into two (i) constants and (ii) Variables

The quantitity which does not change in its value is called constant.

Constants can be divided (i) Absolute constants and (ii) Arbitrary constants

Absolute constants are those which retain in values at any time and at any place.

Examples:

Numbers 5, – 7, 3, $\frac{-11}{27}$, π, e, $\sqrt{2}$, $\sqrt{3}$, ... etc. are absolute constants.

Arbitrary constants are those which are constants in a particular problem but changes from problem to problem. They are represented as a, b, c, f, g, h, ... etc.

Examples:

(i) ax + by + c = 0 — a, b, c

(ii) $x^2 + y^2 + 2gx + 2fy + c = 0$ — f, g, h.........ex.

Variables are those which vary in their values. They are represented as u, v, w, x, y, z, θ, ...

Variables can be divided into two (i) Independent variable and (ii) Dependent varaible.

In the equation $y = x^2$,x is the independent variable and y is the dependent variable depending on x

For example, if x takes the value 2, y takes the value 4. If x takes –3, y takes 9 .... etc.

The relation or equation or connection between two variables is called a function and is denoted as f(x).

Example: $y = f(x) = x^2$ or sin x.

The value of f(x) at x = a is f(a) and is called the functional value of f(x) at x = a.

If for every function, if the functional value at every point of its domain exists, there is no need to study Limit chapter and from that the famous CALCULS.

For example, $f(x) = \frac{x^2-4}{x-2}$ at x = 2 i.e, f(2) does not exist, since $f(2) = \frac{0}{0}$, the indeterminant quantity. Similarly there are two more indeterminant quantities $\frac{\infty}{\infty}$ and ∞ – ∞.

But, as x approaches 2, $f(x) = \frac{x^2-4}{x-2}$ will approach 4.

x	1	1.5	1.9	1.99	1.999	1.9999	......	→ 2
$y = \frac{x^2-4}{x-2}$	3	3.5	3.9	3.99	3.999	3.9999	......	→ 4

Hence the approachment value 4 of the function $\frac{x^2-4}{x-2}$ as x approaches 2 is called the limit value of the function.

Definition of Limit:

When the variable x approaches a constant a and if the function f(x) approaches a constant *l*, then *l* is called the limit value of f(x) as x approaches a and is denotd as

$$\boxed{\underset{x\to a}{\text{Lt}}\, f(x) = l}$$

Results :

1) $\underset{x\to a}{\mathrm{Lt}}[f(x)+g(x)] = \underset{x\to a}{\mathrm{Lt}} f(x) \pm \underset{x\to a}{\mathrm{Lt}} g(x)$

2) $\underset{x\to a}{\mathrm{Lt}}[k\, f(x)] = k \underset{x\to a}{\mathrm{Lt}} f(x)$, where k is a constant

3) $\underset{x\to a}{\mathrm{Lt}}[f(x).g(x)] = \underset{x\to a}{\mathrm{Lt}} f(x). \underset{x\to a}{\mathrm{Lt}} g(x)$

4) $\underset{x\to a}{\mathrm{Lt}}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\underset{x\to a}{\mathrm{Lt}} f(x)}{\underset{x\to a}{\mathrm{Lt}} g(x)}$, provided $\underset{x\to a}{\mathrm{Lt}} g(x) \neq 0$

Formulae:

1) $\underset{x\to a}{\mathrm{Lt}} \dfrac{x^n - a^n}{x-a} = na^{n-1}$, for all values of x.

2) $\underset{x\to a}{\mathrm{Lt}} \dfrac{\sin\theta}{\theta} = 1$

3) $\underset{x\to a}{\mathrm{Lt}} \dfrac{\tan\theta}{\theta} = 1$, θ in radians

Now,

$$\underset{\theta\to 0}{\mathrm{Lt}} \frac{\sin n\theta}{\theta} = \underset{\theta\to 0}{\mathrm{Lt}}\left[\frac{\sin n\theta}{n\theta}.\, n\right]$$

$$= n \underset{\theta\to 0}{\mathrm{Lt}}\left(\frac{\sin n\theta}{n\theta}\right) \qquad [\text{By } R_2$$

$$= n\,.\,1 \qquad [\text{By } F_2$$

$$= n$$

Similarly, $\underset{\theta\to 0}{\mathrm{Lt}} \dfrac{\tan n\theta}{\theta} = n$

WORKED EXAMPLES

PART – A

1. Evaluate $\underset{\theta\to 0}{\mathrm{Lt}} \dfrac{3x^2+2x+1}{5x^2+6x+7}$.

Solution:

$$\underset{\theta\to 0}{\mathrm{Lt}} \frac{3x^2+2x+1}{5x^2+6x+7} = \frac{3(0)^2+2(0)+1}{5(0)^2+6(0)+7}$$

$$= \frac{1}{7}$$

2. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \dfrac{3x^2+4x}{5x-7x^2}$

Solution:

$$\underset{x\to 0}{\mathrm{Lt}} \frac{3x^2+4x}{5x-7x^2} = \underset{x\to 0}{\mathrm{Lt}} \frac{x(3x-4)}{x(5-7x)}$$

$$= \frac{3.0-4}{5-7.0}$$

$$= \frac{-4}{5} = -\frac{4}{5}$$

3. Evaluate: $\underset{\theta\to 0}{\text{Lt}}\dfrac{\sin 7\theta}{0}$

Solution:

$$\underset{\theta\to 0}{\text{Lt}}\frac{\sin 7\theta}{\theta}=\underset{\theta\to 0}{\text{Lt}}\left[\frac{\sin 7\theta}{7\theta}.7\right]$$

$$=7\,\underset{\theta\to 0}{\text{Lt}}\frac{\sin 7\theta}{7\theta}$$

$$= 7.1$$

$$= 7$$

PART – B

1. Evaluate: $\underset{x\to 1}{\text{Lt}}\dfrac{x^2+x-2}{x^2-4x+3}$

Solution:

$$\underset{x\to 1}{\text{Lt}}\frac{x^2+x-2}{x^2-4x+3}=\frac{1+1-2}{1-4(1)+3}=\frac{2-2}{4-4}=\frac{0}{0}\text{ Indeterminent}$$

$$\underset{x\to 1}{\text{Lt}}\frac{x^2+x-2}{x^2-4x+3}=\underset{x\to 1}{\text{Lt}}\frac{(x+2)(x-1)}{(x-1)(x-3)}$$

$$=\underset{x\to 1}{\text{Lt}}\frac{x+2}{x-3}$$

$$=\frac{1+2}{1-3}=\frac{3}{-2}=-\frac{3}{2}$$

2. Evaluate $\underset{x\to 2}{\text{Lt}}\dfrac{x^{10}-2^{10}}{x-2}$.

Solution:

$$\underset{x\to 2}{\text{Lt}}\frac{x^{10}-2^{10}}{x-2}=10\,.\,2^{10-1}\qquad\left[\text{By F1 }\underset{x\to a}{\text{Lt}}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=10.2^9$$

$$=10\times 512$$

$$=5120$$

3. Evaluate $\underset{x\to a}{\text{Lt}}\dfrac{\sqrt{x}-\sqrt{a}}{x-a}$.

Solution:

$$\underset{x\to a}{\text{Lt}}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\underset{x\to a}{\text{Lt}}\frac{x^{1/2}-a^{1/2}}{x-a}$$

$$=\frac{1}{2}\,.\,a^{1/2-1}$$

$$=\frac{1}{2}a^{-1/2}$$

$$=\frac{1}{2}\,.\,\frac{1}{a^{1/2}}=\frac{1}{2\sqrt{a}}$$

4. Evaluate $\underset{x\to 0}{\text{Lt}} \frac{\sin 5x}{4x}$

Solution:

$$\underset{x\to 0}{\text{Lt}} \frac{\sin 5x}{4x} = \frac{1}{4} \underset{x\to 0}{\text{Lt}} \frac{\sin 5x}{x} = \frac{1}{4} \underset{x\to 0}{\text{Lt}} \left[\frac{\sin 5x}{5x} \times 5\right]$$

$$= \frac{1}{4} \times 5 \underset{x\to 0}{\text{Lt}} \frac{\sin 5x}{5x} \qquad \left[\begin{array}{l}\text{By } F_2 \\ \underset{x\to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1\end{array}\right.$$

$$= \frac{5}{4} . 1 = \frac{5}{4}$$

5. Evaluate $\underset{x\to 0}{\text{Lt}} \frac{\tan^2 5x}{x^2}$.

Solution:

$$\underset{x\to 0}{\text{Lt}} \frac{\tan^2 5x}{x^2} = \underset{x\to a}{\text{Lt}} \left(\frac{\tan 5x}{x}\right)^2$$

$$= \underset{x\to 0}{\text{Lt}} \left[\frac{\tan 5x}{5x}.5\right]^2$$

$$= \underset{x\to 0}{\text{Lt}} \left(\frac{\tan 5x}{5x}\right)^2 \times 25 \qquad \left[\begin{array}{l}\text{By } F_3 \\ \underset{\theta\to 0}{\text{Lt}} \frac{\tan\theta}{\theta} = 1\end{array}\right.$$

$$= 25 \underset{x\to a}{\text{Lt}} \left(\frac{\tan 5x}{5x}\right)^2$$

$$= 25.1^2 = 25$$

PART – C

1. Evaluate $\underset{x\to 3}{\text{Lt}} \frac{x^5 - 243}{x^2 - 9}$.

Solution:

$$\underset{x\to 3}{\text{Lt}} \frac{x^5 - 243}{x^2 - 9} = \underset{x\to 3}{\text{Lt}} \frac{x^5 - 3^5}{x^2 - 9}$$

$$= \underset{x\to 3}{\text{Lt}} \left[\frac{x^5 - 3^5}{x - 3} \times \frac{x - 3}{x^2 - 3^2}\right]$$

$$= \underset{x\to 3}{\text{Lt}} \left[\frac{x^5 - 3^5}{x - 3} \bigg/ \frac{x^2 - 3^2}{x - 3}\right]$$

$$= \frac{\underset{x\to 3}{\text{Lt}} \frac{x^5 - 3^5}{x - 3}}{\underset{x\to 3}{\text{Lt}} \frac{x^2 - 3^2}{x - 3}} \qquad [\text{By R4}$$

$$= \frac{5.3^{5-1}}{2.3^{2-1}} = \frac{5.3^4}{2.3^1}$$

$$= \frac{5}{2} \times 3^3 = \frac{5}{2} \times 27 = \frac{135}{2}$$

2. Evaluate : $\underset{x\to 4}{\text{Lt}} \dfrac{x^{7/3} - 4^{7/3}}{x^{2/3} - 4^{2/3}}$

Solution:

$$\underset{x\to 4}{\text{Lt}} \frac{x^{7/3} - 4^{7/3}}{x^{2/3} - 4^{2/3}} = \underset{x\to 4}{\text{Lt}} \left[\frac{x^{7/3} - 4^{7/3}}{x - 4} \times \frac{x - 4}{x^{2/3} - 4^{2/3}} \right]$$

$$= \underset{x\to 4}{\text{Lt}} \left[\frac{x^{7/3} - 4^{7/3}}{x - 4} \bigg/ \frac{x^{2/3} - 4^{2/3}}{x - 4} \right]$$

$$= \frac{\underset{x\to 4}{\text{Lt}} \dfrac{x^{7/3} - 4^{7/3}}{x - 4}}{\underset{x\to 4}{\text{Lt}} \dfrac{x^{2/3} - 4^{2/3}}{x - 4}}$$

$$= \frac{\frac{7}{3} . 4^{\frac{7}{3}-1}}{\frac{2}{3} . 4^{\frac{2}{3}-1}} = \frac{\frac{7}{3} . 4^{4/3}}{\frac{2}{3} . 4^{-1/3}}$$

$$= \frac{7}{3} \times \frac{3}{2} \times 4^{4/3 + 1/3} = \frac{7}{2} 4^{5/3}$$

3. Evaluate $\underset{\theta\to 0}{\text{Lt}} \dfrac{\sin 8\theta}{\tan 3\theta}$.

Solution:

$$\underset{\theta\to 0}{\text{Lt}} \frac{\sin 8\theta}{\tan 3\theta} = \underset{\theta\to 0}{\text{Lt}} \left[\frac{\sin 8\theta}{8\theta} \times \frac{3\theta}{\tan 3\theta} \right] \times \frac{8}{3}$$

$$= \underset{\theta\to 0}{\text{Lt}} \left[\frac{\sin 8\theta}{8\theta} \bigg/ \frac{\tan 3\theta}{3\theta} \right] \times \frac{8}{3}$$

$$= \frac{8}{3} \frac{\underset{\theta\to 0}{\text{Lt}} \dfrac{\sin 8\theta}{8\theta}}{\underset{\theta\to 0}{\text{Lt}} \dfrac{\tan 3\theta}{3\theta}}$$

$$= \frac{8}{3} . \frac{1}{1} = \frac{8}{3}$$

4.3 DIFFERENTIATION

Let y = f(x) —— (1) be a function of x.

Let Δx be a small increment in x and let Δy be the corresponding increment in y.

$\therefore\ y + \Delta y = f(x + \Delta x)$ ——— (2)

(2) – (1) $\quad \Delta y = f(x + \Delta x) - f(x)$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Taking the limit as $\Delta x \to 0$,

$$\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x}$ is called the differential coefficient of y with respect to x and is denoted as $\frac{dy}{dx}$.

$$\therefore \boxed{\frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

List of formulae:

(1) $\frac{d}{dx}(x^n) = n x^{n-1}$, n a real number

(2) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$ (3) $\frac{d}{dx}(e^x) = e^x$ (4) $\frac{d}{dx}(\log x) = \frac{1}{x}$

(5) $\frac{d}{dx}(\sin x) = \cos x$ (6) $\frac{d}{dx}(\cos x) = -\sin x$

(7) $\frac{d}{dx}(\tan x) = \sec^2 x$ (8) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(9) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (10) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

Results:

1) If u and v are functions of x,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

2) $\frac{d}{dx}(ku) = k\frac{du}{dx}$, where k is a constant.

3) $\frac{d}{dx}$(any constant) = 0

4) Product Rule of Differentiation

If u & v are two functions of x,

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

5) $\boxed{\frac{d}{dx}(u\ v\ w) = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}}$, where u, v and w are function of x.

6) Quotient Rule of Differentiation

If u & v are functions of x,

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

WORKED EXAMPLES

PART – A

1. Find $\frac{dy}{dx}$ if (i) $y = \frac{1}{x^2}$ (ii) $y = x^3 + 2$ (iii) $y = \frac{1}{\sqrt{x}}$ (iv) $y = \sqrt{x}$ (v) $y = \frac{1}{\sin x}$ (vi) $y = 9\sqrt{x} + x^2$

Solution:

(i) $y = \frac{1}{x^2} = x^{-2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^{-2}) = -2x^{-3} = -\frac{2}{x^3}$$

(ii) $y = x^3 + 2$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^3 + 2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(2) = 3x^2 + 0 = 3x^2$$

(iii) $y = \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}}$$

(iv) $y = \sqrt{x} = x^{1/2}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2}\cdot\frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$$

(v) $y = \frac{1}{\sin x} = \operatorname{cosec} x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

(vi) $y = 9\sqrt{x} + x^2$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(9\sqrt{x} + x^2) = 9\frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}(x^2)$$

$$= 9.\frac{1}{2\sqrt{x}} + 2x$$

$$= \frac{9}{2\sqrt{x}} + 2x$$

PART –B

1. Find $\frac{dy}{dx}$ if $y = \frac{3}{x^2} + \frac{2}{x} + \frac{1}{4}$.

Solution:

$$y = \frac{3}{x^2} + \frac{2}{x} + \frac{1}{4} = 3x^{-2} + 2x^{-1} + \frac{1}{4}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^{-2} + 2x^{-1} + \frac{1}{4}\right) = 3(-2)x^{-3} + 2(-1)x^{-2} + 0$$

$$= \frac{-6}{x^3} - \frac{2}{x^2}$$

2. Find $\frac{dy}{dx}$ if $y = e^x \sin x$.

Solution:

$$y = e^x \sin x$$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin x) = "u\frac{dv}{dx} + v\frac{du}{dx}"$$

$$= e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$= e^x \cos x + \sin x . e^x$$

3. Find $\frac{dy}{dx}$ if $y = x^2 e^x \sin x$.

Solution:

$$y = x^2 e^x \sin x$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 e^x \sin x) = "uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}"$$

$$= x^2 e^x \frac{d}{dx}(\sin x) + e^x \sin x \frac{d}{dx}(x^2) + \sin x . x^2 \frac{d}{dx}(e^x)$$

$$= x^2 e^x \cos x + e^x \sin x . 2x + \sin x \;.\; x^2 e^x$$

4. Find $\frac{dy}{dx}$ if $y = \frac{\cos x}{\sqrt{x}}$.

Solution:

$$y = \frac{\cos x}{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\sqrt{x}\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x}(-\sin x) - \cos x . \frac{1}{2\sqrt{x}}}{x}$$

PART – C

1. Find $\frac{dy}{dx}$ if y = $(x^2 + 3)\cos x \log x$.

Solution:

$$y = (x^2 + 3)\cos x \log x$$

$$\frac{dy}{dx} = \frac{d}{dx}[(x^2+3)\cos x \log x] = \text{"}uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}\text{"}$$

$$= (x^2+3)\cos x\frac{d}{dx}(\log x) + \cos x \log x\frac{d}{dx}(x^2+3) + \log x(x^2+3)\frac{d}{dx}(\cos x)$$

$$= (x^2+3)\cos x.\frac{1}{x} + \cos x \log x\ .\ 2x + \log x\ .\ (x^2+3)(-\sin x)$$

$$= \frac{(x^2+3)\cos x}{x} + 2x\cos x \log x - (x^2+3)\sin x \log x$$

2. If $y = \frac{x^3 \tan x}{e^x + 1}$, find $\frac{dy}{dx}$.

Solution:

$$y = \frac{x^3 \tan x}{e^x + 1}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(e^x+1)\frac{d}{dx}(x^3\tan x) - (x^3\tan x)\frac{d}{dx}(e^x+1)}{(e^x+1)^2}$$

$$= \frac{(e^x+1)\left[x^3\frac{d}{dx}(\tan x) + \tan x\frac{d}{dx}(x^3)\right] - x^3\tan x.e^x}{(e^x+1)^2}$$

$$= \frac{(e^x+1)[x^3\sec^2 x + \tan x.3x^2] - x^3\tan x\ .\ e^x}{(e^x+1)^2}$$

3. Find $\frac{dy}{dx}$ if $y = \frac{1+x+x^2}{1-x+x^2}$.

Solution:

$$y = \frac{1+x+x^2}{1-x+x^2}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1-x+x^2)\frac{d}{dx}(1+x+x^2) - (1+x+x^2)\frac{d}{dx}(1-x+x^2)}{(1-x+x^2)^2}$$

$$= \frac{(1-x+x^2)(1+2x) - (1+x+x^2)(-1+2x)}{(1-x+x^2)^2}$$

$$= \frac{1+2x-x-2x^2+x^2+2x^3+1-2x+x-2x^2+x^2-2x^3}{(1-x+x^2)^2}$$

$$= \frac{2-2x^2}{(1-x+x^2)^2} = \frac{2(1-x^2)}{(1-x+x^2)^2}$$

EXERCISE

PART –A

1. Find the principal value of

(i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (ii) $\cos^{-1}(0)$ (iii) $\tan^{-1}(\sqrt{3})$ (iv) $\cot^{-1}(-\sqrt{3})$ (v) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(vi) $\text{cosec}^{-1}\left(-\sqrt{2}\right)$ (vii) $\sec^{-1}(-1)$ (viii) $\cos^{-1}\left(-\frac{1}{2}\right)$ (ix) $\sin^{-1}(-1)$ (x) $\tan^{-1}(-1)$

2. Prove that $\sin^{-1}x + \sec^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$.

3. Prove that $\sec^{-1}x + \sin^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$.

4. Evaluate : $\underset{x\to 1}{\text{Lt}}\ \frac{px^2 + qx + r}{ax^2 + bx + c}$

5. Evaluate: $\underset{x\to 0}{\text{Lt}}\ \frac{5x - 8x^2}{2x^2 - 3x}$

6. Evaluate : $\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin 9\theta}{\theta}$

7. Evaluate : $\underset{\theta\to 0}{\text{Lt}}\ \frac{\tan 5\theta}{\theta}$

8. Evaluate : $\underset{x\to 2}{\text{Lt}}\ \frac{x^2 + 2x}{4x - 3x^2}$

9. Find $\frac{dy}{dx}$ if $y = 3x^4 - 7$.

10. Find $\frac{dy}{dx}$ if $y = \frac{1}{x^3}$.

11. Find $\frac{dy}{dx}$ if $y = \frac{1}{\cot x}$.

12. Find $\frac{dy}{dx}$ if $y = 8e^x - 4\ \text{cosec}\ x$.

13. Find $\frac{dy}{dx}$ if $y = 3\log x - 4\sqrt{x}$.

14. Find $\frac{dy}{dx}$ if $y = \frac{1}{x^{5/7}}$.

15. Find $\frac{dy}{dx}$ if $y = \frac{8}{x^{57}}$.

PART – B

1. Prove that $\cos^{-1}\left(\sqrt{1 - x^2}\right) = \sin^{-1}x$.

2. Prove that $\text{cosec}^{-1}\left(\sqrt{1 + x^2}\right) = \cot^{-1}x$.

3. Prove that $\sec^{-1}\left(\sqrt{1 + x^2}\right) = \tan^{-1}x$.

4. Prove that $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$.

5. Evaluate : $\underset{x\to 1}{\text{Lt}}\frac{x^2-3x+2}{x^2-5x+4}$

6. Evaluate: $\underset{x\to 1}{\text{Lt}}\frac{x^2+4x-5}{x^2+x+4}$

7. Evaluate : $\underset{x\to 3}{\text{Lt}}\frac{x^6-3^6}{x-3}$

8. Evaluate: $\underset{x\to 0}{\text{Lt}}\frac{\tan 7x}{13x}$

9. Evaluate : $\underset{x\to 0}{\text{Lt}}\frac{\sin^2 13x}{x^2}$

10. Evaluate : $\underset{x\to 0}{\text{Lt}}\frac{4\sin 3x}{5x}$

11. Evaluate : $\underset{x\to 0}{\text{Lt}}\frac{\sin\left(\frac{x}{2}\right)}{x}$

12. Evaluate : $\underset{x\to 0}{\text{Lt}}\frac{\tan\left(\frac{x}{3}\right)}{x}$

13. Find $\frac{dy}{dx}$ if $y = x^3 - 4x^2 + 7x - 11$.

14. Find $\frac{dy}{dx}$ if $y = x^3 + \frac{2}{x^2} - \frac{1}{x} + \frac{3}{2}$.

15. Find $\frac{dy}{dx}$ if $y = 2\sqrt{x} - 3e^x + 7\sin x - 8\cos x + 11$

16. Find $\frac{dy}{dx}$ if $y = \sqrt{x}\log x$.

17. Find $\frac{dy}{dx}$ if $y = x^2\,\text{cosec}\,x$.

18. Find $\frac{dy}{dx}$ if $y = (x^3 - 4)\tan x$.

19. Find $\frac{dy}{dx}$ if $y = \frac{x-7}{x+3}$.

20. Find $\frac{dy}{dx}$ if $y = \sqrt{x}\log x\,\text{cosec}\,x$.

21. Find $\sqrt{x}$ if $y = x^3\cot x\log x$.

22. Find $\frac{dy}{dx}$ if $y = \frac{e^x}{\sin x}$.

23. Find $\frac{dy}{dx}$ if $y = \frac{\tan x}{x^4}$.

24. Find $\frac{dy}{dx}$ if $y = \frac{1-\cos x}{1+\sin x}$.

25. Find $\frac{dy}{dx}$ if $y = \frac{4}{x} + 7\cos x - 9\log x + \frac{8}{\text{cosec}\,x} - 3$.

PART – C

1. Show that $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1} x$.
2. Show that $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1} x$.
3. Prove that $\sin^{-1}(3x - 4x^3) = 3\sin^{-1} x$.
4. Prove that $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x$.
5. Prove that $\tan^{-1}x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$.
6. Prove that $\sin^{-1}x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$.
7. Prove that $\cos^{-1}x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$.
8. Show that $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$.
9. Show that $\tan^{-1}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}\right] = \frac{x}{2}$.
10. Show that $2\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}\left(\frac{24}{7}\right)$.
11. Evaluate : $\cos\left[\sin^{-1}\left(\frac{5}{13}\right)\right]$
12. Evaluate : $\cos\left[\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right]$
13. Evaluate the following:

(i) $\underset{x\to 6}{\text{Lt}} \frac{x^3 - 6^3}{x^5 - 6^5}$ (ii) $\underset{x\to 4}{\text{Lt}} \frac{x^8 - 4^8}{x^5 - 4^5}$ (iii) $\underset{x\to 3}{\text{Lt}} \frac{x^{-\frac{7}{11}} - 3^{-\frac{7}{11}}}{x^{\frac{4}{11}} - 3^{\frac{4}{11}}}$ (iv) $\underset{x\to 2}{\text{Lt}} \frac{x^{32} - 2^{32}}{x^{17} - 2^{17}}$

(v) $\underset{\theta\to 0}{\text{Lt}} \frac{\sin 7\theta}{\sin 2\theta}$ (vi) $\underset{\theta\to 0}{\text{Lt}} \frac{\tan 6\theta}{\tan 11\theta}$ (vii) $\underset{\theta\to 0}{\text{Lt}} \frac{\sin 9\theta}{\tan 7\theta}$ (viii) $\underset{x\to 0}{\text{Lt}} \frac{1-\cos x}{x \sin x}$

14. Differentiate the following w.r.t. x.

(i) $y = (3x^2 + 2x + 1)\, e^x \tan x$.

(ii) $y = (2x + 1)(3x - 7)(4 - 9x)$.

(iii) $y = \frac{2\sin x + 3\cos x}{2\cos x + 3\sin x}$

(iv) $y = \frac{x^2 + 3}{x\cos x}$

(v) $y = \cos x - \frac{\sin x}{x + 7}$

(vi) $y = \frac{\sqrt{x} + \log x}{e^x + x^3}$

(vii) $y = \frac{e^x \sin x}{x^2 + 1}$

(viii) $y = \frac{\sqrt{x} + e^x}{x^3 \sin x}$

(ix) $y = \frac{x^2 e^x}{(x+2)\tan x}$

(x) $y = \frac{x + \tan x}{e^x - \sin x}$

14. (i) $(3x^2 + 2x + 1)\, e^x \sec^2 x + e^x \tan x\,(6x + 2) + \tan x\,(3x^2 + 2x + 1)\, e^x$

(ii) $(2x + 1)(3x - 7)(-9) + (3x - 7)(4 - 9x) . 2 + (49x)(2x + 1) 3$

(iii) $\dfrac{(2\cos x + 3\sin x)(2\cos x - 3\sin x) - (2\sin x + 3\cos x)(-2\sin x + 3\cos x)}{(2\cos x + 3\sin x)^2}$

(iv) $\dfrac{x\cos x(2x) - (x^2 + 3)(-x\sin x + \cos x)}{x^2\cos^2 x}$ (v) $-\sin x - \left[\dfrac{(x+7)\cos x - \sin x.1}{(x+7)^2}\right]$

(vi) $\dfrac{(e^x + x^3)\left(\dfrac{1}{2\sqrt{x}} + \dfrac{1}{x}\right) - (\sqrt{x} + \log x)\,(e^x + 3x^2)}{(e^x + x^3)^2}$ (vii) $\dfrac{(x^2+1)(e^x\cos x + \sec x.e^x) - e^x\sin x.2x}{(x^2+1)^2}$

(viii) $\dfrac{x^3\sin x\left(\dfrac{1}{2\sqrt{x}} + e^x\right) - (\sqrt{x} + e^x)(x^3\cos x + 3x^2\sin x)}{x^6\sin^2 x}$

(ix) $\dfrac{(x+2)\tan x\,(x^2e^x + 2xe^x) - (x^2e^x)[(x+2)\sec^2 x + \tan x\,.\,1]}{(x+2)^2\tan^2 x}$

(x) $\dfrac{(e^x - \sin x)(1 + \sec^2 x) - (x + \tan x)(e^x - \cos x)}{(e^x - \sin x)^2}$

ANSWERS

PART – A

1) (i) 60^o (ii) 90^o (iii) 60^o (iv) -30^o (v) 30^o (vi) -45^o

(vii) -180^o (viii) 120^o (ix) -90^o (x) -45^o

4) $\dfrac{p+q+r}{a+b+c}$ 5) $\dfrac{-5}{3}$ 6) 9 7) 5 8) –2 9) $12x^3$ 10) $-\dfrac{3}{x^4}$

11) $\sec^2 x$ 12) $8ex + 4\,\text{cosec}\,x\ \cot x$ 13) $\dfrac{3}{x} - \dfrac{2}{\sqrt{x}}$ 14) $\dfrac{-5}{7}x - \dfrac{12}{7}$ 15) $-\dfrac{456}{x^{58}}$

PART – B

5) $\dfrac{1}{3}$ 6) 2 7) 6.3^5 8) $\dfrac{7}{13}$ 9) 169 10) $\dfrac{12}{5}$ 11) $\dfrac{1}{2}$ 12) $\dfrac{1}{3}$ 13) $3x^2 - 8x + 7$

14) $3x^2 - \dfrac{4}{x^3} + \dfrac{1}{x^2}$ 15) $\dfrac{1}{\sqrt{x}} - 3e^x + 7\cos x + 8\sin x$ 16) $\dfrac{\sqrt{x}}{x} + \log x\,.\,\dfrac{1}{2\sqrt{x}}$

18) $(x^3 - 4)\sec^2 x + 3x^2\tan x$ 19) $\dfrac{10}{(x+3)^2}$ 20) $-\sqrt{x}\log x\ \text{cosec}\,x\cot x + \log x\ \text{cosec}\,x\,.\,\dfrac{1}{2\sqrt{x}} + \text{cosec}\,x\,.\,\dfrac{\sqrt{x}}{x}$

21) $\dfrac{x^3\cot x}{x} + \cos x\log x\,.\,3x^2 + \log x\ x^3(-\text{cosec}^2 x)$ 22) $\dfrac{\sin x e^x - e^x\cos x}{\sin^2 x}$ 23) $\dfrac{x^4\sec^2 x - \tan x\,.\,4x^3}{x^8}$

24) $\dfrac{(1+\sin x)\sin x - (1-\cos x)\cos x}{(1+\sin x)^2}$ 25) $-\dfrac{4}{x^2} - 7\sin x - \dfrac{9}{x} + \sin x$

PART – C

11) $\dfrac{12}{13}$ 12) $\dfrac{16}{65}$ 13) (i) $\dfrac{1}{60}$ (ii) $\dfrac{512}{5}$ (iii) $-\dfrac{7}{12}$ (iv) $\dfrac{3.2^{15}}{17}$ (v) $\dfrac{7}{2}$

(vi) $\dfrac{6}{11}$ (vii) $\dfrac{9}{7}$ (vii) $\dfrac{1}{2}$

UNIT – V

DIFFERENTIAL CALCULUS-II

5.1 DIFFERENTIATION METHODS

Differentiation of function functions (chain rule), Inverse Trigonometric functions and Implict functions. Simple Problems.

5.2 SUCCESSIVE DIFFERENTIATION

Successive differentiation up to second order (parametric form not included). Definition of differential equation, order and degree, formation of differential equation. Simple problems.

5.3 PARTIAL DIFFERENTIATION

Definition – Partial differentiation of two variables up to second order only. Simple Problems.

5.1 DIFFERENTIATION METHODS

Function of Funtions Rule:

If 'y' is a function of 'u' and 'u' is a function of 'x' then the derivative of 'y' w.r.t 'x' is equal to the product of the derivative of 'y' w.r.t 'u' and the derivative of 'u' w.r.t 'x'.

i.e. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

It is called function of function rule. This rule can be extended which is known as 'Chain rule'.

Chain rule:

If 'y' is a function of 'u' and 'u' is a function of 'v' and 'v' is a function of 'x' then.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \quad \frac{dv}{dx}$$

WORKED EXAMPLES

PART –A

Find $\frac{dy}{dx}$ if

1) $y = (2x + 5)^3$ 2) $y = \sqrt{\sin x}$ 3) $y = \cos^4 x$ 4) $y = e^{\tan x}$ 5) $y = \log(\sec x)$

6) $y = \sin mx$ 7) $y = \sec\sqrt{x}$ 8) $y = \cos(2 - 3x)$

Solution:

1) $y = (2x + 5)^3$

$y = u^3$ where $u = 2x + 5$

$\frac{dy}{du} = 3u^2 \quad \Big| \quad \frac{du}{dx} = 2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$= (3u^2)(2)$

$= 6(2x + 5)^2$

2) $y = \sqrt{\sin x}$

$y = \sqrt{u}$ where $u = \sin x$

$\frac{dy}{du} = \frac{1}{2\sqrt{u}} \quad \Big| \quad \frac{du}{dx} = \cos x$

$$\frac{dy}{dx} = \left(\frac{1}{2\sqrt{u}}\right)(\cos x)$$

$$= \frac{\cos x}{2\sqrt{\sin x}}$$

3) $y = \cos^4 x$

$y = u^4$ where $u = \cos x$

$\frac{dy}{du} = 4u^3 \quad \Big| \quad \frac{du}{dx} = -\sin x$

$$\frac{dy}{dx} = 4u^3(-\sin x)$$

$= -4\sin x \cos^3 x$

4) $y = e^{\tan x}$

$y = e^u$ where $u = \tan x$

$\frac{dy}{du} = e^u \quad \Big| \quad \frac{du}{dx} = \sec^2 x$

$$\frac{dy}{dx} = (e^u)(\sec^2 x)$$

$= e^{\tan x}\sec^2 x$

5) y = log (sec x)

y = log u where u = sec x

$\frac{dy}{du} = \frac{1}{u}$ | $\frac{du}{dx} = \sec x \ \tan x$

$$\frac{dy}{dx} = \left(\frac{1}{u}\right)(\sec x \tan x)$$

$$= \frac{1}{\sec x}(\sec x \tan x)$$

$$= \tan x$$

6) y = sin mx

y = sin u where u = mx

$\frac{dy}{du} = \cos u$ | $\frac{du}{dx} = m$

$$\frac{dy}{dx} = (\cos u)\ (m)$$

$$= m \cos mx$$

7) y = cos (2 – 3x)

y = cos u where u = 2 – 3x

$\frac{dy}{du} = -\sin u$ | $\frac{du}{dx} = 0 - 3(1) = -3$

$$\frac{dy}{dx} = (-\sin u)\ (-3)$$

$$= 3\sin(2 - 3x)$$

8) $y = \sec\sqrt{x}$

y = sec u where u = $\sqrt{x}$

$\frac{dy}{du} = \sec u \ .\tan u$ | $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$$\frac{dy}{dx} = (\sec u \tan u)\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x}} \ . \sec\sqrt{x} \ . \ \tan\sqrt{x}$$

Note:

1. $\frac{d}{dx}(e^{mx}) = me^{mx}$
2. $\frac{d}{dx}(\sin mx) = m \cos mx$
3. $\frac{d}{dx}(\cos mx) = -m \sin mx$
4. $\frac{d}{dx}(\tan mx) = m \sec^2 mx$

5. $\frac{d}{dx}$ (cot mx) = – m $cosec^2$ mx

6. $\frac{d}{dx}$ (sec mx) = m sec mx tan mx

7. $\frac{d}{dx}$ (cosec mx) = – m cosec mx cot mx

PART – B

Differentiate the following w.r.t 'x'.

1) $(2x^2 - 3x + 1)^3$ 2) $\cos(e^{5x})$ 3) $e^{\sin^2 x}$ 4) log (sec x + tan x) 5) $y = cosec^3(5x+1)$

Solution:

1) $y = (2x^2 - 3x + 1)^3$

$y = u^3$ where $u = 2x^2 - 3x + 1$

$\frac{dy}{du} = 3u^2 \;\Big|\; \frac{du}{dx} = 4x - 3$

$\frac{dy}{dx} = (3u^2)(4x-3)$

$= 3(2x^2 - 3x + 1)^2(4x - 3)$

2) Let $y = \cos(e^{5x})$

$y = \cos u$ where $u = e^{5x}$

$\frac{dy}{du} = -\sin u \;\Big|\; \frac{du}{dx} = 5e^{5x}$

$\frac{dy}{dx} = (-\sin u)(5e^{5x})$

$= -5e^{5x}\sin(e^{5x})$

3) Let $y = e^{\sin^2 x}$

$y = e^u$ where $u = \sin^2 x$

$\frac{dy}{du} = e^u \;\Big|\; u = v^2$ where $v = \sin x$

$\frac{du}{dv} = 2v \;\Big|\; \frac{dv}{dx} = \cos x$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

$= (e^u)(2v)(\cos x)$

$= 2e^{\sin^2 x}\sin x \cos x$

$= \sin 2x \cdot e^{\sin^2 x}$

4) Let y = log (sec x + tan x)

y = log u where u = sec x + tan x

$$\frac{dy}{du} = \frac{1}{u} \quad \Bigg| \quad \frac{du}{dx} = \sec x \,.\, \tan x + \sec^2 x$$
$$= \sec x(\tan x + \sec x)$$

$$\frac{dy}{dx} = \left(\frac{1}{u}\right)[\sec x(\tan x + \sec x)]$$

$$= \frac{1}{(\sec x + \tan x)} \,.\, \sec x(\tan x + \sec x)$$

$$= \sec x$$

5) Let $y = \text{cosec}^3(5x+1)$

$y = u^3$ where $u = \text{cosec}\,(5x + 1)$

$$\frac{dy}{du} = 3u^2 \quad \Bigg| \quad u = \text{cosec}\, v \text{ where } v = 5x + 1$$

$$\frac{du}{dv} = -\text{cosec}\, v \cot v \quad \Bigg| \quad \frac{dv}{dx} = 5$$

$$\frac{dy}{dx} = (3u^2)(-\text{cosec}\, v \cot v)\,(5)$$

$$= -15\ \text{cosec}^2(5x+1) \,.\, \text{cosec}\,(5x + 1) \cot (5x + 1)$$

$$= -15\ \text{cosec}^3(5x+1)\cot(5x+1)$$

PART – C

Differentiate the following w.r.t x.

1) $y = \sin\,(e^x \log x)$

2) $(x^2 + 5)\, e^{-2x} \sin x$

3) $y = (x \cos x)^3$

4) $y = \log\left(\dfrac{1-\cos x}{1+\cos x}\right)$

5) $y = \sqrt{\dfrac{1+x^3}{1+x^3}}$

6) $y = e^{4x} + \sin\,(x^2 + 5)$

7) $y = x^3e^{-5x} \log(\sec x)$

Solution:

1) $y = \sin\,(e^x \log x)$

$y = \sin u$ where $u = e^x \log x$

$$\frac{dy}{du} = \cos u \quad \Bigg| \quad \frac{du}{dx} = e^x \frac{1}{x} + \log x \times e^x$$
(using product rule)
$$= e^x\left(\frac{1}{x} + \log x\right)$$

$$\frac{dy}{dx} = (\cos u)\left[e^x\left(\frac{1}{x} + \log x\right)\right]$$

$$= \cos(e^x \log x)\left[e^x\left(\frac{1}{x} + \log x\right)\right]$$

2) $y = (x^2 + 5)\, e^{-2x} \sin x$

$$u = x^2 + 5 \quad \Big| \quad v = e^{-2x} \quad \Big| \quad w = \sin x$$

$$\frac{du}{dx} = 2x \quad \Bigg| \quad \frac{dv}{dx} = e^{-2x}(-2) = -2e^{-2x} \quad \Bigg| \quad \frac{dw}{dx} = \cos x$$

$$\frac{dy}{dx} = u\,v\frac{dw}{dx} + v\,w\frac{du}{dx} + u\,w\frac{dv}{dx}$$

$$= (x^2 + 5)e^{-2x}\cos x + e^{-2x}\sin x(2x) + (x^2 - 5)\sin x(-2e^{-2x})$$

3) $y = (x \cos x)^3$

$y = u^3$ where $u = x \cos x$

$$\frac{dy}{du} = 3u^2 \quad \Bigg| \quad \frac{du}{dx} = x(-\sin x) + \cos x(1)$$

(using product rule)

$$= -x\sin x + \cos x$$

$$\frac{dy}{dx} = (3u^2)(-x\sin x + \cos x)$$

$$= 3(x\cos x)^2(-x\sin x + \cos x)$$

4) $y = \log\left(\frac{1-\cos x}{1+\cos x}\right)$

$y = \log u$ where $u = \frac{1-\cos x}{1+\cos x}$

$$\frac{dy}{du} = \frac{1}{u} \quad \Bigg| \quad \frac{du}{dx} = \frac{(1+\cos x)(+\sin x) - (1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

(using Quotient rule)

$$\frac{dy}{dx} = \left(\frac{1}{u}\right)\left[\frac{\sin x(1+\cos x) + \sin x(1-\cos x)}{(1+\cos x)^2}\right]$$

$$= \left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{2\sin x}{(1+\cos x)^2}\right]$$

$$= \frac{2\sin x}{(1+\cos x)(1-\cos x)} = \frac{2\sin x}{1-\cos^2 x}$$

$$= \frac{2\sin x}{\sin^2 x}$$

$$= \frac{2}{\sin x}$$

5) $y = \sqrt{\frac{1+x^3}{1+x^3}}$

$y = \sqrt{u}$ where $u = \frac{1-x^3}{1+x^3}$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

$$\frac{du}{dx} = \frac{(1+x^3)(-3x^2) - (1-x^3)(3x^2)}{(1+x^3)^2}$$

$$= 3x^2\left[\frac{-(1+x^3) - (1-x^3)}{(1+x^3)^2}\right]$$

$$= \frac{-6x^2}{(1+x^3)^2}$$

$$\frac{dy}{dx} = \left(\frac{1}{2\sqrt{u}}\right)\frac{-6x^2}{(1+x^3)^2}$$

$$= \frac{1}{2\sqrt{\frac{1-x^3}{1+x^3}}}\left[\frac{-6x^2}{(1+x^3)^2}\right]$$

$$= \frac{-3x^2}{(1+x^3)^2}\sqrt{\frac{1+x^3}{1-x^3}}$$

6) $y = e^{4x} + \sin(x^2 + 5)$

$$\frac{dy}{dx} = e^{4x} \times 4 + \cos(x^2+5) \times (2x+0)$$

$$= 4e^{4x} + 2x\cos(x^2+5)$$

7) $y = x^3 e^{-5x} \log(\sec x)$

$u = x^3$	$v = e^{-5x}$	$w = \log(\sec x)$
$\frac{du}{dx} = 3x^2$	$\frac{dv}{dx} = e^{-5x}(-5)$	$\frac{dw}{dx} = \frac{1}{\sec x}\sec x \cdot \tan x$
	$= -5e^{-5x}$	$= \tan x$

$$\frac{dy}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}$$

$$= x^3e^{-5x}\tan x + e^{-5x}\log(\sec x)(3x^2) + x^3\log(\sec x)(-5e^{-5x})$$

5.1.2 DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

If x = sin y then we get a unique real value of x to the given angle y. But if x is given y has many values.

We can express y interms of x as $y = \sin^{-1}x$ "Inverse sine of x". The Quantities $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$ are called Inverse Trigonometric Functions.

Differential co-efficient of $\sin^{-1}x$:

Let $y = \sin^{-1}x$

$\Rightarrow \sin y = x$

Differentiating w.r.t x

$$\cos y.\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\text{i.e., } \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

Similarly we get

(i) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(iii) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(iv) $\frac{d}{dx}(\sec^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

(v) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

WORKED EXAMPLES

PART – A

Find $\frac{dy}{dx}$ if

(i) $y = \sin^{-1}\sqrt{x}$

(ii) $y = \cos^{-1}(2x)$

(iii) $y = \tan^{-1}(\log x)$

Solution:

(i) $y = \sin^{-1}\sqrt{x}$

$$\frac{dy}{dx} = \sin^{-1} u \text{ where } u = \sqrt{x}$$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \quad \bigg| \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-u^2}}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x(1-x)}}$$

(ii) $y = \cos^{-1}(2x)$

$y = \cos^{-1}u$ where $u = 2x$

$$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}} \quad \bigg| \quad \frac{du}{dx} = 2$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}.2$$

$$= \frac{-1}{\sqrt{1-(2x)^2}} \cdot 2$$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-4x^2}}$$

(iii) $y = \tan^{-1}(\log x)$

$y = \tan^{-1}u$ where $u = \log x$

$$\frac{dy}{du} = \frac{1}{1+u^2} \quad \Big| \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \left(\frac{1}{1+u^2}\right)\left(\frac{1}{x}\right)$$

$$= \frac{1}{x[1+(\log x)^2]}$$

PART – B

Find $\frac{dy}{dx}$ if

(i) $y = \sin^{-1}(\cos x)$ (ii) $y = \cos^{-1}(2x + 3)$ (iii) $y = \cos^{-1}(2x^2 - 1)$ (iv) $y = \sin^{-1}(3x - 4x^3)$

Solution:

(i) $y = \sin^{-1}(\cos x)$

$y = \sin^{-1}u$ where $u = \cos x$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \quad \Big| \quad \frac{du}{dx} = -\sin x$$

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-u^2}}\right)(-\sin x)$$

$$= \frac{1}{\sqrt{1-\cos^2 x}}(-\sin x)$$

$$= \frac{-\sin x}{\sqrt{\sin^2 x}} = -1$$

(ii) $y = \cos^{-1}(2x + 3)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(2x+3)^2}} \cdot \frac{d}{dx}(2x+3)$$

$$= \frac{-1}{\sqrt{1-(2x+3)^2}} \times 2$$

$$= \frac{-2}{\sqrt{1-(2x+3)^2}}$$

(iii) $y = \cos^{-1}(2x^2 - 1)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(2x^2-1)^2}} \cdot \frac{d}{dx}(2x^2)$$

$$= \frac{-1}{\sqrt{1-(2x^2-1)^2}} \cdot 4x$$

$$= \frac{-4x}{\sqrt{1-(2x^2-1)^2}} = \frac{-4x}{\sqrt{1-4x^4+4x^2-1}} = \frac{-4x}{\sqrt{4x^2-4x^4}} = \frac{-4x}{2x\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

ALITER

$y = \cos^{-1}(2x^2 - 1)$

Put $x = \cos\theta$

$y = \cos^{-1}(2\cos^2\theta - 1)$

$= \cos^{-1}(\cos 2\theta)$

$y = 2\theta$

$y = 2\cos^{-1}x \qquad x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$$\frac{dy}{dx} = 2 \times \frac{-1}{\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

(iv) $y = \sin^{-1}(3x - 4x^3)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x-4x^3)^2}}.(3-12x^2)$$

ALITER

Put $x = \sin\theta$

$y = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \sin^{-1}(\sin 3\theta)$

$y = 3\theta$

$y = 3\sin^{-1}x \qquad x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$$\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$$

PART – C

(i) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(ii) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(iii) $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(iv) $y = (1+x^2)\tan^{-1}x$

Solution:

(i) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$= \cos^{-1}(\cos 2\theta)$

$= 2\theta$

$y = 2\tan^{-1}x$

$$\frac{dy}{dx} = 2\frac{1}{1+x^2}$$

(ii) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Let x = tan θ ⇒ θ = tan⁻¹x

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$
$$= \sin^{-1}(\sin 2\theta)$$
$$= 2\theta$$
$$y = 2\tan^{-1} x$$
$$\frac{dy}{dx} = 2\,.\,\frac{1}{1+x^2}$$

(iii) $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Let x = tan θ ⇒ θ = tan⁻¹x

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$
$$= \tan^{-1}(\tan 2\theta)$$
$$y = 2\theta$$
$$y = 2\tan^{-1} x$$
$$\frac{dy}{dx} = 2\,.\,\frac{1}{1+x^2}$$

(iv) $y = (1 + x^2)\tan^{-1} x$

$$\frac{dy}{dx} = (1+x^2)\frac{1}{1+x^2} + \tan^{-1} x(0+2x) \quad \text{(using product rule)}$$
$$= 1 + 2x\tan^{-1} x$$

5.1.3 IMPLICIT FUNCTION

If the variables x and y are connected by a relation f (x, y) = 0 then it is called an implicit function.

i.e If the dependent variable (y) cannot be expressed explicitly in terms of the independent variable x.

Example: (i) $x^2 + y^2 - 4x + y + 12 = 0$

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are implicit function

PART –B

1. Find $\frac{dy}{dx}$ if

(i) $x^2 + y^2 + 2y = 0$ (ii) $x^2 + y^2 = a^2$ (iii) $xy = c^2$ (iv) $\sqrt{y} + \sqrt{x} = \sqrt{a}$

Solution:

(i) $x^2 + y^2 + 2y = 0$

Differentiate w.r.t x

$$2x + 2y\frac{dy}{dx} + 2\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$(y+1)\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{-x}{1+y}$$

(ii) $x^2 + y^2 = a^2$

Differentiate w.r.t x

$$2x + 2y\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

(iii) $xy = c^2$

Differentiate w.r.t x

$$x\,.\frac{dy}{dx} + y(1) = 0 \text{ (using product rule)}$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

(iv) $\sqrt{y} + \sqrt{x} = \sqrt{a}$

Differentiate w.r.t x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}.\frac{dy}{dx} = 0$$

Multiply by 2

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}.\frac{dy}{dx} = 0$$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} = \frac{-1}{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}$$

PART – C

1. Find $\frac{dy}{dx}$ if

(i) $\cos x + \sin y = c$ (ii) $x^2 \sin y = c$ (iii) $y = a + xe^y$

Solution:

(i) $\cos x + \sin y = c$

Differentiate w.r.t x

$$-\sin x + \cos y \cdot \frac{dy}{dx} = 0$$

$$\cos y \cdot \frac{dy}{dx} = \sin x$$

$$\frac{dy}{dx} = \frac{\sin x}{\cos y}$$

(ii) $x^2 \sin y = c$

Differentiate w.r.t x

$$x^2 \cos y . \frac{dy}{dx} + \sin y(2x) = 0 \quad \text{(using product rule)}$$

$$x^2 \cos y . \frac{dy}{dx} = -2x \sin y$$

$$\frac{dy}{dx} = \frac{-2x \sin y}{x^2 \cos y}$$

$$\frac{dy}{dx} = \frac{-2}{x} \tan y$$

(iii) $y = a + xe^y$

Differentiate w.r.t x

$$\frac{dy}{dx} = 0 + x\ e^y . \frac{dy}{dx} + e^y$$

$$\frac{dy}{dx} - xe^y \frac{dy}{dx} = e^y$$

$$\frac{dy}{dx}(1 - xe^y) = e^y$$

$$\frac{dy}{dx} = \frac{e^y}{1 - xe^y}$$

2. Find $\frac{dy}{dx}$ if

(i) $x^3 + y^3 = 3axy$ (ii) $x^2 + y^2 - 4x + 6y - 5 = 0$ (iii) $x^2 + y^2 = xy$

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(i) $x^3 + y^3 = 3axy$

Differentiate w.r.t x

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y(1)\right]$$

Divide by 3 (using product rule)

$$x^2 + y^2 \frac{dy}{dx} = a\left[x\frac{dy}{dx} + y\right]$$

$$y^2 \frac{dy}{dx} - ax\frac{dy}{dx} = ay - x^2$$

$$(y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

(ii) $x^2 + y^2 - 4x + 6y - 5 = 0$

Differentiating w.r.t x

$$2x + 2y\frac{dy}{dx} - 4 + 6\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} + 6\frac{dy}{dx} = -2x + 4$$

$$\frac{dy}{dx}(2y + 6) = -2x + 4$$

$$\frac{dy}{dx} = \frac{-2x + 4}{2y + 6} = \frac{2(-x + 2)}{2(y + 3)}$$

$$= \frac{-x + 2}{y + 3}$$

(iii) $x^2 + y^2 = xy$

uv rule

Differentiating w.r.t x,

$$2x + 2y\frac{dy}{dx} = x\frac{dy}{dx} + y(1)$$

$$2y\frac{dy}{dx} - x\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx}(2y - x) = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Differentiate w.r.t x

$$2ax + 2h\left[x.\frac{dy}{dx} + y(1)\right] + b(2y).\frac{dy}{dx} + 2g(1) + 2f.\frac{dy}{dx} + 0 = 0$$

Divide by 2

$$ax + hx\frac{dy}{dx} + hy + by\frac{dy}{dx} + g + f\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(hx + by + f) = -ax - hy - g$$

$$\frac{dy}{dx} = \frac{-ax - hy - g}{hx + by + f}$$

5.2.1 SUCCESSIVE DIFFERENTIATION

If $y = f(x)$ then $\frac{dy}{dx} = f'(x)$ is a function of x or constant which can be differentiated once again.

$\frac{dy}{dx} = f'(x)$ is the first order derivative of y w.r.t x.

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the second order derivative of y w.r.t x

i.e the derivate of $\frac{dy}{dx}$ w.r.t x.

$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is the third order derivative of y w.r.t x and so on.

Notations of successive derivatives

$$\frac{dy}{dx} = y_1 = y' = f'(x) = D(y)$$

$$\frac{d^2y}{dx^2} = y_2 = y'' = f''(x) = D^2(y)$$

$$\frac{d^ny}{dx^n} = y_n = y^{(n)} = f^{(n)}(x) = D^n(y)$$

WORKED EXAMPLES

PART – A

1. If $y = e^{3x}$ find $\frac{d^2y}{dx^2}$.

Solution:

$$\frac{dy}{dx} = 3e^{3x}$$

$$\frac{d^2y}{dx^2} = 3(3e^{3x})$$

$$= 9e^{3x}$$

2. If $y = \sin 3x$ find $\frac{d^2y}{dx^2}$.

Solution:

$$\frac{dy}{dx} = 3\cos 3x$$

$$\frac{d^2y}{dx^2} = 3(-3\sin 3x)$$

$$= -9\sin 3x$$

PART –B

1. If $y = a \cos nx + b \sin nx$ find y_2.

Solution:

$$
\begin{aligned}
y_1 &= a(-n \sin nx) + b(n \cos nx)\\
&= n(-a \sin nx + b \cos nx)\\
y_2 &= n[-a(+n \cos nx) + b(-n \sin nx)]\\
&= n^2 (a \cos nx + b \sin nx)\\
&= -n^2 y
\end{aligned}
$$

2. If $y = Ae^{3x} + Be^{-3x}$ find $\frac{d^2y}{dx^2}$.

Solution:

$$
\begin{aligned}
\frac{dy}{dx} &= A(3e^{3x}) + B(-3e^{-3x})\\
&= 3(A.e^{3x} - Be^{-3x})\\
\frac{d^2y}{dx^2} &= 3[A(3e^{3x}) - B(-3e^{-3x})]\\
&= 3[3A\,e^{3x} + 3B\,e^{-3x}]\\
&= 9[Ae^{3x} + Be^{-3x}]\\
&= 9y
\end{aligned}
$$

3. If $y = \tan x$ find y_2.

Solution:

$$
\begin{aligned}
\frac{dy}{dx} &= \sec^2 x\\
\frac{d^2y}{dx^2} &= 2(\sec x) \times \sec x \tan x\\
&= 2 \sec^2 x \tan x
\end{aligned}
$$

PART – C

1. If $y = x^2 \cos x$ prove that $x^2y_2 - 4xy_1 + (x^2 + 6)y = 0$.

Solution:

$$
\begin{aligned}
y &= x^2 \cos x\\
y_1 &= x^2(-\sin x) + \cos x\,(2x) \qquad \text{(using product rule)}\\
&= -x^2 \sin x + 2x \cos x\\
y_2 &= -x^2(\cos x) + \sin x(-2x) + 2x(-\sin x) + \cos x\,(2)\\
&= -x^2 \cos x - 2x \sin x - 2x \sin x + 2 \cos x\\
&= -x^2 \cos x - 4x \sin x + 2 \cos x
\end{aligned}
$$

$$
\begin{aligned}
x^2y_2 - 4xy_1 + (x^2 + 6)y &= x^2[-x^2 \cos x - 4x \sin x + 2 \cos x]\\
&\quad - 4x[-x^2 \sin x + 2x \cos x] + (x^2 + 6)(x^2 \cos x)\\
&= -x^4 \cos x - 4x^3 \sin x + 2x^2 \cos x + 4x^3 \sin x - 8x^2 \cos x + x^4 \cos x + 6x^2 \cos x\\
&= 0
\end{aligned}
$$

2. If $y = e^x \sin x$, prove that $y_2 - 2y_1 + 2y = 0$.

Solution:

$y = e^x \sin x$

$y_1 = e^x (\cos x) + \sin x\, (e^x)$ (using product rule)

$y_1 = e^x \cos x + y$ ——— (1)

Differentiating w.r.t x

$y_2 = e^x (-\sin x) + \cos x\, e^x + y_1$

$y_2 = -y + (y_1 - y) + y_1$ from (1)

$y_2 = -y + y_1 - y + y_1$

$= -2y + 2y_1$

$y_2 - 2y_1 + 2y = 0$

3. If $y = a \cos (\log x) + b \sin (\log x)$ prove that $x^2y_2 + xy_1 + y = 0$.

Solution:

$y = a \cos (\log x) + b \sin (\log x)$

Differentiating w.r.t x

$$y_1 = a\left[-\sin(\log x).\frac{1}{x}\right] + b\left[\cos(\log x).\,\frac{1}{x}\right]$$

$$y_1 = \frac{1}{x}[-a\sin(\log x) + b\cos(\log x)]$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating w.r.t x

$$xy_2 + y_1(1) = -a\cos(\log x)\,.\,\frac{1}{x} + b\left(-\sin(\log x)\,.\,\frac{1}{x}\right)$$

$$xy_2 + y_1 = \frac{-1}{x}[a\cos(\log x) + b\sin(\log x)]$$

$$x[xy_2 + y_1] = -y$$

$$x^2y_2 + xy_1 + y = 0$$

5.2.2 DIFFERENTIAL EQUATION

Definition: An equation containg differential co-efficients is called differential equation.

Examples:

1. $\dfrac{dy}{dx} + y = x$
2. $x\dfrac{d^2y}{dx^2} + 1 = 0$
3. $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = d$

Order of the differential equation:

The order of the highest derivative in the differential equation is called order of the differential equation.

Degree of the differential equation:

The power of the highest derivative in the differential equation is called degree of the differential equation.

Example:

(i) $7\left(\frac{d^2y}{dx^2}\right)^2 + 5\left(\frac{dy}{dx}\right)^5 + 7y = \sin x$

Order – 2, degree –2

Formation of Differential Equation:

A differential equation is obtained by differentiating the function f (x, y) = 0 as many times as the number of arbitrary constants, followed by elimination of arbitrary constants.

PART –A

1. Find the order and degree of the following differential equations.

(i) $\left(\frac{d^3y}{dx^3}\right)^4 + \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 7y = 0$

(ii) $\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} = k$

(iii) $\sqrt{1+\frac{dy}{dx}} = \frac{d^2y}{dx^2}$

Solution

(i) order = 3, degree = 4

(ii) order =1, degree = 2

(iii) $\sqrt{1+\frac{dy}{dx}} = \frac{d^2y}{dx^2}$

Squaring on both sides

$$\left(1+\frac{dy}{dx}\right) = \left(\frac{d^2y}{dx^2}\right)^2$$

Order –2, degree – 2.

2. From the differential equation by eliminating the constant.

(i) xy = c (ii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Solution:

(i) xy = c

$x.\frac{dy}{dx} + y.1 = 0$ (using product rule)

$$x\frac{dy}{dx} + y = 0$$

(ii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

Multiply by 2

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$$

PART –B

From the different equation by eliminating the constants.

(i) $y = ax + b^2$ (ii) $y = m\,e^{5x}$ (iii) $\frac{x}{a} + \frac{y}{b} = 1$ (iv) $y^2 = 4ax$

Solution:

(i) $y = ax + b^2$

$$\frac{dy}{dx} = a(1) + 0$$

$$\frac{d^2y}{dx^2} = 0$$

(ii) $y = m\,e^{5x}$ —— (1)

$$\frac{dy}{dx} = m.(5e^{5x})$$

$$\frac{dy}{dx} = 5m(e^{5x}) \qquad \text{from (1) } me^{mx} = y$$

$$\frac{dy}{dx} = 5y$$

(iii) $\frac{x}{a} + \frac{y}{b} = 1$

Differentiating w.r.t x

$$\frac{1}{a} + \frac{1}{b} \quad \frac{dy}{dx} = 0$$

$$0 + \frac{1}{b}\frac{d^2y}{dx^2} = 0$$

$$\therefore \frac{d^2y}{dx^2} = 0$$

(iv) $y^2 = 4ax$

$$2y.\frac{dy}{dx} = 4a(0)$$

$$2y.\frac{dy}{dx} = 4\left(\frac{y^2}{4x}\right) \qquad \left(\because y^2 = 4ax,\ \frac{y^2}{4x} = a\right)$$

$$2y\frac{dy}{dx} = \frac{y^2}{x}$$

5.3 PARTIAL DIFFERENTIATION

Functions of two or more variables:

In many applications, we come across function involving more than one independent variable. For example, Area of the rectangle depends on its length and breadth, volume of cuboid depends on its length, breadth and height.

Hence one variable u depends on more than one variable.

i.e. $u = f(x, y)$ $\quad v = \varphi(x, y, z)$

Definition of partial differentiation:

Let $u = f(x, y)$ then partial differentiation of u w.r.t x is defined as differentiation of u w.r.t x treating y as constant and is denoted by $\frac{\partial u}{\partial x}$.

$$\therefore \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

Similarly partial differentiation of u w.r.t y is defined as differentiation of u w.r.t y treating x as constant and is denoted by $\frac{\partial u}{\partial y}$.

$$\therefore \frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

Second order partial derivatives:

In General, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are function of x and y. They can be further dfferentiated partially w.r.t x and y as follows.

(i) $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial^2 u}{\partial x^2}\right)$

(ii) $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)$

(iii) $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial^2 u}{\partial y\, \partial x}\right)$

(iv) $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial^2 u}{\partial y^2}\right)$

For all ordinary functions $\left(\frac{\partial^2 u}{\partial x\, \partial y}\right) = \left(\frac{\partial^2 u}{\partial y\, \partial x}\right)$.

WORKED EXAMPLES

PART – A

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following:

(i) $u = x + y$ (ii) $u = x^3 + y^3$ (iii) $u = e^{x+y}$ (iv) $u = e^{2x+3y}$

Solution:

i) $u = x + y$

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = 1$$

ii) $u = x^3 + y^3$

$$\frac{\partial u}{\partial x} = 3x^2$$

$$\frac{\partial u}{\partial y} = 3y^2$$

(iii) $u = e^{x+y}$

$$\frac{\partial u}{\partial x} = e^{x+y}(1) = e^{x+y}$$

$$\frac{\partial u}{\partial y} = e^{x+y}(1) = e^{x+y}$$

(iv) $u = e^{2x+3y}$

$$\frac{\partial u}{\partial x} = e^{2x+3y}(2) = 2e^{2x+3y}$$

$$\frac{\partial u}{\partial y} = e^{2x+3y}(3) = 3e^{2x+3y}$$

PART – B

(i) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following:

(i) $u = x^3 + 4x^2y + 5xy^2 + y^3$

(ii) $u = 5 \sin x + 4 \tan y$

(iii) $u = y^2 \sec x$

(iv) $u = \sin 4x \cos 2y$

Solution:

(i) $u = x^3 + 4x^2y + 5xy^2 + y^3$

$$\frac{\partial u}{\partial x} = 3x^2 + 8xy + 5y^2 + 0$$

$$\frac{\partial u}{\partial y} = 0 + 4x^2 + 10xy + 3y^2$$

(ii) $u = 5 \sin x + 4 \tan y$

$$\frac{\partial u}{\partial x} = 5\cos x + 0$$

$$\frac{\partial u}{\partial y} = 0 + 4\sec^2 y$$

(iii) $u = y^2 \sec x$

$$\frac{\partial u}{\partial x} = y^2 \sec x \tan x$$

$$\frac{\partial u}{\partial y} = \sec x.2y$$

(iv) $u = \sin 4x \cos 2y$

$$\frac{\partial u}{\partial x} = 4\cos 4x.\cos 2y$$

$$\frac{\partial u}{\partial y} = 2\sin 4x \sin 2y$$

2. Find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ for the following:

(i) $u = x^3 + y^3$ (ii) $u = x^3 \tan y$

Solution:

(i) $u = x^3 + y^3$

$$\frac{\partial u}{\partial x} = 3x^2 + 0$$

$$\frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = 0 + 3y^2$$

$$\frac{\partial^2 u}{\partial y^2} = 6y$$

(ii) $u = x^3 \tan y$

$$\frac{\partial u}{\partial x} = 3x^2 \tan y$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \tan y$$

$$\frac{\partial u}{\partial y} = x^3 \sec^2 y$$

$$\frac{\partial^2 u}{\partial y^2} = x^3 (2\sec y \,.\, \sec y \,\tan y)$$

$$= 2x^3 \sec^2 y \,.\, \tan y$$

3. Find $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ for $u = x^2e^{5y}$.

Solution:

$u = x^2e^{5y}$

$$\frac{\partial u}{\partial x} = 2xe^{5y}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 2x \,.\, (5e^{5y}) = 10xe^{5y}$$

$$\frac{\partial u}{\partial y} = 5x^2e^{5y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 10xe^{5y}$$

PART – C

1. If $u = \log(x^3 + y^3)$ find $\frac{\partial^2 u}{\partial x^2}$.

Solution:

$u = \log(x^3 + y^3)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y3}(3x^2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^3 + y^3)(6x) - (3x^2)(3x^2 + 0)}{(x^3 + y^3)^2} \qquad \text{(using quotient rule)}$$

$$= \frac{6xy^3 - 3x^4}{(x^3 + y^3)^2}$$

2. If $u = \frac{x}{y^2} - \frac{y}{x^2}$ find $\frac{\partial^2 u}{\partial x \partial y}$.

Solution:

$u = \frac{x}{y^2} - \frac{y}{x^2}$

$$\frac{\partial u}{\partial y} = x.\left(\frac{-2}{y^3}\right) - \frac{1}{x^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{-2}{y^3}\right) - \left(\frac{-2}{x^3}\right)$$

$$= \frac{2}{x^3} - \frac{2}{y^3}$$

3. If $u = x^3 + 3x^2y + 3xy^2 + y^3$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u$.

Solution:

$$u = x^3 + 3x^2y + 3xy^2 + y^3 \quad \text{.............(A)}$$

$$\frac{\partial u}{\partial x} = 3x^2 + 6xy + 3y^2 + 0$$

$$x\frac{\partial u}{\partial x} = x(3x^2 + 6xy + 3y^2)$$

$$x\frac{\partial u}{\partial x} = 3x^3 + 6x^2y + 3xy^2 \quad \text{............(1)}$$

$$u = x^3 + 3x^2y + 3xy^2 + y^3$$

$$\frac{\partial u}{\partial y} = 0 + 3x^2 + 3x(2y) + 3y^2$$

$$= 3x^2 + 6xy + 3y^2$$

$$y\frac{\partial u}{\partial y} = y(3x^2 + 6xy + 3y^2)$$

$$= 3x^2y + 6xy^2 + 3y^3 \quad \text{.............(2)}$$

$$(1) + (2) \Rightarrow$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3 + 6x^2y + 3xy^2 + 3x^2y + 6xy^2 + 3y^3$$

$$= 3x^3 + 9x^2y + 9xy^2 + 3y^3$$

$$= 3(x^3 + 3x^2y + 3xy^2 + y^3)$$

$$= 3u \text{ using (A)}$$

EXERCISE

PART – A

1. Find $\dfrac{dy}{dx}$ if

   (i) $y = (2x + 5)^3$ (ii) $y = \sqrt{ax + b}$

   (iii) $y = \sin 2x$ (iv) $y = \cos^3 x$

   (v) $y = \sin^2 5x$ (vi) $y = \tan\sqrt{x}$

   (vii) $y = \sin(\cos x)$ (viii) $y = \cos(\log x)$

   (ix) $y = \log(2x + 3)$ (x) $y = e^{x^2}$

2. Find $\dfrac{dy}{dx}$ if

   (i) $y = \sin^{-1}(2x)$ (ii) $y = \cos^{-1}(x^2)$ (iii) $y = \tan^{-1}(\sqrt{x})$ (iv) $y = \cos^{-1}(e^x)$

3. Find $\dfrac{dy}{dx}$ from the following equations

   (i) $y^2 = 4ax$ (ii) $2x^2 - y^2 - 9 = 0$ (iii) $x^3 + xy = 0$ (iv) $x \sin y = c$

4. If (i) $y = 2x^5 - 4x^2 + 3$

   (ii) $y = \sin^3 x$ (iii) $y = x + \cos x$ find $\dfrac{d^2y}{dx^2}$.

5. From the differential equation by eliminating the arbitrary constants from the following equations.

(i) $y = ax + b$ (ii) $y^2 = 4ax$ (iii) $y = mx$

(iv) $y = mx + \frac{1}{m}$ (v) $x^2 + y^2 = a^2$ (vi) $y = cx + c^3$

6. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ if $u = x^3 + 5x^2y + y^3$.

7. If $u = e^{x^2+y^2}$ find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

8. If $u = \tan(ax + by)$ find $\frac{\partial u}{\partial y}$.

9. If $u = \tan(ax + by)$ find $\frac{\partial u}{\partial x}$.

PART – B

1. Find $\frac{dy}{dx}$ if

(i) $y = \text{cosec}^3(5x + 1)$ (ii) $y = \tan(x^2 \log x)$

(iii) $y = \cos\left[\frac{\log x}{x}\right]$ (iv) $y = e^{\cos^3 x}$

2. Find $\frac{dy}{dx}$ if

(i) $y = \sqrt{x}\tan^{-1}x$ (ii) $y = x^3 + \tan^{-1}(x^2)$ (iii) $y = (\sin^{-1}x)^2$ (iv) $y = \cos^{-1}(4x^3 - 3x)$

3. If $x^2 - y^2 = 5y - 3x$ find $\frac{dy}{dx}$.

4. If $y^2 = x \sin y$ find $\frac{dy}{dx}$.

5. If $\sin y = x \sin(a + y)$ find $\frac{dy}{dx}$.

6. If $y = ae^x + be^{-x}$ that $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$.

7. If $y = \frac{\cos x}{x}$ prove that $xy_2 + 2y_1 + xy = 0$.

8. If $u = x^3 - 2x^2y + 3xy^2 + y^3$ find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$.

9. If $u = x^y$ find $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$.

10. If $u = \sqrt{x^2 + y^2}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

PART – C

1. Find $\frac{dy}{dx}$ if

(i) $y = e^{3x} \log x \sin 2x$ (ii) $y = \frac{1+\sin 2x}{1-\sin 2x}$

(iii) $y = \frac{\sin(\log x)}{x}$ (iv) $(3x^2 - 2)^2 \cot x \,.\, \log \sin x$.

2. Find $\frac{dy}{dx}$ if

(i) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ (ii) $y = x \sin^{-1}(\tan x)$ (iii) $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$

3. If $y = x^2 \sin x$ prove that $x^2y_2 - 4xy_1 + (x^2 + 6)\, y = 0$.

4. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the following functions:

(i) $u = \frac{x}{y^2} - \frac{y}{x^2}$ (ii) $u = x \sin y + y \sin x$ (iii) $u = \log\left(\frac{x^2 + y^2}{xy}\right)$

5. If $u = x^4 + 4x^3y + 3x^2y^2 + y^4$ find $x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2}$.

6. If $u = \log (x^2 + y^2)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$.

ANSWERS

PART – A

1. (i) $6\,(2x + 5)^2$ (ii) $\frac{a}{\sqrt[2]{ax+b}}$ (iii) $2 \cos 2x$ (iv) $-3 \cos^2 x \sin x$

(v) $10 \sin 5x \cos 5x$ (vi) $\frac{1}{2\sqrt{x}} \sec^2 \sqrt{x}$ (vii) $-[\cos (\text{cois } x)] \sin x$

(viii) $-\frac{1}{x}\sin(\log x)$ (ix) $\frac{2}{2x+3}$ (x) $2x\, e^{x^2}$

2. (i) $\frac{2}{\sqrt{1-4x^2}}$ (ii) $-\frac{2x}{\sqrt{1-4^2}}$ (iii) $\frac{1}{2\sqrt{x}(1+x)}$ (iv) $\frac{-e^x}{\sqrt{1-e^{2x}}}$

3. (i) $\frac{dy}{dx} = \frac{2a}{y}$ (ii) $\frac{dy}{dx} = \frac{2x}{y}$ (iii) $\frac{-(3x^2 + y)}{x}$ (iv) $\frac{-\tan y}{x}$

4. (i) $40x^3 - 8$ (ii) $3 \sin x\, [2 \cos^2 x - \sin^2 x]$ (iii) $-\cos x$

5. (i) $\frac{d^2y}{dx^2} = 0$ (ii) $y^2 = 2xy\frac{dy}{dx}$ (iii) $y = x\frac{dy}{dx}$ (iv) $y = x\frac{dy}{dx} + \frac{dx}{dy}$

(v) $x + y\frac{dy}{dx} = 0$ (vi) $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3$

6. $3x^2 + 10xy,\ 5x^2 + 3y^2$

8. $2x\, e^{x^2+y^2},\ 2y\, e^{x^2+y^2}$

9. $a \sec^2(ax + by)$

PART – B

1. (i) $15\ \text{cosec}^3(5x+1)\cot(5x+1)$ (ii) $x(1+2\log x)\sec^2(x^2\log x)$

   (iii) $\dfrac{-(1+\log x)}{x^2}\sin\left(\dfrac{\log x}{x}\right)$ (iv) $-3\cos^2 x\sin x\, e^{\cos^3 x}$

2. (i) $\dfrac{1}{2\sqrt{x}}\tan^{-1}x+\dfrac{\sqrt{x}}{1+x^2}$ (ii) $3x^2+\dfrac{2x}{1+x^4}$ (iii) $\dfrac{2\sin^{-1}x}{\sqrt{1-x^2}}$ (iv) $\dfrac{-3(4x^2-1)}{\sqrt{1-(4x^3-3x)^2}}$

3. $\dfrac{2x+3}{2y+5}$ 4. $\dfrac{\sin y}{2y-x\cos y}$ 5. $\dfrac{\sin(a+y)}{\cos y-x\cos(a+y)}$ 8. $6x-4y,\ 6x+6y$

9. $yx^{y-1},\ y(y-1)x^{y-2}$

PART – C

1. (i) $e^{3x}\left[2\log x\cos 2x+3\log x\sin 2x+\dfrac{\sin 2x}{x}\right]$ (ii) $\dfrac{4\cos 2x}{(1-\sin 2x)^2}$

   (iii) $\dfrac{\cos(\log x)-\sin(\log x)}{x^2}$

   (iv) $12x(3x^2-2)\cot x\log(\sin x)+(3x^2-2)^2\cot^2 x-(3x^2-2)\log(\sin x)\,\text{cosec}^2 x$

2. (i) $\dfrac{2}{1+x^2}$ (ii) $\dfrac{x\sec^2 x}{\sqrt{1-\tan^2 x}}$ (iii) $\dfrac{\sqrt{1-x^2}-x\sin^{-1}x}{(1-x^2)^{3/2}}$

10. $6(2x^4+4x^3y+2x^2y^2-xy^3+2y^4)$

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